Optimization of Impermanent Loss

White paper
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Impermanent loss is a well-known issue that arises when trading on liquidity pools. It represents the opportunity cost of participating in a liquidity pool versus a simple asset holding strategy. To learn more about liquidity pools and how they work, check out this article.

1 Formalization of a liquidity pool

1.1 Introduction

As a reminder, a liquidity pool is a pool of assets that are available for trading. Liquidity providers (LPs) deposit of pair of assets in the pool and are rewarded depending on the number of transactions that are made. Each transaction increases a *pool fee envelope*, and each LP receives a reward from this envelope that depends on the number of assets that were deposited in the pool. In addition, they often receive a reward token linked to the platform that increases the overall yield.

In order to present the formalization of a liquidity pool, in what follows, we assume it contains two assets X and Y. We also assume that the pool is built on a 50/50 ratio, which means that when an LP deposits assets, they must deposit the same value of assets X as Y.

Asset Y will be used to compute prices. We will denote by p_t and call *price of* X the quantity of asset Y with the same value as $1X$ ($1X = p_tY$). This means that if the liquidity pool contains a quantity x_t of asset X and y_t of asset Y, then necessarily $p_t = \frac{x_t}{y_t}$. The no-arbitrage hypothesis implies that this price is the same for the pool and for the entire market.

1.2 Evolution of asset quantities

In practice, the quantities x_t and y_t of assets in the pool are governed by a *constant market making formula*. In this paper we consider the *constant product formula*, which is the formula used in *Uniswap* pools:

$$\forall t \; x_t y_t = K$$

In other words, after every transaction the new quantities of each asset must respect the constraint above. This formula, along with the liquidity pool ratio, permits to deduce the quantities of each asset after every price shift.

At time t_0, we have the following:

- The current price is p_0.
- The pool contains x_0 assets X and $y_0 = p_0 x_0$ assets Y.
- The constant K is $K = x_0 y_0 = (x_0)^2 p_0$.

At time t_1, the price has shifted and is now $p_1 = p_0(1+\delta)$. We compute the new quantities of assets X and Y as follows.

1. The *constant product* constraint imposes $x_1 y_1 = x_0 y_0 = K$, so $x_1 y_1 = x_0^2 p_0$.
2. The *50/50 ratio* imposes $y_1 = p_1 x_1 = p_0(1+\delta)x_1$.

We deduce that

$$x_1 y_1 = x_1^2 p_0(1+\delta) = x_0^2 p_0,$$

so that

$$\begin{cases} x_1 = \frac{x_0}{\sqrt{1+\delta}} \\ y_1 = x_0 p_0 \sqrt{1+\delta} \end{cases} \tag{1.2.1}$$

2 Impermanent loss

2.1 Formalization of the problem

As explained above, *impermanent loss* represents the amount that would be lost by participating in a liquidity pool instead of simply holding the asset. In the crypto ecosystem, this holding strategy is often called *HODL* (in reference to a typo made by a crypto fan in a forum in 2013). More precisely, this loss can be evidenced by an LP who borrows the assets to deposit in the pool and has to return them at the end of the investment period.

Formally, the net asset value (expressed in asset Y) of a portfolio following the *HODL* strategy at time t_1 is:

$$\begin{aligned} V_H &= x_0 p_1 + y_0 \\ &= x_0 p_0(1+\delta) + x_0 p_0 \\ &= x_0 p_0(2+\delta) \end{aligned}$$

Using Equation (1.2.1), we also compute the net asset value of the pool at time t_1:

$$\begin{aligned} V_P &= x_1 p_1 + y_1 \\ &= x_1 p_0(1+\delta) + y_1 \end{aligned}$$

$$
\begin{aligned}
&= x_0 p_0 \sqrt{1+\delta} + x_0 x \sqrt{1+\delta} \\
&= 2 x_0 p_0 \sqrt{1+\delta}
\end{aligned}
$$

Finally, the impermanent loss representing the percentage of loss relative to a price shift is:

$$IL = \frac{V_P - V_H}{V_H} = \frac{2\sqrt{1+\delta}}{2+\delta} - 1$$

The LP who borrowed the quantities x_0 and y_0 of assets X and Y will need to find $IL \cdot V_H$ to buy back enough assets and return the original quantities to the lender.

