

Life, the universe and the hidden meaning of everything

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It is hard to look at the universe and not wonder about the meaning, of, well, everything. A natural question is whether what we see is a sign of intelligent design. The antithesis of design would be a random universe or, assuming laws of physics, one whose fundamental physical parameters were randomly selected, but conditioned on life (ourselves) being here to observe it. In unpublished work, the British physicist Dennis Sciama argued that such a randomly selected universe would display a statistical signature. He concluded that a random universe would almost certainly have parameters only just allowing for the possibility of life. Here we consider whether this signature is definitive. We find that with plausible additional assumptions Sciama's signature would appear to reverse: Were our universe random, it could give the false impression of being intelligently designed, with the fundamental constants appearing to be fine tuned to a strong probability for life to emerge and be maintained.

Whatever might be eventually concluded about a universal definition for life, we can certainly agree that the universe we inhabit has so far supported the emergence, evolution and continued sustenance of human beings. Despite our having grown collectively more powerful than most known species, within the universe we are very fragile and maintain a precarious hold on existence. We are carbon based, requiring a planetary body to live on, which follows a comfortable and steady orbit around a single and not too energetic star.

These constraints already place tight bounds on the fundamental constants of the universe. To ensure that a population of yellow dwarf stars like our sun exist, the fine structure constant must be tuned to within a percent or two of its current value (outside this narrow range almost all stars would either be blue giants, or red dwarfs).¹ The cosmological constant must be between 120-124 orders-of-magnitude smaller than its naive quantum field-theoretic value. The upper bound ensures that bodies can form gravitationally^{2,3} and the lower bound ensures that nascent life will not be extinguished by proximity to gamma ray bursts.⁴ To ensure that nuclear reactions within stars can form carbon, but not have the process bypassed leaving only oxygen, a remarkable set of coincidences is required among the fundamental constants so that one resonant energy level exists, yet another level just fails to be resonant.⁵ Many of the fundamental constants therefore seem to be boxed into a narrow range of values compatible with our existence.⁶ Of course, a less anthropocentric view would considerably broaden this range.⁷

How can we understand our being in such a human-compatible universe? It has been suggested⁸ that the fundamental constants may have been selected 'randomly' among all possible values. If that were the case, then such compatibility is merely a condition consistent with our being here to observe it. The conditional probability for such a fact would be one. Equivalently, if these

'random' selections were individual universes within a multiverse, then our universe being human-compatible would be the same as us being located in one of the universes within the multiverse where humans are possible. Such 'explanations' are said to invoke the weak anthropic principle,⁶ yet they explain nothing and fail to provide any real resolution. Are they at least predictive?

Dennis Sciama, considered to be one of the fathers of modern cosmology, argued that were our universe random it would almost certainly have a low probability for life as we know it.⁹

Sciama assumed that the feature distinguishing different potential universes was the set of specific values taken by the fundamental constants; the underlying physical laws themselves being fixed. We can then envision the human-compatible universes as an 'island' within a 'sea' of more general possibilities. Each point on the island or in the sea describes a unique universe that is described by a distinct set of fundamental constants. The dimensionality of this space of points is naively given by the number of fundamental constants. Thus the human-compatible island of universes corresponds to some shape in a high dimensional space. The shoreline of the island corresponds to the boundary separating universes with a chance for human life to form from those where this is impossible. Thus, the shoreline itself will be made up of universes with an exactly vanishing probability for such life. Assuming continuity, as one moves inland, this probability will increase, reaching a maximum presumably somewhere far in from the shoreline.

This probability landscape is different from the chance of randomly selecting a universe. Because the range of parameters consistent with human life is quite small, one might expect any smooth measure for randomly selecting universes, to be approximately uniform across the island. Sciama's argument now follows from a well-known concentration-of-measure phenomenon.¹⁰ For any uniform-density shape in a high-dimensional space, the

weight is almost entirely concentrated within a thin layer at the surface. Figuratively, a high-dimensional elephant is essentially all skin. Applied to the high-dimensional island of human-compatible universes, a randomly selected universe will then almost certainly be found in a narrow band on the shore, where the probability for life would be expected to be low, since it vanishes at the shoreline.

This prediction is in contrast to that of intelligent design where one might expect a universe further inland closer to, or possibly achieving, the greatest chance for human life.

