

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 0: AN OVERVIEW OF THE SERIES

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ABSTRACT. This is a guide to the series *Cross-energy of Dirichlet partial sums* (Parts 1–17). The series studies $C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma}$, the interaction of the newest term of $X_N(s) = \sum_{n \leq N} n^{-s}$ with all the previous ones, and relates it to the classical reformulations of the Riemann hypothesis (RH). For each part we state the main theorems with their exact status (proved, computer-assisted, certified in interval arithmetic, numerical, heuristic or conditional), say what is new and what is classical, list the new formulas, and record how each result relates to RH: equivalent to it, implied by it, conditional on it, unconditional, or a control. The organising results are Theorem A (Part 1: $\lim_N C_2(N, s)$ is $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ as $\sigma =, >, < \frac{1}{2}$), the Fresnel transition law (Part 2), Theorem B (Part 5: for simple zeros the mean energy of the prime number theorem error equals $\sum_\rho \lim_N C_2(N, \rho)$; it is finite if and only if RH holds, and it is at least $C_B = 2 + \gamma_0 - \log 4\pi$), the C_2 form of the Balazard–Saias–Yor criterion and Weil’s functional (Parts 10–12), the bound $\Lambda \leq 0.174$ for the de Bruijn–Newman constant (Part 13) and, if $\zeta(\frac{1}{2} + \delta + i\gamma) = 0$ with $\delta > 0$, $\Lambda \geq \delta^2/(70 \log \gamma \log \log \gamma)$ (Part 15), and the Epstein obstruction (Part 16). The criteria are tested on the Davenport–Heilbronn function, an Epstein zeta function, Beurling systems and function fields (Part 17). Closing sections collect, across the series, the map of RH reformulations, the main formulas, the key constants, the certified computations, the obstructions and the open problems. No part of the series proves RH, and none claims to: every equivalence with RH is a reformulation. We do not prove the Riemann hypothesis.

CONTENTS

1. Introduction	2
2. The series in five blocks	4
3. Part 1: The C-transform and the cross-energy	6
4. Part 2: The transition at the critical line and Hardy’s function	9
5. Part 3: The reflected cross-energy	11
6. Part 4: Zeros of truncations: spirals, sections and the circles of the primes	13
7. Part 5: The energy of the primes	16
8. Part 6: Second-order asymptotics and limiting fluctuations of the prime energy	20
9. Part 7: Multiplicative resonance and large values of Dirichlet polynomials	23
10. Part 8: The Euler product inside the critical strip	27
11. Part 9: Zeros of the prime zeta function in its half-plane of absolute convergence	30
12. Part 10: Jensen’s formula, the energy slope and Weil’s functional	33
13. Part 11: The Euler–Caceres bound	35
14. Part 12: Weil positivity at the windows of the zeros	38
15. Part 13: Optimised mollifiers and the bound $\Lambda \leq 0.174$ for the de Bruijn–Newman constant	40
16. Part 14: The constant in the $C/\log X$ law for triangle-inequality upper bounds on the de Bruijn–Newman constant	41

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17. Part 15: How long can an off-line zero survive the de Bruijn–Newman heat flow? An explicit unconditional lower bound for Λ	44
18. Part 16: Structural limits of the cross-energy method	47
19. Part 17: Controls for the Riemann hypothesis: Davenport–Heilbronn, Beurling systems and function fields	49
20. Synthesis across the series	52
Conclusion	57
References	57

1. INTRODUCTION

1.1. The object. Let $X_N(s) = \sum_{n \leq N} n^{-s}$, $s = \sigma + it$, be the partial sums of the Dirichlet series of ζ . The *cross-energy* of consecutive partial sums is

(1)

$$C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} (X_{N-1}(s) N^{-\bar{s}}) = 2N^{-\sigma} \sum_{n < N} n^{-\sigma} \cos \left(t \log \frac{N}{n} \right).$$

It records how the newest term N^{-s} interacts with all the previous ones; summed over N it rebuilds the energy, $|X_N(s)|^2 = \sum_{n \leq N} C_2(n, s) + \sum_{n \leq N} n^{-2\sigma}$. Its companion $S_2(N, s) = 2 \operatorname{Im}(X_{N-1}(s) N^{-\bar{s}})$ makes $W_N = C_2 + iS_2$ a complex wave. The series [CE1]–[CE17] studies this one object at every point of the critical strip, at the zeros, along the critical line, in the primes, in Euler products, in positivity criteria and in the heat flow of de Bruijn and Newman, and it tests every statement against functions that share some of the structure of ζ but violate the Riemann hypothesis (RH).

Throughout, $\rho = \beta + i\gamma$ is a nontrivial zero of ζ , $\Theta = \sup \beta$, $\lambda = s(1-s)$ (so $\lambda = \frac{1}{4} + t^2$ on the critical line), γ_0 is Euler’s constant, $\Lambda(n)$ is von Mangoldt’s function (and Λ alone the de Bruijn–Newman constant; the meaning is always clear from context), and

$$C_B = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots,$$

the sum taken in symmetric order. Under RH, $\rho(1-\rho) = \frac{1}{4} + \gamma^2$, and C_B is the sum over the zeros of the limits of C_2 given by Theorem A below.

1.2. Purpose of this overview. This paper is a map of the series. It does not contain new results. For each of the seventeen parts it gives, in a fixed format (Sections 3–19):

- (1) the origin of the part and a short summary;
- (2) the main results, each with its statement number in that part and its exact status;
- (3) what is new and what is classical, as the part itself says;
- (4) the new formulas, numbered $(N.k)$ for Part N ;
- (5) the relation of each result to RH;
- (6) the key numbers.

Section 2 tells the story of the series in five blocks, and Section 20 collects, across all parts, the RH reformulations, the main formulas, the constants, the certified computations, the controls, the obstructions and the open problems. Citations of the form [CE k , Theorem $x.y$] refer to the numbering of Part k .

1.3. Status labels. Each result is labelled with one of the following.

- *proved*: a complete proof is given in the part;
- *proved, computer-assisted*: the proof contains a finite computation, done in interval arithmetic or with outward rounding, with the code in the ancillary files;

- *certified in interval arithmetic*: a numerical value or inequality certified in interval or ball arithmetic, mostly with the Arb library through python-flint;
- *numerical*: computed in floating point or high precision, not certified;
- *heuristic*: an argument or a fitted law, not a proof;
- *conditional on RH* or *conditional on* a named hypothesis (simplicity of zeros, linear independence LI, the spacing hypothesis (H^*), a barrier condition, and so on);
- *classical, restated*: a known result rewritten in the coordinates of C_2 .

Relations with RH are labelled *equivalent to RH*, *implied by RH*, *conditional on RH*, *unconditional*, *control* (a test on a function that is not ζ) or *caveat* (a limitation stated by the part itself).

1.4. What the series does not do. No part of the series proves RH, and none claims to. Every statement in the series that is equivalent to RH is a *reformulation*: proving it is as hard as RH, and the parts say so each time. The series has two kinds of unconditional content. The first is exact structure: identities, laws and asymptotics of C_2 that hold at every point and do not distinguish zeros on the line from zeros off it. The second is explicit, certified bounds that do not presuppose RH, such as $\Lambda \leq 0.174$ in Part 13 and $\Lambda \geq \delta^2/(70 \log \gamma \log \log \gamma)$ for a hypothetical off-line zero in Part 15. Part 16 explains, by an explicit counterexample, why the cross-energy alone cannot reach RH.

1.5. The seventeen parts. Table 1 gives the plan. The parts replace the author's earlier numbered preprints and the consolidated paper *The cross-energy of Dirichlet partial sums and the zeros of the Riemann zeta function* [CP]; version 4 of the latter is divided into Parts 1, 2, 5, 10 and 16.

TABLE 1. The series. Pages refer to the compiled preprints.

Part	title	pp.	headline result
<i>I. Foundations</i>			
1	The C-transform and the cross-energy	34	Theorem A (trichotomy); crossing law
2	The transition at the critical line and Hardy's function	25	Fresnel law, $\kappa = -0.2077978646\dots$
3	The reflected cross-energy	30	C_2^{FR} ; Hankel–Jacobi form
4	Zeros of truncations: spirals, sections and the circles of the primes	24	spiral of a zero; prime circles
<i>II. Primes and Euler products</i>			
5	The energy of the primes	25	Theorem B; RH $\iff \sum C_2^\Lambda = 4N + O(\sqrt{N})$
6	Second-order asymptotics and limiting fluctuations of the prime energy	22	$K = 2.512479790006\dots$
7	Multiplicative resonance and large values of Dirichlet polynomials	15	Steinhaus model vs. Guth
8	The Euler product inside the critical strip	32	product form of RH; Euler helix
9	Zeros of the prime zeta function in its half-plane of absolute convergence	11	sharp edge $\sigma_P = 1.7795446535\dots$
<i>III. Positivity criteria</i>			
10	Jensen's formula, the energy slope and Weil's functional	29	C_2 form of Balazard–Saias–Yor
11	The Euler–Caceres bound	45	law $(\log X)B_X \rightarrow \Omega(s_0)$ under RH
12	Weil positivity at the windows of the zeros	25	DH threshold $3.42 < L^* \leq 3.43$
<i>IV. The de Bruijn–Newman constant</i>			
13	Optimised mollifiers and the bound $\Lambda \leq 0.174$	19	$\Lambda \leq 0.174$ (certified)
14	The constant in the $C/\log X$ law for triangle-inequality upper bounds	16	$C_{\text{opt}} = 4w_0 = 4.9146\dots$; floor 4
15	How long can an off-line zero survive the de Bruijn–Newman heat flow?	33	$\Lambda \geq \delta^2/(70 \log \gamma \log \log \gamma)$
<i>V. Limits and controls</i>			
16	Structural limits of the cross-energy method	16	Epstein obstruction
17	Controls for the Riemann hypothesis: Davenport–Heilbronn, Beurling systems and function fields	24	the control catalogue

2. THE SERIES IN FIVE BLOCKS

I. Foundations (Parts 1–4). Part 1 starts from Euler–Maclaurin summation in the form of a *C-transform*, $C_N(f) = \sum_{n \leq N} f(n) - \int_1^N f$. In the strip, $\zeta(s) = C(x^{-s}) - 1/(1-s)$, so ζ is the additive balance between the discrete sum X_N and its continuous envelope $Y_N = N^{1-s}/(1-s)$, and a zero is a point of exact balance. Its central result is Theorem A [CE1, Theorem 3.2]: at every point of the strip,

$$\lim_{N \rightarrow \infty} C_2(N, s) = \begin{cases} +\infty, & \sigma < \frac{1}{2}, \\ 1/(\frac{1}{4} + t^2), & \sigma = \frac{1}{2}, \\ 0, & \sigma > \frac{1}{2}, \end{cases}$$

whether or not $\zeta(s) = 0$. The trichotomy is the trace of the pole of ζ at $s = 1$; the zeros enter only through a wave $2\operatorname{Re}[\zeta(s)N^{-\bar{s}}]$ of size $N^{-\sigma}$. This single observation organises the series: everything C_2 says about the line comes from the pole, and everything it says about the zeros comes from the wave. Part 1 also proves the exact rotation–contraction–forcing dynamics of W_N , a crossing law with rational coefficients at a zero on the line, the same anatomy for any Dirichlet series with a simple pole, and the Mellin transform of C_2 .

Part 2 zooms in on the critical line. At a zero on the line, $C_2(\cdot, \rho)$ is eventually monotone, and its last decrease follows a Fresnel law $N_{\text{last}} = t/2\pi + \kappa\sqrt{t} + O(1)$ with a universal constant κ . On the line the helix of $X_{N-1}N^{-\bar{s}}$ has a signed radius that is Hardy’s $Z(t)$, and the Riemann–Siegel formula and Hardy’s theorem read naturally in these coordinates. Part 3 pairs s with $1-s$ through the reflected cross-energy C_2^{FR} , which brings in the functional equation, Weil’s form in a dual range and a Hankel–Jacobi positivity criterion. Part 4 follows the zeros of truncations of ζ : the spiral by which the zeros of the truncations $F_n = X_{n-1} - n^{1-s}/(1-s)$ converge to a zero of ζ , the right edge of the zeros of the sections X_N , and the circles of the primes, on which, under RH, Landau’s formula makes the zeros of ζ lean toward the angle π .

II. Primes and Euler products (Parts 5–9). Part 5 contains Theorem B: for simple zeros the mean energy of the normalised error of the prime number theorem equals $\sum_{\rho} \lim_N C_2(N, \rho)$ (in general both are finite or both infinite); it is finite if and only if RH holds, and it is at least $C_B = 2\lambda_1$ (twice Li’s first coefficient), with equality if and only if RH holds and the zeros are simple. It reads the explicit formula, the Nyman–Beurling criterion and an arithmetic cross-energy C_2^{Λ} in the same coordinates; RH is equivalent to $\sum_{n \leq N} C_2^{\Lambda}(n, \frac{1}{2}) = 4N + O(\sqrt{N})$. Part 6 goes to second order: the constant $K = 2.512479790006\dots$ (defined under RH, its value certified in interval arithmetic) and the limiting distribution of the fluctuations under RH, simple zeros and a spacing hypothesis (H^*), with its variance computed under LI. Part 7 reads the diagonal energy of C_2 as the budget behind Montgomery’s large value conjecture, carries out two of Guth’s numerical checks of that conjecture and a search for worst cases, and calibrates them against the Steinhaus random multiplicative model. Part 8 removes the pole from the partial Euler product inside the strip and proves that RH holds if and only if the corrected product E_x^* converges to ζ in $\frac{1}{2} < \sigma < 1$, a product form of the classical equivalence of RH with the analyticity of $\log \zeta$ in $\sigma > \frac{1}{2}$. Part 9 locates the zeros of the prime zeta function $P(s) = \sum_p p^{-s}$ in $\operatorname{Re} s > 1$: the edge σ_P beyond which Belovas, Čepaitytė and Sabaliauskas proved $P \neq 0$, shown here to be sharp, a Jessen density and certified counts.

III. Positivity criteria (Parts 10–12). Part 10 writes the Balazard–Saias–Yor criterion as a Jensen formula for the C_2 wave, reads Lagarias’s energy slope $\partial_{\sigma} \log |\xi|$ and truncates Weil’s functional exactly. Part 11, starting from the hybrid Euler–Hadamard product of Gonek, Hughes and Keating, compares ζ with a smoothed Euler product by a Jensen moment, the *Euler–Caceres bound* $B_X(s_0)$, and proves under RH an exact law $(\log X)B_X(s_0) \rightarrow \Omega(s_0)$ with universal multiplicity constants κ_m . Part 12 computes Weil’s quadratic form at the windows of the zeros and finds the threshold $3.42 < L^* \leq 3.43$ (upper bound certified, lower bound numerical) beyond which positivity fails for the Davenport–Heilbronn function.

IV. The de Bruijn–Newman constant (Parts 13–15). RH is equivalent to $\Lambda \leq 0$ (de Bruijn, Newman), and Rodgers and Tao proved $\Lambda \geq 0$. Part 13 improves the bound $\Lambda \leq 0.2$ of Polymath and Platt–Trudgian to the unconditional bound $\Lambda \leq 0.174$, using Polymath’s criterion, the verification of RH up to $3 \cdot 10^{12}$ by Platt and Trudgian and an optimised mollifier, with a certificate in Arb and outward-rounded floating point. Part 14 shows that for triangle-inequality certificates the constant in $\Lambda \leq (C + o(1))/\log X$ is exactly $4w_0 = 4.9146\dots$ without a mollifier, and at least 4 with any mollifier of length at most $N/2$: a limitation of the method, not of Λ . Part 15 goes the

other way: a hypothetical zero at $\frac{1}{2} + \delta + i\gamma$ forces $\Lambda \geq \delta^2/(70 \log \gamma \log \log \gamma)$, with every constant explicit and certified.

V. Limits and controls (Parts 16–17). Part 16 explains why C_2 alone cannot prove RH. The Epstein zeta function of $m^2 + 100n^2$ has a pole at $s = 1$, nonnegative coefficients, a functional equation and the full trichotomy of Theorem A, and still has a real zero $\beta_0 = 0.8133678\dots$ off the line (real zeros of such Epstein functions go back to Bateman and Grosswald). So any argument that uses only what C_2 sees fails. What the Epstein function lacks is the Euler product; the Beurling controls of Part 17 show that the Euler product must be used together with the functional equation. Part 17 is the catalogue of controls—Davenport–Heilbronn, Beurling systems, function fields—against which every criterion of the series is tested.

3. PART 1: THE C-TRANSFORM AND THE CROSS-ENERGY

Origin. Part 1 [CE1] (34 pages), with Parts 2, 5, 10 and 16, replaces version 4 of the consolidated paper (doi:10.5281/zenodo.23243641); the series as a whole replaces the earlier preprints [C1]–[C8]. Part 1 also contains the plan of the series and an appendix of formulas.

Summary. With the C-transform $C_N(f) = \sum_{n \leq N} f(n) - \int_1^N f$, the zeta function in the strip is the additive balance of the partial sum $X_N(s) = \sum_{n \leq N} n^{-s}$ and its continuous envelope $Y_N(s) = N^{1-s}/(1-s)$, and a zero is a point of exact balance. The ratio X_N/Y_N tends to 1 everywhere and cannot see the critical line. The X – Y energies agree to all orders at every zero, and their first correction equals $-B_2$ exactly on the line. The C-transform of Λ converges exactly in $\sigma > \Theta = \sup \beta$, so RH is its convergence in $\sigma > \frac{1}{2}$.

Theorem A splits $C_2(N, s) = |X_N|^2 - |X_{N-1}|^2 - N^{-2\sigma}$ into three parts: a structural mode coming from the pole, one wave $\zeta(s)n^{-\bar{s}}$, and Bernoulli terms. Hence $\lim_N C_2(N, s)$ is $+\infty$, $1/(\frac{1}{4} + t^2)$ or 0 at every point of the strip, zero or not. The paper also proves:

- the exact rotation–contraction–forcing dynamics of $W_N = C_2 + iS_2$;
- the crossing law $n^* = \lambda + \frac{1}{12} + \frac{1}{180\lambda} + \dots$ at a zero on the line, a universal series with rational coefficients;
- the same anatomy for every Dirichlet series with a simple pole;
- the Mellin transform of C_2 ;
- an explicit bound for the last sign change at non-zeros.

The main statements are tested on the Davenport–Heilbronn function and on a planted zero $\zeta(s) - \zeta(s_0)$.

Main results.

- (1) [CE1, Proposition 2.3] (*classical, restated*). The paper’s label is “proved (classical in substance)”. For $\sigma > 0$, $s \neq 1$,

$$X_N^C(s) - Y_N(s) = \zeta(s) - \sum_{k=1}^K c_k(s)_{2k-1} N^{-s-2k+1} + O_{s,K}(N^{-\sigma-2K-1}),$$

with $X_N^C = X_N - \frac{1}{2}N^{-s}$ and $c_k = B_{2k}/(2k)!$. Moreover $\zeta(\rho) = 0$ if and only if $X_N(\rho) - Y_N(\rho) \rightarrow 0$.

- (2) [CE1, Propositions 2.4 and 2.5] (*proved*). $X_N/Y_N \rightarrow 1$ at every point of the open strip. At every zero, $N^{2\beta}(|X_N^C(\rho)|^2 - |Y_N(\rho)|^2) \rightarrow \kappa(\rho)$, given in (1.2), and $\kappa(\rho) = -\frac{1}{6}$ if and only if $\beta = \frac{1}{2}$. On the line, at a zero, $|X_N^C|^2 = N/\lambda + \sum_m Q_m(\lambda)N^{1-2m}$, with Q_m exact rational polynomials.

- (3) [CE1, Proposition 2.6] (*classical, restated*). For $\sigma > \Theta$, $C_N(s; \Lambda) \rightarrow -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$. Convergence at one point with $\operatorname{Re} s_0 = \sigma_0 > 0$ excludes zeros with $\beta > \sigma_0$. Hence RH holds if and only if the C-transform of the primes converges in $\sigma > \frac{1}{2}$.
- (4) [CE1, Proposition 3.1] (*proved*). The energy identity $|X_N(s)|^2 = \sum_{n \leq N} C_2(n, s) + X_N(2\sigma)$ holds, and W_N evolves by the exactly solvable dynamics (1.3).
- (5) [CE1, Theorem 3.2 (Theorem A)] (*proved*). For $0 < \sigma < 1$ the anatomy (1.4) holds, and at every point of the strip, whether or not $\zeta(s) = 0$,

$$\lim_{n \rightarrow \infty} C_2(n, s) = +\infty, \quad \frac{1}{\frac{1}{4} + t^2}, \quad 0 \quad \text{for } \sigma < \frac{1}{2}, \quad \sigma = \frac{1}{2}, \quad \sigma > \frac{1}{2}.$$

- (6) [CE1, Corollary 3.5] (*proved*).
- (i) On the line, the straight-line law $|X_n(\rho)|^2 = (n + \frac{1}{2})/(\frac{1}{4} + \gamma^2) + O_\rho(n^{-1})$ holds.
 - (iii) Each of five conditions on $C_2(\cdot, \rho)$, imposed at every zero, is equivalent to RH; for example $\sup_n C_2(n, \rho) < \infty$, or $C_2(n, \rho) = O(n^\varepsilon)$. The paper calls (iii) “a reformulation of RH”.
 - (iv) Unconditionally, the limit is $1/(\gamma^2 + \frac{1}{4})$ for all zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ and for more than $\frac{5}{12}$ of all zeros.
- (7) [CE1, Propositions 3.6 and 3.8] (*proved*). For $|t| \geq t_0$, $g_s(x) = -\operatorname{Re}[x^s \zeta(s, x)]$ has exactly one zero $n^*(s) = \nu(s) + \frac{\sigma}{6} + c(\sigma)t^{-2} + O(t^{-4})$ in $x \geq |t|^{3/2}$. At a zero, $C_2(\cdot, \rho)$ changes sign there exactly once, upwards. On the line this point is given to all orders by (1.5), with exact coefficients.
- (8) [CE1, Theorems 3.11 and 3.12] (*proved*). If $\sum_{n \leq x} a_n = cx + O(x^\theta)$ with $\theta < \frac{1}{2}$, the same trichotomy holds, whether or not $F(s) = 0$. The Mellin transform is (1.6), and a zero makes the pole at $q = 1 - \sigma + it$ removable, with value $\zeta'(\sigma + it)$.
- (9) [CE1, Propositions 3.13 and 3.14] (*proved*). At a point of the line that is not a zero, every $x \geq 2\lambda$ with $C_2(x, s) \leq 0$ obeys (1.7). Also, $\zeta(\frac{1}{2} + it) = 0$ if and only if the accumulated deficit $\mathcal{H}(t, n) - H_n$ converges; at a zero its limit is $-\frac{1}{2}(\frac{1}{4} + t^2)^{-1}$, so that $\mathcal{H}(t, n) - \log n \rightarrow \gamma_0 - \frac{1}{2}(\frac{1}{4} + t^2)^{-1}$.

What is new.

- New in this form: the C-transform representation, Theorem A, the crossing law and its series to all orders, the exact dynamics, the general anatomy, the Mellin transform and the sign-change bound. All are elementary consequences of Euler–Maclaurin summation, with complete proofs.
- Classical, cited as such: Euler–Maclaurin summation, Backlund’s bound and Landau’s theorem. Propositions 2.3, 2.6, 2.8 and 2.9 are classical in substance. Remark 3.9 (the spectral variable λ , Pólya, Jensen polynomials) reads a classical reformulation.
- Corrections of claims of the author’s earlier preprints, among them [C1] and [E24]:
 - the multiplicative three-regime mechanism based on X_N/Y_N is withdrawn;
 - a crossing $C_2 = 0$ does not imply $\sigma = \frac{1}{2}$;
 - the crossing point is $\gamma^2 + \frac{1}{3}$, not $\gamma^2 + \frac{1}{4}$;
 - the “non-collapse lemma” of [C1] is RH itself;
 - the two-zero claim of [E24] fails, since the difference tends to -0.001369 ;
 - the sign-change bound improves that of [C1] by a factor 4.

New formulas. The C-transform of x^{-s} , and ζ as its regular part [CE1, Lemma 2.1, Proposition 2.3]:

$$(1.1) \quad C_N(x^{-s}) = X_N(s) - Y_N(s) + \frac{1}{1-s}, \quad \zeta(s) = C(x^{-s}) - \frac{1}{1-s}.$$

The normalized X - Y gap at a zero [CE1, Proposition 2.5]:

$$(1.2) \quad \kappa(\rho) = -\frac{1}{6} \frac{\operatorname{Re} \lambda(\rho)}{|1 - \rho|^2} = -\frac{1}{6} - \frac{(1 - \beta)(2\beta - 1)}{6|1 - \rho|^2}.$$

The dynamics of the wave, with $r_N = (N/(N+1))^\sigma$ and $\delta_N = t \log(1 + 1/N)$ [CE1, Proposition 3.1]:

$$(1.3) \quad W_{N+1} = r_N e^{i\delta_N} W_N + 2[N(N+1)]^{-\sigma} e^{i\delta_N}.$$

Theorem A, with $A_s = 2(1 - \sigma)/|1 - s|^2$ [CE1, Theorem 3.2]:

$$(1.4) \quad \begin{aligned} C_2(n, s) &= A_s n^{1-2\sigma} + 2n^{-\sigma} \operatorname{Re} [\zeta(s) n^{it}] - n^{-2\sigma} \\ &\quad - 2 \sum_{j=1}^m c_j \operatorname{Re} [(s)_{2j-1}] n^{-2\sigma-2j+1} + E_m. \end{aligned}$$

The crossing law to all orders at a zero on the line [CE1, Proposition 3.8]:

$$(1.5) \quad n^*(\rho) = \lambda + \frac{1}{12} + \frac{1}{180\lambda} - \frac{53}{5040\lambda^2} - \frac{683}{129600\lambda^3} + \frac{232349}{39916800\lambda^4} + O(\lambda^{-5}).$$

The Mellin transform of the step extension, for $\operatorname{Re} q > 2 - 2\sigma$ [CE1, Theorem 3.12]:

$$(1.6) \quad \int_1^\infty \tilde{C}_2(x, s) x^{-q} dx = \zeta(2\sigma + q - 1) \left[\frac{1}{q - (1 - \sigma + it)} + \frac{1}{q - (1 - \sigma - it)} \right].$$

The sign-change bound at non-zeros on the line [CE1, Proposition 3.13]:

$$(1.7) \quad x \leq 4|\zeta(\tfrac{1}{2} + it)|^2 \lambda^2 \left(1 + \frac{c}{\lambda}\right) + 2\lambda, \quad \lambda = \tfrac{1}{4} + t^2.$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** Each of the following is equivalent to RH:
 - convergence of the C-transform of Λ in $\sigma > \frac{1}{2}$ (Proposition 2.6), and its cross-energy form $\sup_N N^\sigma |W_N^\Lambda - W_N| < \infty$ for $\sigma > \frac{1}{2}$, which the paper calls “not a step towards it”;
 - $R_N(\frac{1}{2} + i\gamma_0) = O(N^\varepsilon)$ for a single γ_0 (Proposition 2.9);
 - the conditions of Corollary 3.5(iii).
- **Unconditional.** Theorem A, the crossing law, the Mellin transform, the sign-change bound, and Corollary 3.5(iv), which also gives $C_2(n, \rho) \leq (A_\rho + o(1))n^{1-2/(R_0 \log \gamma)}$ at every zero.
- **Control.** The Davenport–Heilbronn function has no pole, and therefore no trichotomy and no crossing. Its wave collapses as a shrinking pentagon at the on-line zero $\frac{1}{2} + 43.0813i$ and at the off-line zero $0.808517 + 85.699348i$, with slopes -1.000 and $-1.617 = -2\beta$. The winding count finds both of its off-line zeros near height 85.7.
- **Control.** The planted zero $\zeta(s) - \zeta(s_0)$ shows that “an argument that sees only the wave and the envelope cannot exclude off-line zeros”. The Epstein function of $m^2 + 100n^2$ [CE16] keeps the pole and the trichotomy but has a real zero off the line.
- **Caveat.** The trichotomy and the crossing law characterize the line, not the zeros. RH is the statement that every collapse has rate n^{-1} , and “no finite computation of C_2 can establish such a rate uniformly in the height”. The paper states: “We do not prove the Riemann hypothesis.”

Key numbers.

- $C_B = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots$ and $R_0 = 5.573412$.
- At γ_1 the crossing is at 200.1238157, against $\gamma_1^2 + \frac{1}{3} + \frac{1}{180\gamma_1^2} = 200.1238159$.
- At $\frac{1}{2} + 20i$: $|\zeta| = 1.1478424$, and (1.7) gives about 845 082, against 3 377 125 for the bound of [C1]. There are 78 integer sign changes for $N \leq 10^7$, the last between 682 701 and 682 702.

- $K = C_2(n^*, \cdot)$ has 278 zeros on the line for $10 < t < 100$, against 29 zeros of ζ .

4. PART 2: THE TRANSITION AT THE CRITICAL LINE AND HARDY'S FUNCTION

Origin. Part 2 [CE2] (25 pages) is one of the parts that replace version 4 of the consolidated paper. It takes up the Fresnel law of the preprint [C1] and the monotonicity windows of “the old Part III” [C3], with corrections.

Summary. At a zero $\rho = \frac{1}{2} + it$, Part 1 proved monotonicity of $C_2(\cdot, \rho)$ only for $n \gtrsim 0.057t^{5/3}$, while numerically it starts just below $n = t/2\pi$. At a zero the head of the series is minus its tail, and Poisson summation of the tail has a stationary point exactly when $n < t/2\pi$. Above that threshold every Fourier mode is non-stationary and C_2 is monotone (Theorem 2.5). In a $\sqrt{t}$ -window the mode $k = 1$ gives a Fresnel profile, which fixes the last decrease at $t/2\pi + \kappa\sqrt{t} + O(1)$ (Theorem 2.10); beyond it the crossing is unique. The paper then locates the maximum a hypothetical off-line zero would produce. It also proves a window law near a zero: the windows measure $|\zeta^{(m)}(\rho)|$ and the multiplicity, and under RH and the simplicity of the zeros their total is Gonek's sum $J_{-1}(T)$. A pole-free difference energy motivates three “filters” for rigidity statements.