Note. The loss is said to be *impermanent* because it is not effective until the LP withdraws their assets from the pool. If the price returns to its initial state (i.e., if $\delta = 0$), then the loss goes back down to zero.

In what follows we will consider the *impermanent loss* as a function of the initial and final prices:

$$IL(p_0, p_1) = \sqrt{\frac{p_1}{p_0}} \frac{2}{1 + \frac{p_1}{p_0}} - 1$$

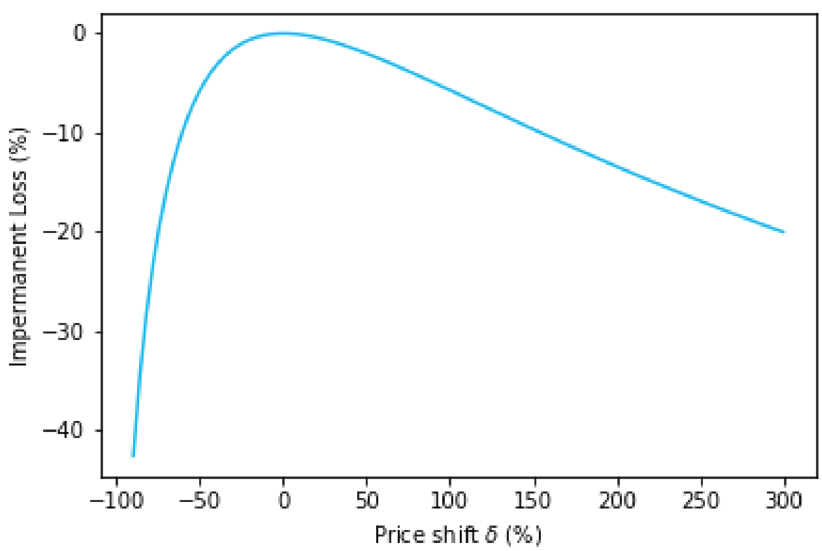

Figure 1: Impermanent Loss relative to the price shift

As can be seen on Figure 1, when the price shift is small, so is the *impermanent loss*. But when the price spikes (up or down), there is a risk of significant loss. The question is: how can an LP collect as much reward as possible and at the same time avoid an important loss?

2.2 Portfolio rebalancing

Intuitively, one way of avoiding this problem is to *reset* the position when the price shifts *too much*, in order to always remain in the upper part of the Figure 1. Resetting the position means rebalancing the portfolio to get as close as possible to the initial position (x_0, y_0). There are several methods to do this, the one we propose here is to withdraw some of the assets from the pool and swap one asset for the other to rebalance the quantities.

2.2.1 Example

Let's consider a pool ETH/DAI with a current price of 1 ETH = 2400 DAI. With our previous notations, we have X = ETH, Y = DAI and $p_0 = 2400$.

An LP deposits $x_0 = 10$ ETH and $y_0 = 24\ 000$ DAI into the pool at time t_0. At time t_1, the price is 2450. Equations (1.2.1) permit to compute the asset quantities withheld by the LP:

$$\begin{aligned} \delta &= \tfrac{p_1 - p_0}{p_0} &\approx 0.021 \\ x_1 &= \tfrac{x_0}{\sqrt{1+\delta}} &\approx 9.89 \\ y_1 &= x_0 p_0 \sqrt{1+\delta} &\approx 24249 \end{aligned}$$

Time	y (DAI qty)	x (ETH qty)	Price p (DAI)	Total (DAI)
t_0	24 000	10	2 400	48 000
t_1	24 249	9.89	2 450	**48 497**

If all the assets are withdrawn from the pool then the total value of the portfolio is 48 497 DAI. If the assets had not been deposited in the pool, then the value of the portfolio would have been $24000 + 10 \cdot 2450 = 48500$ DAI. The current impermanent loss amount is therefore 3 DAI.

We now assume the price at t_2 is $p_2 = 2500$, the portfolio for the *HODL* strategy has a value of 49 000 DAI. We investigate the impact of rebalancing the portfolio on the impermanent loss.