Is this space of universes really high dimensional? In 1936 Eddington counted four fundamental constants.¹¹ This count excludes Newton’s gravitational constant, the speed of light, Planck’s constant and the permittivity of the vacuum, all used to provide scales for dimensional quantities like length, time, mass and electric charge.¹¹ Just a few years ago this count had grown to 26 for the ‘standard model’ including the cosmological constant for gravity.¹² Today, if we add three neutrino masses, the count would be 29. However, our current model of the universe hardly explains everything. There remain numerous long-standing open questions, many cosmological in nature, such as matter-antimatter asymmetry, dark matter, dark energy and more. Thus it would be surprising if the total number of the fundamental constants in a complete theory of the universe were not much larger.

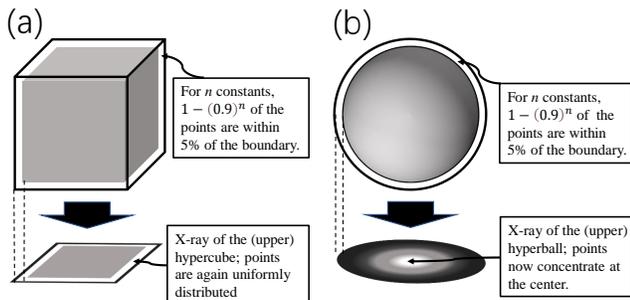


FIG. 1: Random sampling from the human-compatible island can look different dependent on its shape, if one has only limited access to the fundamental constants. The high-weight region (typically the shoreline) is shown in white, with the remaining low-weight contribution in gray. The island of the accessible fundamental constants is obtained by integrating out those constants which are unknown or unobserved; this is visualized as an ‘X-ray’ of the actual island. (a) For an island which is an n -dimensional hypercube (upper), an X-ray reduces to a uniform-measure hypercube in a lower dimension (lower). (b) For an island which is a uniformly-distributed n -dimensional ball (upper) with many unknown constants, its X-ray is well approximated by a narrow Gaussian concentrated at the center of the human-compatible island (lower).

Although it did not figure into Sciamia’s original argument, we shall see that the shape of the island plays a crucial role in the possible apparent reversal of Sciamia’s conclusion. Note that the ‘elephant skin’ result itself is essentially independent of this shape, which follows simply from the scaling of the ‘hypervolume’ with dimensionality. Thus, there is no question about a randomly

selected universe compatible with human life having a set of fundamental constants that almost certainly lie on the narrow shore, with a low chance for life like us.

Notwithstanding this, where on the island the universe appears to lie can depend on the island’s shape. This can be the case whenever our knowledge of the list of fundamental constants is incomplete. In this case, we would consider the island and its surrounding sea to be a lower-dimensional space than it actually is. Our view of the island would be one that projects out the unknown constants. This may be visualized as an ‘X-ray’ of the actual n -dimensional island onto a lower m -dimensional island describing the known constants. See Fig. 1.

In the case that the island has the shape of a uniform-weight n -dimensional cube (a hypercube), with independent bounds on each constant, the X-ray is simply a lower-dimensional uniform-weight hypercube. See Fig. 1(a). Again the lower-dimensional shore contains the greatest weight. However, for a uniform-weight hyperball shaped island, the X-ray, integrating out many dimensions, leads to a narrow Gaussian with the weight concentrated at the center of the island, far inland from the shoreline (see Appendix). See Fig. 1(b).

Surprisingly, the result we find for a hyperball-shaped island shown in Fig. 1(b) may well be the generic result. Indeed, the coincidences found among the fundamental constants when determining the human-compatible island’s shoreline,^{5,6,15} suggest that sharp corners or edges are unlikely, leaving a smooth and potentially convex island. Further, the projective central limit theorem^{13,14} ensures that the projection of any such high n -dimensional uniform-weight shape will be well approximated by a Gaussian with variance scaling as $1/n$.

