Maxima and Minima

Relative Extrema

Relative maxima and minima (also called relative extrema) may exist wherever the derivative of a function is either equal to zero or undefined. However, these conditions are not sufficient to establish that an extreme exists; we must also have a change in the direction of the curve, i.e., from increasing to decreasing or from decreasing to increasing.

Note: relative extrema cannot exist at the endpoints of a closed interval.

First Derivative Test

If

- a function, f, is continuous on the open interval (a, b), and
- c is a critical number $\in (a, b)$ (i.e., $f'(c)$ is either zero or does not exist),
- f is differentiable on the open interval (a, b), except possibly at c,

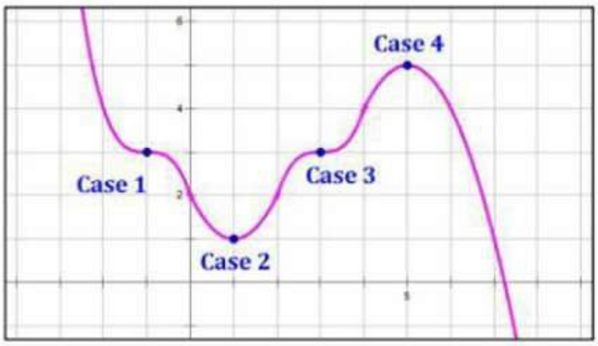
Then

- If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum.
- If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative minimum.

The conclusions of this theorem are summarized in the table below:

	First Derivative	**Sign of $\frac{dy}{dx}$ left of $x = c$**	**Sign of $\frac{dy}{dx}$ right of $x = c$**	**Type of Extreme**
Case 1	$\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.	$-$	$-$	None
Case 2		$-$	$+$	Minimum
Case 3		$+$	$+$	None
Case 4		$+$	$-$	Maximum

Illustration of First Derivative Test for Cases 1 to 4:



Second Derivative Test

If

- a function, f, is continuous on the open interval (a, b), and
- $c \in (a, b)$, and
- $f'(c) = 0$ and $f''(c)$ exists,

Then

- If $f''(c) < 0$, then $f(c)$ is a relative maximum.
- If $f''(c) > 0$, then $f(c)$ is a relative minimum.

The conclusions of the theorem are summarized in the table below:

	First Derivative	Second Derivative	Type of Extreme
Case 1	$\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.	$\frac{d^2y}{dx^2} < 0$	Maximum
Case 2		$\frac{d^2y}{dx^2} > 0$	Minimum
Case 3		$\frac{d^2y}{dx^2} = 0$ or does not exist	Test Fails

In the event that the second derivative is zero or does not exist (Case 3), we cannot conclude whether or not an extreme exists. In this case, it may be a good idea to use the First Derivative Test at the point in question.

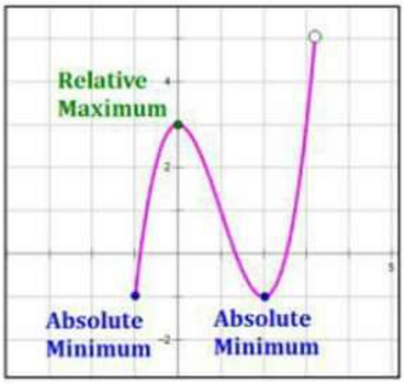
Absolute Extrema

Absolute extrema (also called "global extrema" or simply "extrema") are the highest or lowest values of the function on the interval in question. If a function is continuous, its absolute extrema exist at the locations of either its relative extrema or the endpoints of the interval.

Note that if an interval is open, the endpoint does not exist and so it cannot be an absolute extreme. This means that in some cases, a function will not have an absolute maximum or minimum on the interval in question. Discontinuities in a function can also cause a function to not have a relative maximum or minimum.

A function may have 0, 1 or multiple absolute maxima and/or absolute minima on an interval. In the illustration to the right, the function has:

- Two absolute minima, at $(-1, -1)$ and $(2, -1)$.
- No absolute maximum (due to the discontinuity).
- One relative maximum, at $(0, 3)$.
- One relative minimum – The point located at $(2, -1)$ is both a relative minimum and an absolute minimum.



Inflection Points

Definition

An *inflection point* is a location on a curve where concavity changes from upward to downward or from downward to upward.

At an inflection point, $f''(x) = 0$ or $f''(x)$ does not exist.

However, it is not necessarily true that if $f''(x) = 0$, then there is an inflection point at $x = c$.