Section 3 reads Hardy's $Z(t)$ as the signed radius of the C_2 helix. Phase-locking extracts $2Z(t)$ from the real trajectory, and the Riemann–Siegel formula and Hardy's theorem are restated in C_2 form. A scan of Z up to 10^5 finds no extremum without a crossing. The mollified cross-energy isolates the Levinson–Conrey mechanism.

Main results.

- (1) [CE2, Lemma 2.4 and Theorem 2.5] (*proved*). Uniformly for zeros and $x \geq t/(2\pi - \eta)$, $C_2(x, \rho) = \frac{1}{\lambda} - \frac{1}{x} + \Upsilon(t/x)x^{-2} + O_\eta(x^{-3})$, with Υ given by (2.1). Consequently $C_2(N+1, \rho) > C_2(N, \rho)$ for all $N \geq \max(N_\eta, t/(2\pi - \eta))$; N_η is not explicit.
- (2) [CE2, Proposition 2.7] (*proved*). The two-term Fresnel expansion (2.3) holds in the window $|w| \leq W$.
- (3) [CE2, Lemma 2.8] (*numerical*). The largest zero of Φ' is $w^* = -0.9232215874\dots$, it is simple, and $\Phi' < 0$ on (w^*, ∞) . This is proved for $w \geq 1.6$. On $[w_2, 1.6]$ it is verified on a controlled double-precision grid, “not in interval arithmetic”.
- (4) [CE2, Theorem 2.10 and Corollary 2.13] (*conditional on Lemma 2.8, which is verified numerically*). The transition law (2.4) and the floor law (2.5) hold. The minimum beyond N_{last} is $C_2(N_{\text{last}} + 1) = -4.6292\dots t^{-1/2} + O(t^{-1})$. For $t \geq t_0$ there is exactly one sign change for $N > N_{\text{last}}$, at $n^*(\rho)$. These are asymptotic statements; t_0 is not explicit.
- (5) [CE2, Proposition 2.15] (*proved*). At a hypothetical zero with $\beta = \frac{1}{2} + a$ and $\gamma \geq \gamma_0$, $C_2(\cdot, \rho)$ has exactly one maximum in $x \geq \gamma^{3/2}$, at $x_m = (1 + \frac{1}{2a})\nu(\rho)(1 + O(\gamma^{-2}))$, with $C_2(x_m, \rho) = \frac{x_m^{-2\beta}}{2a}(1 + O(\gamma^{-2}))$. This describes an off-line zero and does not exclude one.
- (6) [CE2, Theorem 2.18 and Corollary 2.19] (*proved*). The window law (2.6) holds at a zero of multiplicity m . Hence $\log N^*(\varepsilon)/\log(1/|\varepsilon|) \rightarrow 2m$.
- (7) [CE2, Proposition 2.20] (*conditional on RH and on the simplicity of the zeros*). The total windows satisfy $\mathcal{W}_\pm(T) = (1 + o(1))J_{-1}(T)$ with $J_{-1}(T) = \sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2}$. So Gonek's conjecture $J_{-1}(T) \sim \frac{3}{\pi^3}T$ is equivalent to $\mathcal{W}_\pm(T) \sim \frac{3}{\pi^3}T$.
- (8) [CE2, Propositions 2.21 and 2.25]. Proposition 2.21 (*classical, restated*): $E_N^- = |X_N - X_N^\Lambda|^2$ is free of the pole and is bounded for every $\sigma > \frac{1}{2}$ if and only if RH holds. Proposition 2.25 (*proved*): no nonnegative series that is analytic near $[1, \infty)$ can restore the positivity of the coefficients of $1 - \zeta - \zeta'/\zeta$.
- (9) [CE2, Proposition 3.1, Theorems 3.2, 3.6 and 3.9]. Proposition 3.1 and Theorem 3.2 (*proved*): the signed radius (2.7) and phase-locking, $\mathcal{R}_{N_k}(t) \rightarrow 2Z(t)$. Theorem 3.6 (*classical, restated*): Hardy's theorem in C_2 form, where “ C_2 enters only in the statement of

the conclusion, not in the proof”. Theorem 3.9 (*proved*): the mollified anatomy, in which the structural mode is scaled by $M_y(1)$, the wave ζM_y is flattened, and the endpoint term becomes a Möbius-weighted sawtooth.

What is new.

- New in this form: the Fresnel transition law, the windows and their link with Gonek’s sum, the orientation of the helix, phase-locking and the mollified cross-energy. Proofs are complete except for Lemma 2.8, which is numerical.
- Classical, cited as such: Hardy’s function, Riemann–Siegel (Proposition 3.3), Turing’s method, Hardy’s theorem, and the mollifiers of Selberg, Levinson and Conrey. Remarks 3.11–3.12 are a heuristic “dictionary for the mechanism, not an improved estimate”.
- Relative to prior work:
 - Theorem 2.5 sharpens [CE1, Corollary 3.5], and Proposition 2.15 proves [CE1, Remark 3.7];
 - C3 used $6T/\pi^3$, whereas the correct Gonek constant is $3/\pi^3$;
 - an ordinate used in [C1] was off by 10^{-9} , which does not affect any conclusion.

New formulas. The resummed correction above the threshold [CE2, Lemma 2.4]:

$$(2.1) \quad \Upsilon(\tau) = -\left[\frac{1}{4\sin^2(\tau/2)} + \frac{1}{\tau^2} - \frac{\tau \cos(\tau/2)}{4\sin^3(\tau/2)}\right] = -\frac{1}{12} - \frac{\tau^2}{80} + O(\tau^4).$$

The Fresnel profile, with $F(w) = \int_w^\infty e^{iv^2} dv$ [CE2, (2.10)]:

$$(2.2) \quad e^{-iw^2} F(w) = \Phi(w) + i\tilde{\Phi}(w), \quad \Phi_1(w) = 2w\Phi(w) + \frac{8}{3}w^3\tilde{\Phi}(w) - \frac{4}{3}w^2.$$

The two-term Fresnel expansion, with $w = \pi\sqrt{2}(x - t/2\pi)/\sqrt{t}$ [CE2, Proposition 2.7]:

$$(2.3) \quad C_2(x, \rho) = \frac{1}{\lambda} - \frac{1}{x} - \frac{2\sqrt{2}}{\sqrt{t}}\Phi(w) + \frac{\Phi_1(w)}{t} + O_W(t^{-3/2}).$$

The transition law [CE2, Theorem 2.10(b)]:

$$(2.4) \quad N_{\text{last}}(t) = \frac{t}{2\pi} + \kappa\sqrt{t} + O(1), \quad \kappa = \frac{w^*}{\pi\sqrt{2}} = -0.2077978646\dots$$

The floor law, with $c_1 = 0.0513\dots$ [CE2, Theorem 2.10(c)]:

$$(2.5) \quad N_{\text{last}}(t) = \left\lfloor \frac{t}{2\pi} + \kappa\sqrt{t} - \frac{1}{2} + c_1 \right\rfloor.$$

The window law at a zero of multiplicity m , with $s_\varepsilon = \frac{1}{2} + i(\gamma + \varepsilon)$ [CE2, Theorem 2.18]:

$$(2.6) \quad (1 - \delta)N_0(\varepsilon) \leq N^*(\varepsilon) \leq (1 + \delta)e^{2\pi/\gamma}N_0(\varepsilon), \quad N_0(\varepsilon) = \frac{(m!)^2}{4|\zeta^{(m)}(\rho)|^2|s_\varepsilon|^2|\varepsilon|^{2m}}.$$

The signed radius of the helix [CE2, Proposition 3.1]:

$$(2.7) \quad e^{i(\theta(t) - t \log n)} \sqrt{n} \left(\frac{1}{2} W_n(s) - \frac{1}{\frac{1}{2} - it} \right) = Z(t) - \frac{1}{2} e^{i(\theta(t) - t \log n)} n^{-1/2} + O(|s|n^{-3/2}).$$

Relation to the Riemann hypothesis.

- **Unconditional.** Theorems 2.5 and 2.18, Proposition 2.15 and Section 3. Theorem 2.10 and Corollary 2.13 also hold without RH but use the numerical Lemma 2.8. These results “do not distinguish the real part of a hypothetical zero off the line”.
- **Conditional on RH.** Proposition 2.20, which also assumes that the zeros are simple.
- **Equivalent to RH.** Boundedness of E_N^- for $\sigma > \frac{1}{2}$ (Proposition 2.21). Turing’s count read on the helix. The energy law $\sum_{n \leq N} C_2^\Lambda(n, \frac{1}{2}) = 4N + O(N^{1/2+\epsilon})$ of [CE5, Theorem 2.23].
- **Control.** The filters (F1)–(F3) of Definition 2.22, with the heuristic verdicts of Remark 2.24. Structure-only arguments fail (F1), because the Epstein function of $m^2 + 100n^2$ has a real zero $\beta_0 = 0.8134\dots$. For the Davenport–Heilbronn function the signed radius has 287 crossings on $0 < t < 400$, against 303 zeros: the deficit of 16 comes from its eight off-line pairs. Its Z_f has exactly eight extrema without crossing.
- **Caveat.** Theorem 3.6 “does not advance” Hardy’s result, and density one on the line is compatible with infinitely many off-line zeros (Remark 3.12). The paper states: “We do not prove the Riemann hypothesis.”

Key numbers.

- $w^* = -0.9232215874$, $\kappa = -0.2077978646$, $c_1 = 0.05134$.
- The floor law is exact for 119/120 zeros; the single miss is a boundary case.
- For 25 zeros the mean window exponent is 2.0004, so all are simple.
- $J_{-1}(T)/(3T/\pi^3) = 1.115, 1.057, 0.987, 0.965$ at $T = 100, 200, 400, 600$.
- The scan of Z on $10 < t < 10^5$ finds 138069 sign changes, against $\theta(10^5)/\pi + 1 = 138068.6$. The lowest extremum is $Z = -5.5 \cdot 10^{-4}$ at $t = 71732.909$, a Lehmer pair.

5. PART 3: THE REFLECTED CROSS-ENERGY

Origin. Part 3 [CE3] (30 pages) names no replaced paper. Its precursor is the reflected normalized defect of [C1, Sect. 1], whose “Reflected Asymptotic Rigidity Problem” Proposition 1 shows to be RH restated.

Summary. C_2 pairs s with $\bar{s}$. Part 3 pairs s with $1 - \bar{s}$ and with $1 - s$ instead, the two reflections of the functional equation.

At a reflected pair, the product $C_2(n, \rho)C_2(n, 1 - \bar{\rho})$ has a finite positive limit for every β and is a square on the line. Off the line, above height 16 and beyond $|\gamma|^{3/2}$, it is negative exactly between two crossings around $n = \gamma^2$. In two variables this window is where the real part of the Wronskian of consecutive partial sums dominates.

The reflected cross-energy C_2^{FR} is entire. It equals a function of $\lambda = s(1 - s)$ plus $\zeta(s)$ times a carrier of the functional equation, so at every zero it is a fixed function of λ_ρ . It reproduces $Z(t)$ from the integers below N and is a fixed-height Gonek–Montgomery truncation. For a Beurling system it detects off-line zeros, which drift with N unless $1 - \rho$ is also a zero.

Weil’s criterion restricted to the test functions $f * \mu_n$ remains equivalent to RH. Its positivity lives in the dual range of primes $[n, n^2]$, which $\Lambda * 1 * 1 = \log * 1$ trades for the primes up to ne^{2a} and the divisor function. As $n \rightarrow \infty$ it becomes positivity of the Hankel matrices of the power sums of $1/\lambda_\rho$, computable from ξ at $s = 1$; this criterion is classical in substance.

Main results.

- (1) [CE3, Lemma 2] (*proved*). The crossing law of [CE1, Proposition 3.6] holds with explicit constants for $|t| \geq 16$: $n^*(s) = A(s)(1 + 0.34\theta t^{-2})$ with $|\theta| \leq 1$ and $A(s) = |1 - s|^2/(2(1 - \sigma))$. The one numerical step evaluates closed-form bounds at $|t| = 16$.

- (2) [CE3, Proposition 3] (*proved*). Let $|t| \geq 16$ and $\sigma > \frac{1}{2}$. The wave-free product Π_s is negative on $n \geq |t|^{3/2}$ exactly on $(n^*(1 - \bar{s}), n^*(s))$, with the depth (3.3), and $\Pi_s \rightarrow 1/(AA') \leq (\frac{1}{4} + t^2)^{-2}$. For the zeros above height 16, $\beta = \frac{1}{2}$ if and only if $C_2(n, \rho)C_2(n, 1 - \bar{\rho}) \geq 0$ for all $n \geq |\gamma|^{3/2}$. With integers, detection is guaranteed if $|\beta - \frac{1}{2}| > \delta_{\mathbb{Z}}(\gamma)$, where $\delta_{\mathbb{Z}}(\gamma) < 1.006/(4\gamma^2)$.
- (3) [CE3, Proposition 5] (*proved*). The two-variable identity (3.4) holds. At an off-line zero with $|\gamma| \geq 16$ the product equals $-(2\beta - 1)^2\gamma^{-4}(1 + O(\gamma^{-2}))$ at the point $N_0 = \gamma^2(1 + O(\gamma^{-2}))$.
- (4) [CE3, Theorem 8] (*proved*). The decomposition (3.2) is exact, with $V_N = \zeta(s)\mathcal{K}_N(s)$ and $\mathcal{K}_N(s) = N^{s-1} + \chi(1-s)N^{-s}$. Hence $C_2^{\text{FR}}(N, \rho) = \mathcal{G}_N^*(\lambda_\rho)$ at every zero. On the line, $N^{1/2}(C_2^{\text{FR}} - \mathcal{G}_N) = 2Z(t) \cos(t \log N - \vartheta(t)) + O((|t| + 8)^6 N^{-11/2})$. For $|t| \geq 2$ the zeros of $\mathcal{K}_N$ with ordinate t in $0 < \sigma < 1$ lie on the line once $N^2 > \frac{|t|}{2\pi} e^{1/|t|}$.
- (5) [CE3, Proposition 10] (*conditional on* $N_B(x) = cx + c'x^b + O(x^\theta)$ with $\theta < \frac{1}{2}$). The construction detects every zero of ζ_B with real part in $(\frac{1}{2}, 1 - \theta)$. A detected zero is fixed in N if and only if $\zeta_B(1 - \rho) = 0$; otherwise it drifts by $-N^{1-2\rho}\zeta_B(1 - \rho)/\zeta'_B(\rho)$. For the Beurling control the drift at 0.6 is checked under a measured θ , and “it is not a theorem”.
- (6) [CE3, Proposition 11] (*proved*). RH implies $Q(f * \mu_n) \geq 0$ for all f and n . Positivity for a single n forces $\mathcal{G}_n^*(\lambda_\rho) = 0$ at every zero with $\beta > \frac{1}{2}$. For any n_0 , RH holds if and only if $Q(f * \mu_n) \geq 0$ for all real $f \in C_c^\infty(\mathbb{R})$ and all $n \geq n_0$.
- (7) [CE3, Proposition 12] (*proved*). The prime term of $Q(f * \mu_n)$ can be written with $\Lambda(m)$ only for $m \leq ne^{2a}$, the divisor function near n^2 , and an edge term $\max(j, k) \geq n$; the direct form uses Λ up to $n^2 e^{2a}$.
- (8) [CE3, Proposition 14] (*proved*). $\lim_n Q(f * \mu_n) = \sum_\rho \Phi(\gamma_\rho) t_\rho^2$ with $t_\rho = 1/\lambda_\rho$, and RH holds if and only if this limit is ≥ 0 for all f .
- (9) [CE3, Propositions 15 and 16] (*classical, restated*). Let $s_k = S_{k+2}$ with $S_k = \sum t_\rho^k$. RH holds if and only if all Hankel determinants $D_m = \det(s_{i+j})$ are positive, if and only if all $\beta_m = D_{m+1}D_{m-1}/D_m^2 > 0$. Heine’s formula (3.6) shows that off-line quadruples give negative families. Proposition 15 is proved and “classical in substance”; Proposition 16 is Heine’s classical formula. The checks up to $m = 43$ are numerical and verify RH to no height.

What is new.

- New in this series: Propositions 3 and 5; Theorem 8, a “new object; related to” Gonek–Montgomery; Propositions 10, 12 (“new in this form”) and 14; and the explicit $t_0 = 16$ for the crossing law of [CE1].
- Restriction of a classical criterion: Proposition 11 restricts Weil’s criterion, using the explicit formula and Bochner–Schwartz.
- Classical: Proposition 15 is Hermite’s theorem and the Hamburger moment problem applied to $\Xi(\lambda)$, in the circle of Csordas–Norfolk–Varga and Griffin–Ono–Rolen–Zagier. Proposition 16 is Heine’s formula.
- Prior work: the problem stated in [C1] is shown to be RH restated.

New formulas. The reflected cross-energy [CE3, Sect. 1.1]:

$$\begin{aligned}
 (3.1) \quad C_2^{\text{FR}}(N, s) &= X_N(s)X_N(1-s) - X_{N-1}(s)X_{N-1}(1-s) - \frac{1}{N} \\
 &= N^{-s}X_{N-1}(1-s) + N^{s-1}X_{N-1}(s).
 \end{aligned}$$

The exact detector decomposition, with $|\mathcal{G}_N^* - \mathcal{G}_N| \leq 1.1 \cdot 10^{-4}(|s| + 7)^6 N^{-6}$ [CE3, Theorem 8]:

$$(3.2) \quad \begin{aligned} C_2^{\text{FR}}(N, s) &= \mathcal{G}_N^*(\lambda) + N^{s-1}\zeta(s) + N^{-s}\zeta(1-s), \\ \mathcal{G}_N(\lambda) &= \frac{1}{\lambda} - \frac{1}{N} - \frac{1}{12N^2} + \left(\frac{1}{120} - \frac{\lambda}{80}\right)N^{-4}. \end{aligned}$$

The depth of the reflected-pair window, with $A' = |s|^2/(2\sigma)$ [CE3, Proposition 3(iii)]:

$$(3.3) \quad \min_{n \geq |t|^{3/2}} \Pi_s(n) = -\frac{(2\sigma-1)^2}{4\sigma(1-\sigma)} \frac{1}{AA'} (1 + O(t^{-2})).$$

The two-variable identity, with $\mathcal{W}(N; s, w) = N^{-s}X_{N-1}(w) - N^{-w}X_{N-1}(s)$ [CE3, Proposition 5]:

$$(3.4) \quad C_2(N, s)C_2(N, 1-\bar{s}) = (\operatorname{Re} C_2(N; s, 1-s))^2 - (\operatorname{Re} \mathcal{W}(N; s, 1-s))^2.$$

Weil's form on the test functions $f * \mu_n$ [CE3, Sect. 5.2]:

$$(3.5) \quad Q(f * \mu_n) = \sum_{\rho} \Phi(\gamma_{\rho}) C_2^{\text{FR}}(n, \rho)^2 = \sum_{\rho} \Phi(\gamma_{\rho}) \mathcal{G}_n^*(\lambda_{\rho})^2.$$

Heine's formula, summed over the sets J of m pairs [CE3, Proposition 16]:

$$(3.6) \quad D_m = \sum_{|J|=m} \prod_{j \in J} t_j^2 \prod_{i < j, i, j \in J} (t_i - t_j)^2.$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** The sign of the reflected-pair product, which is equivalent to RH above height 16 (Proposition 3(v)). The absence of zeros of V_N off the line. Propositions 11(c), 14 and 15. The paper says these are equivalent to RH and “not a step towards a proof”.
- **Implied by RH.** $Q(f * \mu_n) \geq 0$ for all f and n , and $D_m > 0$ for all m .
- **Conditional on RH.** Remark 9, Conrad's resonance seen through the primes. The Beurling controls are also constructed under RH.
- **Unconditional.** Lemma 2, Propositions 3, 5, 12 and 16, and Theorem 8. The objects “characterize the line, not the zeros”; the detector “detects the zeros; it does not place them”.
- **Control.** The off-line zero $0.8085171825 + 85.6993484854i$ of the Davenport–Heilbronn function is a fixed zero of the wave, the same to $3 \cdot 10^{-15}$ for $N = 200, 10^3, 10^4, 10^5$. The real zero of the Beurling system at 0.6 drifts: 0.5817, 0.5918, 0.5938 at $N = 1.12 \cdot 10^6, 1.07 \cdot 10^7, 3 \cdot 10^7$. With that system's data, Weil's functional “is not a criterion”.
- **Caveat.** The precision remark (Remark 13) is heuristic. The margin is of order T^{-2} to T^{-4} of the Gamma term, so positivity “has to come from an exact identity”. The paper states: “It does not prove RH.”

Key numbers.

- At $0.7 + i\gamma_1$ the product is negative exactly for the integers $144 \leq n \leq 333$. Its minimum is $-3.9801 \cdot 10^{-6}$, against $-3.9768 \cdot 10^{-6}$ predicted.
- At the window, the zero side is $4 \cdot 10^{-8}$ to 10^{-5} of the Gamma term. Chebyshev's identity used as an inequality misses it by a factor $1.4 \cdot 10^7$ to $4.3 \cdot 10^7$.
- $S_2 = 3.7100636 \dots \cdot 10^{-5}$.
- $\beta_m > 0$ for $m \leq 43$, and an inserted off-line pair at index j is seen at $m \approx 2j$.

6. PART 4: ZEROS OF TRUNCATIONS: SPIRALS, SECTIONS AND THE CIRCLES OF THE PRIMES

Origin. Part 4 [CE4] (24 pages) revises Section 2 of version 3 of the consolidated paper and Part VIII of the earlier series [C8]. Its Section 5 is the erratum to the 2024 “ellipse” preprint [E24].

Summary. Part 4 studies three truncations of ζ : $F_n = X_{n-1} - Y_n$, the corrected sum $D_N = X_N^C - Y_N$, and the sections $S_N = X_N$. It also places the zeros of ζ on the circles of single Euler factors.

- Near a simple zero, the zero of F_n moves on a logarithmic spiral of radius $n^{-\beta}/(2|\zeta'(\rho)|)$. On the line it crosses the line $\gamma \log 2/\pi + \vartheta$ times per dyadic range, $|\vartheta| \leq 1$; off the line it stops crossing.
- The zeros of D_N form helices of radius $|\rho|N^{-\beta-1}/(12|\zeta'(\rho)|)$, with explicit hypotheses certified for 100 zeros.
- An off-line pair converges at two rates whose product does not depend on β : the spirals measure β but do not force it.
- For S_N , Bohr and Kronecker turn the right edge $e(N)$ of the real parts of the zeros into a polygon problem on a torus, with one free direction per prime.

The 2024 “ellipse” is an exact circle, the Euler factor of a “prime” e . On the circle of a true prime, under RH, Landau’s formula makes the zeros lean toward the angle π , or toward $\pi + \arg \chi(p)$ for $L(s, \chi)$ under GRH. This bias lives in the exact positions of the zeros and is blind to β .

Main results.

- (1) [CE4, Proposition 1] (*proved*). For a simple zero ρ with $0 < \beta < 1$ and $n \geq n_1$, F_n has exactly one zero ρ_n near ρ , given by (4.1). It moves at the angle $-\gamma \log n - \arg \zeta'(\rho)$.
- (2) [CE4, Theorems 5 and 6] (*proved*). For a simple zero on the line and $N \geq N_0(\rho)$, the function $u(x) = \operatorname{Re} \rho_x - \frac{1}{2}$ has exactly $\gamma \log 2/\pi + \vartheta$ zeros in $(N, 2N]$, $|\vartheta| \leq 1$. For integer $N \geq \max(N_0(\rho), \gamma)$, sampling at the integers sees every crossing. No uniform bound for $N_0(\rho)$ is known.
- (3) [CE4, Proposition 7] (*proved*). If $\beta \neq \frac{1}{2}$, then $\operatorname{Re} \rho_x - \frac{1}{2}$ eventually has the sign of $\beta - \frac{1}{2}$. To leading order this holds for $x > (2|\zeta'(\rho)||\beta - \frac{1}{2}|)^{-1/\beta}$.
- (4) [CE4, Proposition 10] (*conditional on the hypotheses (H1)–(H3)*). Under (H1)–(H3), D_N has exactly one zero near ρ , given by (4.2), and the hypotheses persist for larger N . Separately, (H1)–(H3) are *certified in interval arithmetic* (Arb) for the first 100 zeros and all $N \geq N_0^{\text{cert}}$, with $0.391|\rho| \leq N_0^{\text{cert}} \leq 0.636|\rho|$.
- (5) [CE4, Propositions 12–14]. Proposition 12 (*classical, restated*), after Bohr and Kronecker: the closure of the set of real parts of the zeros of S_N is the set $\mathcal{A}_N$ of exponents a at which the polygon of sides k^{-a} closes. Propositions 13–14 (*proved*): $e(p) \geq e(p-1)$ for every prime p , and the polygon reduction (4.3).
- (6) [CE4, Proposition 15] (*proved, computer-assisted*). $e(2^k) < e(2^k - 1)$ for $k = 2, 3, 4$, using interval arithmetic: $e(8) < 0.93 < 0.97 \leq e(7)$ and $e(16) < 0.975 < 0.99 \leq e(15)$. The case of general k is a conjecture.
- (7) [CE4, Propositions 16 and 18] (*proved*). The curve $DC(\alpha + i\beta) = \sum_{k \geq 1} e^{-k(\alpha + i\beta)}$ of [E24] is the circle (4.4), and $1 + DC = (1 - e^{-s})^{-1}$. By Hlawka’s theorem, asymptotically a fraction $1 - \operatorname{gd}(\alpha)/\pi$ of the zeros lies left of its centre.
- (8) [CE4, Theorem 19 and Corollary 20]. Theorem 19 (*classical, restated*): Landau’s formula $\sum_{0 < \gamma \leq T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O_x(\log T)$. Corollary 20 (*conditional on RH*): the bias (4.5) on the circle of a prime power, and a mean $O_x(1/T)$ for any other base.
- (9) [CE4, Proposition 21] (*classical, restated*), proved in outline: Landau’s formula for $L(s, \chi)$, with the main term $\Lambda(x)\chi(x)$. The heat-flow Landau law (decay, rotation by $-\frac{\pi}{8}t \log p$, and memory of composites) is proved in [CE15, Theorem 11.1] for fixed n and $0 < t \leq \frac{1}{2}$ and is only summarized here.

What is new.

- Classical: work on the zeros of sections by Turán, Montgomery, Montgomery–Vaughan and Platt–Trudgian; Landau’s formula, Hlawka and Ford–Zaharescu; Turing’s method; Backlund’s bound (Lemma 9); and the Bohr–Kronecker torus.
- New, several of them “elementary once formulated”:
 - the spiral description of the convergence of the zeros, with an explicit, checkable version;
 - the fixed product of the two rates of an off-line pair;
 - a Turing-type crossing count with weight $\gamma \log 2/\pi$ per zero;
 - the polygon reduction, the prime step and the behaviour of prime powers;
 - the erratum to [E24];
 - the comparison of the various circles.
- Relative to prior work:
 - version 3 of the consolidated paper had the spiral angle off by π and lacked the factor $\log n$ in the error term;
 - the [E24] “ellipse” is an exact circle;
 - the [E24] two-zero claim fails.

New formulas. The spiral of a simple zero [CE4, Proposition 1]:

$$(4.1) \quad \rho_n = \rho + \frac{n^{-\rho}}{2\zeta'(\rho)} + O_\rho(n^{-2\beta} \log n + n^{-\beta-1}).$$

The explicit spiral, with $q_N = |\rho|N^{-\beta-1}/12$ and $r_N = 3q_N/|d|$ [CE4, Proposition 10]:

$$(4.2) \quad \begin{aligned} \rho_N &= \rho + \frac{\rho N^{-\rho-1}}{12\zeta'(\rho)}(1 + \theta_N), \\ |\theta_N| &\leq \left(1 + \frac{r_N}{|\rho|}\right) N^{r_N}(1 + \bar{\eta}) - 1 + \frac{9M_2q_N}{2|d|^2}. \end{aligned}$$

The polygon reduction for the sections [CE4, Proposition 14]:

$$(4.3) \quad P_N(a, w) = R_N(w') + \sum_{\sqrt{N} < p \leq N} w_p p^{-a} C_p(w').$$

The circle of e , with $r = e^{-\alpha}$ [CE4, Propositions 16 and 18]:

$$(4.4) \quad \begin{aligned} h &= \frac{r^2}{1 - r^2}, \quad R = \frac{r}{1 - r^2}, \\ \mathbb{P}(\operatorname{Re} z < h) &= 1 - \frac{\operatorname{gd}(\alpha)}{\pi}, \quad \operatorname{gd}(\alpha) = \arctan(\sinh \alpha). \end{aligned}$$

The bias on the circle of a prime power x , under RH [CE4, Corollary 20]:

$$(4.5) \quad \frac{1}{N(T)} \sum_{0 < \gamma \leq T} \cos(\gamma \log x) = -\frac{\Lambda(x)}{\sqrt{x} \log \frac{T}{2\pi e}} (1 + o(1)).$$

Relation to the Riemann hypothesis.

- **Unconditional.** Propositions 1, 7, 12–16 and 18, Theorems 5, 6 and 19, and Proposition 10 given (H1)–(H3). “None of these results bears on RH beyond what is said: each truncation reads off β for a zero that has been located, and none excludes $\beta \neq \frac{1}{2}$.” The rates of an off-line pair multiply to n^{-1} whatever β is.
- **Conditional on RH.** Corollary 20; Proposition 21 needs GRH for the twisted bias. Without RH, Landau’s formula weights each angle by $x^{\beta-1/2}$, and the angles “do not see β ”.

- **Control.** At the off-line pair of the Davenport–Heilbronn function, the spiral exponents are -0.809 and -0.191 , against -0.8085 and -0.1915 predicted. The pair gives no crossings. Its prime spectrum takes both signs, whereas the one-signed spectrum of ζ “is the Euler product seen from the zeros”.
- **Caveat.** The crossing count would verify RH only up to a height, and only with an explicit uniform N_0 (Remark 8, heuristic). “The geometry therefore says where a zero is, not why it must lie on the line.” The abstract states: “We do not prove the Riemann hypothesis.”

Key numbers.