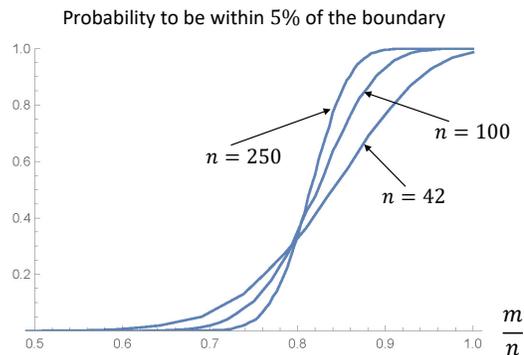


FIG. 2: Probability for the m accessible constants of a random universe to be within 5% of the life-denying boundary, versus the fraction m/n . The calculation assumes the island has the shape of a hyperball with n fundamental constants, though only m are known about. We consider $n \in \{42, 100, 250\}$. In each case, unless at least 80% of the total number of constants are accessible ($m/n \geq 0.8$), the chance of being near the boundary is less than $\simeq 0.35$.

However, the projective central limit theorem only tells us that the distribution is peaked far inland from the accessible (projected) shoreline. What about the probability for the accessible parameters nevertheless being found

on the projected shore near the boundary? In Fig. 2 we compute the probability to be within 5% of the projected shoreline versus the fraction of accessible constants, m/n , when an n -dimensional hyperball is projected down to m dimensions. For $n \in \{42, 100, 250\}$, we see that if less than 80% of the total number of fundamental constants are accessible, then the chance of being near the boundary is less than around 0.35. Thus, taking as null hypothesis that our universe is random, there would be a low chance for finding the fundamental constants of the universe to be near the shoreline until we had knowledge of the vast majority of all the universe’s parameters.

In summary, Sciama’s reasoning suggests that were our universe random, there would be a statistical signature on the fundamental constants and their location within the island of human-compatible universes.

One might view Sciama’s result to be solely that a random universe would lead to a scenario where life as we know it is only barely possible (in that it will almost certainly be close to the human life-denying boundary). This ‘skin of the elephant’ argument stands firm and may even explain the apparent scarcity of life in the universe, potentially resolving Fermi’s paradox.⁹

However, when our knowledge of the fundamental constants is incomplete, we have shown that the signature for a random universe can be reversed. For example, were our universe random with 42 fundamental constants, and taking the human-compatible island of parameters to have the shape of a hyperball, there would be only $\simeq 5.5\%$ chance of the set of 29 currently known constants to lie within 5% of the boundary where human

life becomes impossible. Instead, the greatest likelihood would be to find these known constants to be far within the human-compatible island of universes, mimicking a universe built by intelligent design.

If we consider the intelligent design of the universe as an artful act, whatever else it might be, then we have uncovered a mechanism whereby even a random universe may appear artful; or loosely speaking, whereby even an atheist might say “life imitates art.”¹⁶ Recalling that our analysis is based on concentration-of-measure phenomena in high-dimensional spaces, it is natural to ask whether this mechanism for imitating art may not have wider application. For example, could one enhance the artfulness in say a computer-generated piece of ‘art’ to make it mimic a work of art; not by slavish copying, but perhaps by so constraining the work around with interconnections and correlations – the coincidences constraining our fundamental constants – that one may begin to find it harder to distinguish between such a computer-generated piece and an intelligently crafted work? For something complex, having a sufficient number of working parts, it may not even be necessary to hide part of the work; or as with our analysis here, it may be crucial to first build the work and then make large portions inaccessible; be they backstory, foundation, milieu, history, whatever. And if anything like this can succeed, why not look further, perhaps toward the imitation of intelligence itself. Maybe the magic behind creating meaning has simply been a matter of hiding much of the supporting artifice from the audience, and even from ourselves.

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¹⁵ A. S. Eddington, *The Philosophy of Physical Science* (Cambridge Univ. Press, Cambridge, 1939).
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APPENDIX

Consider a smooth convex n -dimensional geometric body with a uniform probability distribution across and within it. The projective central limit theorem claims that as $n \rightarrow \infty$, if we project such a body to lower dimensions, the probability will concentrate to a ‘center’ of the lower dimensional object.^{13,14,17} This phenomenon has been proved for all smooth convex geometric bodies

and the limiting probability distribution is claimed to be a Gaussian distribution with variance scaling as $1/n$.