Case 1: no rebalancing. Assume the portfolio is not rebalanced at all. The evolution of the portfolio until t_2 is summarized in the table below:

Time	y (DAI qty)	x (ETH qty)	Price p (DAI)	Total (DAI)
t_0	24 000	10	2 400	48 000
t_1	24 249	9.89	2 450	48 497
t_2	24 495	9.79	2 500	**48 990**

Case 2: rebalancing. Assume the portfolio is rebalanced at time t_1. The LP:

- Withdraws 1% of the assets from the pool and receives 0.0989 ETH and 242.49 DAI.
- Swaps 242.49 DAI for 0.0989 ETH (at the current price of 2450).

The assets withheld by the LP are summarized in the following table:

	y (DAI qty)	x (ETH qty)
In the pool	24 006	9.7984
Outside the pool	0	0.1979
Total	24 006	9.9964

The LP therefore owns a quantity of ETH and DAI close to the initial position $(x_0, y_0) = (24000, 10)$. The table below recaps the quantities of assets inside and outside the pool, before and after the rebalancing at time t_1, and at time t_2.

Time	y (DAI qty)	x (ETH qty)	Price p (DAI)	Total (DAI)
t_0	24 000	10	2 400	48 000
t_1	24 249	9.89	2 450	48 497
t_1^+	24 006 | 0	9.79 | 0.1979	2 450	48 497
t_2	24 249 | 0	9.70 | 0.1979	2 500	**48 994**

Conclusion: At time t_2, here are the different portfolios values :

- HODL strategy: 49 000 DAI
- Pool strategy without rebalancing: 48 990 DAI (IL = -10 DAI)
- Pool strategy with rebalancing: 48 994 DAI (IL = -6 DAI)

We managed to reduce the impermanent loss by resetting our position at time t_1.

2.3 Rebalancing costs

One disadvantage of the rebalancing process detailed in **case 2** is that it has a cost. Two different transaction fees are to be taken into account:

- Deposit and withdrawal pool fees.
- Transaction fees for swaps (ETH/DAI or DAI/ETH).

If the portfolio is rebalanced too often, a large amount of fees will be paid and if it is not rebalanced often enough, the impermanent loss may be important. Finding the right balance between reasonable risk and reasonable fees is critical.

2.4 Optimization of the risk/fees tradeoff

We formalize the optimization problem to be solved. Our aim is to devise a strategy that guarantees that the LP gains as much as possible *regardless of the price evolution of the assets*. The gains made by the LP depend on:

- The rewards earned by depositing the assets (*block percentage yield*, BPY).
- The amount lost due to the impermanent loss.
- The transaction costs (TC) that are paid to rebalance the portfolio.

More formally, we assume that the portfolio can be rebalanced at times $(t_0, \ldots, t_N)$. The HODL portfolio value at time t_i is $V_{t_i} = x_{t_i} p_{t_{i+1}} + y_{t_i}$. The gains[1] made between times t_i and t_{i+1} are written as:

$$\begin{aligned} P\&L(t_i, t_{i+1}) &= x_{t_{i+1}} p_{t_{i+1}} + y_{t_{i+1}} + BPY(t_i, t_{i+1}) + TC(t_i) - (x_{t_i} p_{t_{i+1}} + y_{t_i}) \\ &= IL(p_{t_i}, p_{t_{i+1}}) \cdot V_{t_i} + BPY(t_i, t_{i+1}) + TC(t_i) \end{aligned}$$

Note that $IL(p_{t_i}, p_{t_{i+1}})$ and $TC(t_i)$ are both negative.

Since the impermanent loss is not linear we cannot simply sum the $IL(p_{t_i}, p_{t_{i+1}})$ terms. The total amount earned by the LP at time t_N is:

$$\begin{aligned} P\&L &= IL(p_{t_0}, p_{t_N}) + \sum_{i=0}^{N-1} BPY(t_i, t_{i+1}) + TC(t_i) \\ &= \sum_{i=0}^{N-1} P\&L(t_i, t_{i+1}) + x_{t_0}(p_{t_1} - p_{t_N}) + \sum_{i=1}^{N-1} x_{t_i} p_{t_{i+1}} - x_{t_i} p_{t_i}. \end{aligned}$$

This amount is of course random: the block percentage yield may change depending of the number of investors in the pool and the impermanent loss depends on the (random) price evolution, which also impacts the transaction costs. Our goal is thus to devise a strategy that maximizes the earnings *on average*, i.e., that maximizes $\mathbb{E}[P\&L]$.

