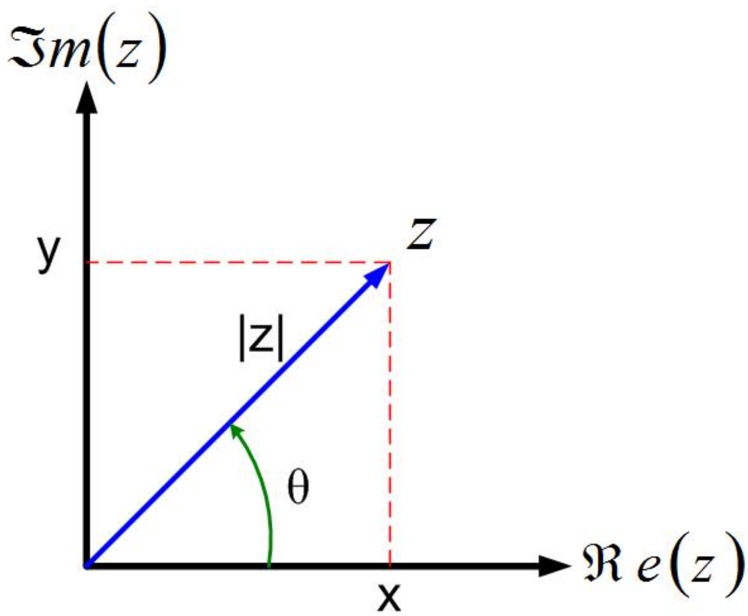
HL: Complex numbers notes and guide



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1 Complex Numbers

1.1 Complex numbers into and its algebra

Complex numbers are numbers of the form $z = a + \mathrm{i}b$, where $a, b \in \mathbb{R}$.

a is called the real part of z, and we write $\mathrm{Re}(z) = a$. b is called the imaginary part of z, and we write $\mathrm{Im}(z) = b$. The set of complex numbers is denoted by $\mathbb{C}$. Given the complex number $z = x + \mathrm{i}y, x, y \in \mathbb{R}$, the modulus of z is given by

$$|z| = |x + \mathrm{i}y| = \sqrt{x^2 + y^2} = \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}.$$

Two complex numbers $z_1 = a + b\mathrm{i}$ and $z_2 = c + d\mathrm{i}, a, b, c, d \in \mathbb{R}$, are equal if and only if their real parts are equal and their imaginary parts are equal, $a = c$ and $b = d$.

Addition and multiplication by a scalar:

$$z_1 + z_2 = (a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}, \quad a, b, c, d \in \mathbb{R}$$
$$\lambda z = \lambda(a + b\mathrm{i}) = (\lambda a) + (\lambda b)\mathrm{i}, \quad \lambda, a, b \in \mathbb{R}$$

Multiplication of complex numbers:

$$z_1 \times z_2 = (a + b\mathrm{i}) \times (c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}, \quad a, b, c, d \in \mathbb{R}$$

For any complex number $z = a + b\mathrm{i}, a, b \in \mathbb{R}$, there is a **conjugate** complex number of the form $z* = a - b\ \mathrm{i}$. Their real parts are equal, $\mathrm{Re}(z) = \mathrm{Re}(z^*)$, and their imaginary parts are opposite, $\mathrm{Im}(z) = -\mathrm{Im}(z^*)$. Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1 \times z_2^*}{|z_2|^2} \quad \text{Powers of i :} \quad \mathrm{i}^n = \begin{cases} 1, n = 4k \\ \mathrm{i}, n = 4k + 1 \\ -1, n = 4k + 2 \\ -\mathrm{i}, n = 4k + 3 \end{cases}, k \in \mathbb{Z}$$

1.2 Polar and Euler forms of complex numbers and De Moivre's Theorem

Cartesian form of a complex number

$$z = x + y\mathrm{i}, \quad x, y \in \mathbb{R}$$

Modulus-argument or polar form of a complex number Given a complex number with the modulus $r, r \in \mathbb{R}, r \geq 0$ and the argument $\theta, \theta \in \mathbb{R}$

$$z = r \operatorname{cis} \theta.$$

Euler's or exponential form of a complex number

Given a complex number with the modulus $r, r \in \mathbb{R}, r \geq 0$ and the argument $\theta, \theta \in \mathbb{R}$

$$z = r\mathrm{e}^{\mathrm{i}\theta}.$$

- Relationship between the rectangular and polar coordinates

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan \frac{y}{x} \end{cases} \Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Two equal complex numbers

Given $z_1 = r_1 \operatorname{cis} \theta_1 = r_1 \mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = r_2 \operatorname{cis} \theta_2 = r_2 \mathrm{e}^{\mathrm{i}\theta_2}$, if $z_1 = z_2$ then $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi, k \in \mathbb{Z}$.

Conjugate, opposite and conjugate opposite complex numbers

$$z = r\operatorname{cis}\theta \Rightarrow \begin{cases} -z = r\operatorname{cis}(\pi + \theta) \\ z^* = r\operatorname{cis}(\pi - \theta) \\ -z^* = r\operatorname{cis}(2\pi - \theta) \end{cases}$$

Multiplication

Given $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, if $z_1 \times z_2 = z_3 = r_3 \operatorname{cis} \theta_3$ then $r_3 = r_1 r_2$ and $\theta_3 = \theta_1 + \theta_2 + 2k\pi, k \in \mathbb{Z}$

You multiply two complex numbers in polar form by multiplying the moduli and adding the arguments.