- Crossings for the first 29 zeros: 397, against 398.1 predicted.
- $N_0^{\text{cert}} = 9$ at ρ_1 and 129 at ρ_{100} .
- Sections: $e(3) = 0.78788$, $e(4) = 0.62629$, and the first edge above 1 is $e(19) = 1.0070$. The only prime power up to 128 that raises the edge is 11^2 , from 1.03714 to 1.03720.
- Circle of e : $h = 0.5819767$ and $R = 0.9595174$. The predicted fraction is 84.709%, against $290/341 = 85.04\%$ observed.
- The mean of $\cos(\gamma \log 2)$ over 202 zeros is -0.1465 , against -0.1545 predicted.

7. PART 5: THE ENERGY OF THE PRIMES

Origin. Part 5 (25 pages) is one of the five parts (1, 2, 5, 10 and 16) that replace version 4 of the consolidated paper (doi:10.5281/zenodo.23243641); its arithmetic cross-energy comes from the earlier preprint “Arithmetic cross-energy of prime partial sums: positivity, phase-locking, and the pole barrier (Part II)” and the author’s manuscript on the asymptotic balance of partial sums, and Remark 2.15 is an erratum to the 2024 preprint “Ellipse symmetry in Dirichlet series”.

Summary. Part 5 connects the cross-energy with the mean square of the normalized error of the prime number theorem, $E(u) = e^{-u/2}(\psi(e^u) - e^u + \log 2\pi + \frac{1}{2} \log(1 - e^{-2u}))$, whose explicit-formula expansion is $-\sum_{\rho} e^{(\rho - \frac{1}{2})u} / \rho$. Theorem B states that the mean energy of this prime signal equals the sum over the zeros of the limits $\lim_n C_2(n, \rho)$ (for simple zeros), that both are finite exactly under RH, and (Corollary 2.3) that the energy is at least $C_B = 2 + \gamma_0 - \log 4\pi = 0.0461914 \dots$, with equality if and only if RH holds and all zeros are simple. The same constant is twice Li’s first coefficient. The paper also writes the explicit formula in the cross-energy coordinates $C_2(\infty, \rho)$, $S_2(\infty, \rho)$, proves an unconditional local energy law in a Gaussian window as an absolutely convergent double sum over pairs of zeros, and places the Nyman–Beurling distance and the Gram data of the Báez-Duarte criterion in the same picture.

The second half introduces the arithmetic cross-energy C_2^Λ of the partial sums of $\Lambda(n)n^{-s}$. It satisfies a 3-4-1 positivity at every finite cut, which fails for the Epstein zeta function of $m^2 + 5n^2$. Phase-locking extracts Mertens’ combination and gives $\zeta(1 + it) \neq 0$ by a new route, and a “pole barrier” explains why positivity reaches only $\sigma = 1$. RH is equivalent to $\sum_{n \leq N} C_2^\Lambda(n, \frac{1}{2}) = 4N + O(\sqrt{N})$. The numerics use the prime powers up to 10^8 (ψ up to 10^9), with the Davenport–Heilbronn function and a Dirichlet L -function of conductor 5 as controls.

Main results.

- (1) [CE5, Theorem 2.2] (*proved*) Growth rate. Let $\Theta = \sup \operatorname{Re} \rho$ and $F(U) = \int_0^U E(u)^2 du$. If $\Theta > \frac{1}{2}$, then $\limsup_{U \rightarrow \infty} \frac{1}{U} \log F(U) = 2\Theta - 1$. If $\Theta = \frac{1}{2}$, then $F(U) \sim C_B U$ when the zeros are simple, and $F(U) = O(U)$ in general. Hence RH holds if and only if $F(U) = O(U)$, and if and only if F grows subexponentially. The converse half (bounded running energy implies RH) is Cramér’s classical Proposition 2.1(b).

- (2) [CE5, Theorem B] (*proved*) The C_2 energy law: in $[0, \infty]$,

$$\limsup_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)^2 du = \sum_{\rho} \lim_{n \rightarrow \infty} C_2(n, \rho)$$

if all zeros are simple, and in general both sides are finite or both are infinite. RH holds if and only if they are finite, and then the right side equals $\sum_{\rho} |\rho|^{-2} = C_B$. The paper calls this “a reformulation, not a new constraint”.

- (3) [CE5, Corollary 2.3] (*proved*) With $\mathcal{E} = \limsup_U \frac{1}{U} \int_0^U E^2 \in [0, \infty]$, we have $\mathcal{E} \geq 2 + \gamma_0 - \log 4\pi = 0.0461914\dots$, with equality if and only if RH holds and all zeros are simple. By [CE5, Proposition 2.4] (classical, Li; the C_2 form is proved), $\lambda_1 = \sum_{\rho} 1/\rho = C_B/2 = 0.0230957\dots$ unconditionally, and $\frac{1}{2} \sum_{\rho} \lim_n C_2(n, \rho) \geq \lambda_1$, with equality if and only if RH holds.
- (4) [CE5, Proposition 2.5] (*proved*) The non-holomorphic gap: $\sum_{\rho} |\rho|^{-2} = C_B + \mathcal{Q}$ with $\mathcal{Q} \geq 0$ an absolutely convergent sum, and $\mathcal{Q} = 0$ if and only if RH holds. The identity is elementary; the equivalence is known.
- (5) [CE5, Theorem 2.7] (*proved*) The explicit formula in cross-energy coordinates. By [CE5, Proposition 2.6], on the critical line $1/s = \frac{1}{2}(C_2(\infty, s) - iS_2(\infty, s))$, and at a zero on the line $C_2(n, \rho) = 1/|\rho|^2 - 1/n - 1/(12n^2) + O(|\rho|^4 n^{-4})$. Part (a) is unconditional:

$$\psi_0(x) = x - \frac{1}{2} \sum_{\rho} x^{\rho} \overline{W_{\infty}(1 - \bar{\rho})} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}).$$

Part (b) is conditional on RH. In it, a distinct ordinate $\gamma > 0$ of multiplicity $m(\gamma)$ contributes one cosine in $\log x$ with amplitude $2m(\gamma)\sqrt{C_2(\infty, \rho)} = 2m(\gamma)/|\rho|$ and phase $\arctan 2\gamma$.

- (6) [CE5, Theorem 2.11] (*proved*) Local energy law. The Gaussian-window energy $\mathcal{E}(U, H)$ is an absolutely convergent double sum over all pairs of zeros, for all U and $H > 0$; see (5.5) below. By [CE5, Corollary 2.12] (*proved*):
- (a) under RH, $|\mathcal{E}(U, H) - C_B| \leq B(H)$;
 - (b) RH holds if and only if $\sup_U \mathcal{E}(U, H) < \infty$;
 - (c) an isolated off-line pair $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$ costs $\sqrt{\mathcal{E}(U, H)} \geq e^{\delta U + \delta^2 H^2 / (2\pi)} / |\rho_0| - D_{\gamma_0}(H) - T_{\gamma_0}(U, H, \Gamma)$.
- (7) [CE5, Theorem 2.17] (*proved*) 3-4-1 positivity at every finite cut: for all real $x > 1$, σ and t , $3C_2^{\Lambda}(x, \sigma) + 4C_2^{\Lambda}(x, \sigma + it) + C_2^{\Lambda}(x, \sigma + 2it) \geq 0$. Its analogue for the Epstein zeta function of $m^2 + 5n^2$ is false ([CE5, Proposition 2.18], numerical): at $x = 127$ the normalized combination has minima from -26.35 ($\sigma = 0.5$) to -0.70 ($\sigma = 1.2$).
- (8) [CE5, Proposition 2.21] (*proved*) The pole barrier. For every nonnegative trigonometric polynomial $k(\theta) = \sum_{j \leq J} a_j \cos j\theta$, not identically zero, and every $t \neq 0$, the combination $\frac{1}{2} \sum_j a_j C_2^{\Lambda}(x, \frac{1}{2} + ijt)$ is nonnegative for every $x > 1$. By the prime number theorem alone it tends to $\int_0^{\infty} e^{-u/2} k(tu) du > 0$, “whatever the location of the nontrivial zeros”. The phase-locked Mertens limit [CE5, Theorem 2.19] and its consequence $\zeta(1 + it) \neq 0$ [CE5, Corollary 2.20] are proved; the conclusion is the theorem of Hadamard and de la Vallée Poussin, and only the route is new.
- (9) [CE5, Theorem 2.23] (*proved*) The arithmetic energy criterion for $S^{\Lambda}(N) = \sum_{n \leq N} C_2^{\Lambda}(n, \frac{1}{2})$. Parts (a), (b) and (d) are proved; part (c) is conditional on RH and proved in outline. The following are equivalent: RH; $S^{\Lambda}(N) = 4N + O(\sqrt{N})$; and $S^{\Lambda}(N) = 4N + O_{\varepsilon}(N^{1/2+\varepsilon})$ for every $\varepsilon > 0$. The paper stresses that this “is a reformulation, not a proof”.

What is new.

- *New, as stated in the paper:* “Theorem B in this form, its local version, the cross-energy coordinates of the explicit formula and the arithmetic energy criterion are, as far as we know, new.”
- *New route only:* the proof of $\zeta(1+it) \neq 0$ through positivity of the finite trajectory of C_2^Λ , with phase-locking before the limit $\sigma \rightarrow 1^+$. The kernel is Mertens’.
- *New readings of classical objects:*
 - $\lim_n C_2(n, \rho)$ is the contribution of ρ to the Nyman–Beurling distance;
 - the energy matrix $M(U)$ of the zeros has the C_2 limits on its diagonal and trace C_B ;
 - $\mathcal{Q}$ is a weighted sum of squared growth exponents of C_2 ;
 - the harmonic offset of a zero is half its C_2 limit.
- *Classical, cited as such:* Cramér’s theorem; the equivalence of bounded prime energy with RH; the criteria of Li and of Nyman–Beurling–Báez-Duarte; Vasyunin’s Gram entries; the conjecture of Báez-Duarte, Balazard, Landreau and Saias; Burnol’s lower bound; Ng’s Möbius energy.
- *Correction of prior work:* Remark 2.15 is an erratum to the author’s 2024 preprint on ellipse symmetry. That preprint claimed that the difference of the harmonic offset at two zeros tends to 0; it tends to $\frac{1}{2}[L(\gamma_2) - L(\gamma_1)]$, which is -0.001369 for the first two zeros.

New formulas. The prime signal, used throughout Section 2 of [CE5]:

$$(5.1) \quad E(u) = e^{-u/2} \left(\psi(e^u) - e^u + \log 2\pi + \frac{1}{2} \log(1 - e^{-2u}) \right).$$

The non-holomorphic gap [CE5, Proposition 2.5], with $w_\rho = 1 - \frac{1}{\rho}$:

$$(5.2) \quad \sum_{\rho} \frac{1}{|\rho|^2} = 2 + \gamma_0 - \log 4\pi + \mathcal{Q},$$

$$\mathcal{Q} = 2 \sum_{\rho} \frac{(\beta - \frac{1}{2})^2}{|\rho|^2 |1 - \rho|^2} = \frac{1}{2} \sum_{\rho} (|w_\rho| - |w_\rho|^{-1})^2 \geq 0.$$

The dictionary on the critical line [CE5, Proposition 2.6(ii)], for $s = \frac{1}{2} + it$:

$$(5.3) \quad \frac{1}{s} = \frac{C_2(\infty, s) - iS_2(\infty, s)}{2},$$

$$C_2(\infty, s) = \frac{1}{|s|^2} = \frac{1}{\frac{1}{4} + t^2}, \quad S_2(\infty, s) = 2t C_2(\infty, s).$$

The explicit formula under RH in amplitude–phase form [CE5, Theorem 2.7(b)]:

$$(5.4) \quad \psi_0(x) = x - 2\sqrt{x} \sum_{\gamma > 0} \sqrt{C_2(\infty, \rho)} \cos(\gamma \log x - \arctan 2\gamma)$$

$$- \log 2\pi - \frac{1}{2} \log(1 - x^{-2}).$$

The local energy law [CE5, Theorem 2.11, eq. (2.2)], the sum over all pairs of zeros converging absolutely:

$$(5.5) \quad \mathcal{E}(U, H) = \frac{1}{H} \int_{-\infty}^{\infty} E(u)^2 e^{-\pi(u-U)^2/H^2} du$$

$$= \sum_{\rho, \rho'} \frac{e^{zU + z^2 H^2 / (4\pi)}}{\rho \bar{\rho}'}, \quad z = \rho + \bar{\rho}' - 1.$$

The arithmetic cross-energy [CE5, eq. (2.3)]:

$$(5.6) \quad \begin{aligned} W^\Lambda(x, s) &= C_2^\Lambda(x, s) + iS_2^\Lambda(x, s) = 2x^{-s} \sum_{n < x} \Lambda(n)n^{-s}, \\ C_2^\Lambda(x, s) &= 2x^{-\sigma} \sum_{n < x} \Lambda(n)n^{-\sigma} \cos\left(t \log \frac{x}{n}\right). \end{aligned}$$

The second-order form of the arithmetic energy criterion under RH [CE5, Theorem 2.23(c)], where the sum over the zeros converges absolutely:

$$(5.7) \quad S^\Lambda(N) = 4N - 4\frac{\zeta'}{\zeta}\left(\frac{1}{2}\right)\sqrt{N} - 2\sqrt{N} \sum_{\rho} \frac{N^{i\gamma}}{\rho(\rho - \frac{1}{2})} + O(\log N).$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.**

- Finiteness of the mean prime energy, equivalently of $\sum_{\rho} \lim_n C_2(n, \rho)$ (Theorem B).
- $F(U) = O(U)$, and subexponential growth of F (Theorem 2.2).
- Equality in Proposition 2.4.
- $\mathcal{Q} = 0$ (Proposition 2.5, known).
- $\sup_U \mathcal{E}(U, H) < \infty$ (Corollary 2.12(b)).
- $d_N \rightarrow 0$ (Nyman–Beurling, Báez-Duarte; classical).
- $S^\Lambda(N) = 4N + O(\sqrt{N})$ (Theorem 2.23(d)).

Equality $\mathcal{E} = C_B$ is equivalent to RH together with simplicity of the zeros (Corollary 2.3).

- **Conditional on RH.**

- Proposition 2.1(a) (Cramér; RH and simple zeros).
- Theorem 2.7(b).
- Remarks 2.8 and 2.9 (Cramér’s mean square; Schoenfeld’s constant $\frac{1}{8\pi}$ as a sum of amplitudes).
- Corollary 2.12(a).
- Proposition 2.13, first part (RH and simple zeros; proved in outline).
- Remark 2.22 (proved in outline).
- Theorem 2.23(c).
- The Möbius constant C_M , which also needs Ng’s hypothesis.
- The conjecture $d_N^2 \log N \rightarrow C_B$, under RH and simple zeros.

- **Unconditional.** Theorem 2.2, Theorem 2.7(a), Theorem 2.11, Theorems 2.17 and 2.19, Corollary 2.20, Proposition 2.21, and the truncated explicit formula (2.6) for the prime sum. Because (2.6) is unconditional, “the known zeros extrapolate the arithmetic cross-energy in length, not in height”.

- **Control.**

- Davenport–Heilbronn: the running energy of its signal grows (0.21 to 0.61 up to $x = 1.2 \cdot 10^6$), with local growth rate 0.58 on $12 \leq U \leq 16$, between the exponents $2\beta - 1 = 0.617$ and 0.739 of its zeros $0.8085 + 85.70i$ and $0.8695 + 240.40i$.
- The Dirichlet L -function of conductor 5 behaves like ζ .
- Epstein: the 3-4-1 positivity fails for $m^2 + 5n^2$ (Proposition 2.18), so Theorem 2.17 passes the filter (F1) of [CE2, Definition 2.22].

- **Caveat.** “We do not prove the Riemann hypothesis.” Proving that the right side of Theorem B is finite is RH. Every finite window of primes controls only a finite range of heights: Corollary 2.12(c) excludes only isolated off-line pairs with, for example, $\beta > 0.600$ at $\gamma_0 = 14.13$ and $\beta > 0.921$ at 1900, which is far weaker than the direct verification of RH.

Key numbers. $C_B = 0.0461914\dots$ and $\lambda_1 = 0.0230957\dots$. The prime energy from the prime powers up to 10^8 is 0.0453, against the truncated zero sum 0.04324 ($|\gamma| < 600$); the autocorrelation from the primes matches the zero side with correlation 0.9997. All λ_n with $n \leq 120$ are positive ($\lambda_{100} = 118.60377538$). Richardson extrapolation gives $J(\delta) \rightarrow 0.148$, against $\pi C_B = 0.1451$. For $300 \leq N \leq 1200$, $d_N^2 \log N \in [0.0449, 0.0464]$ with mean 0.0456. Möbius energy: $C_M = 0.02914$, $K_M = 1.2603 \pm 0.0023$. The maxima of $|\psi(x) - x|/(\sqrt{x} \log^2 x)$ fall from 0.0365 (at $x = 97$) to 0.0023, against $\frac{1}{8\pi} = 0.0398$. Phase-locked Mertens limit: $M(1.5, 10) = 5.0891$. Theorem 2.23: $-4\frac{\zeta'}{\zeta}(\frac{1}{2}) = -10.74437$, against measured values of $(S^\Lambda(N) - 4N)/\sqrt{N}$ between -10.730 and -10.755 for $10^4 \leq N \leq 10^7$. Cross-references: Theorem A of [CE1] is the foundation; [CE6] treats the second-order constant $K = 2.51248\dots$, the fluctuations, the random models and the orbits, and [CE7] the large values; harmonic measure and Jensen weights are from [CE10]; Epstein and pole-barrier material is in [CE16].

8. PART 6: SECOND-ORDER ASYMPTOTICS AND LIMITING FLUCTUATIONS OF THE PRIME ENERGY

Origin. Part 6 (22 pages) and Part 7 together supersede Parts II–IV of the preprint “The cross-energy of Dirichlet partial sums: from the energy law of the primes to large values of Dirichlet polynomials” and the note “The mean-square prime energy: constant term, limiting distribution and variance of its fluctuations”; it also contains the constant term, the random models and the orbits from version 3 of the consolidated paper, and an earlier version of it, titled “Fluctuations and large values of the prime energy”, also contained the material now in Part 7.

Summary. Part 6 studies the terms that follow Cramér’s slope in the energy of the primes $I(X) = \int_1^X (\psi(x) - x)^2 x^{-2} dx$. It assumes RH, simple zeros and the spacing condition (H*) below. Under these hypotheses it proves $I(X) = C_B \log X + K + \Phi(\log X) + O(X^{-1/2})$, where Φ is an absolutely convergent almost periodic sum over pairs of zeros at the frequencies $\gamma - \gamma'$. The constant K has a closed form without zeros, and under RH its value is certified in interval arithmetic. Under RH and simple zeros alone, K is the Cesàro mean of the deviation if and only if a weak small-gap condition holds: a pair of zeros at distance λ acts like a double zero for $\log X \ll 1/\lambda$. Under (H*), $I(X) - C_B \log X$ has a limiting logarithmic distribution with mean K ; under linear independence (LI) its variance is an explicit double sum over the zeros.

The numerics have three parts. The primes up to 10^9 agree with the zero side to $1.3 \cdot 10^{-5}$ (r.m.s.). Random models of the primes and of μ carry 8 to 150 times more energy, a measure of the rigidity of the arithmetic. A Gaussian demodulation of the prime signal isolates the principal mode of each zero: a closed ellipse on the line, an expanding spiral off it.

Main results.

- (1) [CE6, Lemma 1] (*proved*) Local L^2 convergence of the zero expansion, unconditional: $\sum_{\rho, \rho'} J(\rho, \rho') < \infty$, and $S_T \rightarrow S$ in $L^2(0, U)$ for every $U > 0$. The companion statements [CE6, Lemmas 2 and 3] (uniform window energy $\int_V^{V+1} |S_F|^2 \leq e^{\pi/4} M$, and the cross term $\int_0^\infty S r = -\sum_\rho R(1 - \rho)/\rho$) are proved under RH.
- (2) [CE6, Theorem 5] (*conditional on (H*)*) Second-order energy law. With $u = \log X$,

$$I(X) = C_B u + K + \Phi(u) + O(e^{-u/2}).$$

By [CE6, Proposition 4], $H = \sum_{\rho \neq \rho'} |c_{\rho \rho'}| < \infty$ under (H*), so Φ is Bohr almost periodic with mean zero and $|\Phi| \leq H$. Only Φ uses the spacing condition; the three terms of K are defined under RH alone.

- (3) [CE6, Proposition 8] (*conditional on RH*) K without zeros. With $\sigma_2 = \sum_\rho \rho^{-2}$ and $P = \sum_{n \geq 2} \Lambda(n) \varphi(1/n)$, the paper proves $\sigma_2 = 1 + \gamma_E^2 + 2\gamma_1 - \pi^2/8$ and the closed formula (6.4)

below. RH enters twice: through $|\rho|^2 = \rho(1 - \rho)$, and through the absolute convergence of $\int_1^\infty (\psi(x) - x)x^{-2}dx$. The preceding [CE6, Proposition 7] expresses the three terms as convergent sums; its identities for $G(a)$ and for the trivial term are stated unconditionally.

- (4) [CE6, Corollary 9] (*computer-assisted, conditional on RH*) Assume RH. Then $K \in [2.512479790006 \pm 6 \cdot 10^{-13}]$, evaluated in ball arithmetic (Arb, 128 bits). The paper labels this “computer-assisted, under RH”, and the abstract describes it as certified in interval arithmetic.
- (5) [CE6, Proposition 11] (*conditional on RH and simple zeros*) The constant as a Cesàro mean, without (H^*) . With $\mathcal{D}(u) = I(e^u) - C_B u$ and $U \geq 1$,

$$\frac{1}{U} \int_0^U \mathcal{D}(u) du = K + \mathcal{E}(U) + O\left(\frac{1}{U}\right),$$

with $\mathcal{E}(U)$ as in (6.5) below. Moreover $\lim_{U \rightarrow \infty} \frac{1}{U} \int_0^U (I(e^u) - C_B u) du = K$ if and only if $\nu(\varepsilon) = o(\varepsilon)$, where

$$\nu(\varepsilon) = \sum_{\substack{\rho \neq \rho' \\ 0 < |\gamma - \gamma'| < \varepsilon}} \frac{1}{|\rho||\rho'|}.$$

- (6) [CE6, Proposition 12] (*conditional on RH, simple zeros and (Q)*) Under the mean-square approximation (Q), $\mathcal{D}$ is B^2 -almost periodic and has a limiting distribution with mean K and variance $\lim_T \lim_U \frac{1}{U} \int_0^U |\Phi_T|^2$. Moreover $(H^*) \Rightarrow (Q) \Rightarrow \nu(\varepsilon) = o(\varepsilon)$. Under LI, (Q) also forces the quadratic spacing condition (H_2) . [CE6, Problem 13] (the main open problem) asks for a proof of (Q) under RH, simple zeros, (H_2) and $\nu(\varepsilon) = o(\varepsilon)$.
- (7) [CE6, Theorem 14] (*conditional on (H^*)*) Limiting distribution. Here $a_\lambda = \sum_{\gamma - \gamma' = \lambda} c_{\rho\rho'}$ collects the coefficients of Φ by frequency.
- (1) $\mathcal{D}$ has a limiting distribution μ .
 - (2) $\int x d\mu = K$, $\int (x - K)^2 d\mu = V := \sum_\lambda |a_\lambda|^2$, and $\text{supp } \mu \subset [K - H, K + H]$.
 - (3) Under LI in addition, V is the explicit double sum (6.6) below, and μ is the law of $K + \sum_{\rho \neq \rho'} c_{\rho\rho'} e^{i(\theta_\gamma - \theta_{\gamma'})}$ with independent uniform phases.

Existence of the distribution is standard; the identification of the mean and the explicit variance are new. In [CE6, Remark 15], if some $\beta > \frac{1}{2}$, then no limiting distribution exists; this is “a consequence of RH, not a criterion that can be verified in finite range”.

- (8) [CE6, Proposition 16] (*proved*) The orbit identity: the Gaussian demodulation $A_{\gamma_0}(u)$ of the prime signal at a frequency γ_0 is an absolutely convergent sum over all zeros, given by (6.7) below. By [CE6, Corollary 17] (proved), it splits into a principal mode and an interference term with explicit bounds:
- for $\beta_0 = \frac{1}{2}$ the principal mode is $-1/\rho_0$, a closed ellipse;
 - for $\beta_0 = \frac{1}{2} + \delta$ it is $-e^{\delta u + \delta^2 s^2/2}/\rho_0$, an expanding spiral.

What is new.

- *New, proved here:*
 - Theorem 5 with its constant K ;
 - the closed formula of Proposition 8 and the certified value of K ;
 - Proposition 11, which characterizes when K is the Cesàro mean without (H^*) ;
 - Proposition 12, described as a direct consequence of a theorem of Akbary, Ng and Shahabi;
 - the identification of the mean and the explicit variance in Theorem 14;
 - the orbit identity (Proposition 16, elementary).

- *New formulations of known objects:* the zero-zero part of K as the second elementary symmetric function of the C_2 limits; the variance as a weighted Rodgers–Tao energy with weights $w_\rho = \lim_n C_2(n, \rho) = |\rho|^{-2}$; single zeros read through their principal modes.
 - *Classical, cited:*
 - Cramér’s theorem and the equivalence of bounded mean energy with RH [CE5];
 - the explicit formula and the Hadamard product;
 - the theorems of Wintner and Jessen–Wintner on limiting distributions, and the B^2 theory of Akbary–Ng–Shahabi;
 - the Montgomery–Vaughan inequality and Trudgian’s bound for $N(T)$.
- Lemmas 1–3 are “standard in nature”.
- *Prior work:* Brent, Platt and Trudgian give the leading term of the logarithmic mean but no constant term. A targeted search did not find K , its closed form or the distribution of $I(X) - C_B \log X$. The values 2.481, $\dots$, 2.511 reported in [CE11] are values of $K + \Phi(\log N)$.
 - *Numerical observations, not theorems:* the values of V , the shape of the distribution, the agreement with the primes up to 10^9 , the random models, and the orbits and phases.

New formulas. The spacing hypothesis of Theorem 5, written (H^*) in [CE6, Sect. 3], where δ_ρ is the distance from γ to the nearest other ordinate:

$$(6.1) \quad \text{RH holds, all zeros are simple, and} \quad \sum_{\rho} \frac{\log(|\gamma| + 2)}{|\rho|^2 \delta_\rho} < \infty.$$

The oscillating term [CE6, eq. (5)]:

$$(6.2) \quad \Phi(u) = \sum_{\rho \neq \rho'} c_{\rho\rho'} e^{i(\gamma - \gamma')u}, \quad c_{\rho\rho'} = \frac{1}{i(\gamma - \gamma') \rho \bar{\rho}'}.$$

The constant of the second-order law [CE6, Theorem 5, eq. (6)]:

$$(6.3) \quad K = \frac{1}{2} \left(C_B^2 - \sum_{\rho} \frac{1}{|\rho|^4} \right) - 2 \sum_{\rho} \frac{R(1 - \rho)}{\rho} + \frac{1}{2\pi} \int_{-\infty}^{\infty} |R(\tfrac{1}{2} + it)|^2 dt.$$

K without zeros under RH [CE6, Proposition 8, eq. (13)], with $\varphi(y) = y - \operatorname{artanh} y - \frac{y}{2} \log(1 - y^2)$ and $\sigma_2 = 1 + \gamma_E^2 + 2\gamma_1 - \frac{\pi^2}{8}$:

$$(6.4) \quad K = \frac{C_B^2}{2} - C_B - \sigma_2 + 2(1 + \gamma_E) \log 2\pi - (\log 4\pi - 1)^2 - 1 + 2 \sum_{n \geq 2} \Lambda(n) \varphi(1/n).$$

The error term in the Cesàro mean [CE6, Proposition 11], a sum over unordered pairs of distinct zeros:

$$(6.5) \quad \mathcal{E}(U) = \sum_{\{\rho, \rho'\}} \frac{2}{(\gamma - \gamma')^2 U} \operatorname{Re} \left(\frac{1 - e^{i(\gamma - \gamma')U}}{\rho \bar{\rho}'} \right).$$

The variance under LI [CE6, Theorem 14, eq. (15)]:

$$(6.6) \quad V = 2 \sum_{\substack{\rho \neq \rho' \\ \rho' \neq \bar{\rho}}} \frac{1}{(\gamma - \gamma')^2 |\rho|^2 |\rho'|^2} + \sum_{\rho} \frac{1}{4\gamma^2 |\rho|^4}.$$

The orbit identity [CE6, Proposition 16, eq. (16)], valid for every $s > 0$ and real u, γ_0 :

$$(6.7) \quad A_{\gamma_0}(u) = - \sum_{\rho} \frac{e^{z_\rho u + z_\rho^2 s^2 / 2}}{\rho}, \quad z_\rho = \rho - \frac{1}{2} - i\gamma_0.$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** No new equivalence. The paper recalls from [CE5, Corollary 2.3] that proving finiteness of the energy is RH.
- **Implied by RH.** If RH and (H^*) hold, then the deviation $I(X) - C_B \log X$ has a limiting distribution with bounded support (Theorem 14). If some zero has $\beta > \frac{1}{2}$, then no limiting distribution exists (Remark 15). This is “what RH looks like at second order”.
- **Conditional on RH.**
 - RH alone: Lemmas 2–3, Proposition 8, Corollary 9.
 - RH and simple zeros: Proposition 11.
 - (H^*) : Proposition 4, Theorem 5, Theorem 14 (and LI for part 3).
 - RH, simple zeros and (Q) : Proposition 12.

The hypotheses (H^*) , (H_2) , (Q) and LI are unproved. Heuristically (H^*) is expected under the GUE model.

- **Unconditional.** Lemma 1; the identities for $G(a)$ and for the trivial term in Proposition 7; the orbit identity (Proposition 16); and Corollary 17, under local on-line hypotheses on the nearby zeros.
- **Control.** The orbits of the Davenport–Heilbronn function are computed from its coefficients up to $2 \cdot 10^7$.
 - The on-line zero $\frac{1}{2} + 43.0813i$ gives a closed ellipse.
 - The off-line zeros $0.8085 + 85.6993i$ and $0.8695 + 240.4047i$ give spirals with measured growth rates 0.3057 and 0.3716, against $\beta - \frac{1}{2} = 0.3085$ and 0.3695.
 - The phase reading gives $\beta = 0.930 \pm 0.054$ for $0.8085 + 85.699i$, a biased miss of more than two standard deviations, attributed to the partner zero $1 - \bar{\rho}_0$.