Here we compute this exactly for an $(n-1)$ -sphere (the surface of an n -dimensional ball) projected to m dimensions. We find for large $n-m$ that the resulting distribution is Gaussian with variance scaling as $1/n$. This is in agreement with the more general, though looser claim of the projective central limit theorem when $n \gg m$. Finally, combining the ‘skin of the elephant’ concentration of measure result, we argue that the same limit to a Gaussian with variance scaling as $1/n$ will hold for the projection of an n -dimensional ball onto m dimensions.

Because the general calculation is rather complicated, we start with the simpler case of projecting an $(n-1)$ -sphere onto a single dimension.

Projection of an $(n-1)$ -sphere to one dimension

An $(n-1)$ -sphere with unit radius in n -dimensional Euclidean space (Cartesian coordinates) may be described

as satisfying $x_1^2 + x_2^2 + \dots + x_n^2 = 1$. It may be transformed into the hyper-spherical coordinate by

$$\begin{aligned} x_1 &= -\cos \varphi_1 \\ x_2 &= -\sin \varphi_1 \cos \varphi_2 \\ &\vdots \\ x_{n-1} &= -\sin \varphi_1 \sin \varphi_2 \cdots \cos \varphi_{n-1} \\ x_n &= -\sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-1}, \end{aligned} \quad (1)$$

where $\varphi_1, \varphi_2, \dots, \varphi_{n-2} \in [0, \pi]$ and $\varphi_{n-1} \in [0, 2\pi]$. Here the minus sign is just for future convenience; note that each x_i is not sensitive to such a minus sign.

Then in hyper-spherical coordinates, the volume element of such an $(n-1)$ -sphere may be written

$$d\Omega_{n-1} = \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \cdots \sin^2(\varphi_{n-3}) \sin(\varphi_{n-2}) d\varphi_1 d\varphi_2 \cdots d\varphi_{n-2} d\varphi_{n-1}, \quad (2)$$

as is easily checked by computing the Jacobian for this transformation. Integrating this volume element over the entire sphere yields the standard result

$$\begin{aligned} S_{n-1} &= \int d\Omega_{n-1} \\ &= \int \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \cdots \sin^2(\varphi_{n-3}) \sin(\varphi_{n-2}) d\varphi_1 d\varphi_2 \cdots d\varphi_{n-2} d\varphi_{n-1} \\ &= \pi^{\frac{n-2}{2}} \frac{\Gamma(\frac{n-1}{2}) \Gamma(\frac{n-2}{2})}{\Gamma(\frac{n}{2}) \Gamma(\frac{n-1}{2})} \cdots \frac{\Gamma(\frac{2}{2})}{\Gamma(\frac{3}{2})} 2\pi \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \end{aligned} \quad (3)$$

where $\Gamma(n)$ is the gamma function so $\Gamma(1) = 1$, and in moving from the second to the third line we have used the result that

$$\int_0^\pi \sin^m(\varphi) d\varphi = \sqrt{\pi} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}. \quad (4)$$

We now consider projecting the uniformly distributed $(n-1)$ -sphere onto a single dimension (i.e., the case $m=1$). Normalizing the measure of Eq. (2) by S_{n-1} , we may compute the expectation of a general function of φ_1 as

$$\begin{aligned} \langle f(\varphi_1) \rangle &= \frac{1}{S_{n-1}} \int f(\varphi_1) d\Omega_{n-1} \\ &= \frac{1}{S_{n-1}} \int_0^\pi f(\varphi_1) \sin^{n-2}(\varphi_1) \pi^{\frac{n-3}{2}} \frac{\Gamma(\frac{n-2}{2}) \Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2}) \Gamma(\frac{n-2}{2})} \cdots \frac{\Gamma(\frac{2}{2})}{\Gamma(\frac{3}{2})} 2\pi d\varphi_1 \\ &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_0^\pi f(\varphi_1) \sin^{n-2}(\varphi_1) \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} d\varphi_1 \\ &= \int_0^\pi f(\varphi_1) \sin^{n-2}(\varphi_1) \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n-1}{2})} d\varphi_1. \end{aligned} \quad (5)$$

Consequently, the distribution on φ_1 is given by

$$P(\varphi_1) d\varphi_1 = \sin^{n-2}(\varphi_1) \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n-1}{2})} d\varphi_1, \quad \varphi_1 \in [0, \pi] \quad (6)$$