Testing for an Inflection Point

To find the inflection points of a curve in a specified interval,

- Determine all x-values ($x = c$) for which $f''(c) = 0$ or $f''(c)$ does not exist.
- Consider only c-values where the function has a tangent line.
- Test the sign of $f''(x)$ to the left and to the right of $x = c$.
- If the sign of $f''(x)$ changes from positive to negative or from negative to positive at $x = c$, then $(c, f(c))$ is an inflection point of the function.

	Second Derivative	Sign of $\frac{d^2y}{dx^2}$ left of $x = c$	Sign of $\frac{d^2y}{dx^2}$ right of $x = c$	Inflection Point?
Case 1	$\frac{d^2y}{dx^2} = 0$ or $\frac{d^2y}{dx^2}$ does not exist	$-$	$-$	No
Case 2		$-$	$+$	Yes
Case 3		$+$	$+$	No
Case 4		$+$	$-$	Yes

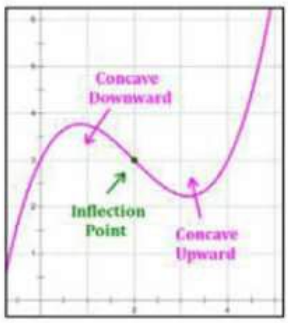
Note: inflection points cannot exist at the endpoints of a closed interval.

Concavity

A function, f, is *concave upward* on an interval if $f'(x)$ is increasing on the interval, i.e., if $f''(x) > 0$.

A function, f, is *concave downward* on an interval if $f'(c)$ is decreasing on the interval, i.e., if $f''(x) < 0$.

Concavity changes at inflection points, from upward to downward or from downward to upward. In the illustration at right, an inflection point exists at the point $(2, 3)$.



Special Case: Extrema and Inflection Points of Polynomials

For a polynomial, $f(x)$, critical values of exist at all x-values for which $f'(x) = 0$. However, critical values do not necessarily produce extrema. Possible inflection points exist at all x-values for which $f''(x) = 0$. However, not all of these x-values produce inflection points.

To find the extrema and inflection points of a polynomial we can look at the factored forms of $f'(x)$ and $f''(x)$, respectively. Every polynomial can be factored into linear terms with real roots and quadratic terms with complex roots as follows:

$$P(x) = k(x - r_1)^{a_1} \cdot (x - r_2)^{a_2} \dots (x - r_n)^{a_n} \cdot Q_1(x) \cdot Q_2(x) \dots Q_m(x)$$

where, k is a scalar (constant), each r_i is a real root of $f(x)$, each exponent a_i is an integer, and each Q_j is a quadratic term with complex roots.

Extrema

The exponents (a_i) of the linear factors of $f'(x)$ determine the existence of extrema.

- An odd exponent on a linear term of $f'(x)$ indicates that $f'(x)$ crosses the x-axis at the root of the term, so $f(x)$ has an extreme at that root. Further analysis is required to determine whether the extreme is a maximum or a minimum.
- An even exponent on a linear term of $f'(x)$ indicates that $f'(x)$ bounces off the x-axis at the root of the term, so $f(x)$ does not have an extreme at that root.

Example 3.1: Consider $f'(x) = (x+3)^3(x+2)^2\left(x+\sqrt{3}\right)^3\left(x-\sqrt{3}\right)^3(x-4)^2(x-7)$.

The original polynomial, $f(x)$, has critical values for each term: $CV = \{-3, -2, -\sqrt{3}, \sqrt{3}, 4, 7\}$.

However, extrema exist only for the terms with odd exponents: $Extrema = \{-3, -\sqrt{3}, \sqrt{3}, 7\}$.

Inflection Points (PI)

The exponents (a_i) of the linear factors of $f''(x)$ determine the existence of inflection points.

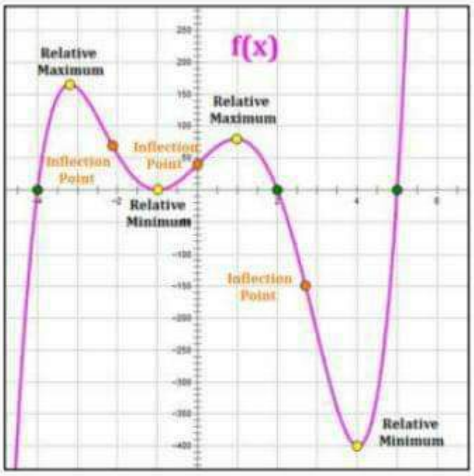
- An odd exponent on a linear term of $f''(x)$ indicates that $f(x)$ has an inflection point at the root of that term.
- An even exponent on a linear term of $f''(x)$ indicates that $f(x)$ does not have an inflection point at the root of that term.