The random models (Cramér’s, Granville’s, independent signs, random multiplicative functions) are the other controls.

- **Caveat.** “We do not prove the Riemann hypothesis”; “Nothing in this paper bears on the truth of RH.” The orbit identity “cannot constrain a zero that has not been located”. A zero with $\beta = 0.75$ at height $3 \cdot 10^{12}$ would add an amplitude of only about $4 \cdot 10^{-11}$ at $x = 10^7$.

Key numbers. $K = 2.512479790006 \dots$ (under RH, certified to $6 \cdot 10^{-13}$), with zero–zero term $0.00102972290885806 \dots$, cross term $-0.010118144286 \dots$ and trivial term $2.52156821138377 \dots$; the interval $1 \leq x < 2$ contributes exactly 1. $\sum_{\rho} |\rho|^{-4} = 7.42012728749 \dots \cdot 10^{-5}$, $\sigma_2 = -0.0461543172958 \dots$, $P = 0.029014156649 \dots$. From the primes up to 10^9 , $K_{\text{primes}} = 2.512484$, agreeing with K to $4 \cdot 10^{-6}$; the r.m.s. residual is $1.2\text{--}1.3 \cdot 10^{-5}$ with the zeros below height 3000. Under LI, $V = 1.2502 \cdot 10^{-5}$ and $\sqrt{V} = 0.0035358$; the lower bound $V_{10^4} = 1.25018 \cdot 10^{-5}$ is numerical, the upper bound $V \leq 1.31 \cdot 10^{-5}$ holds under the gap assumption $\delta_{\rho} \geq |\gamma|^{-0.4}$, and the 138 zeros below height 300 carry 99.4% of V . Logarithmic mean of the energy at $X = 10^7$: primes 0.0488 against a Cramér-model median of 7.24 (ratio 148); μ 0.0289 against independent signs 0.570 (ratio 20) and random multiplicative functions 0.242 (ratio 8.4). At γ_1 the mean of $|A_{\gamma_1}|$ on $8 \leq u \leq 12$ is 0.07069, against $1/|\rho_1| = 0.07070$. Cross-references: [CE5] (Theorem B, the local energy law and the energy matrix), [CE1] (Theorem A), [CE7] (large values), [CE8] (heat flow, Sect. 6), [CE4] (zeros of truncations, used in Remark 19), [CE17] (Davenport–Heilbronn control).

9. PART 7: MULTIPLICATIVE RESONANCE AND LARGE VALUES OF DIRICHLET POLYNOMIALS

Origin. Part 7 (15 pages) contains the material on large values (Sects. 7–9) of the earlier version of Part 6 titled “Fluctuations and large values of the prime energy” and, with Part 6, supersedes Parts II–IV of the preprint “The cross-energy of Dirichlet partial sums: from the energy law of the primes to large values of Dirichlet polynomials”; Check 2, part of Check 3 and the two figures of Sect. 2 are taken from that preprint, and the statistics of Sect. 4.2 are new.

Summary. The diagonal term $n^{-2\sigma}$ that C_2 subtracts is the energy that mean value theorems control. Read as a budget for zeros off the critical line, it gives the classical zero-density theorems of Bohr–Landau and Carlson, and a mollifier empties it. What remains is the largest eigenvalue of a cross-energy matrix of the peaks of a short Dirichlet polynomial $D(t) = \sum_{N < n \leq 2N} b_n n^{-it}$, the object of Montgomery’s large value conjecture. The paper studies this obstruction numerically: the top eigenvector for the peaks of $\sum n^{-it}$ and its behaviour on smooth, random and rough supports; two of Guth’s polynomial-time checks of the conjecture; a direct ascent search for worst cases with $|b_n| \leq 1$ up to $N = 8000$, $T = N^2$; and a statistical calibration against Steinhaus random completely multiplicative functions.

The worst cases found are close to completely multiplicative. At $\sigma = 0.7$, random multiplicative coefficients give even more large values, exceeding $N^{2-2\sigma}$ by a mean factor that grows from 1.85 to 9.2; at $\sigma = 0.75$ the search gives more than the model. Harper’s high-moment asymptotics suggest heuristically a log-normal regime that is consistent with Montgomery’s conjecture but dominates until $\log N$ is of order 10^2 . The conclusion is methodological: numerical tests should be measured against the random multiplicative model, not against $N^{2-2\sigma}$.

Main results.

- (1) [CE7, Sect. 2] (*classical, restated*) The Bohr–Landau theorem in energy form, proved in [CE10, Proposition 3.9]: for $\frac{1}{2} < \sigma < 1$,

$$N(\sigma; T, 2T) \leq \frac{\log \zeta(\sigma + \frac{1}{2})}{2\pi(\sigma - \frac{1}{2})} T + o(T).$$

With the mollifier $M_X(s) = \sum_{d \leq X} \mu(d) d^{-s}$ and $X = T^{2\sigma_0-1}$, the emptied diagonal gives $\int_T^{2T} |\zeta M_X - 1|^2 (\sigma_0 + it) dt \ll T^{1-(2\sigma_0-1)^2+\varepsilon}$, and hence Carlson’s $N(\sigma; T, 2T) \ll T^{4\sigma(1-\sigma)+\varepsilon}$. The reading of the mollifier as emptying the diagonal energy is the author’s.

- (2) [CE7, Lemma 1] (*proved*) Duality, which the paper calls elementary. For every b , $\sum_r |D(t_r)|^2 \leq \lambda_{\max}(G) \|b\|^2$. Hence if $|D(t_r)| \geq Y$ for all r , then $R \leq \lambda_{\max}(G) \|b\|^2 / Y^2$. Moreover $\lambda_{\max}(G) \leq (\text{tr } G^k)^{1/k}$ for every $k \geq 1$. Gershgorin’s theorem then gives the Halász–Montgomery inequality $\lambda_{\max}(G) \ll (N + RT^{1/2}) \log T$.
- (3) [CE7, Remark 2] (*numerical*) Peaks of $\sum n^{-it}$, with $T = N^2$ and $R = N$. For random points $\lambda_{\max}(G)/(N + R)$ stays between 1.8 and 2.0; for peak points it grows from 4.8 to 27 (gap 1) and from 2.5 to 17 (gap 10). The top eigenvector is not flat (mean of $|b_n|/\max |b_n|$ is 0.18). Its largest coordinates sit on smooth numbers: the median largest prime factor of the 20 largest is 7, against 61 for the whole range. This is the resonance mechanism of Soundararajan and of Bondarenko–Seip.
- (4) [CE7, Sect. 2.1] (*numerical*) Smooth, random and rough half-supports, by a local adversarial search. At $N = 800$, $\lambda_{\max}(G)/(M + R)$ is 18.21 (smooth half), 11.53 (random half) and 7.48 (rough half). The fitted growth is $N^{0.60}$ or $(\log N)^{3.3}$ for the smooth half, against $N^{0.38}$ or $(\log N)^{2.1}$ for the rough half. Removing the smooth numbers does not stop the growth.
- (5) [CE7, Sect. 3.1] (*numerical*) Guth’s Check 1, fluctuations of the difference-set density. At $T = N^{3/2}$ the Dirichlet ratio to Poisson noise falls from 0.661 ($N = 500$) to 0.391 ($N = 4000$), with 12 to 376 spikes at $\log(p/q)$. Each spike carries a Farey exclusion zone of half-width about $1/(2qN)$ with satellite peaks, a fine structure not mentioned by Guth. At $T = N^2$ the ratio stays at 0.912–0.915, and “smooth plus sparse spikes” no longer applies. Check 2 (flattened Schatten norms, [CE7, Sect. 3.2], numerical) shows no clear separation at $N \leq 150$: after span matching the Dirichlet norm is smaller by only about 10%.
- (6) [CE7, Sect. 3.3] (*numerical*) Check 3, worst cases with $|b_n| \leq 1$. At $T = N^2$ and $\sigma = 0.7$ the ratio of the search count to $N^{2-2\sigma}$ grows from 1.3 ($N = 250$) to 4.8 ($N = 4000$) and 6.8 ($N = 8000$); at $\sigma = 0.75$ it stays around 2. The coefficients found are close to completely

multiplicative: for $N = 1000$, $|\text{mean}_k b_{kp} \overline{b_{kq}}|$ lies between 0.92 and 0.98, against 0.09 for random pairs.

- (7) [CE7, Proposition 3] (*proved*) The diagonal of the fourth moment is the multiplicative energy. For completely multiplicative f with $|f(n)| = 1$ and $D_f(t) = \sum_{N \leq n \leq 2N} f(n)n^{it}$, the diagonal part of $|D_f(t)|^4$ is

$$\mathcal{M}(N) = \#\{n_1 n_2 = n_3 n_4, N \leq n_i \leq 2N\},$$

the same for every such f . For a Steinhaus random multiplicative function, $\mathbb{E}_f |D_f(t)|^4 = \mathcal{M}(N)$ for every t .

- (8) [CE7, Sect. 4.1] (*heuristic*) Taking Harper’s high-moment asymptotics as a model for the t -moments of one realization, and applying Markov’s inequality, gives the bound (7.5) below. This is explicitly “not a consequence of Harper’s theorem for the sampled t -process”. Asymptotically it is consistent with Montgomery’s conjecture. At computable sizes it allows an excess over $N^{2-2\sigma}$ growing like a power of N with exponent below 1.
- (9) [CE7, Remark 4] (*numerical*) Calibration, with a heuristic model. Over $250 \leq N \leq 8000$, Steinhaus coefficients exceed $N^{2-2\sigma}$ at $\sigma = 0.7$ by a mean factor growing from 1.85 to 9.2, with small dispersion across realizations. “Numerical tests of the conjecture must therefore be measured against the random multiplicative model, not against $N^{2-2\sigma}$.” At $\sigma = 0.75$ the comparison goes the other way: the search exceeds the model, and the model does not exceed $N^{1/2}$.

What is new.

- *New, as claimed:*
 - the reading of the mollifier as emptying the diagonal energy of C_2 (a formulation);
 - the experiments on the cross-energy matrix of peaks and on smooth and rough supports;
 - the implementation of Guth’s checks, including the Farey structure of the spikes in Check 1;
 - the multiplicative structure of the worst cases found by the search;
 - the statistical calibration against the Steinhaus model.
- “All of these are numerical observations or heuristics, not theorems.”
- *Classical, cited:*
 - the density theorems of Bohr–Landau and Carlson, and the explicit density estimate of Kadiri, Lumley and Ng;
 - the Halász–Montgomery method and Montgomery’s conjecture;
 - Bourgain’s counterexample;
 - the resonance method of Soundararajan and Bondarenko–Seip;
 - the large value estimates of Guth and Maynard;
 - Harper’s moment asymptotics.
- *Prior work:* Guth proposed the polynomial-time checks; the paper reports that it found no earlier numerical tests of Montgomery’s conjecture.

New formulas. The cross-energy matrix of the peaks [CE7, Sect. 2.1], with $v_r = (n^{-it_r})_{N < n \leq 2N}$:

$$(7.1) \quad G_{rs} = \langle v_r, v_s \rangle = \sum_{N < n \leq 2N} n^{i(t_s - t_r)}, \quad G_{rr} = N, \quad \text{tr } G = RN.$$

The duality bound [CE7, Lemma 1]:

$$(7.2) \quad \sum_r |D(t_r)|^2 \leq \lambda_{\max}(G) \|b\|^2, \quad \lambda_{\max}(G) \leq (\text{tr } G^k)^{1/k} \quad (k \geq 1).$$

The fourth-moment expansion [CE7, Proposition 3]:

$$(7.3) \quad |D_f(t)|^4 = \sum_m \left| \sum_{\substack{n_1 n_2 = m \\ N \leq n_i \leq 2N}} 1 \right|^2 + \sum_{\substack{n_1 n_2 \neq n_3 n_4 \\ N \leq n_i \leq 2N}} f(n_1 n_2) \overline{f(n_3 n_4)} \left(\frac{n_1 n_2}{n_3 n_4} \right)^{it}.$$

Harper’s theorem for $1 \leq q \leq c \log x / \log \log x$, as quoted in [CE7, eq. (2)]. It is a known result, not new, and is included because the heuristic rests on it:

$$(7.4) \quad \mathbb{E}_f \left| \sum_{n \leq x} f(n) \right|^{2q} = e^{-q^2 \log q - q^2 \log \log(2q) + O(q^2)} x^q (\log x)^{(q-1)^2}.$$

The heuristic Markov bound [CE7, eq. (3)], with $L = \log N$, $\ell = \log \log N$ and $T = N^2$:

$$(7.5) \quad \log \frac{\#\{t : |D_f(t)| > N^\sigma\}}{N^{2-2\sigma}} \lesssim L - \frac{(\sigma - \frac{1}{2})^2 L^2}{\ell}.$$

Relation to the Riemann hypothesis.

- **Unconditional.** The Bohr–Landau and Carlson density bounds (classical) and the duality lemma. Montgomery’s large value conjecture would give the density hypothesis $N(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon}$ (classical). The ℓ^2 form $\lambda_{\max}(G) \ll T^\varepsilon(N+R)$ for arbitrary sets of peaks is false, by Bourgain’s counterexample.
- **Conditional on RH.** No result of this part is conditional on RH. The values it quotes from [CE6], standard deviation 0.0035 and $K = 2.51248\dots$, are quoted under RH, simple zeros and (H^*) , the standard deviation also under LI.
- **Control.** No Davenport–Heilbronn or Epstein test. The controls are:
 - Guth’s almost counterexample $\{\sqrt{n/N}\}$, also in its span-matched form;
 - random frequencies;
 - independent random phases, which produce essentially no large values;
 - the Steinhaus random multiplicative model.
- **Caveat.** “We do not prove the Riemann hypothesis or Montgomery’s conjecture.” “Nothing in this paper bears on the truth of RH or of Montgomery’s conjecture.”
 - The search is local and gives lower bounds for the worst case, not certificates.
 - The sizes are modest ($N \leq 8000$).
 - The use of Harper’s theorem is heuristic: the transfer from averages over f to averages over t for one realization is not proved.
 - The fitted growth $N^{0.46}$ is pre-asymptotic.

Key numbers. Search counts of 10-separated $t \in [0, N^2)$ with $|D(t)| > (N+1)^{0.7}$ are 37, 72, 155, 314, 508, 696, 1488 for $N = 250, 500, 1000, 2000, 3000, 4000, 8000$, against $N^{2-2\sigma} = 28, 42, 63, 96, 122, 145, 220$. Steinhaus means (64, 32, 24 realizations) are 51.0, 109.7, 232.8, 484.7, 1003.0, 2022.7 for $N = 250, 500, 1000, 2000, 4000, 8000$, with mean ratios 1.85, 2.63, 3.69, 5.07, 6.92, 9.21; fitted exponent 0.46 (bootstrap [0.458, 0.468]), local exponents falling from 0.51 to 0.41, relative standard deviation from 8% to 1.4%. At $\sigma = 0.75$ the search finds 62 ($N = 1000$) and 151 ($N = 8000$) points, against model means 21.0 and 52.9. $\mathcal{M}(N)/(N+1)^2$ grows from 2.895 to 3.837, against the constant 2 of random phases, and the variance of $\log |D_f/\sqrt{N}|$ is 0.47–0.49, against $\pi^2/24 = 0.411$. Cross-references: the rigidity of the integrated energy in [CE6] is the “other face” of the same Euler product. The energy matrix $M(U)$ is from [CE5], and the Bohr–Landau bound in energy form from [CE10, Proposition 3.9].

10. PART 8: THE EULER PRODUCT INSIDE THE CRITICAL STRIP

Origin. Part 8 (32 pages) supersedes the preprint “Cross-energy of Dirichlet partial sums VI: the Euler product inside the critical strip” (doi:10.5281/zenodo.23070519) and contains the Euler-product material of Section 6 of version 3 of the consolidated paper, with the function-field, parity and transport topics only summarized from [CE17] and [CE12].

Summary. Inside the critical strip the partial Euler product $E_x(s) = \prod_{p \leq x} (1 - p^{-s})^{-1}$ diverges because of the pole. Removing the pole with the exponential integral, $E_x^* = E_x \exp(E_1((s - 1) \log x))$, gives an exact product form of the explicit formula, in which each zero is a mode of size $x^{\beta - \sigma} / \log x$. Consequently, RH holds if and only if $E_x^* \rightarrow \zeta$ in $\frac{1}{2} < \sigma < 1$. At a simple zero, $E_x^*(\rho) \log x$ equals $\zeta'(\rho)e^{-\gamma_0}$ times the modes of the other zeros; its decay is a statement of Deep-Riemann-Hypothesis type.

The central identity is that a partial Euler product is the cross-energy C_2 of the partial sums of ζ over the smooth integers. The cross-energy of the prime zeta function obeys the trichotomy of Theorem A with the normalization $\log p$. As functions of $u = \log x$ the modes form the “Euler helix”: a circle for each zero on the line and a spiral for each zero off it. With the primes up to 10^8 , the helix detects symmetric displacements of about $4 \cdot 10^{-5}$ of a zero near a given height. Sieving the partial sums turns the cross-energy wave into the helix. The Laguerre expressions of ξ are moments of the helix, with a corrected second identity. Further topics are an apparent abscissa read from the smoothed product, the off-line zero of the Davenport–Heilbronn function as an interference of two Euler products, a damped energy with abscissa $2 \sup \beta - 1$ and its discrete matrix form, collisions under the heat flow, the pairing of the 1-points of ζ with the zeros of ζ' , and the Euler product as a quantum state whose coherence does not depend on t .

Main results.

- (1) [CE8, Proposition 2] (*proved*) Explicit formula in product form, a classical reorganization with explicit domain and branches. On the slit half-plane $\mathcal{D}$ and for $x > 1$ not a prime power, $\log E_x^*$ equals $\log \zeta$ plus one exponential-integral mode per nontrivial and trivial zero plus R_x (formula (8.2)). It rests on [CE8, Lemma 1], the explicit formula with a parameter s .
- (2) [CE8, Corollary 3] (*proved*) Let $\Omega = \{\frac{1}{2} < \sigma < 1\}$. The following are equivalent:
 - (i) RH;
 - (ii) $E_x^* \rightarrow \zeta$ locally uniformly on Ω ;
 - (iii) for every compact $K \subset \Omega$, $|E_x^*(s)/\zeta(s) - 1| \leq C_K x^{1/2 - \sigma} \log x$ for $s \in K$ and $x \geq x_K$. This is “the product form of the classical equivalence of RH with the analyticity of $\log \zeta$ in $\sigma > \frac{1}{2}$ ”.
- (3) [CE8, Proposition 4] (*proved*) The corrected product at a simple zero ρ_* (formula (8.3)). Hence $|E_x^*(\rho_*)| \log x \rightarrow |\zeta'(\rho_*)|e^{-\gamma_0}$ if and only if the real part of the sum over the other zeros tends to 0. This limit is conditional on a Deep Riemann Hypothesis at ρ_* , the analogue for ζ of Conrad’s conjecture. It is not proved, and the paper does not claim that it is equivalent to Conrad’s conjecture. [CE8, Remark 5] gives an exact, unconditional smoothed form of the relation $\zeta'(\rho) = P_X(\rho)Z_X'(\rho)(1 + o(1))$ of Bui, Gonek and Milinovich.
- (4) [CE8, Theorem 6] (*proved*) The Euler product as a cross-energy, called elementary and “the central identity”. For $\sigma > 0$ and $P \geq 2$, with $n_1 < n_2 < \dots$ the P -smooth integers,

$$|E_P(s)|^2 = \prod_{p \leq P} (1 - p^{-2\sigma})^{-1} + \sum_{k \geq 2} C_2^{(P)}(n_k, s).$$

For $\sigma > 1$, letting $P \rightarrow \infty$ gives $|\zeta(s)|^2 = \zeta(2\sigma) + \sum_{n \geq 2} C_2(n, s)$. By [CE8, Remark 7], the positivity of the left side “is automatic and gives no information on the zeros of ζ ”.

- (5) [CE8, Proposition 8] (*proved*) Trichotomy for the prime zeta function, from the prime number theorem. For $0 < \sigma < 1$, $\log p \cdot C_2^P(p, s)$ tends to $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$.
- (6) [CE8, Remark 11] (*numerical*) Rigidity of the Euler helix (formula (8.5)). At 45 heights $16.5 \leq t \leq 117.7$, with the prime powers up to 10^8 , the smoothed helix agrees with the sum over the zeros to a relative r.m.s. of $2 \cdot 10^{-7}$ to $8 \cdot 10^{-6}$. A second-order detector for symmetric pairs $\frac{1}{2} \pm \delta$ has threshold $\delta_{\min} \approx 4 \cdot 10^{-5}$. This is “a statistical detection threshold . . . , not a bound for β ”. The energy identity of [CE8, Remark 10] is heuristic, because the interchange of sum and mean is not justified. The sieved-wave identities of [CE8, Remark 12] are proved.
- (7) [CE8, Proposition 14] (*conditional on RH and simple zeros, and on LI for L_2*) Laguerre expressions as helix moments, “proved for the model sum (corrected)”. With $w_\gamma = (t - \gamma)^{-2}$, $L_n(\Xi)/\Xi^2 = e_n(w)$ at $x = t$ and $L_1/\Xi^2 = M_2$. Under linear independence of the positive ordinates, L_2 is given by (8.6). The second term corrects version 3 of the consolidated paper, which overlooked the coincidences between conjugate pairs. Its identification with the helix computed from the primes is the unjustified interchange of Remark 10.
- (8) [CE8, Proposition 15] (*proved*) Damped energy, classical in substance (Landau’s theorem in Cauchy–Schwarz form). The abscissa of convergence of $\mathcal{E}(\varepsilon) = \int_2^\infty e^{-\varepsilon u} S(u)^2 du$, $S(u) = e^{-u/2}(\psi(e^u) - e^u)$, is $2\Theta - 1$. Hence RH holds if and only if $\mathcal{E}(\varepsilon) < \infty$ for every $\varepsilon > 0$. The discrete form, [CE8, Remark 16] (exact identities; equivalence classical in substance), diagonalizes $\mathcal{Q}_N = \sum_{k \leq N} (\psi(k) - k)^2$ explicitly. RH is equivalent to $\mathcal{Q}_N = O_\varepsilon(N^{2+\varepsilon})$.
- (9) [CE8, Proposition 18] (*proved*) The Euler state.
 - (i) $\sum_n n^{-s} |n\rangle \in \ell^2$ if and only if $\sigma > \frac{1}{2}$, and $\psi_s = e^{-itL} \psi_\sigma$.
 - (ii) e^{-itL} factors over the primes, so every coherence or entanglement measure invariant under diagonal or local unitaries takes the same value for every t .
 - (iii) The relative entropy of coherence is $C(\sigma) = \log \zeta(2\sigma) - 2\sigma \frac{\zeta'}{\zeta}(2\sigma)$.
 - (iv) The zeros of ζ in $0 < \sigma < 1$ are the points at which the rescaled overlap with the alternating vector tends to 0.

What is new.

- *New, as claimed:* “the systematic use of the sharp product form as a geometric object (the Euler helix), its relation to the cross-energy . . . (Theorem 6 and Proposition 8, Remark 12), the corrected moment identity for the Laguerre expressions (Proposition 14), and the measurements, with controls”. “None of them constrains the position of the zeros.”
- *The paper’s own fourfold classification:*
 - (1) classical identities, reorganized: Proposition 2, Corollary 3, Proposition 15;
 - (2) new structural connections: Theorem 6, Proposition 8, Remark 12, Proposition 14;
 - (3) heuristic and numerical statements: the helix energy, the reduction of the Laguerre hierarchy, the double-sum picture of the damped energy, the deformed cross-energy under the heat flow, and all measurements;
 - (4) constraints on the zeros: “there are none”.
- *Classical or prior work, cited:*
 - the explicit formula, Landau’s theorem, Hurwitz’s theorem and the Hadamard product;
 - the theory of the de Bruijn–Newman constant (Newman, Rodgers–Tao, Polymath, Platt–Trudgian);
 - Conrad’s Deep Riemann Hypothesis, whose constant $e^{-\gamma_0}$ reappears here;
 - the hybrid Euler–Hadamard product of Gonek, Hughes and Keating;
 - Speiser’s theorem and Turing’s method.

- *Corrections of the sources:* the second term in Proposition 14; the sign of the pole term in Lemma 9 (which does not affect the computations); the pole term in the sum of the prime increments at a zero.

New formulas. The pole-corrected partial product [CE8, Sect. 2]:

$$(8.1) \quad E_x^*(s) = E_x(s) \exp(E_1((s-1)\log x)), \quad R_x(s) = \sum_{k \geq 2} \sum_{x^{1/k} < p \leq x} \frac{p^{-ks}}{k}.$$

The explicit formula in product form [CE8, Proposition 2, eq. (2)]:

$$(8.2) \quad \log E_x^*(s) = \log \zeta(s) + \sum_{\rho} E_1((s-\rho)\log x) + \sum_{k \geq 1} E_1((s+2k)\log x) + R_x(s).$$

The corrected product at a simple zero [CE8, Proposition 4, eq. (4)]:

$$(8.3) \quad E_x^*(\rho_*) \log x = \zeta'(\rho_*) e^{-\gamma_0} \exp\left(\sum_{\rho \neq \rho_*} E_1((\rho_* - \rho)\log x) + \sum_{k \geq 1} E_1((\rho_* + 2k)\log x) + R_x(\rho_*)\right).$$

The trichotomy for the primes [CE8, Proposition 8], as $p \rightarrow \infty$:

$$(8.4) \quad \log p \cdot C_2^P(p, s) = \frac{2(1-\sigma)}{(1-\sigma)^2 + t^2} p^{1-2\sigma} (1 + o(1)) + O(p^{-\sigma} \log p) + o(p^{1-2\sigma}).$$

The Euler helix [CE8, eq. (5)]. The first equality is exact; the second describes the individual terms, not the conditionally convergent sum:

$$(8.5) \quad \begin{aligned} H_s(u) &= \sum_{\rho} u E_1((s-\rho)u) + \sum_{k \geq 1} u E_1((s+2k)u) \\ &= \sum_{\rho} \frac{e^{(\rho-s)u}}{s-\rho} \left(1 + O\left(\frac{1}{|s-\rho|u}\right)\right) + O(e^{-(2+\sigma)u}). \end{aligned}$$

The corrected second Laguerre identity [CE8, Proposition 14], with $c_\gamma = 1/(\gamma^2 - t^2)$:

$$(8.6) \quad \frac{L_2}{\Xi^2} = \frac{1}{2} (M_4 - M_2^2) - 2 \left[\left(\sum_{\gamma > 0} c_\gamma \right)^2 - \sum_{\gamma > 0} c_\gamma^2 \right].$$

The energy matrix of the Chebyshev errors and its eigenvalues [CE8, Remark 16], with $a_n = \Lambda(n) - 1$ and $A(k) = \psi(k) - k$:

$$(8.7) \quad \begin{aligned} \mathcal{Q}_N &= \sum_{k \leq N} A(k)^2 = a^\top W_N a, \quad W_N(m, n) = N + 1 - \max(m, n), \\ \lambda_j &= \frac{1}{4 \sin^2 \frac{(2j-1)\pi}{4N+2}}. \end{aligned}$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** All of the following are classical in substance:
 - $E_x^* \rightarrow \zeta$ locally uniformly on Ω , with or without the rate (Corollary 3);
 - $A_y^*(s) \rightarrow 1$ locally uniformly on Ω (Remark 12);
 - $\mathcal{E}(\varepsilon) < \infty$ for every $\varepsilon > 0$ (Proposition 15);
 - $\mathcal{Q}_N = O_\varepsilon(N^{2+\varepsilon})$, or the same bound for the modes $j \leq N^{1/2+\delta}$ alone (Remark 16).

By [CE1, Proposition 2.9], the open question whether a Dirichlet polynomial with multiplicative coefficients can resonate at one frequency with amplitude $N^{\beta-1/2}$, $\beta > \frac{1}{2}$, is equivalent to the negation of RH.

• **Conditional on RH.**

- The decay at a zero in Proposition 4 is conditional on a Deep Riemann Hypothesis at ρ_* .
- The helix energy of Remark 10 assumes RH and is heuristic.
- Proposition 14 assumes RH and simple zeros, and LI for the identity for L_2 .

• **Unconditional.** Lemma 1, Proposition 2, identity (8.3), Remark 5, Theorem 6, Proposition 8, Lemma 9, Proposition 18, and the sieve and prime-operator identities.

• **Control.** The Davenport–Heilbronn function f , with off-line zero $\rho_f = 0.808517 + 85.699348i$:

- its logarithmic series converges to the right of ρ_f and fails to its left;
- the apparent abscissa has plateaus at 0.806 (against $\beta = 0.8085$) and 0.631 (against 0.6508);
- in the approximate functional equation ρ_f appears as an interference of two Euler products, with $|L(\rho_f, \chi)| = |L(\rho_f, \bar{\chi})| = 0.702$ in opposite phase;
- no prime operator exists for f (relative residual 1.00);
- its 1-points pair with the zeros of f' , so that pairing is “a property of the Dirichlet series, not of the Euler product or of RH”;
- its zeros are maximally mixed states.

Further controls are $\zeta + 5f$, $L(s, \chi)$ modulo 5 and the function-field case (proved in [CE17]).

• **Caveat.** “We do not prove the Riemann hypothesis.” “No statement of the paper excludes a zero with $\beta > \frac{1}{2}$.” The local detection by the helix “does not approach RH”, for three reasons:

- a measurement never gives $\beta = \frac{1}{2}$ without a Turing-type symmetry argument;
- the verification is local;
- a zero outside the list appears only as a residual.

Detecting a zero with $\delta = 0.01$ at height $3 \cdot 10^{12}$ through the damped energy would need primes up to about 10^{1250} .