[1] The gains are commonly named *Profit and Loss*, and denoted by P&L.

3 Rebalancing strategies

In this section, we present two standard rebalancing strategies, and a third one based on the *TEMPO* technology by Kesitys. The results are provided on ETH, DAI data from 2020 in a 50/50 pool. We assume that:

1. Reward fees are modeled by a constant interest rate of 30% per year.
2. Pool deposits/withdrawals fees are 0.1%.
3. Asset swap fees are 0.3%.

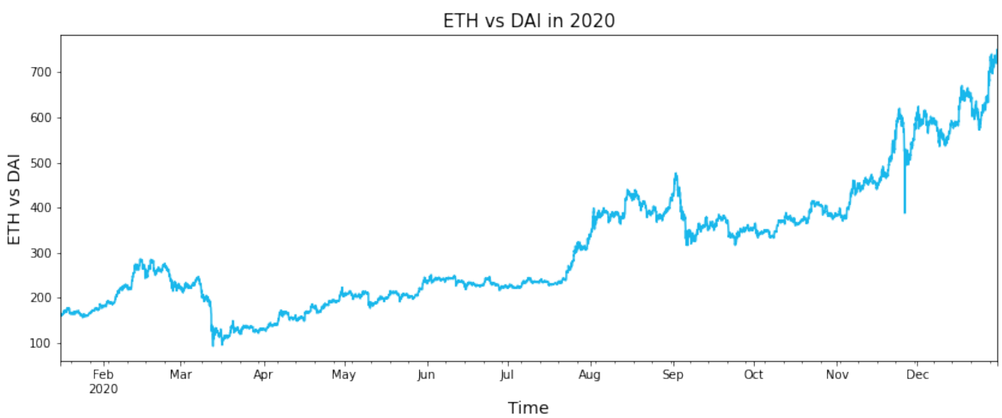

Figure 2: Price ETH vs DAI

On January 1st 2020, the price is 165.87 (1 ETH = 165.87 DAI). The initial portfolio consists of 10 000 DAI and 60.29 ETH for a total value $V_0 = 20000$.

3.1 HODL strategy

The HODL strategy consists in holding the initial assets. At the end of the year, the price is 750.28 and the portfolio is worth 55 232 DAI. Since no transactions were carried out, there are no costs and no rewards.

Portfolio value	*IL* (DAI)	*BPY*	*TC*	P&L
55 232	0	0	0	0

3.2 No rebalancing strategy

This strategy consists in keeping all the assets in the liquidity pool until the end of the year. At the end of the year, all assets are withdrawn from the pool and the portfolio consists of 21 267.78 DAI and 28.35 ETH.

Portfolio value	*IL* (DAI)	*BPY*	*TC*	P&L
42 536	-12 696	7 684	0	-5 012

Note that the P&L is negative, which means the pool rewards did no compensate for the *impermanent loss*.

3.3 Periodic rebalancing strategy

We present a simple rebalancing strategy which consists in rebalancing the pool portfolio on a regular basis. The table below shows the portfolio value for three rebalancing frequencies: daily, weekly and every four weeks.

Frequency	Portfolio value	*IL* (DAI)	*BPY*	*TC*	P&L
1 day	49 044	-6 188	5 416	-204	-976
1 week	49 625	-5 607	5 499	-91	-199
4 weeks	50 701	-4 531	5 674	-54	**1 089**

The P&L of the resulting portfolio obviously depends on the rebalancing period and, as can be seen on the figure below (3), it is not clear at all how a good rebalancing period can be chosen.