Example 3.2: Consider $f''(x) = (x+3)^3(x+2)^2\left(x+\sqrt{3}\right)^3\left(x-\sqrt{3}\right)^3(x-4)^2(x-7)$.

Inflection points exist only for the terms with odd exponents: $PI = \{-3, -\sqrt{3}, \sqrt{3}, 7\}$.

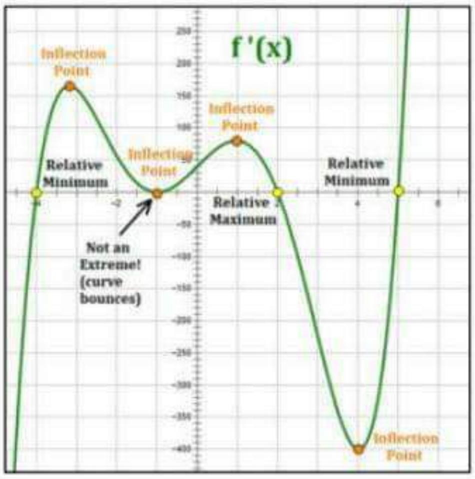
Key Points on $\mathbf{f(x)}$, $\mathbf{f'(x)}$ and $\mathbf{f''(x)}$ – Alauria Diagram

An **Alauria Diagram** shows a single curve as $f(x)$, $f'(x)$ or $f''(x)$ on a single page. The purpose of the diagram is to answer the question: If the given curve is $f(x)$, $f'(x)$ or $f''(x)$, where are the key points on the graph.



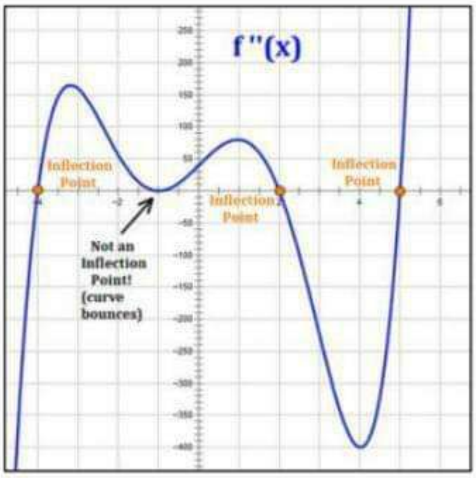
If the curve represents $f(x)$:

- $f(x)$'s x-intercepts (green and one yellow) exist where the curve touches the x-axis.
- Relative maxima and minima (yellow) exist at the tops and bottoms of humps.
- Inflection points (orange) exist where concavity changes from up to down or from down to up.



If the curve represents $f'(x)$ (1st derivative):

- $f(x)$'s x-intercepts cannot be seen.
- Relative maxima and minima of $f(x)$ (yellow) exist where the curve crosses the x-axis. If the curve bounces off the x-axis, there is no extreme at that location.
- Inflection points of $f(x)$ (orange) exist at the tops and bottoms of humps.



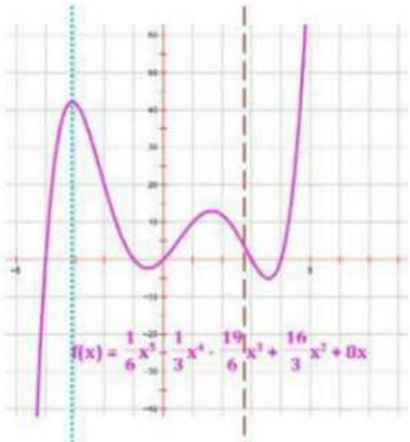
If the curve represents $f''(x)$ (2nd derivative):

- $f(x)$'s x-intercepts cannot be seen.
- Relative maxima and minima of $f(x)$ cannot be seen.
- Inflection points of $f(x)$ (orange) exist where the curve crosses the x-axis. If the curve bounces off the x-axis, there is no inflection point at that location.

Key Points on $\mathbf{f(x), f'(x)}$ and $\mathbf{f''(x)}$

The graphs below show $f(x)$, $f'(x)$ or $f''(x)$ for the <u>same</u> 5th degree polynomial function. The dotted blue vertical line identifies one location of an extreme (there are four, but only one is illustrated) The dashed dark red vertical line identifies one location of a point of inflection (there are three, but only one is illustrated).