Key numbers. At the first zero $\zeta'(\rho_1) = 0.7833 + 0.1247i$ and $|\zeta'(\rho_1)|e^{-\gamma_0} = 0.4453$; at the first five zeros the ratio $|E_x^*(\rho)| \log x / (|\zeta'(\rho)|e^{-\gamma_0})$ stays within 1 ± 0.1 for $10^3 \leq x \leq 2 \cdot 10^6$. Theorem 6 check with the 14672 seven-smooth integers up to 10^{12} : 0.761394 against $|E_7|^2 = 0.761404$ at $s = 0.6 + 20i$. The apparent abscissa of ζ has median 0.500, with 90% of the values in $[0.47, 0.54]$. The mean of $|H_s|^2$ at $s = \frac{1}{2} + 20i$ is 1.045, against $-(\log |\xi|)''(20) = 1.077$; at $t = 20$, $L_2/\xi^2 = e_2(w) = 0.12026$, with correction 0.0017. Heat flow: first collision at $t_c = -0.469477$; the Lehmer pair ($D = 0.03770$) collides at $t_c = -1.778 \cdot 10^{-4}$, within 0.1% of $-D^2/8$. At $N = 10^6$, $\mathcal{Q}_N/N^2 = 0.0219$. Below $t = 350$ there are 169 zeros of ζ , 132 1-points and 132 zeros of ζ' . For $N = 4096$ the relative entropy of coherence 6.066379389410 is the same at $t = 0, 14.13, 20, 37.59$, and the Berry flux/ 2π is 10.35 for 10 zeros. Cross-references: [CE1] (Theorem A, Propositions 2.6, 2.8, 2.9); [CE5] (Theorem B, Proposition 2.5); [CE11] (the smoothed factorization $\zeta = E_g^* Z_g$, Lemma 8, and the Euler–Caceres bound); [CE4] (the truncation helix); [CE9] (zeros of the prime zeta function); [CE10] (energy slope); [CE12] (transport, Sect. 9); [CE13], [CE14], [CE15] (the de Bruijn–Newman constant); [CE16] (pole barrier); [CE17] (function fields, amplification and parity).

11. PART 9: ZEROS OF THE PRIME ZETA FUNCTION IN ITS HALF-PLANE OF ABSOLUTE CONVERGENCE

Origin. Part 9 (11 pages) explains the author’s 2018 numerical computation of the zeros of $P(s)$ (“PrimeZeta”, DNA Primes Math Blog, and the unpublished note “Riemann zeta function – nine

propositions”): the cluster with $1 < \operatorname{Re} s < 1.6$ is genuine and the clusters with $\operatorname{Re} s > 10^9$ or $\operatorname{Im} s > 10^{13}$ are numerical artifacts; the zero-free half-plane it sharpens is due to Belovas, Čepaitytė and Sabaliauskas.

Summary. The prime zeta function $P(s) = \sum_p p^{-s} = \sum_{k \geq 1} \frac{\mu(k)}{k} \log \zeta(ks)$ has zeros in its half-plane of absolute convergence. Belovas, Čepaitytė and Sabaliauskas proved that $P \neq 0$ for $\operatorname{Re} s > \sigma_P = 1.7795446535\dots$, the solution of $P(\sigma) = 2^{1-\sigma}$. Part 9 shows that this half-plane is sharp. There are no zeros on $\operatorname{Re} s = \sigma_P$; by Bohr’s method there are zeros arbitrarily close to every line $\operatorname{Re} s = \sigma$ with $1 < \sigma < \sigma_P$; so the closure of the real parts of the zeros in $\operatorname{Re} s > 1$ is exactly $[1, \sigma_P]$.

By Jessen’s theorem, the density of the real parts is $J''(\sigma)/2\pi$. Here $J(\sigma) = -\sigma \log 2 + \mathbb{E} \log^+(2^\sigma |Y_\sigma|)$, where $Y_\sigma = \sum_{p \geq 3} p^{-\sigma} e^{i\theta_p}$ has independent uniform phases. J is C^∞ , it characterizes the edge, and the density vanishes at σ_P faster than any power. The argument principle in ball arithmetic certifies counts that agree with the prediction, and a heuristic large-deviation extrapolation estimates the frequency of zeros near the edge. Inside the critical strip, single-valuedness of P on $\{\operatorname{Re} s > \frac{1}{2}\} \setminus (\frac{1}{2}, 1]$ is equivalent to RH (a reformulation). Below height $3 \cdot 10^{12}$ the zeros are canonical and are the solutions of $\zeta(s) = e^{A(s)}$ with $|P(s)| < 2\pi$. The zeros are cancellations between 2^{-s} and the remaining primes, the prime-sum analogue of the Euler-product interference behind the off-line zeros of the Davenport–Heilbronn function.

Main results.

- (1) [CE9, Theorem 2.1] (*proved*) Let $\sigma_P > 1$ be defined by $2^{-\sigma_P} = \sum_{p \geq 3} p^{-\sigma_P}$.
 - (a) $P(s) \neq 0$ for $\operatorname{Re} s \geq \sigma_P$.
 - (b) For every $1 < \sigma < \sigma_P$ and every $\varepsilon > 0$, the set of t for which P has a zero in $|s - (\sigma + it)| < \varepsilon$ has positive lower density.
 - (c) Consequently $\{\operatorname{Re} \rho : P(\rho) = 0, \operatorname{Re} \rho > 1\} = [1, \sigma_P]$.

The half-plane $\operatorname{Re} s > \sigma_P$ is due to Belovas, Čepaitytė and Sabaliauskas; the boundary line uses unique factorization. “We do not claim that every such line contains a zero.”
- (2) [CE9, Lemma 2.2] (*proved*) Polygon lemma. If $\ell_1 \geq \ell_2 \geq \dots \geq \ell_n > 0$, $n \geq 2$, with $\ell_1 \leq \ell_2 + \dots + \ell_n$, then $\{\sum_j \ell_j e^{i\theta_j} : \theta_j \in \mathbb{R}\}$ is the closed disc of radius $\sum_j \ell_j$ centred at 0. With the Kronecker–Weyl theorem and Rouché’s theorem it gives Theorem 2.1(b).
- (3) [CE9, Sect. 3] (*classical, restated*) Jessen’s theorem for analytic almost periodic functions. For $1 < \sigma_1 < \sigma_2$, the zero count $N(\sigma_1, \sigma_2; T)$ is $\frac{T}{2\pi}(J'(\sigma_2) - J'(\sigma_1)) + o(T)$. For linearly independent frequencies, $J(\sigma) = \mathbb{E} \log |\sum_p p^{-\sigma} e^{i\theta_p}|$ (Jessen–Tornehave; Bohr–Jessen).
- (4) [CE9, Proposition 3.1] (*proved*) $J(\sigma) = -\sigma \log 2 + \mathbb{E} \log^+(2^\sigma |Y_\sigma|)$. In particular $J(\sigma) = -\sigma \log 2$ exactly for $\sigma \geq \sigma_P$, and $J(\sigma) > -\sigma \log 2$ for $1 < \sigma < \sigma_P$. By [CE9, Corollary 3.2] (*proved*), $\sigma_P = \inf\{\sigma > 1 : J(\sigma) + \sigma \log 2 = 0\}$, and J is strictly convex on $(1, \sigma_P]$ and affine on $[\sigma_P, \infty)$.
- (5) [CE9, Corollary 3.4] (*proved*) Flatness at the edge. By [CE9, Lemma 3.3] (*proved*), $J \in C^\infty(1, \infty)$. Hence for every N , $J(\sigma) + \sigma \log 2$, $J'(\sigma) + \log 2$ and $J''(\sigma)$ are $O((\sigma_P - \sigma)^N)$ as $\sigma \rightarrow \sigma_P^-$, and J is not real-analytic at σ_P .
- (6) [CE9, Sect. 4] (*certified in interval arithmetic*) The argument principle in ball arithmetic (Arb) certifies exactly:
 - 31 zeros with $1.1 \leq \operatorname{Re} s \leq 1.85$, $0 < \operatorname{Im} s < 1000$ (Jessen prediction 35.0);
 - 68 with $1.1 \leq \operatorname{Re} s \leq 1.85$, $0 < \operatorname{Im} s < 2000$ (70.0);
 - 12 with $1.5 \leq \operatorname{Re} s \leq 1.85$, $0 < \operatorname{Im} s < 5000$ (12.0);
 - 0 with $1.6 \leq \operatorname{Re} s \leq 1.85$, $0 < \operatorname{Im} s < 5000$ (1.2).

The assignment of zeros to bins and their locations are numerical (floating point).
- (7) [CE9, Sect. 3] (*heuristic*) Behaviour at the edge. With the Cramér rate $I(\sigma)$, the data are consistent with $\log(J(\sigma) + \sigma \log 2) = -I(\sigma)(1 + o(1))$. A saddle-point analysis suggests,

with $x = \sigma_P - \sigma$,

$$I(\sigma) \asymp x^{-1/(\sigma_P-1)} \left(\log \frac{1}{x} \right)^{-\sigma_P/(\sigma_P-1)}.$$

Extrapolated, zeros with real part above 1.7 occur about once every $6 \cdot 10^6$ units of height, and above 1.72 about once every $3 \cdot 10^8$.

- (8) [CE9, Proposition 5.1] (*proved*) Let $\Omega = \{\operatorname{Re} s > \frac{1}{2}\} \setminus (\frac{1}{2}, 1]$ and $A(s) = \sum_{m \geq 2} P(ms)/m$.
- (a) RH holds if and only if P extends from $\operatorname{Re} s > 1$ to a single-valued analytic function on Ω . This is “a reformulation of the classical fact” and “not progress towards RH”.
 - (b) On $U = \{\operatorname{Re} s > \frac{1}{2}, 0 < \operatorname{Im} s < 3 \cdot 10^{12}\}$, $P = \log \zeta - A$ is single-valued and its zeros are canonical.
- On U , $P(s) = 0 \iff L(s) = A(s) \implies \zeta(s) = e^{A(s)}$, and $\zeta(s) = e^{A(s)}$ if and only if $P(s) \in 2\pi i \mathbb{Z}$ (proved).

What is new.

- *Prior work*: the zero-free half-plane $\operatorname{Re} s > \sigma_P$ is due to Belovas, Čepaitytė and Sabaliuskas. They also computed more than 10^4 zeros and found the largest real part 1.6826... for $|\operatorname{Im} s| < 2 \cdot 10^5$. Fröberg remarked in 1968 that very little was known about the roots of P .
- *New*: Theorem 2.1 adds the boundary line and shows that the half-plane is sharp. It is “an instance of the classical theory of Bohr, Jessen and Tornehave”, with a self-contained proof by Bohr’s argument; Mantovanelli used the same scheme for the Fibonacci zeta function.
- *New*: the Jessen density of the real parts, with its edge characterization and its flatness at σ_P ; the certified counts; the heuristic edge asymptotics; and the analysis inside the critical strip.
- *Consistency with prior work*: the value 1.6826 of Belovas et al. is consistent with the prediction of about 4 zeros above 1.65 and about 0.4 above 1.68 below height $2 \cdot 10^5$.
- *Correction of the author’s 2018 computation*: the zeros with real part below 1 depend on the branch of $\log \zeta$, and the clusters at $\operatorname{Re} s > 10^9$ and $\operatorname{Im} s > 10^{13}$ are floating-point artifacts.

New formulas. The excess of the Jessen function over its affine edge value [CE9, Proposition 3.1]:

$$(9.1) \quad J(\sigma) = -\sigma \log 2 + \mathbb{E} \log^+(2^\sigma |Y_\sigma|), \quad Y_\sigma = \sum_{p \geq 3} p^{-\sigma} e^{i\theta_p}.$$

The edge characterization [CE9, Corollary 3.2]:

$$(9.2) \quad \sigma_P = \inf\{\sigma > 1 : J(\sigma) + \sigma \log 2 = 0\}.$$

The Bessel-product formula used to compute J [CE9, Sect. 3]:

$$(9.3) \quad J(\sigma) = \log 2 + \int_0^\infty \left(e^{-x} - \prod_p J_0(p^{-\sigma} x) \right) \frac{dx}{x}.$$

The Jessen count of zeros in a vertical strip [CE9, Sect. 3], with $\nu = J''/2\pi$:

$$(9.4) \quad N(\sigma_1, \sigma_2; T) = \frac{T}{2\pi} (J'(\sigma_2) - J'(\sigma_1)) + o(T) = T \nu((\sigma_1, \sigma_2)) + o(T).$$

The Cramér rate governing the edge, used heuristically [CE9, Sect. 3]:

$$(9.5) \quad I(\sigma) = \sup_{\lambda > 0} \left(\lambda 2^{-\sigma} - \sum_{p \geq 3} \log I_0(\lambda p^{-\sigma}) \right).$$

The evaluation formula with explicit tail bound used in the certification [CE9, Sect. 4]:

$$(9.6) \quad P(s) = \operatorname{Log} \zeta(s) + \sum_{p \leq M} (\log(1 - p^{-s}) + p^{-s}) + E, \quad |E| \leq \frac{M^{1-2\sigma}}{2(2\sigma-1)(1-M^{-\sigma})}.$$

The zeros inside the strip as prescribed values of ζ [CE9, Sect. 5]:

$$(9.7) \quad P(s) = 0 \iff L(s) = A(s) \implies \zeta(s) = e^{A(s)}, \quad A(s) = \sum_{m \geq 2} \frac{P(ms)}{m}.$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** Single-valued analytic extension of P to Ω (Proposition 5.1(a)). The paper presents this as a reformulation of the classical equivalence of RH with the existence of an analytic branch of $\log \zeta$ on $\operatorname{Re} s > \frac{1}{2}$ minus $(\frac{1}{2}, 1]$.
- **Unconditional.** Everything in $\operatorname{Re} s > 1$ (Sects. 2–4). Proposition 5.1(b) uses only the numerical verification of RH up to height $3 \cdot 10^{12}$ by Platt and Trudgian. No statement of the paper is conditional on RH.
- **Control.** No numerical Davenport–Heilbronn test in this part. The comparison is conceptual: the zeros of P are cancellations between 2^{-s} and the other primes, “the prime-sum analogue of the interference between two Euler products that produces the off-line zeros of the Davenport–Heilbronn function” [CE17, Sect. 4]. Inside the strip the Bohr–Jessen random model is the reference. The observed counts below height 600 agree with it for $\operatorname{Re} s \geq 0.7$ and fall far below it nearer to $\frac{1}{2}$; this is numerical, with a heuristic explanation.
- **Caveat.** The abstract and the Discussion end with “We do not prove the Riemann hypothesis”; the Discussion adds that “Nothing in this part bears on the truth of RH beyond the reformulation in Proposition 5.1(a), which is classical.” The equivalence in Proposition 5.1(a) “is not progress towards RH”. The contrast between the sum $P = \sum_p p^{-s}$ and the product $\zeta = \prod_p (1 - p^{-s})^{-1}$ “gives no new restriction on the zeros of ζ ”. The extension of the Borchsenius–Jessen theory to P inside the strip is not proved.

Key numbers. $\sigma_P = 1.7795446535\dots$ and $1/(\sigma_P - 1) = 1.283$. Certified counts 31, 68, 12, 0, against the predictions 35.0, 70.0, 12.0, 1.2. The real parts of the 68 zeros below height 2000, by bins of width 0.1 from 1.1 (the last bin $[1.7, \sigma_P)$), are observed 21, 18, 13, 11, 5, 0, 0, against predicted 21.8, 18.3, 14.9, 10.2, 4.3, 0.5, 0.0. $J''(\sigma)/2\pi$ is 0.1200 at $\sigma = 1.1$ and 0.0085 at 1.6; the number of zeros per unit height with real part above 1.1 is 0.0350, and above 1.5 it is 0.0024. The first zero with $\operatorname{Re} s \geq 1.1$ is $1.44721094 + 38.10821211i$, and the largest real part below height 5000 is attained at $1.58696272 + 1402.15683346i$. Inside the strip there are 240 zeros with $0.55 \leq \operatorname{Re} s \leq 1.1$ up to height 1000; for $0.55 \leq \operatorname{Re} s < 0.8$ their median distance to the nearest zero of ζ is 0.31, against 0.45–0.51 for random points. Cross-references: below $t = 100$, each of the 18 zeros of P located in [CE8, Sect. 9] lies within 0.08 to 0.59 of a 1-point of ζ , and the 1-points fall typically one per gap between consecutive zeros of ζ [CE8, Remark 17]. Other references: [CE17, Sect. 4] (Davenport–Heilbronn interference) and [CE1] (plan of the series).

12. PART 10: JENSEN’S FORMULA, THE ENERGY SLOPE AND WEIL’S FUNCTIONAL

Origin. Section 2 of [CE10] comes from the earlier preprint IV (a Jensen formula for the C_2 wave) and Section 3 from the earlier Part VIII and version 3 of the consolidated paper; with Parts 1, 2, 5 and 16 it replaces version 4 of the consolidated paper (doi:10.5281/zenodo.23243641).

Summary. Part 10 collects the global positivity criteria for RH in cross-energy form. A locality principle shows that single-point observables of C_2 carry no more than $(\sigma, t, \zeta(s))$, so pointwise criteria, which rest on an asymptotic that holds verbatim for any Dirichlet series with a simple pole of residue 1 and coefficients of mean 1 (for instance the Epstein zeta function of $m^2 + 100n^2$), cannot prove RH. The logarithmic energy of the oscillation of C_2 around its envelope is $2|\zeta(s)|^2$, and the envelope coefficient $A(\sigma, t) = 2(1 - \sigma)/((1 - \sigma)^2 + t^2)$ is 2π times the Poisson kernel of $\{\operatorname{Re} s > \sigma\}$ at the pole; together they give a Jensen formula for C_2 on every vertical line, which on the critical line is the criterion of Balazard, Saias and Yor and as a family in σ recovers $\Theta = \sup \operatorname{Re} \rho$. Moment

results and a moment obstruction explain why arguments through the moments of the C_2 wave cannot close the gap. Section 3 studies the energy slope $S(\sigma, t) = \partial_\sigma \log |\zeta(\sigma + it)|$ (Lagarias' criterion): an off-line pair lowers S exactly inside a disc, the failure set has small measure, and S is Weil's functional at a Poisson kernel with an exact finite truncation in the prime powers $n \leq e^L$. Every criterion is tested on the Davenport–Heilbronn function, which violates each one as predicted.

Main results.

- (1) [CE10, Theorem 2.7] (*proved*) (C_2 –Jensen formula). For $0 < \sigma < 1$ the identity (10.2) holds, every term of its sum being positive.
- (2) [CE10, Corollary 2.8] (*classical, restated*) (Balazard–Saias–Yor in C_2 form). With $d\mu = dt/(2\pi(\frac{1}{4} + t^2))$, $\mathcal{J} = \int \log |\zeta(\frac{1}{2} + it)| d\mu = \sum_{\beta > 1/2} \log |\rho/(1 - \rho)| \geq 0$, and $\text{RH} \iff \mathcal{J} = 0$; the C_2 form, in which integrand and weight are cross-energy quantities, is proved.
- (3) [CE10, Corollary 2.9] (*proved*) (Jensen family). $\mathcal{J}(\sigma)$ is continuous, non-increasing, and $\Theta = \inf\{\sigma \in (0, 1) : \mathcal{J}(\sigma) = 0\}$.
- (4) [CE10, Proposition 2.11] (*proved*) (small moments). For $\mathcal{M}(k) = \int |\zeta(\frac{1}{2} + it)|^{2k} d\mu$: $\mathcal{M}(1) = \log 2\pi - \gamma_0$, $\mathcal{M}(k) \geq e^{2k\mathcal{J}} \geq 1$, and $\text{RH} \iff \mathcal{M}(k) = 1 + o(k)$ as $k \rightarrow 0^+$.
- (5) [CE10, Corollary 2.13] (*proved*) (Jensen gap). Unconditionally $0 \leq \sum_{\beta > 1/2} \log |\rho/(1 - \rho)| \leq \frac{1}{2} \log(\log 2\pi - \gamma_0) = 0.115818\dots$; by [CE10, Proposition 2.14] the mean and variance of the wave energy cannot improve this.
- (6) [CE10, Theorem 3.2] (*proved*) (disc lemma). For $\sigma > \frac{1}{2}$ a critical zero contributes positively to S , and an off-line pair ρ , $1 - \bar{\rho}$ contributes negatively if and only if $(\sigma - \frac{1}{2})^2 + (t - \gamma)^2 < (\beta - \frac{1}{2})^2$, i.e. inside the disc D_ρ centred at $\frac{1}{2} + i\gamma$ through the pair.
- (7) [CE10, Theorem 3.7] (*proved*) (exceptional set). The set where $S(\sigma, \cdot) \leq 0$ on $[0, T]$ has measure at most $2\sqrt{\sigma(1 - \sigma)} N(\sigma, T + 1)$; see (10.4).
- (8) [CE10, Theorem 3.13] (*proved*) (Poisson kernel and exact truncation). $S(\sigma, t) = \sum_\rho h_{\delta, t}(\gamma_\rho)$ with $\delta = \sigma - \frac{1}{2}$, $h_{\delta, t}(r) = \delta/(\delta^2 + (r - t)^2)$; for $\varphi_L(u) = \sqrt{2\delta} e^{-\delta u} e^{-itu} \mathbf{1}_{[0, L]}(u)$, $W_L = W(\varphi_L)$ is given exactly by the explicit formula with the prime powers $n \leq e^L$, satisfies (10.5) if all zeros lie on the line, and a zero with $\beta > \sigma$ contributes a term of size $e^{(\beta - \sigma)L}$ at every t .
- (9) [CE10, Corollary 3.14] (*conditional on RH*) (prime-resolution scale). If $\delta \log P \geq \log(3/\varepsilon)$ then $|W_{\log P}(\sigma, t)/(2S(\sigma, t)) - 1| \leq \varepsilon$ for every t .

What is new.

- New in this form, “as far as we know” ([CE10, Sect. 1.2]): the C_2 form of the Jensen formula and of the Balazard–Saias–Yor criterion, the disc lemma, the exceptional-set bound and the exact truncation of Weil's functional. The paper also proves, without a novelty claim, the wave-energy identity, the harmonic-measure reading of the envelope, the exact finite- N second moment and the Jensen gap.
- Classical and cited as such: the criteria of Balazard–Saias–Yor, Lagarias (with Sondow–Dumitrescu) and Weil; Selberg, Huxley and Guth–Maynard density theorems; Bohr–Landau; de la Vallée Poussin (rewritten for S); Szegő's theorem. The concavity criterion (Proposition 3.4) “is presumably known” (Laguerre–Pólya); the constant $2\sqrt{\sigma(1 - \sigma)}$ was already proved in the earlier Part VIII.
- For Nyman–Beurling (Báez-Duarte, Burnol, Bettin–Conrey–Farmer) the contribution of C_2 is “explanatory, not quantitative”.

New formulas. The logarithmic wave energy, [CE10, Lemma 2.5, (2.3)], for $0 < \sigma < 1$, $t \neq 0$ and $V_N = N^\sigma(C_2(N, \sigma + it) - A(\sigma, t)N^{1-2\sigma} + N^{-2\sigma})$:

$$(10.1) \quad \Omega_\sigma(t) := \lim_{M \rightarrow \infty} \frac{1}{\log M} \sum_{N \leq M} \frac{V_N^2}{N} = 2 |\zeta(\sigma + it)|^2.$$

The C_2 -Jensen formula, [CE10, Theorem 2.7, (2.4)]:

$$(10.2) \quad \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \frac{\Omega_{\sigma}(t)}{2} A(\sigma, t) dt = \sum_{\beta > \sigma} \log \left| \frac{1 - 2\sigma + \bar{\rho}}{1 - \rho} \right| - \log(2(1 - \sigma)).$$

The exact second moment at every $N \geq 2$, $B_N(t) = C_2(N, \frac{1}{2} + it) - L(t) + \frac{1}{N}$, [CE10, Proposition 2.12, (2.6)]:

$$(10.3) \quad \begin{aligned} \int B_N(t)^2 d\mu(t) &= 4 - \frac{4}{N} \log \frac{N^{N-1}}{(N-1)!} - \frac{2H_{N-1}}{N} - \frac{1}{N^2} \\ &= \frac{2(\log 2\pi - \gamma_0)}{N} + O(N^{-2}). \end{aligned}$$

The exceptional set of the energy slope, [CE10, Theorem 3.7]:

$$(10.4) \quad \begin{aligned} \text{meas}\{t \in [0, T] : S(\sigma, t) \leq 0\} &\leq \sum_{\substack{\beta > \sigma \\ 0 < \gamma < T+1/2}} 2\sqrt{(\beta - \tfrac{1}{2})^2 - (\sigma - \tfrac{1}{2})^2} \\ &\leq 2\sqrt{\sigma(1 - \sigma)} N(\sigma, T + 1). \end{aligned}$$

The RH band of the truncated Poisson functional, [CE10, Theorem 3.13(b), (3.5)]:

$$(10.5) \quad (1 - e^{-\delta L})^2 \leq \frac{W_L(\sigma, t)}{2S(\sigma, t)} \leq (1 + e^{-\delta L})^2.$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** $\mathcal{J} = 0$; $\lim_n C_2(n, \rho) > 0$ at every zero (Corollary 2.10); $\mathcal{M}(k) = 1 + o(k)$; the Szegő form of Proposition 2.16; $S > 0$ for $\sigma > \frac{1}{2}$ (Lagarias); concavity in $v = (\sigma - \frac{1}{2})^2$ (Proposition 3.4); Weil positivity.
- **Unconditional.** Theorems 2.7, 3.2, 3.7, Corollaries 2.9, 2.13, Proposition 2.12; Proposition 3.5(b), $S > 0$ for $0 < |t| \leq 3 \cdot 10^{12} - \frac{1}{2}$, from the verification of RH by Platt–Trudgian.
- **Conditional on RH.** Corollary 3.14 and the limits in Proposition 3.4.
- **Control.** Epstein Z_Q (real zero $\beta_0 = 0.813367836786$, $\mathcal{J}_Q \geq 1.47$); the Davenport–Heilbronn function, whose Jensen integral, slope, concavity and truncated W_L all detect its off-line zeros; the twin $L(s, \chi)$ modulo 5, with an Euler product, shows no point with $S_{\chi} < 0$.
- **Caveat.** No computation can decide $\mathcal{J} = 0$ (a zero at height 10^{12} with $\beta - \frac{1}{2} = 10^{-3}$ changes $\mathcal{J}$ by about 10^{-27}). “We do not prove the Riemann hypothesis.”

Key numbers. $C_B = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots$; $\mathcal{M}(1) = 1.2606614\dots$; Jensen gap $0.115818\dots$; $J_{\zeta}(1000) = 3.1 \cdot 10^{-8}$, while for the Davenport–Heilbronn function $J_f(400) - \log |f(1)| = 1.52477 \cdot 10^{-4}$ against $1.52687 \cdot 10^{-4}$ from its eight off-line zeros below 400. $\min_t S(\sigma, t)$ on $20 \leq t \leq 600$ runs from 0.0039 ($\sigma = 0.51$) to 0.1901 ($\sigma = 1$). Primes up to 10^8 give $\delta_{\text{res}} = 0.18$ at $\varepsilon = 0.1$; $\delta = 0.01$ to 1% needs $P \geq 10^{248}$. Over $\mathbb{F}_p$ the window margin is $2 \log p$ [CE10, Proposition 3.15].

13. PART 11: THE EULER–CACERES BOUND

Origin. [CE11] supersedes the earlier preprint on an Euler-product mollifier for the Jensen moment of ζ (source of Theorem 6, Corollary 7 and a weaker Theorem 1), and its factorization lemma and Section 11.1 first appeared in version 3 of the consolidated paper.

Summary. Dirichlet-polynomial mollifiers cannot bring the Jensen bound for the Balazard–Saia–Yor sum below $\frac{1}{2} \log(1 + c_0) = 0.0386$. Part 11 uses instead an exact form $\zeta = E_X^* Z_X$ of the hybrid Euler–Hadamard product of Gonek, Hughes and Keating, E_X^* a smoothed Euler factor carrying the pole and the trivial zeros and $Z_X(s) = \prod_{\rho} e^{-K_X(s-\rho)}$ a product over the zeros. For $\operatorname{Re} s_0 > \frac{1}{2}$ the Euler–Caceres bound $B_X(s_0)$ is the X -independent off-line Blaschke term of the Poisson–Jensen formula plus a nonnegative Jensen gap. Under RH, $(\log X)B_X(s_0)$ tends to $\sum_{\gamma} \kappa_m(\gamma) P_{s_0}(\gamma)$ with universal constants κ_m coming from the well profile $e^{m \operatorname{Ci}(|v|)}$ of $|Z_X|$ at a zero of multiplicity m ; this yields single-point criteria for RH and for RH with simple zeros, while an attained $\Theta > \frac{1}{2}$ makes the bound explode. The paper gives unconditional bounds, under RH a second-order term up to an interaction of the wells (related heuristically to Montgomery’s pair correlation) and an isometry with the energy of the prime error, a function-field model in which the bound holds, and controls (Davenport–Heilbronn, a Beurling system with the zeros of ζ and a real zero at 0.6) showing that a proof must couple the functional equation with the Euler product.

Main results.

- (1) [CE11, Theorem 1] (*proved*) (decomposition). For $\operatorname{Re} s_0 > \frac{1}{2}$, $\zeta(s_0) \neq 0$, $X > e^{a+1}$: $B_X(s_0) = \mathcal{O}(s_0) + J_X(s_0)$, where the off-line term $\mathcal{O}(s_0) \geq 0$ is the sum of $\log(1/|b_{\rho}(s_0)|)$ over the zeros with $\operatorname{Re} \rho > \frac{1}{2}$, $b_{\rho}(s) = (s - \rho)/(s - 1 + \bar{\rho})$, and $J_X \geq 0$ is a Jensen gap; see (11.3).
- (2) [CE11, Theorem 2] (*conditional on RH*) (law of the bound). For $\operatorname{Re} s_0 > \frac{1}{2}$, $(\log X)B_X(s_0)$ tends to $\Omega(s_0)$ as in (11.5).
- (3) [CE11, Corollary 3] (*proved*) (criteria). Unconditionally, (a) $\operatorname{RH} \iff B_X(s_0) \rightarrow 0 \iff B_X(s_0) = O(1/\log X)$; (b) RH with all zeros simple is (11.6), the limit being 0.040186... at $s_0 = 2$; (c) under RH a multiple zero raises the limit by $(\kappa_m - m\kappa_1)P_{s_0}(\gamma)$.
- (4) [CE11, Theorem 5] (*conditional on RH*) (second order, g_0 even). $(\log X)B_X$ equals $\Omega + (\Omega_1 - \Omega^2 + \mathcal{I}_X)/\log X$ plus $o(1/\log X)$, with explicit Ω_1 and the interaction of the wells known only to be $\mathcal{I}_X = o(\log X)$.
- (5) [CE11, Theorem 6] (*conditional on an attained supremum $\Theta > \frac{1}{2}$*) (explosion). For $\operatorname{Re} s_0 > \Theta$ and some $r \in (0, \Theta - \frac{1}{2})$, $B_X(s_0) \geq \frac{1}{2} \operatorname{Ei}(r(\log X - a)) - C \log \log X \geq X^r/(3re^{ra} \log X)$.
- (6) [CE11, Theorem 20] (*proved*) (unconditional bounds). $0 \leq \mathcal{O}(s_0) < 3.1 \cdot 10^{-12}(2\sigma_0 - 1)$ for $|t_0| \leq \frac{1}{2} \cdot 3 \cdot 10^{12}$, $\mathcal{O}(2) < 2.3 \cdot 10^{-12}$; and $B_X(s_0) \ll \sqrt{X}/\log X$ for $\operatorname{Re} s_0 > 1$.
- (7) [CE11, Theorem 30] with [CE11, Corollaries 28 and 31] (*conditional on RH*). $\int |\log Z_X(\frac{1}{2} + it)|^2 P_1(t) dt = \mathcal{E}_{\Delta}(X) + O((\log X)^{-3/2})$, $\mathcal{E}_{\Delta}$ the energy of the normalized prime error beyond $\log X$; for simple zeros $\lim(\log X)B_X(1) = \frac{\kappa_1}{\pi} \lambda_1 = 0.013438\dots$ and $B_X(1)/\mathcal{E}_{\Delta}(X) \rightarrow \kappa_1/(2\pi) = 0.29092\dots$
- (8) [CE11, Proposition 34] (*proved*) (function fields). For $L(u) = \prod_{j \leq D} (1 - \alpha_j u)$: $|\alpha_j| \leq \sqrt{q} \iff \sup_n |c_n| q^{-n/2} < \infty \iff B_N(u_0) \rightarrow 0 \iff \sup_N B_N(u_0) < \infty$, and then $NB_N(u_0) \rightarrow \sum_{\theta} \kappa_m(\theta) P_{u_0}(\theta)$; by Weil’s theorem the bound holds over $\mathbb{F}_q[T]$.
- (9) [CE11, Proposition 46] (*proved*) (Dirichlet floor). For a Dirichlet polynomial M with $M(1) \neq 0$, $R(M) \geq 1 + c_0$, $c_0 = 0.0803270\dots$, so the Jensen bound cannot fall below 0.0386.

What is new.

- Not new ([CE11, Sect. 1.4]): the factorization is the Gonek–Hughes–Keating product in exact form, and the profile of Z_X on the line under RH is theirs; Theorem 1 and Corollary 3(a) are Jensen-type reformulations; Theorem 20 is elementary (Platt–Trudgian, Trudgian’s $N(T)$); Proposition 29 is standard; Proposition 45 rearranges Chebyshev’s identity in a Beurling system.

- **New:** the law of Theorem 2 with the constants κ_m , giving the rate in Corollary 3(a) and the simple-zero criterion; Theorem 6; Theorem 5 up to the interaction of the wells; Theorem 30 and the comparison with the energy of the primes.
- **Heuristic:** the relation of the interaction to Montgomery's pair correlation ("a derivation, not a theorem"), and most conclusions of Section 13 on what a proof would need. "We do not prove that $(\log X)B_X(s_0)$ is bounded; by Corollary 3 this is equivalent to RH."

New formulas. The smoothed Euler factor [CE11, (2)], for which $\zeta = E_X^* Z_X$ [CE11, Lemma 8]:

$$(11.1) \quad \log E_X^*(s) = \sum_{n \geq 2} \frac{\Lambda(n) \psi_X(\log n)}{n^s \log n} + K_X(s-1) - \sum_{k \geq 1} K_X(s+2k),$$

$$K_X(w) = \int g_X(u) E_1(wu) du.$$

The Euler–Caceres bound [CE11, Sect. 1.2], with $\phi_X(t) = \log |Z_X(\frac{1}{2} + it)|$ and P_{s_0} the Poisson kernel of $\operatorname{Re} s > \frac{1}{2}$ at s_0 :

$$(11.2) \quad B_X(s_0) = \frac{1}{2} \log \int e^{2\phi_X(t)} P_{s_0}(t) dt - \log |Z_X(s_0)|.$$

The decomposition [CE11, Theorem 1]:

$$(11.3) \quad B_X(s_0) = \mathcal{O}(s_0) + J_X(s_0), \quad J_X(s_0) = \frac{1}{2} \log \int e^{2\phi_X} P_{s_0} - \int \phi_X P_{s_0} \geq 0.$$

The multiplicity constants, defined before [CE11, Theorem 2]:

$$(11.4) \quad \kappa_m = \int_0^\infty (e^{2m \operatorname{Ci}(v)} - 1 - 2m \operatorname{Ci}(v)) dv, \quad \operatorname{Ci}(v) = - \int_v^\infty \frac{\cos \tau}{\tau} d\tau.$$

The law under RH [CE11, Theorem 2]:

$$(11.5) \quad \lim_{X \rightarrow \infty} (\log X) B_X(s_0) = \Omega(s_0) := \sum_{\gamma} \kappa_m(\gamma) P_{s_0}(\gamma).$$

The single-point criterion for RH with simple zeros [CE11, Corollary 3(b)]:

$$(11.6) \quad \text{RH and all zeros simple} \iff \lim_{X \rightarrow \infty} (\log X) B_X(s_0) = \frac{\kappa_1}{\pi} \operatorname{Re} \frac{\xi'}{\xi}(s_0).$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** $B_X(s_0) = O(1/\log X)$ at any single s_0 ; RH with simple zeros is (11.6). If Θ is attained: $\liminf_X \int |\zeta/E_X^*(\frac{1}{2} + it)|^2 dt / (1+t^2) < \infty$ (Corollary 7) and $\sigma_E = \frac{1}{2}$ (Proposition 40). Corollary 43 restates $\psi(x) - x = O(x^{1/2+\varepsilon})$.
- **Conditional on RH.** Theorems 2, 5, 30 and Corollaries 28, 31.
- **Unconditional.** Theorems 1, 20, Lemma 8, Propositions 34, 45, 46; Theorem 6 under an attained $\Theta > \frac{1}{2}$.
- **Control.** For the Davenport–Heilbronn function the bound grows, from 1.65 ($X = 10$) to 7.58 ($X = 10^4$) at $1 + 85.7i$, above its off-line term 1.4407; a Beurling system of Broucke–Debruyne–Révész type with every zero of ζ , integers better than $\sqrt{x}$ and a real zero at 0.6 shows that no integer-error bound can replace the functional equation.
- **Caveat.** "We do not prove RH."

Key numbers. $\kappa_1 = 1.82792$, $\kappa_2 = 7.34579$, $\kappa_3 = 20.036$, $\kappa_4 = 48.806$, $\kappa_2/\kappa_1 = 4.0187$. At $s_0 = 2$, with $\varpi(s_0) = \pi^{-1} \operatorname{Re}(\xi'/\xi)(s_0)$: $\kappa_1 \varpi(2) = 0.040186$; with zeros $|\gamma| < 800$ the law predicts 0.03816 against computed $(\log X)B_X(2) = 0.0347\text{--}0.0368$ for $X = 10^2\text{--}10^5$; the ζ^2 ratio is within 0.7% of 4.0187. Floor $c_0 = 0.0803270\dots$, 0.0386. Pair correlation at $T = 10^6$: -0.587 ± 0.011 against -0.582 at $\alpha = 0.3$.

14. PART 12: WEIL POSITIVITY AT THE WINDOWS OF THE ZEROS

Origin. [CE12] collects, with corrections, the computations on Weil’s form scattered through version 3 of the consolidated paper and the earlier preprint VII (Weil positivity as a prime cross-energy).

Summary. By Weil’s criterion RH holds if and only if a quadratic form W in the pole, the Gamma factor and the prime powers is nonnegative; on a window of length L only $n \leq e^L$ enter, and the form is a rank-two pole term plus the energy of $\hat{\varphi}$ against the prime partial sum on the line, its prime part being the cross-energy of φ with its translates by $\log n$. Part 12 studies numerically the smallest value $\lambda_{\min}(L)$. In a fixed sine basis it fits $10^{-6.3L^2}$, a law of the basis; in stabilised bases the computed upper bound is positive and close to $10^{-(5.42e^L-12.6)}$ for $2 \leq L \leq 5$. Its size is fixed by the density of the zeros (regular points give the same values; curves over $\mathbb{F}_p$ give $2 \log p$). The primes create the positivity, with no reserve in the windows tested. For the Davenport–Heilbronn function f the form is negative beyond $L^* \approx 3.425$, with $L^* \leq 3.43$ certified in interval arithmetic, and a real zero of the minimiser then locks onto the off-line ordinate. At large height two windows detect an isolated zero $\frac{1}{2} + \delta + i\gamma$ with the prime powers up to $(C \log \gamma)^{1/\delta}$. Prime translations are isometries; “what is missing over $\mathbb{Q}$ is a finite set of lags closed under them.”

Main results.

- (1) [CE12, Proposition 1] (*proved*) (structure; elementary given Weil’s criterion). $W(\varphi) = 2 \operatorname{Re}[\hat{\varphi}(\frac{i}{2})\hat{\varphi}(-\frac{i}{2})] + \frac{1}{2\pi} \int |\hat{\varphi}|^2 \Psi_L$; the pole term has rank at most two; the prime part is $-2 \sum_{n \leq e^L} \Lambda(n) n^{-1/2} \operatorname{Re}\langle \varphi, T_{\log n} \varphi \rangle$; $\lambda_{\min}(L)$ depends only on L and is nonincreasing; RH $\iff \lambda_{\min}(L) \geq 0$ for all L ; finite-dimensional eigenvalues are only upper bounds.
- (2) [CE12, Sect. 3.5] (*numerical*) (margin in stabilised bases). The law (12.3), with $\log_{10} \lambda_{\min} = -792.5$ at $L = 5$; stabilisation observed, not proved.
- (3) [CE12, Remark 3] (*heuristic*) (the law). A function of exponential type $L/2$ follows the zeros up to height R with $\log R \approx L/c$, giving $\log \lambda_{\min} \approx -\alpha L^2/c$ with $\alpha \approx 15$.
- (4) [CE12, Sect. 5] (*numerical*) (prime powers). Without primes the form is negative from $L \approx 0.755$; at $L = 1.25$ the primes 2, 3 raise $\lambda_{\min}$ from -0.571 to $3.47 \cdot 10^{-10}$ and move the minimiser’s zeros to 14.134, 21.022, 25.012; removing the most recent prime power makes the form negative in every window tested.
- (5) [CE12, Proposition 4] (*certified in interval arithmetic*) (Davenport–Heilbronn threshold). At $L = 3.43$ an explicit cosine combination has $W_f(\varphi) < 0$ (value (12.4)), so $L^* \leq 3.43$; $L^* > 3.42$ is numerical [CE12, Sect. 6].
- (6) [CE12, Lemma 5] (*proved*) (two windows). The form on two twisted bumps reduces to the 2×2 matrix (12.5), with minimum $D - |Y|$.
- (7) [CE12, Proposition 6] (*proved; part (b) numerical*) (detection). Under an isolation hypothesis the inequality (12.6) holds; for fixed δ , $A > 1/(10\delta)$, $H = (\log \gamma_0)^A$ and large γ_0 , the prime powers up to $(C_\delta \log \gamma_0)^{1/\delta}$ detect $\frac{1}{2} + \delta + i\gamma_0$; for $\delta \geq 0.1$ this is evaluated at all heights $10^6 \leq \gamma_0 \leq 10^{1000}$.
- (8) [CE12, Sect. 8] (*numerical*) (large height). A two-window positivity holds for ζ at 2400 heights in $[2 \cdot 10^3, 3 \cdot 10^6]$ with margin 6 to 15% and fails for a random multiplicative model at 1.5% of them.
- (9) [CE12, Lemma 7] (*proved*) (isometries). $W(T_{\log p} \varphi) = W(\varphi)$; if $W \geq 0$ the translations act unitarily on the completion.

What is new.

- Prior work: Weil and Bombieri (criterion, explicit formula), Yoshida (short windows), Connes–Consani and Connes–Consani–Moscovici, Connes’ minimiser observation (extended

by Groskin), announced certified bounds (Liu), Suzuki’s screw-function criterion and the computations of Kim and coauthors. Proposition 1 and Lemmas 5, 7 are elementary.

- **New:** “the numerical description of the margin and of who pays for it” (density explanation, role of each prime power, the Davenport–Heilbronn threshold with its certified bound, the Connes minimiser for a function violating RH) and Proposition 6.
- **Corrections to preprint VII:** $L^* \approx 3.6$ becomes $3.42 < L^* \leq 3.43$; $10^{-6.3L^2}$ holds only in a fixed basis.

New formulas. The window margin [CE12, Sect. 1.1]:

$$(12.1) \quad \lambda_{\min}(L) = \inf \left\{ \frac{W(\varphi)}{\|\varphi\|_2^2} : \text{supp } \varphi \subset \left[-\frac{L}{2}, \frac{L}{2}\right] \right\}.$$

The symbol of the form [CE12, Proposition 1]:

$$(12.2) \quad \Psi_L(r) = \text{Re } \psi\left(\frac{1}{4} + \frac{ir}{2}\right) - \log \pi - 2 \text{Re} \sum_{n \leq e^L} \frac{\Lambda(n)}{n^{1/2+ir}}.$$

The numerical margin law [CE12, Sect. 3.5]:

$$(12.3) \quad \log_{10} \lambda_{\min}(L) \approx -(5.42 e^L - 12.6) \quad (2 \leq L \leq 5).$$

The certified negative value for the Davenport–Heilbronn function [CE12, Proposition 4]:

$$(12.4) \quad W_f(\varphi)/\|\varphi\|_2^2 \in [-6.854847504 \cdot 10^{-30} \pm 3 \cdot 10^{-36}].$$

The two-window reduction [CE12, Lemma 5]:

$$(12.5) \quad \begin{aligned} W(\varphi_{a,b}) &= (|a|^2 + |b|^2)D - 2 \text{Re}(a\bar{b}Y), \\ Y &= \sum_n \frac{\Lambda(n)}{\sqrt{n}} K(\log n - L') n^{-i\gamma_0} + R. \end{aligned}$$

The detection inequality [CE12, Proposition 6, (5)]:

$$(12.6) \quad W(\varphi_{1,-1}) \leq -2w^2 \sinh^2 \frac{\delta L'}{2} + 4S_w c(\gamma_0 + H + 1) + \frac{8k^4}{q^2} e^{L/2} \Theta(\gamma_0, H).$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** $\lambda_{\min}(L) \geq 0$ for every L (Weil); in spline bases, all Levinson–Durbin reflection coefficients of modulus below 1 for a sequence of meshes $h \rightarrow 0$.
- **Conditional on RH.** The Hilbert space of Lemma 7.
- **Unconditional.** Proposition 1, Lemma 5, Proposition 6 ((12.6) and part (a)); the certified negativity of Proposition 4.
- **Control.** Davenport–Heilbronn: negative beyond L^* , its second off-line pair seen from $3.72 < L \leq 3.725$ once the first pair is moved to the line; its positivity before L^* rests on coefficients of opposite sign. Curves over $\mathbb{F}_p$, regular point sets and a random multiplicative model are further controls.
- **Caveat.** “No computation here certifies positivity”; positivity on a fixed window and the accuracy of the Connes minimiser are compatible with off-line zeros. “We do not prove the Riemann hypothesis.”

Key numbers. Sine basis: $\lambda_{\min} = 1.079 \cdot 10^{-6}$ ($L = 1$), $1.7 \cdot 10^{-26}$ ($L = 2$); stabilised bases: $\lambda_{\min} \approx 10^{-793}$ at $L = 5$. At $L = 3$ the minimising direction splits as pole $+1.5413$, archimedean -1.4722 , primes -0.0691 . Davenport–Heilbronn: $3.42 < L^* \leq 3.43$ (coefficients up to $n \approx 31$), eigenvalue -0.70 at $L = 4$. Curves: $9.23 = 2 \log 101$.

15. PART 13: OPTIMISED MOLLIFIERS AND THE BOUND $\Lambda \leq 0.174$ FOR THE DE BRUIJN–NEWMAN CONSTANT

Origin. [CE13] names no earlier paper as its source; its proof uses only Polymath’s criterion and the verification of RH to $3 \cdot 10^{12}$ by Platt and Trudgian.

Summary. The heat-flow deformation H_t of Ξ has only real zeros exactly when $t \geq \Lambda$ (de Bruijn, Newman); RH is equivalent to $\Lambda \leq 0$, and Rodgers and Tao proved $\Lambda \geq 0$. Polymath reduced upper bounds to three verifiable hypotheses (RH up to $X/2$, a zero-free region for H_{t_0} right of X , a barrier at X), which with Platt–Trudgian gave $\Lambda \leq 0.2$; the bottleneck is the zero-free region, handled by Polymath with Euler-product mollifiers. Part 13 replaces the Euler product by $\sum_{d \leq D} \lambda_d d^{-s}$ with weights chosen by an ℓ^1 regression; since any weights with $\lambda_1 = 1$ give a valid lower bound, only its evaluation must be certified. At $X \approx 6 \cdot 10^{12}$, $y_0 = 0.2$, the lower bound improves from -0.12 (Euler, $p \leq 13$) to $+0.05$ ($D = 120$, $t_0 = 0.155$), and is $+0.04$ with $D = 240$ at $t_0 = 0.154$. A chunking argument with an explicit tail covers all $x \geq X$, the barrier uses Polymath’s Taylor scheme, and the certificates (Arb ball arithmetic and outward-rounded floating point, reproduced on a second platform) give the computer-assisted upper bound $\Lambda \leq 0.174$. A by-product gives a Poisson-weighted count of the zeros of H_{t_0} in windows above X with error less than 3.001.

Main results.

- (1) [CE13, Theorem 1] (*proved, computer-assisted*). $\Lambda \leq 0.17399 < 0.174$, see (13.1); with Rodgers–Tao, $0 \leq \Lambda \leq 0.174$.
- (2) [CE13, Theorem 2] (*cited: Polymath 2019, Theorem 1.2*) (Polymath’s criterion). If $\zeta \neq 0$ for $\frac{1+y_0}{2} \leq \sigma \leq 1$, $0 \leq T \leq X/2$, if $H_{t_0}(x+iy) \neq 0$ for $x \geq X + \sqrt{1-y_0^2}$, $y_0 \leq y \leq \sqrt{1-2t_0}$, and if the barrier condition holds, then $\Lambda \leq t_0 + \frac{1}{2}y_0^2$.
- (3) [CE13, Lemma 3] (*proved*) (mollified lower bound). For any real $\lambda_1 = 1, \lambda_2, \dots, \lambda_D$, (13.3) holds; by [CE13, Remark 4] only $\lambda_1 = 1$ is used, and maximising the concave $\mathcal{L}$ is an ℓ^1 regression.
- (4) [CE13, Lemma 6] (*proved*) (covering). The chunk bound (13.4) holds for all $x \geq x_a$, and F is strictly decreasing.
- (5) [CE13, Lemma 7] (*proved*) (tail). With $V = e^{60}$, $F(v) \leq F(V)V^2v^{-2}$ for $v \geq V$.
- (6) [CE13, Sect. 6] (*certified in interval arithmetic*) (certificate). For $t_0 = 0.15399$, $y_0 = 0.2$, $X = 5\,998\,313\,329\,980.3135$, $D = 240$: $|f_{t_0}| \geq 0.00594$ on region (ii) against error terms $\leq 2.98 \cdot 10^{-7}$; barrier in 73 steps with margin ≥ 0.345 ; identical output on two platforms.
- (7) [CE13, Proposition 8] (*proved*) (zero count). For Poisson weights w_k of the zeros of H_{t_0} in a window $[a, b]$ above X , (13.5) holds.
- (8) [CE13, Sect. 9] (*numerical*) (limit of the method). With $D \leq 240$ re-optimised at $t_0 = 0.153$ the bound at $x = X$ is only $+0.0009$, so 0.174 is “essentially the limit of the method” with these weights (“not a proven optimality statement”); longer mollifiers might gain about 0.002 (not certified).

What is new.

- New ([CE13, Sect. 9]): the optimisation of the mollifier in Polymath’s criterion (Lemma 3 for arbitrary weights and its ℓ^1 form); the covering of all $x \geq X$ with an explicit tail (Lemmas 6, 7); the certificate reproduced on two platforms.
- Prior work: de Bruijn, Newman; Rodgers–Tao ($\Lambda \geq 0$); Polymath (criterion, effective approximation, error bounds, Taylor scheme, $\Lambda \leq 0.22$); Platt–Trudgian ($\Lambda \leq 0.2$). The result does not depend on unrefereed claims (Gomila 0.1787854, Gordon, conditional bounds of Hakimoglu-Brown and Planat).
- The proof uses no result of the series; the series supplied the approach: optimised weights as for the Jensen moment in [CE11], and the torus of prime phases of [CE1] and [CE6].

New formulas. The main bound [CE13, Theorem 1]:

$$(13.1) \quad \Lambda \leq 0.15399 + \frac{1}{2}(0.2)^2 = 0.17399 < 0.174.$$

The mollified coefficients [CE13, Lemma 3], with $b_m = e^{\frac{t}{4} \log^2 m}$:

$$(13.2) \quad c_n = \sum_{\substack{d|n, d \leq D \\ n/d \leq N}} \lambda_d b_{n/d}, \quad c'_n(y) = \sum_{\substack{d|n, d \leq D \\ n/d \leq N}} \lambda_d d^{-y} b_{n/d}.$$

The certified functional and lower bound [CE13, Lemma 3]:

$$(13.3) \quad \mathcal{L} = 1 - \sum_{n \geq 2} |c_n| n^{-\sigma} - |\gamma| \sum_{n \geq 1} (|c'_n(y)| + \epsilon_n) n^{y-\sigma+|\kappa|},$$

$$|f_t| \geq |A| - |\gamma| |B| \geq \frac{\mathcal{L}}{\sum_{d \leq D} |\lambda_d| d^{-\sigma}} \quad (\mathcal{L} > 0).$$

The chunk bound [CE13, Lemma 6(a)]:

$$(13.4) \quad |f_t(x + iy)| \geq \frac{\mathcal{L}_{N_a}(x_a) - \sum_{N_a < m \leq N(x)} F(m)}{M^+(x_a)}.$$

The zero count [CE13, Proposition 8]:

$$(13.5) \quad \left| \sum_k w_k - \frac{\theta(a) - \theta(b)}{\pi} \right| < 3.001.$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** $\Lambda \leq 0$ (classical); the paper proves nothing equivalent to RH.
- **Unconditional.** $\Lambda \leq 0.174$ is an unconditional computer-assisted upper bound; it uses RH only in the verified range $X/2 < 3 \cdot 10^{12}$.
- **Control.** In Section 8 the Davenport–Heilbronn function shows how an off-line zero appears in the prime error (spiral orbit, growth 0.308 against $\beta - \frac{1}{2} = 0.3085$); this “does not, however, help with Λ ”.
- **Caveat.** By [CE15] an off-line zero $\frac{1}{2} + \delta + i\gamma$ forces $\Lambda \geq \delta^2 / (70 \log \gamma \log \log \gamma)$; by [CE14] mollified triangle bounds of length at most $N/2$ cannot give $C < 4$ in $\Lambda \leq C / \log X$, while Theorem 1 has $C = 4.68$. “We do not prove the Riemann hypothesis.”

Key numbers. Lower bound $\mathcal{L}$ at $x = 6 \cdot 10^{12}$, $y = 0.2$: triangle inequality $-1.66/-1.79$ and Euler on $p \leq 13$ $-0.048/-0.124$ at $t_0 = 0.157/0.155$; optimised $d \leq 120$: $+0.122, +0.051, -0.027$ at $t_0 = 0.157, 0.155, 0.153$; $d \leq 240$: $+0.040$ at $t_0 = 0.154$. Certificate: $N = 690\,891$ on the barrier, worst certified $|f_{t_0}| \geq 0.00594$, error terms $\leq 2.977 \cdot 10^{-7}$, about one hour on a laptop. Context: Polymath 0.22, Platt–Trudgian 0.2, Gomila (preprint) 0.1787854, this paper 0.174; $D = 120$ gives 0.175.

16. PART 14: THE CONSTANT IN THE $C/\log X$ LAW FOR TRIANGLE-INEQUALITY UPPER BOUNDS ON THE DE BRUIJN–NEWMAN CONSTANT

Origin. Part 14 (16 pages, October 2026) names no earlier paper of the author as its source. It builds on the Polymath method (arXiv:1904.12438v2) and on the mollified inequality and the bound $\Lambda \leq 0.174$ of Part 13 [CE13].

Summary. Polymath's criterion bounds the de Bruijn–Newman constant by $\Lambda \leq t_0 + \frac{1}{2}y_0^2$ under three hypotheses: (i) RH verified for ζ up to $X/2$; (ii) H_{t_0} has no zeros with $\text{Im } z \geq y_0$ to the right of X ; (iii) a barrier condition at X . Polymath observed that this gives $\Lambda = O(1/\log X)$. Part 14 determines the constant C in bounds $\Lambda \leq (C + o(1))/\log X$ for one class of arguments, the *triangle certificates*, in which (ii) is obtained by a triangle inequality applied to Polymath's main term f_t , with or without a mollifier of length D . The paper stresses that C is a property of this class of proofs, not of Λ , and keeps apart the constant Λ , the bound $B(X) = t_0 + \frac{1}{2}y_0^2$ that a method produces, and the coefficient $C(X) = B(X) \log \frac{X}{4\pi}$.

Without a mollifier the optimal coefficient is exactly $4w_0$, where $\zeta(\frac{1}{2} + w_0) = 2$; for mollifiers of length $D \leq N/2$ ($N \approx \sqrt{X/4\pi}$ the length of the sum) it lies in $[4, 4w_0]$. The value $4w_0$ comes from an explicit criterion in which each term keeps its own effective exponent $\alpha_n = \frac{1+y}{2} + \frac{t}{4}(L - \log n)$; the floor 4, a limitation of the method, comes from the primes in $(N/2, N]$, whose coefficients no such mollifier can change. The triangle inequality gives every cross-energy $|F_n|^2 - |F_{n-1}|^2 - |a_n|^2$ its most negative value; this is the link with the series, and where a phase-sensitive certificate would have to gain.

Main results.

- (1) [CE14, Theorem A] (*proved; part (c) conditional on the verification of RH up to $X/2$ and on the barrier*). (a) Without a mollifier, $C_{\text{opt}}(\{1\}) = 4w_0$. (b) For mollifier lengths $D \leq N/2$, $4 \leq C_{\text{opt}}(\{D \leq N/2\}) \leq 4w_0$. (c) The families attaining $4w_0$ establish hypothesis (ii) on the whole region, including Polymath's error terms; hence $\Lambda \leq (4w_0 + o(1))/\log X$ along any sequence of heights at which RH is verified up to $X/2$ and the barrier holds for the parameters of Corollary 7. By [CE14, Definition 1], a triangle certificate of length D at height X consists of real weights $\lambda_1 = 1, \dots, \lambda_D$ and admissible σ, g, k with $\mathcal{L} > 0$ at $t = t_0$, $x = X + \sqrt{1 - y_0^2}$, $y = y_0$; its value is $B = t_0 + \frac{1}{2}y_0^2$, and $C_{\text{opt}}(\mathcal{D})$ is the infimum of $\liminf_{X \rightarrow \infty} B(X) \log X$ over families with lengths in $\mathcal{D}$. Where $C_{\text{opt}}(\{D \leq N/2\})$ lies in $[4, 4w_0]$ is open.
- (2) [CE14, Theorem 2] (*proved*). For $0 < t \leq \frac{1}{2}$, $x \geq 200$ and $0 \leq y \leq 1 - \frac{4}{x}$, $|f_t(x + iy)| \geq G(t, x, y)$, formula (14.1); hence $H_t(x + iy) \neq 0$ whenever $G > e_A + e_B + e_{C,0}$. Moreover $G \leq \mathcal{L}$ for the certificate of length $D = 1$ with $\sigma = A$.
- (3) [CE14, Theorem 6 and Corollary 7] (*proved; conditional on the verification of RH up to $X/2$ and on the barrier*). Under hypotheses (i) and (iii) and the inequalities (H1), (H2) at the single point $x = X$ (formula (14.3)), $\Lambda \leq t_0 + \frac{1}{2}y_0^2$. For every $\varepsilon > 0$ there is X_ε such that, for $X \geq X_\varepsilon$, the parameters chosen in the proof satisfy (H1), (H2) and $(t_0 + \frac{1}{2}y_0^2) \log X \leq 4w_0 + \varepsilon$; so $\Lambda \leq (4w_0 + \varepsilon)/\log X$ at every such X where (i) and (iii) hold. The table of admissible parameters is computed in double precision and is not certified; its margins are at least 0.024.
- (4) [CE14, Proposition 9 and Corollary 10] (*proved*). For $D \leq N/2$ and $N \geq 2 \cdot 10^5$, every certificate satisfies (14.4); hence $\liminf(t_0 + \frac{1}{2}y_0^2) \log X \geq 4$, that is, $C_{\text{opt}}(\{D \leq N/2\}) \geq 4$. This is a limitation of a class of proofs, not a lower bound for Λ . The prime count $\pi(N) - \pi(N/2) \geq \frac{N}{3 \log N}$ for $N \geq 2 \cdot 10^5$ [CE14, Lemma 8] rests on Rosser–Schoenfeld; its numerical step is checked in ball arithmetic and by an exact sieve on $[2 \cdot 10^5, 2 \cdot 10^7]$.
- (5) [CE14, Proposition 11 and Corollary 12] (*proved*). If $D = 1$ and $\mathcal{L} > 0$, then for every $2 \leq n_0 \leq N$

$$\sum_{n=2}^{n_0} n^{-\alpha_n^+} < 1, \quad \alpha_n^+ = \frac{1+y}{2} + \frac{t}{4}(L - \log n) + \frac{3t}{2x^2};$$

hence $C_{\text{opt}}(\{1\}) \geq 4w_0$.

- (6) [CE14, Observation 14] (*numerical*). For $D = 30$ and $y_{\min} \gtrsim 0.1$, the data satisfy $w := \frac{y_{\min}}{2} + \frac{t}{8}L \in [0.614, 0.631]$, with mean 0.624, for all t and x computed; the paper calls this “a numerical regularity, not a law”. Longer mollifiers go below this level: $w_D = 0.6149$ at $D = 240$, already at the level $w_0/2 = 0.6143$. The envelope built on it, [CE14, Remark 15], is *heuristic*.

What is new. *Not new:* the law $\Lambda = O(1/\log X)$ is due to Polymath (§10 of their paper), together with their heuristic that one cannot do substantially better without major new information on RH. *New, by the paper’s own claim:* the explicit coefficient $4w_0$ and its optimality without a mollifier, both from keeping the individual exponents of the terms (Theorem 2, Corollaries 7 and 12; the paper is “not aware of an earlier explicit value”), and the floor 4 for mollified triangle certificates, forced by the terminal primes, a structural limitation of the method that explains why optimising the weights cannot improve such bounds indefinitely. The paper also gives the upper bound (14.2) for $\operatorname{Re} s_*$, which is not in Polymath, the mollifier-length data, and the reading of the triangle inequality through cross-energy. *Prior work:* the mollified inequality and the bound $\Lambda \leq 0.174$ are from Part 13 [CE13], a preprint not yet refereed; the unconditional bounds 0.22 (Polymath) and 0.2 (Polymath’s criterion with the verification of RH by Platt–Trudgian) are quoted.

New formulas. The explicit lower bound for Polymath’s main term, with per-term exponents, is [CE14, Theorem 2]:

$$(14.1) \quad \begin{aligned} |f_t(x + iy)| &\geq G(t, x, y) := 1 - S_A - e^{0.02y} e^{-yL/2} (1 + S_B), \\ S_A &= \sum_{n=2}^{n_0} n^{-\alpha_n} + \int_{\log n_0}^M e^{(1-A)v + \frac{t}{4}v^2} dv. \end{aligned}$$

The upper bound for the effective abscissa is [CE14, (2)]:

$$(14.2) \quad \operatorname{Re} s_* \leq \frac{1+y}{2} + \frac{t}{4}L + \frac{3t}{2x^2}.$$

The single-point hypotheses (H1) and (H2) of [CE14, Theorem 6] are:

$$(14.3) \quad \begin{aligned} \text{(H1)} \quad &\sigma_A(X, y_0) > 1, \quad \sigma_B(X, y_0) \geq 1 + \frac{2}{L} \log \frac{1}{y_0}; \\ \text{(H2)} \quad &G(X, y_0) - R_A > E(t_0, X, y_0). \end{aligned}$$

The floor from the terminal primes is [CE14, Proposition 9]:

$$(14.4) \quad \mathcal{L} \leq 1 - \frac{N^{1-\sigma'}}{3 \log N}, \quad \sigma' = \sigma_A + \frac{t}{4} \log 2 + \frac{t}{N}.$$

The decomposition into cross-energies, which for $c_n \equiv 1$ reduces to $C_2(n, s)$ of Part 1 [CE1], is [CE14, (4)]:

$$(14.5) \quad \begin{aligned} |F_N|^2 &= 1 + \sum_{n=2}^N (|a_n|^2 + \mathcal{C}(n)), \\ \mathcal{C}(n) &:= |F_n|^2 - |F_{n-1}|^2 - |a_n|^2 = 2 \operatorname{Re} (a_n \overline{F_{n-1}}). \end{aligned}$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** $\Lambda \leq 0$, by the theorem of de Bruijn and Newman; Rodgers and Tao proved $\Lambda \geq 0$. Both are quoted, classical results.
- **Conditional on RH verified to $X/2$ and on the barrier.** Every bound for Λ in the paper. The limit $\Lambda = 0$ is reached only as $X \rightarrow \infty$, which presupposes RH.

- **Unconditional.** Theorem A(a),(b), Theorem 2, Lemmas 4, 5 and 8, Propositions 9 and 11 and Corollaries 10 and 12 are analytic statements about the method; hypotheses (i) and (iii) play no role in the definition of C_{opt} .
- **Caveat.** $C(X) \geq 4 - o(1)$ does not mean $\Lambda \geq 4/\log X$, which would contradict RH; the floor is “fully compatible with $\Lambda = 0$ ”. The paper states: “We do not prove the Riemann hypothesis, and none of these bounds says anything about it.”

Key numbers. $w_0 = 1.2286472389981\dots$ and $4w_0 = 4.9145889559\dots$. At the present height the bound 0.174 of [CE13] has $C = 0.17399 \cdot \log(X/4\pi) = 4.68$. The optimum of Theorem 6 (conditional) gives $L(t_0 + \frac{1}{2}y_0^2) = 6.67$ at $X = 10^{10}$, 6.31 at $6 \cdot 10^{12}$ and 5.13 at 10^{100} . The finite-height floor gives only $C \geq 1.44$ at $6 \cdot 10^{12}$, 2.22 at 10^{20} , 3.16 at 10^{50} and 3.53 at 10^{100} . On the same scale, Polymath’s table has $C = 5.42$ at $2 \cdot 10^{12}$ and 4.80 at $9 \cdot 10^{21}$, Platt–Trudgian 5.38, and the bounds announced in 2026 (not refereed) Gomila 4.81, Gordon 4.25, Planat 4.00 (conditional on three analytic premises) and Hakimoglu-Brown 3.75 (assuming an unrefereed zero-free half-plane). Mollifier length (numerical, $t = 0.154$, $x = 6 \cdot 10^{12}$): $w_D = 0.6345, 0.6243, 0.6200, 0.6169, 0.6149$ for $D = 10, 30, 60, 120, 240$, against $w_0/2 = 0.6143$. Phases (numerical, $X = 6 \cdot 10^{12}$, 26262 terminal primes): the unmollified triangle value is -1.19 , while $|F_N| \geq 0.566$; for the terminal primes $\sum |a_p| = 0.0095$, while the median of $|\sum a_p|$ is $4 \cdot 10^{-5}$.

17. PART 15: HOW LONG CAN AN OFF-LINE ZERO SURVIVE THE DE BRUIJN–NEWMAN HEAT FLOW? AN EXPLICIT UNCONDITIONAL LOWER BOUND FOR Λ

Origin. Part 15 (33 pages, October 2026) names no earlier paper as its source. Its certified computations replace the floating-point scripts of an earlier version, which are kept for comparison.

Summary. De Bruijn’s argument shows that an isolated maximal off-line pair reaches the real axis by time $2\delta^2$, an *upper* bound. Part 15 asks the opposite question: how small can Λ be if ζ has a zero $\frac{1}{2} + \delta + i\gamma$ with $\delta > 0$? It proves unconditionally that $\Lambda \geq \delta^2/(70 \log \gamma \log \log \gamma)$, with an explicit bound at each height, by an explicit local zero count for the deformations H_τ (Jensen’s formula), a weighted functional $F_R = \sum \phi(x_k) y_k^p$ over the zeros in the upper half-plane, Hölder’s inequality with $p \asymp \log \log \gamma$, integration across collisions of zeros by a monotonicity lemma, and certification of the numerical steps in ball arithmetic. It bounds Λ , not the survival time of the given zero. For a single pair, the interaction term at $\tau = 0$ is the energy slope $\partial_\sigma \log |\xi|$ of the other zeros (Part 10 [CE10]). The dynamics is tested on the 21 off-line zeros of the Davenport–Heilbronn function with $\delta > 0.005$ and height below 800. A heat-flow Landau formula shows that under the flow the memory of a prime decays and rotates, while composite numbers acquire a memory governed by a deformed von Mangoldt function Λ_t .

Main results.

- (1) [CE15, Theorem 1.1 and Corollary 1.2] (*proved, unconditional; the final step of the local count and the maximisation that gives 70 are certified in Arb ball arithmetic*). If $\zeta(\frac{1}{2} + \delta + i\gamma) = 0$ with $\delta > 0$ and $\gamma > 0$, then $\gamma > 3 \cdot 10^{12}$ and

$$\Lambda \geq \frac{\delta^2}{70 \log \gamma \log \log \gamma};$$

more precisely $\Lambda \geq T(\gamma)\delta^2$ with $T(\gamma)$ given by (15.1). Equivalently, every zero $\beta + i\gamma$ with $\gamma \geq 3 \cdot 10^{12}$ satisfies $|\beta - \frac{1}{2}| \leq (70 \Lambda \log \gamma \log \log \gamma)^{1/2}$.

- (2) [CE15, Lemma 3.2] (*proved, computer-assisted*). For $X \geq 200$ and $0 \leq \tau \leq 1/(100\ell_X)$, H_τ has at most $B(X)$ zeros with $X \leq \text{Re } z \leq X + 1$, formula (15.2). Polymath proved such a count with unspecified constants.

- (3) [CE15, Lemma 4.1 and Remark 4.2] (*proved*). The exact equation of motion of a simple zero $z = x + iy$, $y > 0$, at $\tau = 0$ reads as (15.3), where S is the energy slope of [CE10]. Only the definition of S as a logarithmic derivative is used; no identity between energies is claimed.
- (4) [CE15, Proposition 5.2] (*proved*). If the zeros near the support of ϕ are simple, then $\frac{dF_R}{d\tau} \geq -\sum_{y_k > 0} \kappa_k \phi(x_k) y_k^{p-2} - E_R(\tau)$, with κ explicit for central zeros and $\sup |E_R| \rightarrow 0$ as $R \rightarrow \infty$ [CE15, Lemma 5.3].
- (5) [CE15, Theorem 7.1] (*conditional on the hypotheses (I) of local isolation and (C) of regular spacing along the whole trajectory*). $\tau_{\text{abs}} \geq T(y_0)$ and $\Lambda \geq T(y_0)$; with $L = \log \frac{\gamma}{2\pi}$ and $a = 1 + 2A$, A the local excess in (C),

$$\Lambda \geq \frac{4\delta}{L} - \frac{4a}{L^2} \log \left(1 + \frac{L\delta}{a} \right) \quad (1 + o(1)).$$

Evaluating (I) and (C) only at $\tau = 0$ gives a heuristic prediction, not a bound.

- (6) [CE15, Theorem 9.2] (*computer-assisted*). The Davenport–Heilbronn function satisfies $0.228 < \Lambda_f \leq 3.125$. The upper bound is de Bruijn’s theorem ($|\text{Im } z| < 2.5$ for all zeros of H_0^f); the lower bound is a rigorous Rouché argument giving a zero of $H_{0.228}^f$ in $[480.6116, 480.7716] \times [0.0289, 0.1156]$, verified with outward-rounded interval arithmetic and independently with Arb. The comparison $\tau_{\text{abs}} \geq T_S(\rho)$ of [CE15, Proposition 9.1] is *conditional on the interaction term not increasing along the flow*.
- (7) [CE15, Theorem 11.1 and Corollary 11.2] (*proved, unconditional, asymptotic*). The heat-flow Landau formula (15.5) holds for fixed $n \geq 2$ and $0 < t \leq \frac{1}{2}$; it proves [CE15, Conjecture 10.2]. The main term dominates when $\theta = \frac{t}{4} \log n < 1$ and the window is long enough. The result is not uniform in n , and $T_0 \geq 2\pi n^4 e^{8+64/t}$ is astronomically large. The qualitative erasure of the memory is [CE15, Proposition 10.1] (*proved*).
- (8) [CE15, Proposition 11.8] (*proved*). For $t > 0$: $\Lambda_t(p^k) > 0$ for every prime power; $\Lambda_t(n) > 0$ for every squarefree n , by the connected-graph expansion (15.6); and for every $n \geq 2$, $\Lambda_t(n) = \Lambda(n) + \frac{t}{4} \log n (\Lambda(n) \log n + (\Lambda * \Lambda)(n)) + O_n(t^2)$, so $\Lambda_t(n) > 0$ for small t when n has exactly two distinct prime factors. The other composites are numerical (no sign change found); positivity for all n and t is open.

What is new. *New, by the paper’s own claim:* a lower bound for Λ in terms of a hypothetical off-line zero (δ, γ) , and the link with the energy slope (“We have not found a lower bound for Λ in terms of (δ, γ) in the literature”). *Prior work distinguished:* de Bruijn’s opposite (upper) bound $2\delta^2$; Planat’s use of the interaction term to sharpen *upper* bounds; Tao’s remark that de Bruijn’s bound is sharp only for an isolated zero; Polymath’s local zero count with unspecified constants, here made explicit. *Further contributions:* the certified count (Lemma 3.2), the weighted functional and the integration across collisions (Appendix B); the bounds for Λ_f and the observation that each off-line zero of the Davenport–Heilbronn function sits in a “hole” of two missing on-line zeros; the heat-flow Landau formula and Λ_t . Landau’s formula, in Gonek’s uniform form, is classical and is recovered at $t = 0$.

New formulas. The explicit constant of Theorem 1.1 is [CE15, (3)], with $\mu = \frac{1}{2} \hat{B}(\pi W + 2)$:

$$(15.1) \quad \Lambda \geq T(\gamma) \delta^2, \quad T(\gamma) = \max_{p \geq 2} \frac{2p}{(1 + 10^{-100}) \kappa(p) \mu^{2/p}}.$$

The explicit local count is [CE15, Lemma 3.2]:

$$(15.2) \quad \begin{aligned} \#\{\text{zeros of } H_\tau \text{ with } X \leq \text{Re } z \leq X + 1\} &\leq B(X) := 2.07 \ell_X + 2.5, \\ \ell_X &= \log \frac{X + 1}{4\pi}. \end{aligned}$$

The initial velocity of an off-line zero is given by the energy slope [CE15, Remark 4.2]:

$$(15.3) \quad \begin{aligned} \partial_\tau \delta &= -\frac{1}{4\delta} - \frac{1}{2} S_{\text{rest}} \left(\frac{1}{2} + \delta + i\gamma \right), \\ S(\sigma, t) &= \partial_\sigma \log |\xi(\sigma + it)| = \sum_\rho \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2}. \end{aligned}$$

Freezing S_{rest} in (15.3) gives the energy-slope time [CE15, Section 9]:

$$(15.4) \quad T_S(\rho) = \int_0^\delta \frac{u \, du}{\frac{1}{4} + \frac{1}{2} S_{\text{rest}} u} = \frac{2\delta}{S_{\text{rest}}} - \frac{1}{S_{\text{rest}}^2} \log(1 + 2S_{\text{rest}}\delta).$$

The heat-flow Landau formula is [CE15, Theorem 11.1], with Λ_t defined by $b_n \log n = \sum_{d|n} \Lambda_t(d) b_{n/d}$ and $b_m = e^{(t/4) \log^2 m}$ [CE15, (4)]:

$$(15.5) \quad \begin{aligned} \sum_{T < \text{Re } z_k/2 \leq T+H} n^{iz_k/2} &= -\frac{\Lambda_t(n) n^{-i\pi t/8}}{2\pi\sqrt{n}} \int_T^{T+H} \left(\frac{\gamma}{2\pi} \right)^{-\frac{t}{4} \log n} d\gamma \\ &\quad + O_{n,t} \left(\log T + \frac{H}{T} \log^2 T \right). \end{aligned}$$

The connected-graph expansion for squarefree $n = p_1 \cdots p_k$ is [CE15, (7)]:

$$(15.6) \quad \Lambda_t(n) = \log n \cdot \prod_{i=1}^k b_{p_i} \cdot \sum_G \prod_{ij \in G} w_{ij}, \quad w_{ij} = e^{(t/2) \log p_i \log p_j} - 1.$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** $\Lambda \leq 0$, by de Bruijn and Newman; Rodgers and Tao proved $\Lambda \geq 0$.
- **Unconditional.** Theorem 1.1 and Corollary 1.2. Corollary 1.2 recovers RH from $\Lambda = 0$, “as it must”; a bound $\Lambda \leq \varepsilon$, however, gives only a zero-free region of width $\asymp \sqrt{\varepsilon} \log \gamma \log \log \gamma$, which widens with the height. If the given zero has maximal δ , then $\delta^2/(70 \log \gamma \log \log \gamma) \leq \Lambda \leq 2\delta^2$. The heat-flow Landau formula is unconditional and asymptotic.
- **Control.** On the 21 Davenport–Heilbronn zeros, the frozen energy slope predicts the survival time within 7% (numerical), and the unconditional bound lies 860–1650 times below the measured absorption times. What this suggests for ζ is heuristic: if ζ behaved like this function, one would expect $\Lambda \gtrsim 0.9 \cdot 2\delta^2$ for the off-line zero of maximal δ ; closing that gap would require excluding clusters of off-line zeros.
- **Caveat.** Theorem 7.1 and Proposition 9.1 are conditional on hypotheses about the trajectory, not on RH. The backwards-in-time argument [CE15, Remark 10.3] is heuristic; it is consistent with Rodgers and Tao, but “it is not a proof”. A claimed zero-free half-plane $\text{Re } s > \frac{7}{8}$ in an unverified manuscript is not used. The paper states: “We do not prove RH.”

Key numbers. The certified value behind 70 is $K_0 \leq 68.653$ on $\log(3 \cdot 10^{12}) \leq \log \gamma \leq 10^6$, over 10463 cells, the maximum attained in the first cell; for $\log \gamma \geq 10^6$, $K_0 \leq 52.79$, and the asymptotic limit is $e \cdot 2.07 \cdot 2s_2 = 46.74$ with $s_2 = 4.1533$. In Lemma 3.2 the certified constant is $g \leq 2.4067$, against the margin 2.5088. At $\gamma = 3 \cdot 10^{12}$, $p = 19$, $T = 1.512 \cdot 10^{-4}$ and $K_0 = 68.57$; at $\gamma = 10^{200}$, $K_0 = 57.40$. Davenport–Heilbronn (numerical): $0.932 \leq T_S/\tau_{\text{abs}} \leq 1.000$ (mean 0.969) and $0.80 \leq \tau_{\text{abs}}/2\delta^2 \leq 0.9995$; on average 2.15 on-line zeros are missing near each off-line zero (standard deviation 0.42), against 0.01 ± 0.55 in 207 control windows. Exact zeros of H_t (numerical): 9065 on $200 \leq x \leq 18200$ for each t tested, with $|S - M| \leq 3$ for $n \leq 15$ against main terms up to 837.

18. PART 16: STRUCTURAL LIMITS OF THE CROSS-ENERGY METHOD

Origin. Part 16 (16 pages, October 2026) states that, together with Parts 1, 2, 5 and 10, it replaces version 4 of the consolidated paper (Zenodo, doi:10.5281/zenodo.23243641). Its Epstein obstruction first appeared, in a form corrected here, in the author’s earlier preprint “Cross-energy of Dirichlet partial sums and the zeros of the Riemann zeta function: critical-line detection, a Fresnel transition law, and structural limits” (Zenodo, doi:10.5281/zenodo.23066932).

Summary. Part 16 explains why the cross-energy $C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma}$ alone cannot prove RH. Through the trichotomy of Part 1 [CE1], RH takes the form of a non-collapse statement, $\liminf_N C_2(N, \rho) > 0$ at every zero. The Epstein zeta function Z_Q of $Q = m^2 + 100n^2$ has every property that C_2 sees (nonnegative coefficients and a simple pole, a functional equation, the C-transform representation, and the anatomy and trichotomy of [CE1]), yet it has a real zero at $0.8133678\dots$, where its cross-energy collapses. Hence no argument that uses only these properties can prove RH.

The natural ways of bringing in the Euler product (superpositions, convolution identities, divergences on the integers, linear positivity at the primes and its twists, scale recursions) do not supply the missing input. The paper closes with a comparison of ζ and the Davenport–Heilbronn function under the C_2 criteria, and a list of claims of the author’s earlier preprints that do not hold.

Main results.

- (1) [CE16, (2.1)] (*proved, as a reformulation through the trichotomy of [CE1, Theorem 3.2]*). RH is equivalent to (16.1). The paper calls this “a reformulation, not a reduction”: $\liminf_N C_2(N, s) > 0$ is equivalent to $\sigma \leq \frac{1}{2}$ at every point of the strip, zero or not.
- (2) [CE16, Proposition 2.1] (*classical; the value is numerical*). Z_Q has a real zero $\beta_0 = 0.813367836786\dots$, with reflected zero $1 - \beta_0 = 0.186632163214\dots$. Real zeros in $(\frac{1}{2}, 1)$ for $m^2 + kn^2$ beyond a threshold $k \approx 49.17$ are due to Bateman and Grosswald.
- (3) [CE16, Theorem 2.2] (*proved for (a) and (b); (c) is a methodological consequence*). Epstein collapse and Obstruction Principle: (a) at β_0 the cross-energy collapses, (16.3), while $K^Q(x, 1 - \beta_0) \rightarrow +\infty$; (b) Z_Q has nonnegative coefficients and a simple pole, the C-transform representation, a functional equation with the symmetry $\rho \mapsto 1 - \bar{\rho}$, every recurrence and energy identity, and the anatomy and trichotomy; (c) no argument using only (b) can prove (16.1) for ζ .
- (4) [CE16, Proposition 2.3] (*numerical*). The coefficients $b_Q(n)$ of $-Z'_Q/Z_Q$ are not all nonnegative: for $Q = m^2 + 100n^2$ the first negative one is $b_Q(10000) = -\log 100$, with 183 negative coefficients for $n \leq 2 \cdot 10^5$; for $m^2 + 5n^2$, $b(36) = -7.167038\dots$, with 2604.
- (5) [CE16, Propositions 2.4 and 2.5] (*proved*). For $G_\varepsilon = \xi + \varepsilon\Lambda(\cdot, \chi_5)$ and $0 < \varepsilon \leq 5^{-5/2}$, each rectangle R_k about $s_k = \sigma_\varepsilon + \frac{2\pi i(2k+1)}{\log 5}$, $\sigma_\varepsilon = \frac{2\log(1/\varepsilon)}{\log 5}$, contains exactly one zero, which is simple, and there are no zeros with $\sigma \geq \frac{5}{2}$ outside $\bigcup R_k$; see (16.5). On the line, $\frac{1}{2} + it$ is a zero if and only if $r(t) = \varepsilon$, $r = -\xi(\frac{1}{2} + it)/\Lambda(\frac{1}{2} + it, \chi_5)$, and at a nondegenerate critical value a pair leaves or lands through a square-root bifurcation,

$$\rho_\pm = \frac{1}{2} \pm \sqrt{\frac{2|\varepsilon - \varepsilon_c|}{|r''(t_c)|}} + it_c + O(|\varepsilon - \varepsilon_c|).$$

- (6) [CE16, Proposition 2.6] (*proved*). Convolution identities between partial sums, (16.4), hold for every N and every s , and even when n^{-s} is replaced by an arbitrary $x^{v_p(n)} = \prod_p x_p^{v_p(n)}$; they cannot restrict the position of the zeros.
- (7) [CE16, Proposition 2.7] (*proved*). (i) $Q(f) \geq 0$ for all f if and only if $\hat{T}$ is a positive measure; (ii) if $Q(f) = \int |f|^2 dp$ with $p \geq 0$, then $T = c\delta$. Hence entropy, variance and Fisher-information forms on the integers cannot be Weil’s form.

- (8) [CE16, Propositions 2.8 and 2.9] (*proved*). Twisted positive combinations $\Phi_x(\sigma, t) \geq 0$ always keep the term $c_0 \log E_x(\sigma, \psi^0)$, which is $c_0 x^{1-\sigma} / ((1-\sigma) \log x) (1+o(1))$ for $0 < \sigma < 1$. If $F(x) = \sum_{j=1}^K c_j \cos(t_j x + \phi_j)$, with distinct $t_j > 0$, satisfies $F(\log p) \geq 0$ for all but finitely many primes, then $F \equiv 0$: there is no linear polarization without the pole. The scaling of the pole barrier, which excludes zeros only with $\beta > 1 - 0.427/\log x$, is *heuristic*.
- (9) [CE16, Remark 2.10] (*heuristic conclusion; the Mellin identity is elementary*). Scale recursions are neutral: Vaughan's identity, $\Lambda - 1 = \mu * (\log -\tau)$ and $\sum_{k \leq x} M(x/k) = 1$ are covariant under dilations and gain only logarithms. The measured tests (a)–(f), including the periodicity–multiplicativity table, are numerical, partly from the author's working notes; the reduction in (c) is proved in outline.

What is new. *New, by the paper's own claim:* “The Epstein obstruction in cross-energy form (Theorem 2.2) and the limits of superpositions, convolution identities and twists are, as far as we know, new in this form.” *Classical:* the real zeros of Epstein zeta functions (Bateman and Grosswald), the Bochner–Schwartz theorem, the Mertens–de la Vallée Poussin inequality $3+4 \cos \theta + \cos 2\theta \geq 0$, the Davenport–Heilbronn phenomenon, and Hilberdink's theorem on periodic Beurling systems. *Correction of prior work:* relative to the earlier preprint where the obstruction first appeared, the statement of hypothesis (H) is corrected (it needs $\theta < \sigma$), and the Mellin transform of C_2 is dropped from list (b).

Withdrawn claims of the author's earlier preprints: $C_2(N, \sigma, t) = 0$ does not imply $\sigma = \frac{1}{2}$ (for example $C_2(2, s) = 2^{1-\sigma} \cos(t \log 2)$); the vanishing of $X_N^C - Y_N^C$ does not force the energies to be linear in N ; the multiplicative three-regime mechanism is withdrawn; the crossing point at a zero on the line is $n^* = \gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} + O(\gamma^{-4})$, not $\gamma^2 + \frac{1}{4}$; uniqueness of the crossing for $N \geq K|\gamma|$ rested on a sketch; the transition-law prediction is exact for 16 of the 18 zeros it was claimed for; for the Davenport–Heilbronn function Weil's form becomes negative at $3.42 < L^* \leq 3.43$, not at 3.6, and the margin law $10^{-6.3L^2}$ for ζ holds only in a fixed finite basis.

New formulas. The non-collapse form of RH is [CE16, (2.1)]:

$$(16.1) \quad \text{RH} \iff \liminf_{N \rightarrow \infty} C_2(N, \rho) > 0 \text{ for every nontrivial zero } \rho.$$

The incomplete-gamma expansion of the Epstein function, with $k = 100$, is [CE16, (2.3)]:

$$(16.2) \quad \pi^{-s} \Gamma(s) Z_Q(s) = -\frac{1}{s} - \frac{k^{-1/2}}{1-s} + \sum_{v \neq 0} \frac{\Gamma(s, \pi Q(v))}{(\pi Q(v))^s} + k^{-1/2} \sum_{v \neq 0} \frac{\Gamma(1-s, \pi Q^*(v))}{(\pi Q^*(v))^{1-s}}.$$

The collapse of the Epstein cross-energy at its real zero is [CE16, Theorem 2.2(a)]:

$$(16.3) \quad K^Q(x, \beta_0) = \frac{2c}{1-\beta_0} x^{1-2\beta_0} + O(x^{\theta-2\beta_0}) \longrightarrow 0, \quad c = \frac{\pi}{10}.$$

The convolution identity that does not see s is [CE16, Proposition 2.6], with $h = f * g$, $G_M = \sum_{m \leq M} g(m) m^{-s}$ and $H_N = \sum_{n \leq N} h(n) n^{-s}$:

$$(16.4) \quad \sum_{d \leq N} f(d) d^{-s} G_{\lfloor N/d \rfloor}(s) = H_N(s).$$

The location of the distant zeros of a superposition, with $A(s) = L(s, \chi_5)/\zeta(s)$ and $\delta = \zeta(\sigma_\varepsilon - 1)^2 - 1$, is [CE16, Proposition 2.4(iii)]:

$$(16.5) \quad \rho_k = s_k - \frac{2}{\log 5} \text{Log } A(\rho_k), \quad |\rho_k - s_k| \leq \frac{2}{\log 5} \frac{\delta}{1 - \delta}.$$

The comparison at a zero between the prime sum and C_2 is derived in outline after [CE16, Proposition 2.6], where $K(\rho) = \gamma_0 A_\rho - 2 \operatorname{Re}(1 - \rho)^{-2}$:

$$(16.6) \quad P_N(\rho) - C_2(N, \rho) \sum_{d < N} \frac{\Lambda(d)}{d} = K(\rho) N^{1-2\beta} - 2 \operatorname{Re} [N^{-\bar{\rho}} \zeta'(\rho)] + o(N^{1-2\beta}).$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** The non-collapse statement (16.1), as a reformulation through the trichotomy of [CE1].
- **Implied by RH.** When the zeros are on the line, $\hat{T} = \sum_\rho \delta_{\gamma_\rho}$ up to normalization for Weil's form, which is then $Q(f) = \sum_\gamma |\hat{f}(\gamma)|^2$, positive on the spectral side.
- **Unconditional.** Propositions 2.4–2.9 and Theorem 2.2(a),(b).
- **Control.** The Epstein function of $m^2 + 100n^2$ is the sharpest control for C_2 : it has a pole, nonnegative coefficients and the trichotomy, none of which the Davenport–Heilbronn function has; its logarithmic derivative is not one-signed, which points to the Euler product $-\zeta'/\zeta = \sum \Lambda(n)n^{-s}$, $\Lambda \geq 0$, as the decisive difference. Every superposition $\xi + \varepsilon\Lambda(\cdot, \chi_5)$ has off-line zeros: “In this language RH says that ζ contains no second component, however small.” For the Davenport–Heilbronn function the limit of C_2 is 0 for all $\sigma > 0$, the signed helix radius changes sign at 287 of 303 zeros, and Weil matrices are negative for $L > L^*$.
- **Caveat.** The paper states, several times: “We do not prove the Riemann hypothesis.” None of the withdrawn points affects the exact identities or theorems of the series.