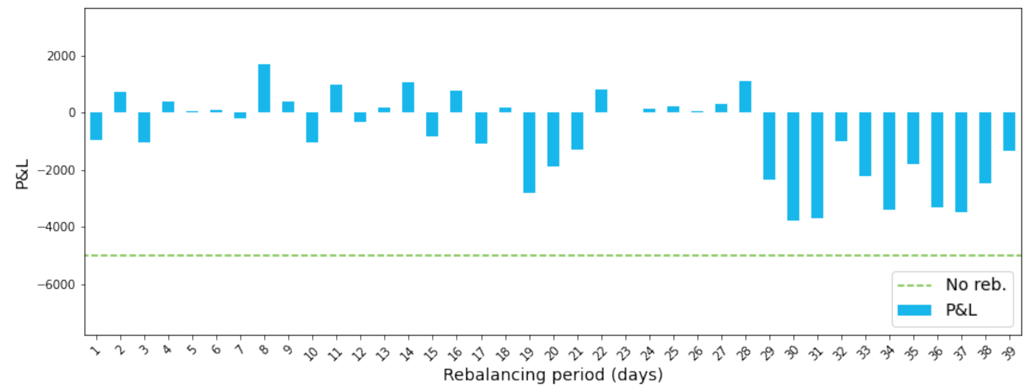

Figure 3: P&L of periodic strategies depending on the rebalancing period

Several comments can be made on these results :

- The *impermanent loss* (see Figure 4) is significantly reduced on average, from -12696 for the *no rebalancing* strategy to -6200.
- The average rewards have decreased compared to the *no rebalancing* strategy (∼5500 compared to 7 684). Also these rewards are quite stable regardless of the strategy (the standard deviation is 74).

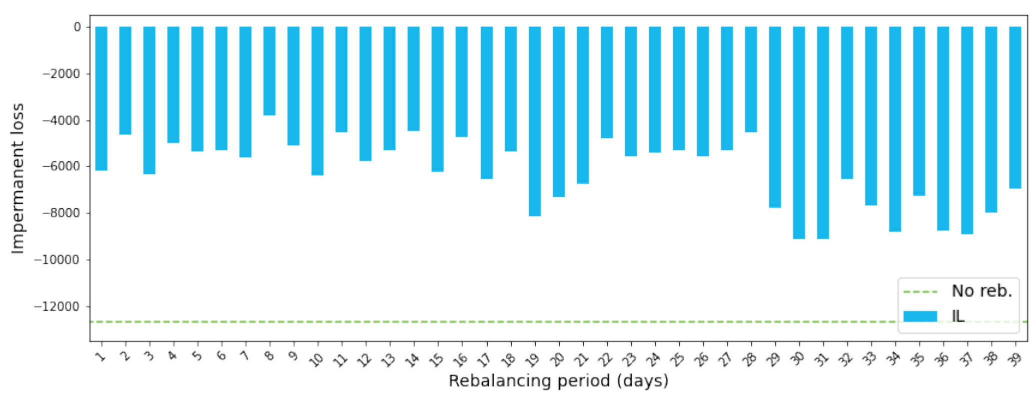

Figure 4: Impermanent loss of periodic strategies depending on the rebalancing period

We can see that, on this dataset, applying a rebalancing strategy, even a simple one, is always better than keeping all the assets in the pool the entire time. However determining the correct frequency is not clear at all and may strongly depend on the dataset.

3.4 Price threshold strategy

It is quite obvious that the periodic rebalancing strategy can be improved by examining the current price. For example if this price has not moved much, then there is no need to rebalance. On the contrary, when the price shifts sharply, it might be better to rebalance immediately rather than wait for the next scheduled rebalancing. In other words, a rebalancing is done every time the price moves a certain percentage in any direction. In the table below, we show a few examples based on the same ETH/DAI dataset.

Threshold	#reb.	Portfolio value	*IL* (DAI)	*BPY*	*TC*	P&L
± 10%	68	49 538	-5 693	5 472	-139	-360
± 25%	16	48 468	-6 764	5 438	-73	-1 398
± 75%	2	50 904	-4 328	6 315	-27	**1 960**
± 100%	2	49 718	-5 514	6 333	-34	**784**