To see the probability distribution in Euclidean space,

we need to transform Eq. (6) back to the coordinate x_1 . Since $x_1 = -\cos \varphi_1$, we have $dx_1 = \sin \varphi_1 d\varphi_1$ and $\sin(\varphi_1) = \sqrt{1-x_1^2}$ and hence we obtain

$$P(x_1) dx_1 = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n-1}{2})} (1-x_1^2)^{\frac{n-3}{2}} dx_1, \quad x_1 \in [-1, 1]. \quad (7)$$

We may also obtain an exact expression for the variance

$$(\Delta x_1)^2 = \frac{1}{n}. \quad (8)$$

It is easy to see that this probability distribution gets narrower and narrower as n increases. Using the result that $\lim_{n \rightarrow \infty} (1 - \frac{x}{n})^n = e^{-x}$ we see that for sufficiently

small x_1 and large n Eq. (7) may be approximated by a Gaussian with variance $1/n$, for the case $m = 1$. This result is in exact agreement with that given in previous work¹⁷, we shall see below that the exact result yields a subtly different outcome for $m > 1$.

Projecting to m -dimension

When projecting an $(n-1)$ -sphere onto m Cartesian dimensions, the logic is similar to that given in the previous section but now we will need to integrate out the angles from the set $\{\varphi_{m+1}, \varphi_{m+2}, \dots, \varphi_{n-1}\}$. Thus, we find

$$\begin{aligned} \langle f(\varphi_1, \dots, \varphi_m) \rangle &= \frac{1}{S_{n-1}} \int f(\varphi_1, \dots, \varphi_m) d\Omega_{n-1} \\ &= \frac{1}{S_{n-1}} \int_0^\pi \dots \int_0^\pi f(\varphi_1, \dots, \varphi_m) \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \dots \sin^{n-m-1}(\varphi_m) \\ &\quad \times \pi^{\frac{n-m-2}{2}} \frac{\Gamma(\frac{n-m-1}{2}) \Gamma(\frac{n-m-2}{2})}{\Gamma(\frac{n-m}{2}) \Gamma(\frac{n-m-1}{2})} \dots \frac{\Gamma(\frac{2}{2})}{\Gamma(\frac{3}{2})} 2\pi d\varphi_1 d\varphi_2 \dots d\varphi_m \\ &= \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_0^\pi \dots \int_0^\pi f(\varphi_1, \dots, \varphi_m) \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \dots \sin^{n-m-1}(\varphi_m) \frac{2\pi^{\frac{n-m}{2}}}{\Gamma(\frac{n-m}{2})} d\varphi_1 d\varphi_2 \dots d\varphi_m \\ &= \int_0^\pi \dots \int_0^\pi f(\varphi_1, \dots, \varphi_m) \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \dots \sin^{n-m-1}(\varphi_m) \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{m}{2}} \Gamma(\frac{n-m}{2})} d\varphi_1 d\varphi_2 \dots d\varphi_m. \end{aligned} \quad (9)$$

Since the volume element described by two different coordinates systems may be connected by the Jacobian, we have

$$dx_1 dx_2 \dots dx_m = J d\varphi_1 d\varphi_2 \dots d\varphi_m, \quad (10)$$

where J is the Jacobian describes this volume transformation. From Eq. (1), we know that J may be written as

$$J = \left| \frac{\partial x_i}{\partial \varphi_j} \right| = \begin{vmatrix} \sin \varphi_1 & 0 & \dots & 0 \\ \cos \varphi_1 \cos \varphi_2 & \sin \varphi_1 \sin \varphi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \cos \varphi_1 \sin \varphi_2 \dots \sin \varphi_{m-1} \cos \varphi_m & \sin \varphi_1 \cos \varphi_2 \dots \sin \varphi_{m-1} \cos \varphi_m & \dots & \sin \varphi_1 \dots \sin \varphi_m \end{vmatrix} \quad (11)$$

Since terms in the upper triangle in the Jacobian, Eq. (11), are all zero, the Jacobian trivially reduces to

$$J = \sin^m(\varphi_1) \sin^{m-1}(\varphi_2) \dots \sin(\varphi_m). \quad (12)$$