Key numbers. The Epstein zero is $\beta_0 = 0.813367836786\dots$; the residue of Z_Q is $\pi/10$, the lattice exponent $\theta = \frac{1}{3}$, the threshold $k \approx 49.17$; there is no real zero in $(\frac{1}{2}, 1)$ at $k = 40$, and there is one at 0.529, 0.680, 0.813 for $k = 50, 60, 100$. Collapse table: $K^Q(x, \beta_0) = 0.002475, 0.000584, 0.000138, 0.000058$ at $x = 10^5, 10^6, 10^7, 4 \cdot 10^7$, each equal to the structural term. Superpositions: ξ has 29 zeros and $\Lambda(\cdot, \chi_5)$ has 54 for $0 < t < 100$; the positive critical values below 1 are 0.0931, 0.1742, 0.6197, 0.8747; at $t_c = 57.668$, $r''(t_c) = 1.7278$, and the predicted coefficient 1.0759 compares with 1.0768 from Newton's method. Pole barrier at $(0.7, 14.13)$, $x = 2 \cdot 10^6$: positive sum 70.77, pole correction -74.45 , total -3.69 , against the limit -3.77 . Zero-sum constant $C_B = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots$

19. PART 17: CONTROLS FOR THE RIEMANN HYPOTHESIS: DAVENPORT–HEILBRONN, BEURLING SYSTEMS AND FUNCTION FIELDS

Origin. Part 17 (24 pages, October 2026) states that its window-inequality section comes from Part V of the author's earlier series, and corrects that source's attribution of the mollification floor. One candidate statement it tests comes from Part III of the earlier series (Zenodo, doi:10.5281/zenodo.23070322).

Summary. A statement meant to imply RH must fail for every function that shares part of the structure of ζ and has zeros off the critical line. Part 17 collects such *controls* and measures them. It classifies thirteen representations and consequences of $\zeta(\rho) = 0$ against the Epstein control of Part 16 [CE16]. At the eight off-line zeros found for the Davenport–Heilbronn function f with $t < 400$, the two Euler products composing f have ordinary size (14th to 38th percentile on the same vertical line): the zeros come from two products meeting in phase. A window inequality bounds the off-line mass in a rectangle by a mollified energy; on short windows its Littlewood sum reproduces the mass of each of the seven zeros of f with $\beta > 0.6$ to four decimals. Beurling controls have an Euler product without the ordinary integers; with f they show that a proof must use the functional equation and the Euler product together. Over $\mathbb{F}_q[T]$, where RH is Weil's theorem, the cross-energy criteria become restatements, and a Davenport–Heilbronn analogue fails at base

change. Elliptic curves, Ramanujan graphs and amplification locate the deciding ingredient in a positivity carried by an operator.

Main results.

- (1) [CE17, Proposition 1] (*classical, restated*). For $0 < \beta < 1$, eight conditions are each equivalent to $\zeta(\rho) = 0$ (partial sums, equivalently vanishing of the C_2 wave energy; alternating series; sawtooth integral; functional equation; theta function; Hadamard product; Euler product, as a logarithmic branch point; Fourier transform Ξ); the Riemann–Siegel bound and the resonance (17.6) are necessary conditions; the Jensen functional, Nyman–Beurling and Weil’s explicit formula are global criteria. The classification against the Epstein control and the table of known theorems are *heuristic*.
- (2) [CE17, Lemma 2] (*proved*). With $x = y = \sqrt{\gamma/2\pi}$, $|\chi(\beta + i\gamma)| = (\gamma/2\pi)^{1/2-\beta}(1 + O(1/\gamma))$, and the trivial bounds of the two Riemann–Siegel halves are both of order $(\gamma/2\pi)^{(1-\beta)/2}$, with constants $\frac{1}{1-\beta}$ and $\frac{1}{\beta}$. The conclusion that no comparison of sizes at a single scale can exclude an off-line zero is *heuristic*.
- (3) [CE17, Proposition 3 and Corollaries 4, 5] (*proved*). For $G = FM$ on $R = [\sigma_0, \sigma_1] \times [T_1, T_2]$, with $G \neq 0$ on ∂R and $|G(\sigma_1 + it) - 1| < 1$, 2π times the off-line mass of F in R is at most $H \log(1 + \sqrt{T}) + B$, (17.1). Hence the floor (17.2), for every Dirichlet polynomial M with its own boundary term B , and conversely the certificate $\sum(\beta - \sigma_0) \leq (H \log(1 + \sqrt{T}) + B)/2\pi$. Global certificates are weak [CE17, Remark 6]: for ζ on $[300, 600]$ the bound is $\lesssim 13.5$, whereas the true value is 0.
- (4) [CE17, Proposition 7] (*proved*). The rounded-prime zeta function $\zeta_\delta(s) = \prod_p (1 - e^{-\delta n_p s})^{-1}$ is a power series in $w = e^{-\delta s}$ with nonnegative integer coefficients and radius of convergence $e^{-\delta}$, periodic in t with period $2\pi/\delta$, and $-\zeta'_\delta/\zeta_\delta(s) \sim c_\delta/(s-1)$ with $c_\delta = 2 \sinh(\delta/2)/\delta \in (1, 2)$; so it has a branch point, is not rational in w , and has no single-valued meromorphic continuation across $|w| = e^{-\delta}$.
- (5) [CE17, Propositions 8, 9 and 10] (*proved*). Over $\mathbb{F}_q[T]$: $C_2(\delta, \rho) = -2(q|\rho|^2)^\delta$ at every zero, so RH for L holds if and only if $C_2(\delta, \rho) = -2$ at every zero, (17.3); the C_2 –Jensen criterion becomes the class-number identity (17.4); and the partial Euler products E_m converge to L uniformly on compact subsets of $|u| < q^{-1/2}$ if and only if L satisfies RH, so that this convergence in $\sigma > \frac{1}{2}$ is exactly Weil’s theorem.
- (6) [CE17, Proposition 11] (*classical, restated*). If $\psi(x) \leq x + Cx^\theta$ for all large x , then ζ has no zeros with $\beta > \theta$; hence RH holds if and only if $\psi(x) \leq x + O(x^{1/2} \log^2 x)$. The proof uses Landau’s theorem and von Koch’s estimate.
- (7) [CE17, Proposition 12] (*proved*). For an elliptic curve $E/\mathbb{F}_q$ the auxiliary wave satisfies the exact formula (17.5), and at a zero $|W(N, \rho) - \text{centre}| = \frac{2N_1}{(q-1)|1-q^{-\rho}|} q^{-2\beta N}$ for every $N \geq 1$. The identity holds equally for $1 - 25T + 101T^2$, which is not the numerator of a curve: the wave measures β but “does not distinguish a curve from a numerator that is not one”.

Other statuses. Numerical: the Davenport–Heilbronn anatomy (percentiles, prime-side resonance, growing inverse coefficients); the simplified Beurling system (“we have not proved that ζ_P has a zero at, or near, $0.75 + 20i$ ”); the function-field Davenport–Heilbronn analogue and its failure at base change; the sieve data and graph examples. Heuristic: the level-set mechanism, the reading of the tower and the filter classification.

What is new. *Classical:* the Davenport–Heilbronn function and its off-line zeros (Spira; Balanzario–Sánchez-Ortiz); the Beurling systems of Diamond–Montgomery–Vorhauer and of Broucke–Debruyne–Révész; Weil’s theorem and the Stepanov–Bombieri proof; the Ihara zeta function and the Ramanujan property; Landau’s theorem on Dirichlet series with nonnegative coefficients; the Selberg and Bohr–Landau heuristic that linear combinations of L -functions have

many off-line zeros. *New, by the paper's own claim:* “explicit, reproducible comparisons in the notation of the cross-energy”: the quantitative anatomy of the Davenport–Heilbronn zeros, the window inequality with its floor, certificate and measurements, the simplified Beurling control and the rounded primes, the exact function-field bench and its Davenport–Heilbronn analogue failing at base change, the wave over a curve and over a graph, and a list of filters for candidate rigidity statements. *Correction:* the mollification floor of f is an observation, not a consequence of Corollary 4; this corrects Part V of the earlier series. *From prior parts:* the analysis of the Broucke–Debruyne–Révész-type system with the zeros of ζ [CE11, Sect. 13.5]; the filters F1–F3 and Landau’s positivity–pole dichotomy [CE2, Def. 2.22, Prop. 2.23]; the blindness of the Euler state [CE8, Prop. 15]; the Epstein and superposition controls [CE16, Sect. 2].

New formulas. The window inequality is [CE17, Proposition 3]:

$$(17.1) \quad \begin{aligned} 2\pi \sum_{\substack{F(\rho)=0 \\ \rho \in R}} (\beta - \sigma_0) &\leq 2\pi \sum_{\substack{G(\rho)=0 \\ \rho \in R}} (\beta - \sigma_0) = \int_{T_1}^{T_2} \log |G(\sigma_0 + it)| dt + B \\ &\leq H \log(1 + \sqrt{I}) + B. \end{aligned}$$

The floor for the mollified energy, with $m = \sum_{F(\rho)=0, \rho \in R} (\beta - \sigma_0)$, is [CE17, Corollary 4]:

$$(17.2) \quad I \geq \left(\max \left\{ \exp \left(\frac{2\pi m - B}{H} \right) - 1, 0 \right\} \right)^2.$$

The function-field cross-energy at a zero is [CE17, Proposition 8]:

$$(17.3) \quad C_2(\delta, \rho) = -2(q|\rho|^2)^\delta, \quad \text{RH for } L \iff C_2(\delta, \rho) = -2 \text{ at every zero.}$$

The C_2 –Jensen criterion as a class-number identity, for $r = 2$ with $h = \#J(\mathbb{F}_q)$, is [CE17, Proposition 9]:

$$(17.4) \quad \text{RH} \iff \int \log |L| d\mu_{1/q} = \log \frac{h}{q^g}.$$

The exact auxiliary wave over an elliptic curve is [CE17, Proposition 12]:

$$(17.5) \quad W(N, s) = 2Z(T) \overline{T^N} - \frac{2N_1}{q-1} \left[\frac{q^{N(1-2\sigma)}}{1-qT} - \frac{q^{-2\sigma N}}{1-T} \right].$$

The resonance of the prime sum at a zero, for N not a prime power, is [CE17, Proposition 1(9)]:

$$(17.6) \quad \sum_{n \leq N} \frac{\Lambda(n)}{n^\rho} = \frac{N^{1-\rho}}{1-\rho} - m \log N + c_\rho - \sum_{\rho' \neq \rho} \frac{N^{\rho'-\rho}}{\rho' - \rho} + o(1).$$

Relation to the Riemann hypothesis.

- **Equivalent to RH.** The global criteria of Proposition 1 (the Jensen functional of Balazard, Saias and Yor; Nyman–Beurling; positivity of Weil’s form); the one-sided bound of Proposition 11; and the candidate rigidity statements of the filter table (non-collapse of C_2 , reflected rigidity, the arithmetic energy law of C_2^Λ , the pole-free difference energy), each “equivalent to RH”.
- **Conditional on RH.** The Broucke–Debruyne–Révész systems are constructed under RH, and von Koch’s estimate is the converse direction in Proposition 11.
- **Unconditional.** Lemma 2, Propositions 3, 7 and 12 and Corollaries 4–5. Over $\mathbb{F}_q[T]$, Propositions 8–10 are unconditional equivalences; with Weil’s theorem they hold, and they “only restate RH”.

- **Control.** The Davenport–Heilbronn function has the functional equation and no Euler product; its eight off-line zeros with $10 \leq t \leq 400$ are “not a certified count”, and its off-line mass grows linearly with the height. Beurling systems have the Euler product, $\Lambda \geq 0$ and $\log = \Lambda * 1$, but not the integers $1, 2, 3, \dots$. The Epstein function is the control of [CE16]. A function-field Davenport–Heilbronn analogue satisfies RH at 7, 4, 2, 0 of 12 moduli on levels 1–4.
- **Caveat.** The paper states: “No result of this paper implies RH, and none is claimed to” and “We do not prove the Riemann hypothesis.” The statement suggested by the experiments, that a single degree-one Euler product cannot cancel itself off the critical line, “is RH itself”.

Key numbers. Davenport–Heilbronn: $\kappa = 0.2840790438\dots$; first off-line zero found $0.808517 + 85.699348i$, highest $0.628508 + 366.640908i$; Littlewood mass in $\sigma > 0.6$ on $[10, 300]$ 0.6551, against 0.6531 from the four zeros located there; resonance slope at Spira’s zero -0.995 , against -1 ; largest inverse coefficients $|g(n)| = 11.6, 52.2, 267, 383$ for $n \leq 10^3, 10^4, 10^5, 2 \cdot 10^5$. Window inequality: $B/2\pi \in [0.041, 0.064]$ on local windows; on $[300, 600]$ ($H = 300, m = 1.302, B \geq 0$) Corollary 4 gives a floor of at most $\approx 7.6 \cdot 10^{-4}$ for f , the value used both in the mollifiability section and in Open problem 2, far below the observed values. Simplified Beurling system: 663 888 generalized primes below 10^7 (against 664 579 primes), $N_P(x) = 2.2715x - 1.65\sqrt{x} + O(10^2)$. Rounded primes: prime energies 0.3205, 0.0933, 0.0738 for $\delta = 0.2, 0.1, 0.05$, against $C_B = 0.0462$. All 10100 curves $y^2 = x^3 + Ax + B$ over $\mathbb{F}_{101}$ have $|a| \leq 20 < 2\sqrt{101}$; the ring of four Petersen graphs has $\lambda_2 = 2.9089 > 2\sqrt{2}$, $\max |\alpha|/\sqrt{q} = 1.2687$. Selberg’s sieve bound exceeds the number of primes in $(10^8, 10^8 + y]$ by 3.07, 2.61, 2.24 for $y = 10^4, 10^5, 10^6$.

20. SYNTHESIS ACROSS THE SERIES

20.1. The map of the Riemann hypothesis. Table 2 lists the main statements of the series that are equivalent to RH, with the part where they appear and with what happens to them on the controls. The equivalences are proved (most of them classically); none of the statements themselves is proved. Many are classical criteria rewritten in the coordinates of C_2 ; the table says so where the part says so. The last column is the reason the series records them: a criterion that also holds for a function with zeros off the line cannot be proved from the structure it shares with that function.

Table 2: Statements equivalent to RH in the series.

statement equivalent to RH	where	nature; behaviour on the controls
the C-transform of $\Lambda(n)$ converges in $\sigma > \frac{1}{2}$	[CE1, Prop. 2.6]	classical in substance (abscissa = Θ); for Davenport–Heilbronn the measured exponent near $t = 85.7$ is $0.8085 - \sigma$ (numerical)
$\liminf_N C_2(N, \rho) > 0$, or $\sup_N C_2(N, \rho) < \infty$, at every zero	[CE1, Cor. 3.5(iii)], [CE16, (2.1)]	reformulation via Theorem A; fails filter (F1) of [CE2, Def. 2.22]: the Epstein function has the same trichotomy and an off-line real zero
boundedness of $E_N^-(s)$ for every $\sigma > \frac{1}{2}$ sign of the reflected pair product $C_2(n, \rho)C_2(n, 1 - \bar{\rho})$ above height 16; Hankel–Jacobi positivity	[CE2, Prop. 2.21] [CE3, Prop. 3(v), Prop. 15]	reformulation; fails filter (F3) exact reformulations; for Davenport–Heilbronn V_N^f has exactly the two off-line zeros of f in $80 < t < 90$, with no drift

statement equivalent to RH	where	nature; behaviour on the controls
finiteness of the mean energy of the prime number theorem error (Theorem B)	[CE5], Theorem B, Cor. 2.3	proved equivalence; the energy equals C_B iff RH holds and the zeros are simple; for Davenport–Heilbronn the energy grows exponentially, at a local rate (0.58) between the values $2\beta - 1$ of its dominant off-line zeros
$\sum_{n \leq N} C_2^\Lambda(n, \frac{1}{2}) = 4N + O(\sqrt{N})$	[CE5, Thm. 2.23(d)]	von Koch in C_2 form; fails filter (F3)
Li’s $\lambda_n \geq 0$; Nyman–Beurling $d_N \rightarrow 0$ convergence of the pole-corrected Euler product $E_x^* \rightarrow \zeta$ on $\frac{1}{2} < \sigma < 1$	[CE5] [CE8, Cor. 3]	classical, restated classical in substance (analyticity of $\log \zeta$); the function-field analogue is Weil’s theorem [CE17]
single-valued extension of the prime zeta function to $\text{Re } s > \frac{1}{2}$ minus the segment $(\frac{1}{2}, 1]$	[CE9, Prop. 5.1(a)]	classical reformulation
$\mathcal{J} = \frac{1}{4\pi} \int \log \frac{\Omega(t)}{2} L(t) dt = 0$ (Balazard–Saias–Yor)	[CE10, Cor. 2.8]	classical, C_2 form proved; Epstein: $\mathcal{J}_Q \geq 1.47$; Davenport–Heilbronn: steps at each off-line zero
energy slope $S(\sigma, t) > 0$ for $\sigma > \frac{1}{2}$	[CE10, Thm. 3.1]	classical (Lagarias; Sondow–Dumitrescu); unconditional for $ t \leq 3 \cdot 10^{12} - \frac{1}{2}$ [CE10, Prop. 3.5(b)], by the verification of Platt–Trudgian; at abscissa σ Davenport–Heilbronn fails on one interval per off-line zero with $\beta > \sigma$
$B_X(s_0) \rightarrow 0$, or $B_X(s_0) = O(1/\log X)$, at a single s_0	[CE11, Cor. 3(a)]	proved equivalence; for Davenport–Heilbronn B_X grows $(1.65 \rightarrow 7.58)$
$\lambda_{\min}(L) \geq 0$ for every L (Weil)	[CE12, Prop. 1(v)]	classical; Davenport–Heilbronn is negative for $L > L^*$, $3.42 < L^* \leq 3.43$
$\Lambda \leq 0$ (de Bruijn–Newman)	[CE13, CE14, CE15]	classical (de Bruijn, Newman); $\Lambda \geq 0$ is Rodgers–Tao and $\Lambda \leq 0.2$ is Polymath with Platt–Trudgian; proved here: $\Lambda \leq 0.174$, and an off-line zero forces an explicit $\Lambda > 0$
$\psi(x) \leq x + O(x^{1/2} \log^2 x)$ (one-sided)	[CE17, Prop. 11]	classical (Landau’s theorem and von Koch), proved in the part

20.2. The main formulas. Table 3 lists the formulas that carry the series. Each is stated in full, with its status, in the section of its part.

Table 3: The main formulas of the series.

formula	where
$\zeta(s) = C(x^{-s}) - \frac{1}{1-s}, \quad 0 < \sigma < 1$	[CE1, Prop. 2.3]
$C_2(n, s) = \frac{2(1-\sigma)}{ 1-s ^2} n^{1-2\sigma} + 2n^{-\sigma} \text{Re} [\zeta(s)n^{it}] - n^{-2\sigma} + O(s n^{-2\sigma-1})$	[CE1, Thm. 3.2]

formula	where
$W_{N+1} = \left(\frac{N}{N+1}\right)^\sigma e^{i\delta_N} W_N + 2[N(N+1)]^{-\sigma} e^{i\delta_N}, \quad \delta_N = t \log\left(1 + \frac{1}{N}\right)$	[CE1, Prop. 3.1]
$n^*(\rho) = \lambda + \frac{1}{12} + \frac{1}{180\lambda} - \frac{53}{5040\lambda^2} - \frac{683}{129600\lambda^3} + \dots$	[CE1, Prop. 3.8]
$N_{\text{last}} = \frac{t}{2\pi} + \kappa\sqrt{t} + O(1), \quad \kappa = -0.2077978646\dots$	[CE2, Thm. 2.10]
$\limsup_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)^2 du = \sum_{\rho} \lim_{n \rightarrow \infty} C_2(n, \rho) \quad (\text{simple zeros})$	[CE5], Theorem B
$\sum_{n \leq N} C_2^\Lambda(n, \frac{1}{2}) = 4N + O(\sqrt{N}) \iff \text{RH}$	[CE5, Thm. 2.23]
$I(X) = C_B \log X + K + \Phi(\log X) + O(X^{-1/2}), \quad K = 2.512479790006\dots$	[CE6, Thm. 5]
$2^{-\sigma_P} = \sum_{p \geq 3} p^{-\sigma_P}, \quad \sigma_P = 1.7795446535\dots$	[CE9, Thm. 2.1]
$\frac{1}{4\pi} \int \log \frac{\Omega(t)}{2} L(t) dt = \sum_{\beta > 1/2} \log \left \frac{\rho}{1-\rho} \right \geq 0$	[CE10, Cor. 2.8]
$\lim_{X \rightarrow \infty} (\log X) B_X(s_0) = \sum_{\gamma} \kappa_{m(\gamma)} P_{s_0}(\gamma) \quad (\text{under RH})$	[CE11, Thm. 2]
$\Lambda \leq 0.15399 + \frac{1}{2}(0.2)^2 = 0.17399$	[CE13, Thm. 1]
$\Lambda \leq \frac{4w_0 + o(1)}{\log X}, \quad \zeta(\frac{1}{2} + w_0) = 2$	[CE14, Thm. A]
$\zeta(\frac{1}{2} + \delta + i\gamma) = 0 \implies \Lambda \geq \frac{\delta^2}{70 \log \gamma \log \log \gamma}$	[CE15, Thm. 1.1]

In the second row the implied constant is absolute for $n \geq (|s| + 2)^2$; in the eighth row, $I(X) = \int_1^X (\psi(x) - x)^2 x^{-2} dx$ is the energy of the primes and the second-order law is conditional on RH, simple zeros and the spacing hypothesis (H*); in the thirteenth row, the bound is conditional on the verification of RH up to $X/2$ and on Polymath's barrier condition at X .

20.3. Key constants.

Table 4: Constants of the series.

quantity	value	part	status
$C_B = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi$	0.0461914...	1, 5	classical
Li's $\lambda_1 = C_B/2$	0.0230957...	5	classical
crossing at γ_1	$n^* = 200.1238\dots$	1	numerical check of the law
Fresnel constant κ	$-0.2077978646\dots$	2	proved, uses a numerically verified lemma
constant in Theorem 2.23, $-4\frac{\zeta'}{\zeta}(\frac{1}{2})$	$-10.74437\dots$	5	classical value
second-order constant K	$2.512479790006\dots$ ($\pm 6 \cdot 10^{-13}$)	6	certified (defined under RH)

quantity	value	part	status
variance of the limiting distribution	$1.2502 \cdot 10^{-5}$	6	numerical, under LI
edge of the zeros of $P(s)$, σ_P	$1.7795446535\dots$	9	zero-free beyond it: Belovas–Čepaitytė–Sabaliauskas; sharpness proved
certified zero counts of P	31, 68, 12, 0	9	certified (argument principle in Arb)
multiplicity constants κ_1, κ_2	1.82792, 7.34579	11	numerical values
Dirichlet-mollifier floor	0.0386	11	proved (Prop. 46)
$\frac{1}{2} \log(1 + c_0)$			
Davenport–Heilbronn Weil threshold	$3.42 < L^* \leq 3.43$	12	upper end certified
de Bruijn–Newman upper bound	$\Lambda \leq 0.174$	13	proved, computer-assisted
triangle constant without mollifier	$4w_0 = 4.9146\dots$	14	proved
floor for mollifiers of length $\leq N/2$	4	14	proved
constant in the lower bound for Λ	70	15	certified in Arb
Epstein real zero, $Q = m^2 + 100n^2$	$\beta_0 = 0.813367836786\dots$	16	existence classical (Bateman–Grosswald), value numerical
first off-line zero of Davenport–Heilbronn	$0.808517 + 85.699348i$	17	numerical; used in most parts

20.4. Certified and computer-assisted results. The following results rest on computations with rigorous error control. Each part ships its scripts in its ancillary files.

- Part 4: the hypotheses of the explicit spiral theorem for the first 100 zeros (Arb, 96 bits), and the inequalities $e(2^k) < e(2^k - 1)$, $k = 2, 3, 4$, for the right edges of the zeros of the sections (Prop. 15, Arb); the other edges $e(N)$ are numerical.
- Part 6: the closed form of K in interval arithmetic, $K = 2.512479790006\dots \pm 6 \cdot 10^{-13}$.
- Part 9: the counts of zeros of the prime zeta function in four rectangles, by the argument principle in ball arithmetic.
- Part 12: negativity of Weil’s form for the Davenport–Heilbronn function at $L = 3.43$.
- Part 13: the full certificate for $\Lambda \leq 0.174$ (Arb plus outward-rounded floating point, reproduced on two platforms).
- Part 14: the prime count in $(N/2, N]$ used for the floor, $h(2 \cdot 10^5)$ in ball arithmetic.
- Part 15: the explicit local zero count and the maximisation giving the constant 70, and the bounds $0.228 < \Lambda_f \leq 3.125$ for the Davenport–Heilbronn analogue of Λ (Theorem 9.2, interval arithmetic with outward rounding).

Everything else that is computed in the series is labelled numerical or heuristic in its part.

20.5. The controls. The criteria of the series are tested on functions that are not ζ . Each control removes one feature of ζ and keeps the others, so a criterion that holds for a control cannot be proved from the features that control shares with ζ .

- *The Davenport–Heilbronn function f* (a combination of two Dirichlet L -functions mod 5) has a functional equation of the same type as ζ , no Euler product and no pole, and zeros off the line, the first at $0.808517 + 85.699348i$. It appears in every part except Parts 7 and 14. It shows, for instance, the decay slope -2β of the wave at an off-line zero (Part 1), eight extrema of Z_f without a crossing below $t = 400$ (Part 2), spirals instead of ellipses

in the prime orbits (Parts 6, 13), the steps of the Jensen integral (Part 10), and the Weil threshold L^* (Part 12).

- *The Epstein zeta function* of $m^2 + 100n^2$ has a pole at $s = 1$, nonnegative coefficients and the trichotomy of Theorem A, but no Euler product, and a real zero $\beta_0 = 0.8133678 \dots$. It is the sharpest control for C_2 (Part 16).
- *Beurling systems* with prescribed zeros keep an Euler product and nonnegative coefficients $\Lambda_B(n) \geq 0$, but replace the integers $1, 2, 3, \dots$ by generalised integers and with them lose the functional equation (Parts 3, 11, 17).
- *Function fields*, where RH is Weil's theorem, show what a proof looks like when one is available (Parts 8, 12, 17).
- $L(s, \chi) \bmod 5$ is an Euler-product twin of the Davenport–Heilbronn function and behaves like ζ throughout.
- *Planted zeros* $\zeta(s) - \zeta(s_0)$ and superpositions $\xi + \varepsilon\Lambda(\cdot, \chi_5)$ show that wave-and-envelope arguments cannot exclude off-line zeros (Parts 1, 16).

20.6. Obstructions. Several parts identify reasons why a given route cannot reach RH.

- (1) *The trace of the pole* (Parts 1, 16). The trichotomy of Theorem A comes from the pole, not from the zeros; it holds for the Epstein function, which has an off-line zero. So non-collapse of C_2 at the zeros cannot follow from the structure C_2 sees.
- (2) *Locality and insensitivity* (Part 10). No pointwise argument suffices. A zero at height 10^{12} with $\beta - \frac{1}{2} = 10^{-3}$ changes the Balazard–Saias–Yor integral by about 10^{-27} , and a finite certificate of the form $W_L > B_L \Rightarrow S > 0$ certifies nothing beyond what is known.
- (3) *Precision* (Parts 3, 12). The positivity margins at height T decay like a power of T (Part 3, a heuristic reading of measured exponents), and on windows of length L like $10^{-6.3L^2}$ in a fixed basis and about $10^{-(5.42e^L - 12.6)}$ in stabilised bases (Part 12, empirical fits). Positivity at large height cannot come from estimates for the primes; it has to come from an exact identity.
- (4) *The pole barrier* (Parts 5, 16). A linear inequality that is positive prime by prime must contain the pole term, which costs a correction of size $x^{1-\sigma}$ inside the strip; such arguments stop at zero-free regions of de la Vallée Poussin type.
- (5) *The floor of a method* (Part 14). Triangle-inequality certificates for Λ cannot beat the constant 4 in $C/\log X$ with mollifiers of length at most $N/2$; going lower needs phase information.
- (6) *Visibility* (Parts 1, 5, 13). An off-line zero becomes visible in the C-transform of the primes, in Li's coefficients or in the prime orbits only at astronomically large N , n or x , for example $N \gtrsim 10^{1600}$ for $\beta = 0.51$ (Part 1).

20.7. Open problems. Each part states its own open problems; we collect the main ones.

- (1) Which property of the Euler product, absent for the Epstein function, can be expressed through the partial sums X_N alone? (Part 16.)
- (2) A structural reason for the Hankel–Jacobi coefficients β_m to be positive for all m (Part 3), and a family of test functions over $\mathbb{Q}$ whose difference set is stable under the prime translations up to an error smaller than the Weil margin (Part 12).
- (3) A proof of the mean-square approximation (Q) for the prime energy under RH, simple zeros and spacing conditions weaker than (H^*) , which would give the limiting distribution without (H^*) (Part 6, Problem 13).
- (4) An unconditional version of the prime-circle statements, which under RH follow from Landau's formula (Part 4).
- (5) Determine C_{opt} for mollified triangle certificates in $[4, 4w_0]$, and go below 4 with a phase-sensitive certificate built from the cross-energies (Part 14).

- (6) The non-attained case of the explosion theorem for the Euler–Caceres bound (Part 11).
- (7) Proofs of the extension of Borchsenius–Jessen to $\frac{1}{2} < \operatorname{Re} s < 1$ for the prime zeta function (Part 9), and of the laws fitted for the Weil margins (Part 12).
- (8) A candidate criterion that passes all three filters of [CE2, Def. 2.22]: it must fail for the controls with off-line zeros (F1), must not follow from the pole at $s = 1$ alone (F2), and must not reduce, through the explicit formula, to “no zero mode has $\operatorname{Re} \rho > \frac{1}{2}$ ” (F3) (Parts 2, 17).

CONCLUSION

The series shows that the cross-energy is a single coordinate system in which many classical objects and reformulations of RH can be written side by side. These include Hardy’s function, Cramér’s mean energy, Li’s coefficients, Nyman–Beurling, Balazard–Saias–Yor, Lagarias’s slope, Weil’s form and the heat flow of de Bruijn and Newman. The series also proves new unconditional structure, among it Theorem A, the crossing and Fresnel laws, the sharpness of the zero-free edge of the prime zeta function found by Belovas, Čepaitytė and Sabaliauskas, the optimal triangle constant without a mollifier, an explicit upper bound for Λ and an explicit lower bound in terms of a hypothetical off-line zero, and it tests every criterion on controls. It does not prove RH. What it indicates is where the difficulty lies: in the property that distinguishes ζ from the Epstein, Davenport–Heilbronn and Beurling controls, which is the Euler product coupled with the functional equation, used in a way that survives at every height.

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