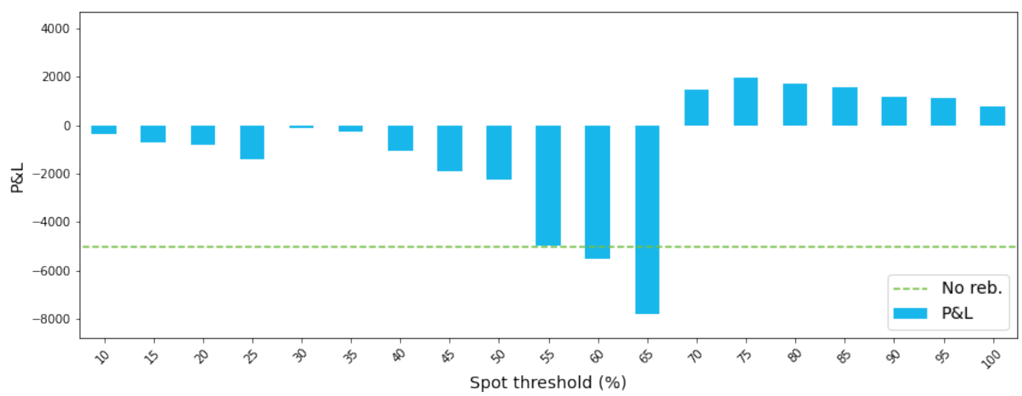

Figure 5: P&L of threshold strategies depending on the spot threshold

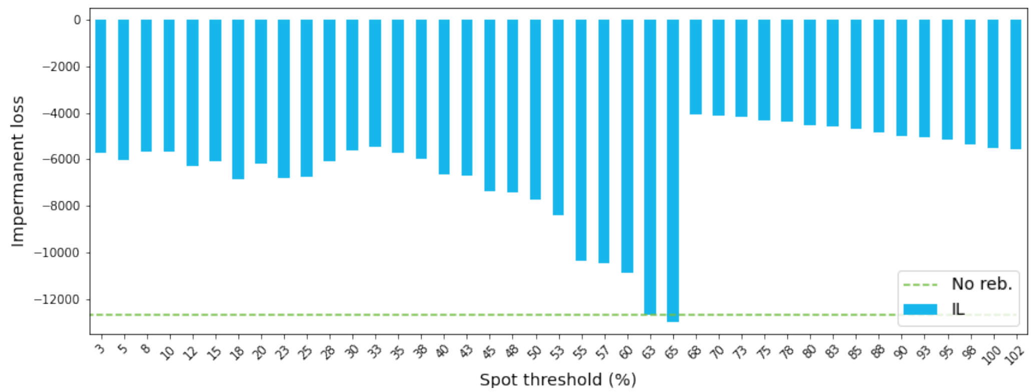

Figure 6: Impermanent loss of threshold strategies depending on the spot threshold

Several questions arise. What is the best threshold to use? How can it be found? Should the threshold be constant or evolve over time? If so, how should it evolve? LPs may rely on their experience and expertise to adjust such a threshold over time depending on market conditions, but this is hard to generalize and it is not based on a quantitative analysis.

3.5 The *TEMPO* strategy

TEMPO is a tool developed by Kesitys that permits to devise a strategy that reduces the *impermanent loss*, and thus maximizes the P&L. More specifically, *TEMPO* sends the LP signals that are **optimal times** at which the portfolio should be rebalanced.

Strategy	#reb.	Portfolio value	*IL* (DAI)	*BPY*	*TC*	P&L
TEMPO	5	52 893	-2 339	5 633	-32	**3 261**

TEMPO outperforms all periodic and threshold strategies. It does so by automatically adapting to market conditions in order to detect the best rebalancing moments. The same data was fed to all strategies: it is important to note that *TEMPO* did *not* require a training phase before being invoked on the dataset.

3.6 How good are these strategies?

The P&Ls of these strategies were measured on December 31. But what if we had stopped our test, e.g., on November 30? Would the results have been the same? To carry out a more robust analysis on the performance of the strategies, we need to employ more sophisticated indicators. One standard indicator in traditional finance is the Sharpe ratio which is an annualized measure of the portfolio returns per unit of risk. More formally, if we denote by $\mathcal{R}^{P\&L}$ the daily returns: $\mathcal{R}_t^{P\&L} = \frac{P\&L_t - P\&L_{t-1}}{P\&L_{t-1}}$, then the Sharpe ratio is given by:

$$Sharpe = \sqrt{365} \cdot \frac{\mathbb{E}[\mathcal{R}^{P\&L}]}{StdDev(\mathcal{R}^{P\&L})}.$$

In other words if two strategies have the same average returns, then the more stable one will have the higher Sharpe ratio. A strategy with a higher Sharpe ratio therefore produces a higher return per unit of risk.