Inserting Eq. (12) into Eq. (10) yields

$$dx_1 dx_2 \dots dx_m = \sin^m(\varphi_1) \sin^{m-1}(\varphi_2) \dots \sin(\varphi_m) d\varphi_1 d\varphi_2 \dots d\varphi_m, \quad (13)$$

and substituting Eq. (13) into Eq. (9) yields

$$\begin{aligned} \langle f(\varphi_1, \dots, \varphi_m) \rangle &= \int_{-1}^1 \int_{-\sin \varphi_1}^{\sin \varphi_1} \dots \int_{-\sin(\varphi_1) \dots \sin(\varphi_{m-1})}^{\sin(\varphi_1) \dots \sin(\varphi_{m-1})} f(\varphi_1, \dots, \varphi_m) \sin^{n-m-2}(\varphi_1) \sin^{n-m-2}(\varphi_2) \dots \sin^{n-m-2}(\varphi_m) \\ &\quad \times \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{m}{2}} \Gamma(\frac{n-m}{2})} dx_1 dx_2 \dots dx_m, \end{aligned} \quad (14)$$

where $\sin(\varphi_i)$ is positive function of x_i .

To further simplify Eq. (14), let us first consider $x_1^2 + x_2^2 + \dots + x_m^2$. From Eq. (1), this may be written as

$$\begin{aligned} x_1^2 + x_2^2 + \dots + x_m^2 &= \cos^2(\varphi_1) + \sin^2(\varphi_1) \cos^2(\varphi_2) + \dots + \sin^2(\varphi_1) \sin^2(\varphi_2) \dots \cos^2(\varphi_m) \\ &= \cos^2(\varphi_1) + \sin^2(\varphi_1)(1 - \sin^2(\varphi_2)) + \dots + \sin^2(\varphi_1) \sin^2(\varphi_2) \dots \cos^2(\varphi_m) \\ &= 1 - \sin^2(\varphi_1) \sin^2(\varphi_2) + \dots + \sin^2(\varphi_1) \sin^2(\varphi_2) \dots \cos^2(\varphi_m). \end{aligned} \quad (15)$$

The above procedure can be repeated until we arrive at

$$x_1^2 + x_2^2 + \dots + x_m^2 = 1 - \sin^2(\varphi_1) \sin^2(\varphi_2) \dots \sin^2(\varphi_m). \quad (16)$$

Applying Eq. (16) to Eq. (14) then gives

$$\begin{aligned} &\langle f(x_1, \dots, x_m) \rangle \\ &= \int_{-1}^1 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \dots \int_{-\sqrt{1-x_1^2-x_2^2-\dots-x_{m-1}^2}}^{\sqrt{1-x_1^2-x_2^2-\dots-x_{m-1}^2}} f(x_1, \dots, x_m) \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{m}{2}} f(x_1, \dots, x_m) \Gamma(\frac{n-m}{2})} \left(1 - \sum_{i=1}^m x_i^2\right)^{\frac{n-m-2}{2}} dx_1 dx_2 \dots dx_m, \end{aligned} \quad (17)$$

By spherical symmetry, it is sufficient to compute the variance on x_1 , but this trivially reduces to the result already obtained, since to compute it we may integrate out all the remaining coordinates x_2, \dots, x_m . Therefore we find exactly

$$\begin{aligned} \langle x_i \rangle &= 0 \\ \langle x_i x_j \rangle &= \delta_{ij} (\Delta x_i)^2 = \frac{\delta_{ij}}{n}. \end{aligned} \quad (18)$$

From similar reasoning, for sufficiently large n the distribution becomes Gaussian.

Therefore, the probability distribution over this reduced m -sphere reduces to

$$P(x_1, x_2, \dots, x_m) = \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{m}{2}} \Gamma(\frac{n-m}{2})} \left(1 - \sum_{i=1}^m x_i^2\right)^{\frac{n-m-2}{2}}, \quad (19)$$

with suitable limits on the x_i . This is for the projection of an $(n-1)$ -sphere to an m -dimensional subspace.

The limit of this distribution as $n-m \rightarrow \infty$ and sufficiently small x_i , $i \in \{1, \dots, m\}$, may be approximated by a Gaussian with mean zero and a variance in every direction of $1/n$. This result agrees with previous work,¹⁷ except on the condition needed, here $n-m$ large as opposed to merely n being large. The difference in this requirement means that as $m \rightarrow n$, where we project out fewer and fewer coordinates and finally none, the Gaussian approximation is found to wholly fail.

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