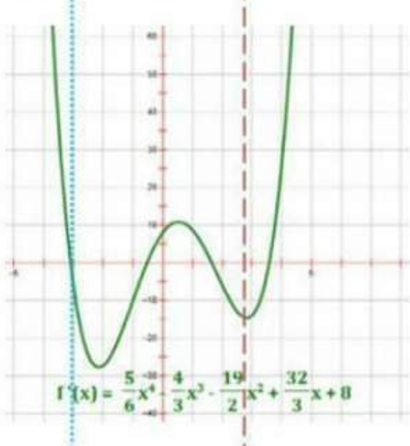
$f(x)$



In a graph of $f(x)$:

- Relative extrema exist at the tops and bottom of humps.
- Inflection points exist at locations where concavity changes from up to down or from down to up.

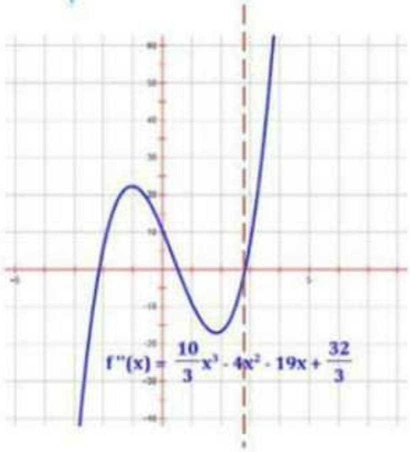
$f'(x)$



In a graph of $f'(x)$:

- Relative extrema of $f(x)$ exist where the curve <u>crosses</u> the x-axis. If the curve bounces off the x-axis, there is no extreme at that location.
- Inflection points of $f(x)$ exist at the tops and bottoms of humps.

$f''(x)$



In a graph of $f''(x)$:

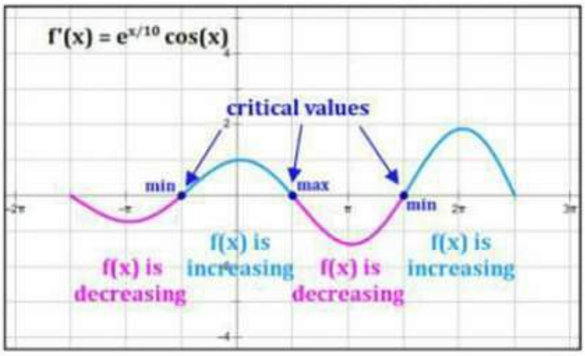
- Relative extrema of $f(x)$ cannot be seen.
- Inflection points of $f(x)$ exist where the curve <u>crosses</u> the x-axis. If the curve bounces off the x-axis, there is no inflection point at that location.

What Does the Graph of $f'(x)$ Tell Us about $f(x)$?

Short answer: a lot! Consider the graph of the derivative of $f(x)$ when $\boldsymbol{f'(x) = e^{x/10} \cdot \cos x}$ on the interval $\left[-\frac{3}{2}\pi, \frac{5}{2}\pi\right]$.

Increasing vs. Decreasing

We can tell if $f(x)$ is **increasing** or **decreasing** based on whether $f'(x)$ is **positive** or **negative**. Critical values exist where $f'(x)$ is zero or does not exist. Relative maxima and minima exist at critical values if the graph of $f'(x)$ crosses the x-axis. See the graph and chart below. *Note that critical values, relative maxima and relative minima do not exist at endpoints of an interval.*

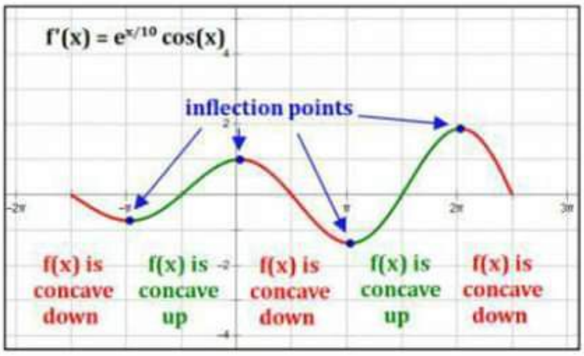


Increasing vs. Decreasing		
$f(x)$	Increasing	Decreasing
$f'(x)$	Positive	Negative

All items in a column occur simultaneously.

Concavity

We can tell if $f(x)$ is **concave up** or **concave down** based on whether $f'(x)$ is **increasing** or **decreasing**. Inflection Points exist at the extrema of $f'(x)$, i.e. at the top and bottom of any humps on the graph of $f'(x)$. See the graph and chart below. *Note that inflection points do not exist at endpoints of an interval.*



Concavity		
$f(x)$	Concave up	Concave down
$f'(x)$	Increasing	Decreasing
$f''(x)$	Positive	Negative

All items in a column occur simultaneously.