Calculus Test 2

Test number 2: Calculus

Topics Covered

1. Entire calculus AA HL syllabus

2. Paper 3 practice test

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1 Paper 1 and paper 2 questions

1. a Write $\frac{2x+5}{x^2-x-2}$ in the form $\frac{A}{x-2}-\frac{B}{x+1}$.

b) Hence find the exact value of

$$\int_3^9 \frac{2x+5}{x^2-x-2}dx$$

c) Use the substitution $x = 2\sec u$ to find $\int \frac{1}{x\sqrt{x^2-4}}\,\mathrm{d}x$.

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2. a) Show that the substitution $y = vx$ transforms the differential equation

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \sqrt{\frac{y}{x}}$

into the equation

$x\frac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{v}$.

b) Hence solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \sqrt{\frac{y}{x}}$ given that $y = 0$ when $x = 1$.

Write y in terms of x.

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3. Variables x and y satisfy the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + 2y$, and $y(0) = A$.

Write $y = \sum_{k=0}^{\infty} a_k x^k$.

a) Express a_1 and a_2 in terms of A and show that, for $k \geq 2, a_{k+1} = \frac{2}{k+1}a_k$.

b) Hence find the Maclaurin series expansion of y.

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4. a) Use l'Hôpital's rule to evaluate

$$\lim_{x\to 0} \frac{e^x - 1}{\sin(2x)}$$

b) Show that $\frac{\ln(x-2)}{\tan\left(\frac{\pi x}{4}\right)}$ diverges to infinity when $x \to 2$.

c) Use Maclaurin series to evaluate

$$\lim_{x\to 0} \frac{\cos(3x)-1}{2x^2}$$

5*. A particle moves such that its velocity v ms^{-1} is related to its displacement s m, by the equation $v(s) = \arctan(\sin s), 0 \leq s \leq 1$. The particle's acceleration is a ms^{-2}.

(a) Find the particle's acceleration in terms of s.

(b) Using an appropriate sketch graph, find the particle's displacement when its acceleration is 0.25 ms^{-2}.

6*. The functions f and g are defined by

$$f(x) = \frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$$
$$g(x) = \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$$

(a) (i) Show that $\frac{1}{4f(x)-2g(x)} = \frac{e^x}{e^{2x}+3}$.

(ii) Use the substitution $u = e^x$ to find $\int_0^{\ln 3} \frac{1}{4f(x)-2g(x)} dx$. Give your answer in the form $\frac{\pi\sqrt{a}}{b}$ where $a, b \in \mathbb{Z}^+$.

Let $h(x) = nf(x) + g(x)$ where $n \in \mathbb{R}, n > 1$.

(b) (i) By forming a quadratic equation in e^x, solve the equation $h(x) = k$, where $k \in \mathbb{R}^+$.

(ii) Hence or otherwise show that the equation $h(x) = k$ has two real solutions provided that $k > \sqrt{n^2-1}$ and $k \in \mathbb{R}^+$.

Let $t(x) = \frac{g(x)}{f(x)}$

(c) (i) Show that $t'(x) = \frac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ for $x \in \mathbb{R}$.

(ii) Hence show that $t'(x) > 0$ for $x \in \mathbb{R}$.

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2 Paper 3- GDC allowed

7. In this question we consider series of the form $\sum_{n=1}^{N} nr^n$.

a) Evaluate $\sum_{n=1}^{4} n \times 3^n$.

b) Find the derivative with respect to r of $\sum_{n=1}^{5} r^n$.

c) i) Show that $\sum_{n=1}^{N} nr^n - r\sum_{n=1}^{N} nr^n = \sum_{n=1}^{N} r^n - Nr^{N+1}$.

ii) Hence or otherwise, show that $\sum_{n=1}^{N} nr^n = \frac{r-(N+1)r^{N+1}+Nr^{N+2}}{(1-r)^2}$.

iii) Use this formula to verify your answer to **a**.

d) i) Show that $\frac{d}{dr}\left(\sum_{n=1}^{N} r^n\right) = \sum_{n=1}^{N} nr^{n-1}$.

ii) Explain why $\sum_{n=1}^{N} nr^n = r\frac{d}{dr}\left(\frac{r-r^{N+1}}{1-r}\right)$.

iii) Hence establish the formula for $\sum_{n=1}^{N} nr^n$ found in c(ii).

e) Use mathematical induction to show that $\sum_{n=1}^{N} nr^n = \frac{r-(N+1)r^{N+1}+Nr^{N+2}}{(1-r)^2}$ for all $N \in \mathbb{Z}^+$.

f) Use l'Hôpital's rule to find $\lim_{r\to 1}\left(\frac{r-(N+1)r^{N+1}+Nr^{N+2}}{(1-r)^2}\right)$.

Verify that this result is consistent with the value of $\sum_{n=1}^{N} n$

g) Now consider the infinite series $\sum_{n=1}^{\infty} nr^n$.

i) Show that $\sum_{n=1}^{\infty} nr^n$ can be written as the sum of the infinite geometric series

$$\left(r+r^2+r^3+r^4+r^5+\dots\right)+\left(r^2+r^3+r^4+r^5+r^6+\dots\right)+\left(r^3+r^4+r^5+r^6+r^7+\dots\right)+\dots$$

ii) Hence show that, for $|r|<1, \sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}$.

iii) Evaluate $\frac{1}{\sqrt{2}}+\frac{2}{2}+\frac{3}{2\sqrt{2}}+\frac{4}{4}+\frac{5}{4\sqrt{2}}+\dots$ Write your answer in the form $a+b\sqrt{2}$, where $a,b \in \mathbb{Z}$.

iv) Suppose the third term of $\sum_{n=1}^{\infty} nr^n$ is the largest in the series.

Find the range of possible values that $\sum_{n=1}^{\infty} nr^n$ can take.