NB: for more stability, we apply a winsorizing on the daily returns consisting in discarding 1% of the most extreme returns.

In the following tables, we show this Sharpe ratio for the *no rebalancing* and *TEMPO* strategies and we output the minimum and maximum values that were measured among all periodic and spot threshold strategies on the ETH/DAI dataset.

Strategies	*Sharpe*
No rebalancing	-0.58
TEMPO	1.20

Table 1: Sharpe ratios

Strategies	min *Sharpe*	max *Sharpe*
Periodic	-2.15	3.03
Spot	-2.70	2.44

Table 2: Minimum and maximum Sharpe ratios

3.7 Adverse case

Now let's take a completely different case: we consider a ETH vs. BTC pool between January 1 2020 and December 31 2020. The main difference with our previous example is the market shape. As we can see in Figure 7, the market is almost *mean reverting*, which means that at the end of the period, the spot price is almost the same as at the beginning.

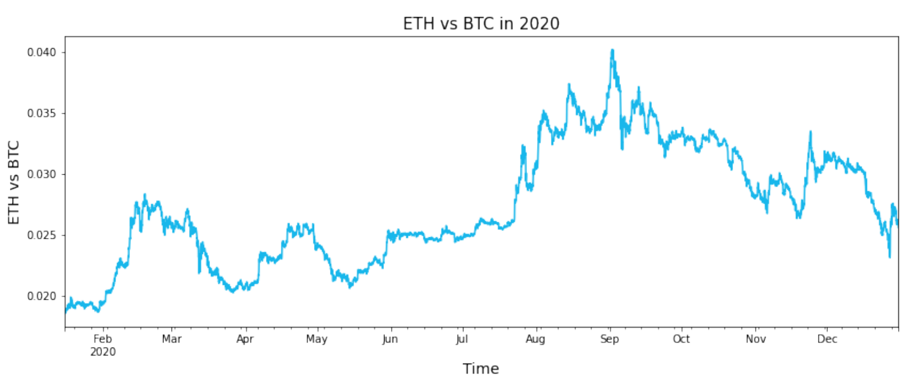

Figure 7: Price ETH vs BTC

On a mean reverting market, if we invest in a pool and we don't rebalance our portfolio, then there will be no impermanent loss at the end of the period. Any other rebalancing strategy leads to a lower P&L. This is therefore an unfavorable case for rebalancing strategies.

We display the best strategy (in terms of P&L) for each strategy type:

Strategy	Portfolio value	*IL* ($)	*BPY*	*TC*	P&L
HODL	9 252	0	0	0	0
No reb.	9 139	-113	2 705	0	2 593
Periodic (4W)	8 997	-255	2 247	-10	1 982
Spot (50%)	9 051	-201	2 210	-4	2 005
TEMPO	9 015	-237	2 239	-9	1 992

Strategies	*Sharpe*
No rebalancing	3.18
TEMPO	5.13

Table 3: Sharpe ratios

Strategies	min *Sharpe*	max *Sharpe*
Periodic	3.15	6.08
Spot	3.03	6.41

Table 4: Minimum and maximum Sharpe ratios

As expected, the *no rebalancing* strategy generates the best P&L, while all rebalancing strategies have similar P&Ls. However if we look at the *Sharpe* ratio, the *no rebalancing* strategy has a much lower ratio than *TEMPO* or *Periodic 4W*. This means that on average the *no rebalancing* strategy generate a lower return than the other ones per unit of risk.

4 Going Forward

Liquidity providing is a risky strategy, but some methods can alleviate the loss in specific markets. As we saw, those methods do not completely cancel the loss and should be used carefully depending on the market type. The next steps will be to characterise the strategy to use, depending on the market conditions and specific indicators.

New AMMs have also emerged, like Curve V2 and Uniswap V3, which change the impermanent loss dynamic. Optimisations on such algorithms remain to be done but are likely to provide better management of the loss or even new ways to use liquidity pools, such as limit orders on concentrated liquidity.