Reciprocal number

Given the complex number in polar form $z = r\operatorname{cis}\theta$ then the reciprocal value in polar form is

$\frac{1}{z} = \frac{1}{r}\operatorname{cis}(2\pi - \theta) = \frac{1}{r}\operatorname{cis}(-\theta)$.

The reciprocal complex number of a given complex number in polar form has the reciprocal modulus and the opposite argument.

Division

Given the two complex numbers $z_1 = r_1\operatorname{cis}\theta_1$ and $z_2 = r_2\operatorname{cis}\theta_2, z_2 \neq 0$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2), z_2 \neq 0, r_2 \neq 0$

You divide two complex numbers in polar form by dividing the moduli and subtracting the arguments.

De Moivre's theorem

Given a complex number in polar or Euler's form $z = r\operatorname{cis}\theta = re^{\mathrm{i}\theta}, z \neq 0$, $r \in \mathbb{R}^+, \theta \in \mathbb{R}$ then

$z^n = r^n\operatorname{cis}n\theta = r^n e^{\mathrm{i}n\theta}, n \in \mathbb{Z}$.

Roots of complex numbers

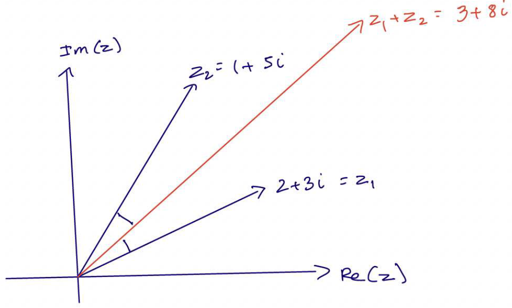
Given the complex number $z = r\operatorname{cis}\theta, r, \theta \in \mathbb{R}, r \geq 0$ then

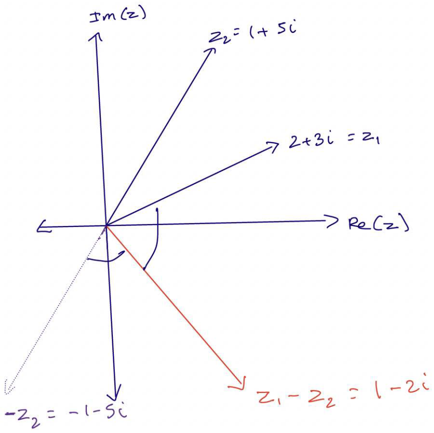
$\sqrt[n]{z} = \sqrt[n]{r}\operatorname{cis}\frac{\theta + 2k\pi}{n}, k = 0, 1, 2, \ldots, n-1$

2 Important points to know

1. Vector representation of the operations of complex numbers

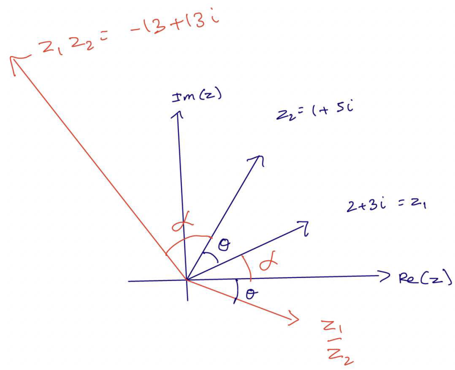
Addition and subtraction of complex numbers can be seen as vector addition and subtraction as well





As you can see above, the resultant vector or complex numbers is right in the middle of the 2 original complex numbers.

Multiplication and division also can be visualised on an Argand diagram.



As you can see above

The product will have and argument as the sum of the arguments of the two complex numbers

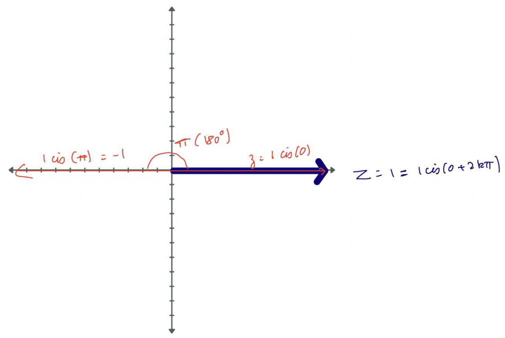
The Division of the 2 complex numbers can also be found. the argument of the ration will be the TOP argument minus the bottom argument.

The product and division diagram also confirms the De Moivre's Theorem.

2. N^{th} root of a number (either real or complex) will be symmetrically apart on an Argand diagram.

Using the Polar or euler form, finding roots become easier. the following examples will make it clear:

For z^2=1, the 2 roots can be visualised using Argand diagram. The 2 roots (both real) are as far from each other as possible making the angle between them 180 degrees or π radians apart.



Of course you don't need to use De moivre's rule or Argand diagram for square root of 1, but the point is that it works. And it works for all roots of 1 or any other number (complex or real) the following examples of other roots of 1 will make it clearer: