

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS.

PART 16: STRUCTURAL LIMITS OF THE CROSS-ENERGY METHOD

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ABSTRACT. This is Part 16 of a series on the cross-energy $C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma}$ of the partial sums of ζ . It explains why the cross-energy alone cannot prove the Riemann hypothesis. The Epstein zeta function of $m^2 + 100n^2$ has nonnegative coefficients, a simple pole, a functional equation, the C-transform representation and the whole anatomy of the cross-energy, and a real zero at $0.8133678\dots$, where its cross-energy collapses: no argument that uses only these properties can prove RH. Superpositions with a Dirichlet L -function move zeros off the line, the convolution identities that express the Euler product through partial sums do not see s , positivity of cosine sums at the primes pays a correction from the pole, and twists and scale recursions do not supply the missing input. The paper ends with the claims of the author's earlier preprints that do not hold, each with a counterexample. We do not prove the Riemann hypothesis.

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1. INTRODUCTION

Let $X_N(s) = \sum_{k \leq N} k^{-s}$ be the partial sums of the Dirichlet series of ζ , with $s = \sigma + it$. The *cross-energy* of consecutive partial sums is

$$C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} (X_{N-1}(s) N^{-\bar{s}}) = 2N^{-\sigma} \sum_{n < N} n^{-\sigma} \cos \left(t \log \frac{N}{n} \right).$$

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It records how the newest term N^{-s} interacts with all the previous ones. This part explains why the cross-energy alone cannot prove RH: every property that C_2 sees is shared by an Epstein zeta function with a real zero off the line, and the natural ways of bringing in the Euler product through partial sums are blind to s .

Throughout, $\rho = \beta + i\gamma$ is a nontrivial zero of ζ , $\Theta = \sup \beta$, $\lambda = s(1-s)$, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, and

$$C_B = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots,$$

the sum taken in symmetric order [Da]; under RH, $\rho(1-\rho) = |\rho|^2 = \frac{1}{4} + \gamma^2$. We write $Y_N(s) = N^{1-s}/(1-s)$ for the continuous envelope of X_N and $X_N^C = X_N - \frac{1}{2}N^{-s}$.

This paper is Part 16 of the series *Cross-energy of Dirichlet partial sums* (seventeen parts; the plan of the series is in Part 1 [CE1]). It uses the notation and the basic results of Part 1, in particular [CE1, Theorem A], which are cited with the prefix [CE1].

1.1. Main results. Structural limits (Section 2). The Epstein zeta function of $m^2 + 100n^2$ has nonnegative coefficients, a pole, a functional equation, the C-transform representation and the whole anatomy of [CE1, Theorem 3.11], and a real zero at $0.8133678\dots$, where its cross-energy collapses (Theorem 2.2): no argument that uses only these properties can prove RH. Superpositions, convolution identities, Bochner duality and twists give further limits, and the section ends with the claims of the earlier preprints that do not hold.

1.2. What is new and what is classical. The Epstein obstruction in cross-energy form (Theorem 2.2) and the limits of superpositions, convolution identities and twists are, as far as we know, new in this form. The real zeros of Epstein zeta functions are classical [BG]. We do not prove the Riemann hypothesis.

2. STRUCTURAL LIMITS

The criteria of the other parts of the series are equivalent to RH or implied by it, and each was tested on functions that share some of the structure of ζ and have zeros off the critical line. This section collects what those tests show about the reach of the cross-energy. It contains four kinds of statement: a control that has every property of ζ used by C_2 except the Euler product (the Epstein zeta function of $m^2 + 100n^2$); constructions in which zeros leave the line (superpositions); identities and positivity principles that provably cannot see the real part of a zero; and the pole barrier, which limits every linear positivity argument. It ends with the comparison table of ζ and the Davenport–Heilbronn function and with the list of statements that do not hold.

2.1. The Epstein obstruction. C_2 for a general Dirichlet series. We use the notation of Part 1 [CE1]: for real coefficients (a_n) with Dirichlet series F , the cross-energy $K^a(x, s)$ of [CE1, (3.18)], Hypothesis (H), [CE1, Lemma 3.10] and [CE1, Theorem 3.11]. [CE1, Lemma 3.10] is the C-transform representation of Part 1 [CE1] in the generality of (H), $F(s) = \lim_x [X_x^a(s) - cx^{1-s}/(1-s)]$, and [CE1, Theorem 3.11] gives the anatomy [CE1, (3.20)] and, for $\theta < \frac{1}{2}$, the trichotomy, whether or not $F(s) = 0$.

For ζ this is the trichotomy of [CE1, Theorem 3.2], and through it RH takes the form of a *non-collapse* statement: since $\lim_N C_2(N, \rho)$ is $+\infty$, $1/(\gamma^2 + \frac{1}{4})$ or 0 according as $\beta < \frac{1}{2}$, $\beta = \frac{1}{2}$ or $\beta > \frac{1}{2}$, and since $1 - \bar{\rho}$ is a zero with real part $1 - \beta$,

$$\text{RH} \iff \liminf_{N \rightarrow \infty} C_2(N, \rho) > 0 \text{ for every nontrivial zero } \rho. \quad (2.1)$$

This is a reformulation, not a reduction: by [CE1, (3.20)], $\liminf_N C_2(N, s) > 0$ is equivalent to $\sigma \leq \frac{1}{2}$ at every point s of the strip, zero or not. The Epstein zeta function below shows that (2.1) cannot be proved from the structure that C_2 sees.

An Epstein zeta function with an off-line zero. Let $Q(m, n) = m^2 + 100n^2$ and

$$Z_Q(s) = \sum_{(m,n) \neq (0,0)} Q(m, n)^{-s} = \sum_{k \geq 1} r_Q(k) k^{-s}, \quad r_Q(k) = \#\{(m, n) : Q(m, n) = k\} \geq 0. \quad (2.2)$$

The following facts are classical. Z_Q is meromorphic on $\mathbb{C}$ with a single simple pole at $s = 1$, of residue $2\pi/\sqrt{400} = \pi/10$. Hypothesis (H) holds with $c = \pi/10$ (the area of the ellipse $Q \leq 1$), with $\theta = \frac{1}{2}$ by Gauss's argument and $\theta = \frac{1}{3}$ by the classical lattice-point estimate for ellipses; we use $\theta = \frac{1}{3}$. With $\Lambda_Q(s) = (10/\pi)^s \Gamma(s) Z_Q(s)$ one has $\Lambda_Q(s) = \Lambda_Q(1-s)$, by the theta transformation and $Z_{Q^*}(s) = 100^s Z_Q(s)$ for the dual form $Q^*(m, n) = m^2 + n^2/100$. The coefficients are real, so the zeros of Z_Q are symmetric under $\rho \mapsto 1 - \bar{\rho}$, exactly as for ζ . Splitting the Mellin integral of the theta function at 1 gives, with $k = 100$,

$$\pi^{-s} \Gamma(s) Z_Q(s) = -\frac{1}{s} - \frac{k^{-1/2}}{1-s} + \sum_{v \neq 0} \frac{\Gamma(s, \pi Q(v))}{(\pi Q(v))^s} + k^{-1/2} \sum_{v \neq 0} \frac{\Gamma(1-s, \pi Q^*(v))}{(\pi Q^*(v))^{1-s}}, \quad (2.3)$$

where $\Gamma(s, y)$ is the incomplete gamma function and both sums converge rapidly for every s .

Proposition 2.1 (a real zero off the line). Z_Q has a real zero $\beta_0 = 0.813367836786\dots$, and the reflected zero $1 - \beta_0 = 0.186632163214\dots$

The existence of real zeros in $(\frac{1}{2}, 1)$ for $m^2 + kn^2$ once k exceeds a threshold $k \approx 49.17$ ($\sqrt{k} \approx 7.0123$) is classical [BG]. The value was computed from (2.3) and validated in three ways: for $k = 1$ the same code reproduces $4\zeta(s)L(s, \chi_{-4})$ to 10^{-8} ; $Z_Q(0.7) = 0.54710$ from (2.3) agrees with the value 0.5476 obtained by lattice summation through [CE1, Lemma 3.10]; and as k varies there is no real zero in $(\frac{1}{2}, 1)$ at $k = 40$, and there is one at 0.529, 0.680 and 0.813 for $k = 50, 60, 100$, consistent with the threshold of [BG]. Status: classical [BG] (existence); the value is numerical (`anc/epstein/epstein_checks.py`).

Theorem 2.2 (Epstein collapse and the Obstruction Principle). (a) At the zero β_0 ,

$$K^Q(x, \beta_0) = \frac{2c}{1 - \beta_0} x^{1-2\beta_0} + O(x^{\theta-2\beta_0}) \longrightarrow 0, \quad c = \frac{\pi}{10}.$$

Hence the analogue of (2.1) for Z_Q is false: Z_Q has a zero at which its cross-energy collapses. At the reflected zero, $K^Q(x, 1 - \beta_0) \rightarrow +\infty$. (b) The following hold for Z_Q , with only the insertion of the weights $r_Q(n)$: the Dirichlet structure with nonnegative coefficients and a simple pole at $s = 1$; the continuous component and the C-transform representation ([CE1, Lemma 3.10]); a functional equation of the same type and the symmetry $\rho \mapsto 1 - \bar{\rho}$ of the zeros; every exact identity that follows from the recurrence $X_n^Q - X_{n-1}^Q = r_Q(n)n^{-s}$, in particular the energy identity $|X_n^Q|^2 - |X_{n-1}^Q|^2 = r_Q(n)^2 n^{-2\sigma} + 2r_Q(n) \operatorname{Re} [X_{n-1}^Q(s) n^{-\bar{s}}]$; and the anatomy and trichotomy ([CE1, Theorem 3.11]). (c) Consequently, no argument that uses only the properties listed in (b) can prove (2.1) for ζ .

Proof. (a) is [CE1, Theorem 3.11] with $F = Z_Q$, $s = \beta_0$, $t = 0$ and $\theta = \frac{1}{3} < \beta_0$; the term $F(\beta_0)$ vanishes. At the real point $1 - \beta_0 < \frac{1}{2}$ all terms of [CE1, (3.18)] are positive, and $\sum_{n \leq x} r_Q(n) n^{-\sigma} \sim c x^{1-\sigma}/(1-\sigma)$ by partial summation from $R(x) \sim cx$, so $K^Q(x, 1 - \beta_0) \asymp x^{2\beta_0-1} \rightarrow \infty$. (b) collects the classical facts above and the results of this subsection. (c) An argument that uses only these properties applies verbatim to Z_Q , and would contradict (a). $\square$ Status: (a) and (b) proved; (c) is a methodological consequence of (a) and (b).

Direct lattice summation confirms (a) (Table 1).

Practical form. Every proposed step towards (2.1) must be tested against Z_Q : a step that remains valid for Z_Q cannot be the step that excludes off-line zeros. Among the controls used

x	$X_x^Q(\beta_0) - cx^{1-\beta_0}/(1-\beta_0)$	$K^Q(x, \beta_0)$	$\frac{2c}{1-\beta_0}x^{1-2\beta_0}$
10^5	0.0010	0.002475	0.002475
10^6	-0.0017	0.000584	0.000584
10^7	0.0002	0.000138	0.000138
$4 \cdot 10^7$	0.0001	0.000058	0.000058

TABLE 1. Collapse of the cross-energy of Z_Q , $Q = m^2 + 100n^2$, at its zero $\beta_0 = 0.8133678$: the first column tends to $Z_Q(\beta_0) = 0$ and K^Q follows the structural term alone. Computed with `anc/epstein/c2_numerics.py` (section `epstein`) and `anc/epstein/epstein_checks.py`.

in this series, Z_Q is the sharpest for C_2 , because it has a pole, nonnegative coefficients and the trichotomy, none of which the Davenport–Heilbronn function has (its Dirichlet series has no pole, and its analogue of C_2 tends to 0 at every point of $\sigma > 0$).

What distinguishes ζ . The decisive difference between ζ and Z_Q is the Euler product, in the form

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n \geq 1} \Lambda(n)n^{-s}, \quad \Lambda(n) \geq 0 \text{ supported on the prime powers.} \quad (2.4)$$

Proposition 2.3 (the logarithmic derivative of Z_Q is not one-signed). The coefficients $b_Q(n)$ of $-Z'_Q/Z_Q = \sum_{n \geq 2} b_Q(n)n^{-s}$ are not all nonnegative. For $Q = m^2 + 100n^2$ the first negative one is $b_Q(10000) = -\log 100$, and there are 183 negative coefficients with $n \leq 2 \cdot 10^5$. For $Q = m^2 + 5n^2$ the first negative one is $b(36) = -7.167038\dots$, and there are 2604 with $n \leq 2 \cdot 10^5$. Status: numerical (`anc/epstein/epstein_checks.py`; b_Q from the recursion $f(n)\log n = \sum_{d|n} b_Q(d)f(n/d)$, $f = r_Q/2$).

Positivity of Λ underlies every known zero-free region, through $3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0$ ([CE10, Proposition 3.6]), and it is not known to reach $\sigma = \frac{1}{2}$. The conclusion here is only that any rigidity principle for C_2 must use this positivity, or another property that Z_Q lacks. The Epstein obstruction first appeared in [C1]; the version above corrects its statement of hypothesis (H) (which needs $\theta < \sigma$) and drops from the list (b) the Mellin transform of C_2 , whose analogue for general weights we have not verified.

2.2. Superpositions and the line as a level set. The Davenport–Heilbronn phenomenon, zeros off the line for a linear combination of L -functions with the same functional equation, can be followed continuously. Let χ_5 be the real (even) character modulo 5 and

$$\Lambda(s, \chi_5) = \frac{1}{2}s(s-1)(5/\pi)^{s/2}\Gamma(\frac{s}{2})L(s, \chi_5), \quad G_\varepsilon = \xi + \varepsilon\Lambda(\cdot, \chi_5) \quad (\varepsilon \in \mathbb{R}),$$

where $\Lambda(s, \chi_5)$, a completed L -function, is not to be confused with the von Mangoldt function. Then $G_\varepsilon(s) = G_\varepsilon(1-s)$, $G_\varepsilon(\bar{s}) = \overline{G_\varepsilon(s)}$, and G_ε is real on the critical line. For $0 < t < 100$, ξ has 29 zeros and $\Lambda(\cdot, \chi_5)$ has 54, all on the line. The next two propositions locate the zeros of G_ε .

Proposition 2.4 (distant zeros of a superposition). Let $0 < \varepsilon \leq 5^{-5/2}$ and put

$$\sigma_\varepsilon = \frac{2\log(1/\varepsilon)}{\log 5} \quad (\geq 5), \quad s_k = \sigma_\varepsilon + \frac{2\pi i(2k+1)}{\log 5},$$

$$R_k = \left\{ |\sigma - \sigma_\varepsilon| \leq 1, |t - \operatorname{Im} s_k| \leq \frac{2\pi}{\log 5} \right\} \quad (k \in \mathbb{Z}).$$

Then: (i) G_ε has no zeros with $\sigma \geq \frac{5}{2}$ outside $\bigcup_k R_k$; (ii) each R_k contains exactly one zero ρ_k of G_ε , and it is simple; (iii) with $\delta = \zeta(\sigma_\varepsilon - 1)^2 - 1$ and $A(s) = L(s, \chi_5)/\zeta(s)$,

$$\rho_k = s_k - \frac{2}{\log 5} \operatorname{Log} A(\rho_k), \quad |\rho_k - s_k| \leq \frac{2}{\log 5} \frac{\delta}{1 - \delta}, \quad \rho_k = s_k + \frac{4}{\log 5} (2^{-s_k} + 3^{-s_k}) + O(4^{-\sigma_\varepsilon}).$$

The points $1 - \rho_k$ are zeros as well.

Proof. The factors $\frac{1}{2}s(s-1)\Gamma(\frac{s}{2})$ are common to ξ and $\Lambda(\cdot, \chi_5)$, so $G_\varepsilon = \xi H$ with $H(s) = 1 + \varepsilon 5^{s/2} A(s)$, and $\xi \neq 0$ for $\sigma > 1$. The Dirichlet series $A(s) = \sum a(n)n^{-s}$ has $a = \chi_5 * \mu$, so $a(1) = 1$, $a(2) = a(3) = -2$ and $|a(n)| \leq 2^{\omega(n)} \leq d(n)$; hence $|A(s)| \leq \zeta(\sigma)^2$ and $|A(s) - 1| \leq \zeta(\sigma)^2 - 1$ for $\sigma > 1$. Note $|\varepsilon 5^{s/2}| = 5^{(\sigma - \sigma_\varepsilon)/2}$.

- (i) The rectangles R_k tile the strip $|\sigma - \sigma_\varepsilon| \leq 1$. If $\frac{5}{2} \leq \sigma \leq \sigma_\varepsilon - 1$, then $|\varepsilon 5^{s/2} A(s)| \leq 5^{-1/2} \zeta(\frac{5}{2})^2 = 0.804$, so $|H| \geq 0.196$. If $\sigma \geq \sigma_\varepsilon + 1 \geq 6$, then $|\varepsilon 5^{s/2} A(s)| \geq 5^{1/2} (2 - \zeta(6)^2) > 2$, so $|H| > 1$.
- (ii) Let $H_0(s) = 1 + \varepsilon 5^{s/2}$, whose zeros are the points s_k , all simple, one in each R_k . On the vertical sides of R_k , $|\varepsilon 5^{s/2}| = 5^{\mp 1/2}$ and $|H_0| \geq 1 - 5^{-1/2} = 0.553$. On the horizontal sides $\frac{t}{2} \log 5 \in 2\pi\mathbb{Z}$, so $\varepsilon 5^{s/2} > 0$ and $H_0 \geq 1$. On R_k , $\sigma \geq \sigma_\varepsilon - 1 \geq 4$ and $|H - H_0| = |\varepsilon 5^{s/2}| |A - 1| \leq 5^{1/2} (\zeta(4)^2 - 1) = 0.383 < 0.553$. By Rouché's theorem H and H_0 have the same number of zeros in R_k , counted with multiplicity, namely one.
- (iii) At ρ_k we have $\varepsilon 5^{\rho_k/2} A(\rho_k) = -1 = \varepsilon 5^{s_k/2}$, so $5^{(\rho_k - s_k)/2} A(\rho_k) = 1$. Since $|\operatorname{Im}(\rho_k - s_k)| \leq 2\pi / \log 5$ and $|A(\rho_k) - 1| \leq \delta < 1$, the principal branch gives the first identity, and $|\operatorname{Log}(1 + w)| \leq |w|/(1 - |w|)$ gives the bound. Finally $\operatorname{Log} A(\rho) = -2 \cdot 2^{-\rho} - 2 \cdot 3^{-\rho} + O(4^{-\beta})$, and $2^{-\rho_k} - 2^{-s_k} = O(2^{-\sigma_\varepsilon} |\rho_k - s_k|) = O(4^{-\sigma_\varepsilon})$. The last assertion follows from the functional equation and $G_\varepsilon(\bar{s}) = \overline{G_\varepsilon(s)}$. $\square$ Status: proved; the checks below are numerical (**anc/quantum/superposition/**).

The proof applies to any two primitive Dirichlet characters of the same parity and conductors $q_1 < q_2$, with 5 replaced by q_2/q_1 and the constants adjusted. The distant zeros, about $\frac{T \log 5}{4\pi}$ with $0 < t < T$ on each side of the line, account for the difference $\frac{T}{2\pi} \log 5$ between the counting functions of $L(s, \chi_5)$ and ζ . Numerically, with $\varepsilon = 10^{-3}, 10^{-4}, 10^{-6}$ and $0 \leq k \leq 5$, the zeros obtained by Newton's method satisfy $|\rho_k - s_k| \leq 6.5 \cdot 10^{-3}, 8.9 \cdot 10^{-4}, 1.7 \cdot 10^{-5}$ (bound (iii): $1.4 \cdot 10^{-2}, 1.8 \cdot 10^{-3}, 3.4 \cdot 10^{-5}$), and after subtracting the correction $\frac{4}{\log 5} (2^{-s_k} + 3^{-s_k})$ the differences are $3.1 \cdot 10^{-5}, 5.6 \cdot 10^{-7}, 2.0 \cdot 10^{-10}$. For $\varepsilon = 5^{-5/2}$ the argument principle gives exactly one zero in each of $R_{-3}, \dots, R_5$.

Proposition 2.5 (the line as a level set). Let $X(t) = \xi(\frac{1}{2} + it)$ and $Y(t) = \Lambda(\frac{1}{2} + it, \chi_5)$, both real for real t , and $r = -X/Y$, a real meromorphic function of t . For real ε and real t with $Y(t) \neq 0$:

- (i) $\frac{1}{2} + it$ is a zero of G_ε if and only if $r(t) = \varepsilon$, and its multiplicity is the order of vanishing of $r - \varepsilon$ at t .
- (ii) If $r(t_0) = \varepsilon_0$ and $r'(t_0) \neq 0$, then for ε close to ε_0 the function G_ε has exactly one zero near $\frac{1}{2} + it_0$, and it lies on the line.
- (iii) If $r'(t_c) = 0 \neq r''(t_c)$ and $\varepsilon_c = r(t_c)$, then for ε close to ε_c the function G_ε has exactly two zeros near $\frac{1}{2} + it_c$. If $(\varepsilon - \varepsilon_c)/r''(t_c) > 0$ they lie on the line, at $t = t_c \pm \sqrt{2(\varepsilon - \varepsilon_c)/r''(t_c)} + O(|\varepsilon - \varepsilon_c|)$; if $(\varepsilon - \varepsilon_c)/r''(t_c) < 0$ they are

$$\rho_\pm = \frac{1}{2} \pm \sqrt{\frac{2|\varepsilon - \varepsilon_c|}{|r''(t_c)|}} + it_c + O(|\varepsilon - \varepsilon_c|).$$

- (iv) Let K be a compact subset of the strip and I an interval such that no G_ε , $\varepsilon \in I$, vanishes on ∂K . Then the number of zeros of G_ε in K off the line is constant on every subinterval of I that contains no critical value $r(t_c)$ with $\frac{1}{2} + it_c \in K$, and it changes by 2 at each nondegenerate one.

Proof. Near the line, $G_\varepsilon(\frac{1}{2} + iz) = X(z) + \varepsilon Y(z) = Y(z)(\varepsilon - r(z))$, with X, Y entire and real on the real axis, which gives (i). The zeros of G_ε are symmetric under $s \mapsto 1 - \bar{s}$; in the variable z this is $z \mapsto \bar{z}$. In (ii), $r - \varepsilon$ has a single zero near t_0 by the implicit function theorem; since the zero set is invariant under conjugation, that zero is real. In (iii), write $r(t_c + w) - \varepsilon_c = \frac{1}{2} r''(t_c) w^2 u(w)$ with u

analytic and $u(0) = 1$, and $\phi(w) = w\sqrt{u(w)}$, which is analytic and invertible near 0 with $\phi'(0) = 1$. Then $r(t_c + w) = \varepsilon$ if and only if $\phi(w) = \pm c$ with $c^2 = 2(\varepsilon - \varepsilon_c)/r''(t_c)$, so $w = \pm c + O(|c|^2)$. If c is real the two zeros are real, that is, on the line; if $c = i|c|$ they are $t_c \pm i|c| + O(|c|^2)$, and $s = \frac{1}{2} + iz$ gives $\rho_{\pm}$. (iv) By the argument principle the number of zeros in K is constant on I . The number on the line changes only where a zero on the line ceases to be simple, which by (ii) and (iii) happens only at critical values, where it changes by 2. $\square$ Status: proved; the computations are numerical (`anc/quantum/superposition/`).

The proposition reduces the zeros on the line to the graph of the single real function $r(t) = -\xi(\frac{1}{2} + it)/\Lambda(\frac{1}{2} + it, \chi_5)$: a horizontal line at height ε meets it at the zeros on the line, a local maximum of r above 0 is a value of ε at which a pair leaves the line, and a local minimum one at which a pair lands (Figure 2). For $0 < t < 100$ the counts of solutions of $r(t) = \varepsilon$ coincide with the zeros found directly for all twenty values of ε tested. The positive critical values below 1 are 0.0931 (maximum at $t = 60.434$, where 29 zeros on the line become 27), 0.1742 (minimum at $t = 3.777$, back to 29), 0.6197 (minimum at $t = 57.668$) and 0.8747 (minimum at $t = 11.020$). For the landing at $t_c = 57.668$, $r''(t_c) = 1.7278$, and (iii) predicts $\beta - \frac{1}{2} = 1.0759\sqrt{\varepsilon_c - \varepsilon}$; the zero followed by Newton's method gives 1.0768, and the ratio stays at 1.0759 for $10^{-6} < \varepsilon_c - \varepsilon < 10^{-3}$.

The global picture is in Figure 1. For every $\varepsilon > 0$, G_ε has the larger zero count, and for small ε the 13 additional pairs with $t < 100$ lie where the two terms have equal size, at $\beta \approx 2 \log(1/\varepsilon)/\log 5$, spaced by $4\pi/\log 5 = 7.81$ in t (Proposition 2.4): the median β is 8.585, 5.733, 3.795 for $\varepsilon = 10^{-3}$, 10^{-2} , 0.05, against 8.584, 5.723, 3.723. As $\varepsilon \rightarrow 0$ these zeros recede to infinity, while the zeros of ξ stay on the line apart from one pair, which leaves it at $\varepsilon = 0.0931$ near $t = 60.43$. As ε grows the distant zeros enter the strip and land on the line in pairs, through square-root bifurcations, and the number of zeros on the line increases from 29 to 54. Together with Proposition 2.4, zeros of G_ε reach the strip off the line either from infinity or through a critical point of r , and at no other place can they leave the line. The two pure functions have all their zeros (in this range) on the line; every superposition has zeros off it. In this language RH says that ζ contains no second component, however small. The same phenomenon appears in the Dirichlet series $\sum_n e^{2\pi i n a/q} n^{-s}$ of an additive character, a superposition of several primitive L -functions when $q \geq 3$: every zero found with $t < 40$ lies off the line, with $\max|\beta - \frac{1}{2}| = 0.66, 0.73, 0.79, 0.77, 0.78, 0.81$ for $a/q = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{5}, \frac{2}{5}, \frac{1}{8}$, in accordance with [DH] (`anc/quantum/`).

2.3. Identities that do not see s , and positivity on the wrong side. Proposition 2.6 (convolution identities do not see s). Let f, g be arithmetic functions, $h = f * g$, $G_M(s) = \sum_{m \leq M} g(m)m^{-s}$ and $H_N(s) = \sum_{n \leq N} h(n)n^{-s}$. Then for every N and every $s \in \mathbb{C}$

$$\sum_{d \leq N} f(d) d^{-s} G_{\lfloor N/d \rfloor}(s) = H_N(s).$$

More generally the identity remains true when n^{-s} is replaced by $x^{v(n)} = \prod_p x_p^{v_p(n)}$ for an arbitrary sequence (x_p) of complex numbers.

Proof. The left side is $\sum_{dm \leq N} f(d)g(m)(dm)^{-s}$, and grouping the terms with $dm = n$ gives $H_N(s)$. Only $(dm)^{-s} = d^{-s}m^{-s}$ was used, which also holds for $x^{v(dm)} = x^{v(d)}x^{v(m)}$. $\square$ Status: proved; the computations below are numerical (`anc/euler_identities/chk.py`).

With $f = \mu$, $g = 1$ this gives $\sum_{d \leq N} \mu(d)d^{-s}X_{\lfloor N/d \rfloor}(s) = 1$; with $f = \Lambda$, $g = 1$ it gives $-X'_N(s) = \sum_{d \leq N} \Lambda(d)d^{-s}X_{\lfloor N/d \rfloor}(s)$; and for $\sqrt{N} < y \leq N$ the decomposition by the largest prime factor gives $X_N(s) = E_{y,N}(s) + \sum_{y < p \leq N} p^{-s}X_{\lfloor N/p \rfloor}(s)$, where $E_{y,N}$ is the sum over the y -smooth $n \leq N$. All three hold at every s , at a zero of ζ or not, and for every value of β ; they are consequences of unique factorization, and by the proposition they cannot restrict the position of the zeros. In Bohr's correspondence [Bh] X_N is the restriction of a polynomial in the infinitely many variables x_p to the points $x_p = p^{-s}$, which wind densely around the torus $|x_p| = p^{-\sigma}$ as

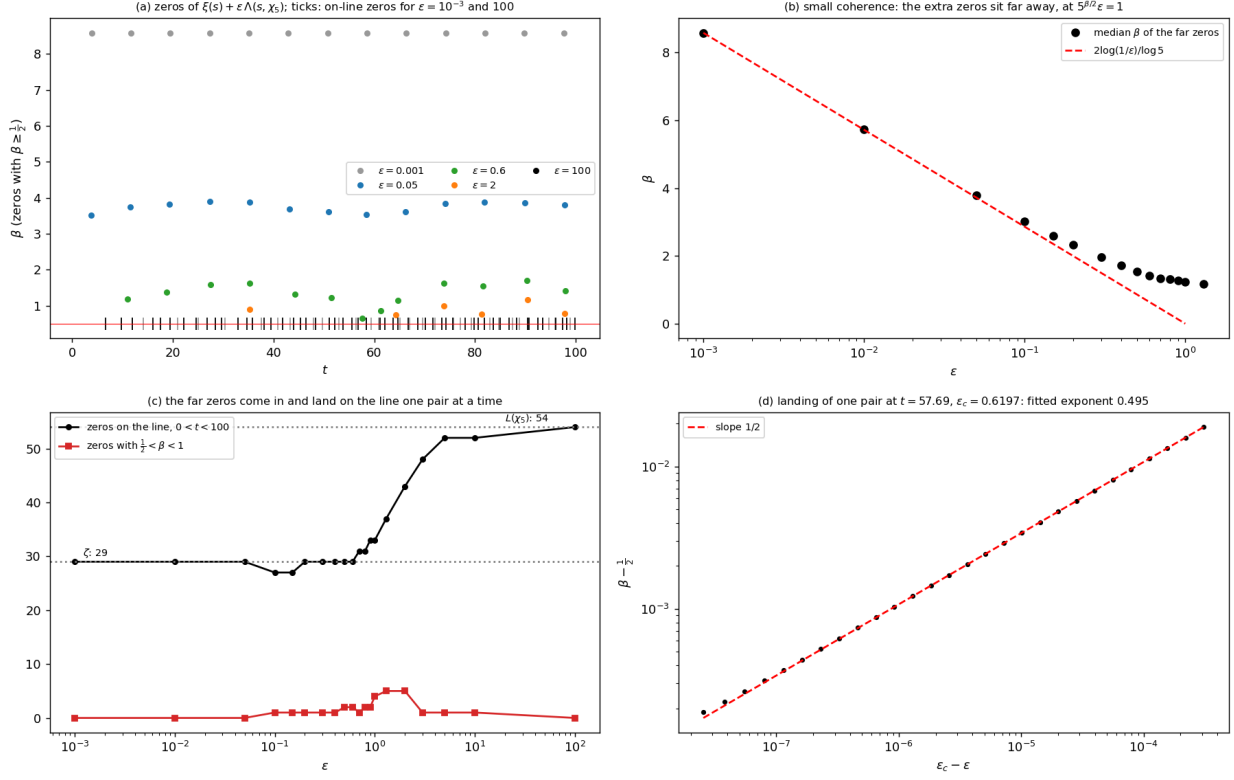


FIGURE 1. Zeros of the superposition $G_\epsilon = \xi + \epsilon\Lambda(\cdot, \chi_5)$. (a) Zeros with $\beta \geq \frac{1}{2}$, $t < 100$, for five values of ϵ ; ticks: zeros on the line. (b) Median β of the distant zeros against $2\log(1/\epsilon)/\log 5$ (Proposition 2.4). (c) Number of zeros on the line and in $\frac{1}{2} < \beta < 1$. (d) Landing of one pair: $\beta - \frac{1}{2}$ against $\epsilon_c - \epsilon$ (Proposition 2.5(iii)).

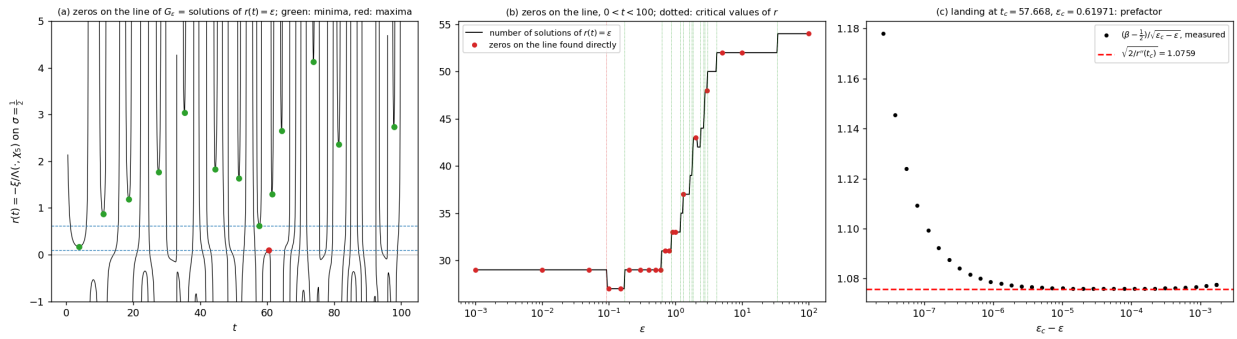


FIGURE 2. The line as a level set (Proposition 2.5). (a) $r(t) = -\xi/\Lambda(\cdot, \chi_5)$ on the critical line, clipped at the poles; dots: positive critical values (green minima, red maxima); dashed: $\epsilon = 0.0931$ and 0.6197 . (b) Number of solutions of $r(t) = \epsilon$ with $0 < t < 100$ and the zeros on the line found directly; dotted: critical values. (c) $(\beta - \frac{1}{2})/\sqrt{\epsilon_c - \epsilon}$ for the pair landing at $t_c = 57.668$, against $\sqrt{2/r''(t_c)}$; the departure for $\epsilon_c - \epsilon < 10^{-6}$ reflects the precision of ϵ_c .

t varies; identities that follow from multiplicativity hold on the whole space of the x_p . The real part β enters only through the radii $p^{-\sigma}$ and through the continuation term $N^{1-s}/(1-s)$ of the Euler–Maclaurin expansion, which is not an identity between Dirichlet polynomials.

The computations confirm this. With the primes up to 10^6 , the prime sum

$$P_N(s) = 2 \operatorname{Re} \sum_{d < N} \Lambda(d) N^{-\bar{s}} d^{-s} X_{[(N-1)/d]}(s)$$

agrees with $-2 \operatorname{Re}[N^{-\bar{s}} X'_{N-1}(s)]$ to 10^{-15} at ρ_1, ρ_2 and at $0.37 + 20i$, which is not a zero, and the Möbius identity holds to 10^{-11} at the same points for $N = 10^4, 10^5, 10^6$. At a zero, Euler–Maclaurin and $\sum_{d < N} \Lambda(d)/d = \log N - \gamma_0 + o(1)$ give

$$P_N(\rho) - C_2(N, \rho) \sum_{d < N} \frac{\Lambda(d)}{d} = K(\rho) N^{1-2\beta} - 2 \operatorname{Re} [N^{-\bar{\rho}} \zeta'(\rho)] + o(N^{1-2\beta}),$$

where $K(\rho) = \gamma_0 A_\rho - 2 \operatorname{Re}(1 - \rho)^{-2}$ and $A_\rho = 2 \operatorname{Re} \frac{1}{1-\rho}$ is the constant of the structural term of C_2 . At ρ_1 , $K = 0.01286$, and the left side is 0.0133, 0.0084, 0.0116 for $N = 10^4, 10^5, 10^6$, the deviations being of the size $2|\zeta'(\rho_1)|N^{-1/2}$ of the second term. The leading terms $A_\rho N^{1-2\beta} \log N$ cancel, and the only arithmetic ingredient of the remaining coefficient is Mertens' constant: the primes reproduce the expansion of C_2 and do not constrain it. Likewise, in the decomposition by the largest prime factor with $y = N^\vartheta$, the main term $N^{1-\rho}/(1-\rho)$ of $X_N(\rho)$ is shared by the large primes and the smooth part in the proportions $-\log \vartheta$ and $1 + \log \vartheta$ [Bu]; for $\vartheta = \frac{1}{2}$ the smooth share at ρ_1 is 0.245, 0.267, 0.270 for $N = 10^4, 10^5, 10^6$, approaching $1 - \log 2 = 0.307$ slowly. A quantity that sees β must compare ρ with $1 - \bar{\rho}$: the product $p^{-\rho} p^{-(1-\bar{\rho})} = p^{-1-2i\gamma}$ does not see β , whereas the ratio $|p^{-\rho}|/|p^{-(1-\rho)}| = p^{1-2\beta}$ does. Summed over the zeros with a test function, comparisons of this kind form Weil's functional of Part 10 [CE10] (the expansion at a zero is derived in outline above; the reflected comparison is the subject of [CE3]).

Proposition 2.7 (entropy forms and Bochner duality). Let T be an even real tempered distribution on $\mathbb{R}$ and $Q(f) = \iint f(x) \overline{f(y)} T(x-y) dx dy$ for $f \in C_c^\infty(\mathbb{R})$.

- (i) $Q(f) \geq 0$ for all f if and only if $\hat{T}$ is a positive measure, and then $Q(f) = \int |\hat{f}(\xi)|^2 d\hat{T}(\xi)$ (with the normalization of the Fourier transform implicit in Parseval's formula).
- (ii) If there is a positive measure p with $Q(f) = \int |f(x)|^2 dp(x)$ for all f , then p is a constant multiple of Lebesgue measure and $T = c\delta$.

Proof. (i) is the Bochner–Schwartz theorem, with Parseval. (ii) By polarization the sesquilinear forms $\iint f(x) \overline{g(y)} T(x-y)$ and $\int f \bar{g} dp$ coincide, so $T(x-y) = p(x)\delta(x-y)$ as distributions on $\mathbb{R}^2$. The left side is invariant under $(x, y) \mapsto (x+a, y+a)$, hence so is p , which is therefore $c dx$, and then $T = c\delta$. $\square$ Status: proved; the comparison below is numerical (anc/weil/).

For ζ , Weil's form of Part 10 [CE10] is Q with

$$T = (\text{pole}) + (\text{archimedean part}) - \sum_n \frac{\Lambda(n)}{\sqrt{n}} (\delta_{\log n} + \delta_{-\log n}),$$

and, when the zeros are on the line, $\hat{T} = \sum_\rho \delta_{\gamma_\rho}$ up to normalization. A second-order expansion of a relative entropy, a variance or a Fisher information, for distributions on the integers or on $x = \log n$, is a form $\int |h|^2 dp$ with $p \geq 0$. By (ii) such a form cannot be Weil's, since T is not a multiple of δ (its masses at the prime powers are negative). By (i) Weil's form is a form of this kind on the spectral side, $Q(f) = \sum_\gamma |\hat{f}(\gamma)|^2$, with the zeros as reference measure. A divergence between positive distributions built from the integers therefore produces positivity on the wrong side of the duality, and on the right side the positive measure is the set of zeros itself. For the same reason the entropies and coherences of the normalized vector $\zeta(2\sigma)^{-1/2} \sum_n n^{-s} |n\rangle$, which is obtained from its value at $t = 0$ by the diagonal unitary $e^{-it \log n}$, depend on σ alone and carry no information on

the zeros. We checked the spectral representation in the basis of cubic B-splines of mesh $D = 0.03$, where the form is a Toeplitz matrix $[w(j-k)]$: the values $w(m)$ computed from the primes, the pole and the archimedean term agree with $\sum_{\gamma} 2\hat{K}(\gamma) \cos(\gamma m D)$, $\hat{K}(r) = (2 \sin(rD/2)/r)^8 \geq 0$, summed over the 341 zeros with $\gamma < 600$ plus the tail from the zero density, to 10^{-10} ($w(0) = 21.0981776757$ from both sides) for $0 \leq m \leq 180$. In this form the matrix is $\sum_{\gamma} 2\hat{K}(\gamma) v_{\gamma} v_{\gamma}^*$ with $v_{\gamma} = (e^{i\gamma k D})_k$, a sum of positive rank-one terms; for the Davenport–Heilbronn function the decomposition has complex γ , and the smallest eigenvalue is -2.90 at window length 4.2 and -7.47 at 5.4.

2.4. The pole barrier. Positivity and the pole. For every x , $\sigma > 0$ and real t , with $E_x(s) = \prod_{p \leq x} (1 - p^{-s})^{-1}$, the inequality of Mertens and de la Vallée Poussin [T, §3.3],

$$3 \log |E_x(\sigma)| + 4 \log |E_x(\sigma + it)| + \log |E_x(\sigma + 2it)| = \sum_{p \leq x} \sum_{k \geq 1} \frac{p^{-k\sigma}}{k} (3 + 4 \cos \theta_{p,k} + \cos 2\theta_{p,k}) \geq 0,$$

with $\theta_{p,k} = kt \log p$, holds exactly. Inside the strip the partial products diverge, and the correction that turns them into ζ [CE8] must be subtracted. At $(\sigma, t) = (0.7, 14.13)$ and $x = 2 \cdot 10^6$ the positive sum is 70.77, the pole correction -74.45 , and the total -3.69 , against the limit $3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)| = -3.77$. Inside the strip the positive sum grows like $3x^{1-\sigma}/((1-\sigma) \log x)$, and the correction from the real point σ cancels it. This cannot be avoided: a nonnegative trigonometric polynomial $\sum a_j \cos j\theta$ has $a_0 > 0$, and the a_0 term sits at $t = 0$, at the pole. Counting only leading terms, a zero at $\beta + it$ lowers the left side by about $4x^{\beta-\sigma}/((\beta-\sigma) \log x)$, while the margin left by the pole is about $3x^{1-\sigma}/((1-\sigma) \log x)$; optimizing over σ , the inequality excludes exactly the zeros with $\beta > 1 - 0.427/\log x$, with the same constant for x from 10^3 to 10^{100} , and with x a power of t this is the shape of the zero-free region of [CE10, Proposition 3.6]. The same happens for the Fejér kernels of degree 2, 4, 8, 16: the combination $\sum_k a_k \operatorname{Re}(-\zeta'/\zeta)(\sigma + ikt)$ at $t = \gamma_1$ is positive for $\sigma > 1$ and already negative at $\sigma = 0.98$. Status: the identity is exact; the scaling is heuristic (leading terms); the computations are numerical (`anc/euler/`).

Twisting by characters does not remove the pole term.

Proposition 2.8 (twists). Let $F(\theta) = \sum_{|j| \leq J} c_j e^{ij\theta}$ be a nonnegative trigonometric polynomial, not identically zero, let ψ be a Dirichlet character mod q , and put

$$\Phi_x(\sigma, t) = \sum_{|j| \leq J} c_j \log E_x(\sigma + ijt, \psi^j), \quad \log E_x(s, \psi) = \sum_{p \leq x} \sum_{k \geq 1} \frac{\psi(p)^k p^{-ks}}{k}.$$

Then $\Phi_x(\sigma, t) \geq 0$ for all x , $\sigma > 0$ and t , and its $j = 0$ term is $c_0 \log E_x(\sigma, \psi^0)$ with $c_0 > 0$ and ψ^0 the principal character mod q . For $0 < \sigma < 1$ this term is $c_0 x^{1-\sigma}/((1-\sigma) \log x) (1 + o(1))$.

Proof. The prime power p^k , $p \nmid q$, contributes

$$\frac{p^{-k\sigma}}{k} \sum_j c_j e^{ijk(\arg \psi(p) - t \log p)} = \frac{p^{-k\sigma}}{k} F(k(\arg \psi(p) - t \log p)) \geq 0,$$

and the primes dividing q contribute nothing. The constant coefficient c_0 is the mean of F , which is positive. The asymptotic is the prime number theorem in the form $\sum_{p \leq x} p^{-\sigma} \sim x^{1-\sigma}/((1-\sigma) \log x)$, the terms $k \geq 2$ being of lower order. $\square$ Status: proved.

The same argument applies to several characters at once, with F a nonnegative trigonometric polynomial on a torus. A twist rotates the phase of each prime by $\arg \psi(p)$ but leaves the mean of F attached to the principal character, which carries the pole. For a single nonprincipal character there is no pole: for the character modulo 5 of the Davenport–Heilbronn function the uncorrected product $E_x(s, \chi)$ approaches $L(s, \chi)$ in the strip, with $|E_x/L - 1| = 0.070, 0.025, 0.017, 0.009, 0.003$ at $s = 0.75 + 85.7i$ for $x = 10^3, \dots, 2 \cdot 10^6$; but no linear inequality with nonnegative coefficients can be built from it alone. The quadratic analogue, averaging $|\sum_p \chi(p) a_p|^2$ over a family of characters, is the large sieve, which is how the density estimates used after [CE10, Theorem 3.7] are proved.

Proposition 2.9 (no linear polarization without the pole). Let $F(x) = \sum_{j=1}^K c_j \cos(t_j x + \phi_j)$ with real c_j, ϕ_j and distinct $t_j > 0$, so that F has no constant term. If $F(\log p) \geq 0$ for all but finitely many primes p , then $F \equiv 0$.

Proof. For real $u \neq 0$ the partial sums $\sum_{p \leq x} p^{-1-iu}$ are bounded as $x \rightarrow \infty$; this is the prime number theorem in the form $\zeta(1+iu) \neq 0$ [T, Ch. III]. Since $\sum_{p \leq x} p^{-1} \rightarrow \infty$, the weighted mean $\langle G \rangle_x = \sum_{p \leq x} G(\log p) p^{-1} / \sum_{p \leq x} p^{-1}$ tends to 0 for every $G(x) = \cos(ux + \phi)$ with $u \neq 0$, and to 1 for $G = 1$. Hence $\langle F \rangle_x \rightarrow 0$. Expanding the square,

$$F^2 = \frac{1}{2} \sum_j c_j^2 + \frac{1}{2} \sum_j c_j^2 \cos(2t_j x + 2\phi_j) + \sum_{j < k} c_j c_k [\cos((t_j - t_k)x + \phi_j - \phi_k) + \cos((t_j + t_k)x + \phi_j + \phi_k)],$$

in which every non-constant term has a nonzero frequency, so $\langle F^2 \rangle_x \rightarrow \frac{1}{2} \sum c_j^2$. If $F(\log p) \geq 0$ for all large p , then $F(\log p)^2 \leq \|F\|_\infty F(\log p)$ there, and the finitely many exceptions do not affect the limits; hence $\frac{1}{2} \sum c_j^2 \leq \|F\|_\infty \lim_x \langle F \rangle_x = 0$. $\square$ Status: proved; the optimization below is numerical (anc/polarization/lp.py).

The proposition explains the term $a_0 > 0$ in [CE10, (3.3)] and in Proposition 2.8: a combination of the values $\operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it_j)$ that is positive prime by prime must contain the point $t = 0$, that is, the pole. The obstruction is the uniform distribution of the angles $t \log p$, which follows from the absence of zeros on $\sigma = 1$. Finitely many primes can be polarized: maximizing $\min_p F(\log p)$ by linear programming over 80 frequencies in $[0.5, 40]$ with $|c_j| \leq 1$ gives a positive minimum 12.6, 5.5 and 1.6 on the primes up to 30, 100 and 1000, but the optimized F is then negative on 57%, 52% and 59% of the primes from there to 10^6 . In the language of Part 10 [CE10] the same barrier appears twice: the Poisson kernel $h_{\delta, t}$ that represents the energy slope is not admissible in the explicit formula for $\sigma < 1$ ([CE10, Theorem 3.13(a)]), and the partition-function reading of ζ has its phase transition at $\sigma = 1$ ([CE10, Remark 3.11]). A positivity without the pole term can only be quadratic, $|\sum a_n n^{-s}|^2 \geq 0$: this is the route of the density theorems ([CE10, Proposition 3.9]), which show that off-line zeros are rare and not that they are absent, and of Weil's functional, whose margin on windows collapses superexponentially [CE12].

2.5. What the cross-energy cannot do by itself. The results of this section and of Part 10 [CE10] can be summarized as a list of no-go statements. Each is proved, or is a direct consequence of a proved statement, unless marked otherwise.

1. *Properties shared with Z_Q .* Nonnegative coefficients, a pole, the C-transform representation, a functional equation, the exact recurrence identities and the trichotomy do not imply the non-collapse statement (2.1) (Theorem 2.2).
2. *Identities from unique factorization.* Convolution identities between partial sums hold at every s and cannot locate zeros (Proposition 2.6).
3. *Positive divergences.* Entropy, variance and Fisher-information forms built from distributions on the integers are positive on the wrong side of the Bochner duality and cannot be Weil's form (Proposition 2.7).
4. *Linear positivity.* Any linear inequality that is positive prime by prime contains the pole, and pays a correction of size $x^{1-\sigma}$ inside the strip; twisting does not help (Propositions 2.8, 2.9). This limits the energy-slope argument to the region of de la Vallée Poussin type ([CE10, Proposition 3.6]).
5. *Quadratic means.* Averaged over t the cross-energy keeps only its diagonal; this gives density theorems but cannot exclude a single zero, and mollification stops at the Jensen gap ([CE10, Proposition 3.9], [CE10, Remark 3.10]).
6. *Finite prime data.* The truncated Poisson functional W_L is exact, but its limit $L \rightarrow \infty$ at fixed $\sigma < 1$ is equivalent to controlling all zeros with $\beta > \sigma$, and any finite certificate certifies nothing beyond the known zero-free regions ([CE10, Theorem 3.13], [CE10, Corollary 3.14]).

Remark 2.10 (scale recursions are neutral). The decompositions of $\Lambda - 1$ into type I and type II pieces (Vaughan's identity, $\Lambda - 1 = \mu * (\log - \tau)$, the recursion $\sum_{k \leq x} M(x/k) = 1$) are covariant under dilations: by Mellin transformation the scale operator $f \mapsto \sum_{n \leq x} f(x/n)$ is multiplication by $\zeta(s)$, a profile x^a is mapped to a multiple of x^a , and a bilinear sum inherits the exponent of its factors (for $\sum_{m \sim U} \mu(m)m^{-s} \sum_{x < mn \leq 2x} (\Lambda(n) - 1)n^{-s}$ the ratio to the product of the ℓ^2 norms stays between 0.3 and 0.5 with no power trend). Such recursions gain only logarithms, which is consistent with the fact that elementary methods stop at savings $\exp(-(\log x)^c)$. Exact periodicity of $N(x) - cx$ and multiplicativity together determine the ordinary primes among Beurling systems up to finitely many primes [Hil]; each alone fails, as the Davenport–Heilbronn function and Beurling systems with large oscillation show [DMV, BDV]. Status: the Mellin identity is elementary; the bilinear ratios are numerical (`anc/scale_recursions/`); the conclusion about proof strategies is heuristic.

Scale recursions, measured. We tested the neutrality of Remark 2.10 on the dyadic block $A(M; s) = \sum_{M < n \leq 2M} (\Lambda(n) - 1)n^{-s}$, whose growth exponent in M is $\sup \beta - \sigma$ ($\frac{1}{2} - \sigma$ under RH, $1 - \sigma$ trivially), comparing ζ with the Beurling system with a zero at $0.75 + 20i$ [CE17] and, in (e), with the Davenport–Heilbronn function.

- (a) *No feedback.* The state $D(x; s) = C_x(s; \Lambda)$, the C-transform [CE1, (2.12)] of Λ , satisfies $D(2x) - D(x) = A(x; s)$ up to a smooth term: the forcing is fixed by the primes in $(x, 2x]$ and does not depend on $D(x)$, so there is no feedback for a Lyapunov function to use, and a bound for D is a bound for the sum of the forcings, which is the original problem. What is visible is coherence. At $s = 0.6 + 20i$ and $x = 2^k$, $10 \leq k \leq 25$, the cosine between forcing and state is 0.99 to 1.00 in every window for the Beurling system, whose zero pumps in phase (the forcing grows with exponent 0.154 against $0.75 - \sigma = 0.15$), while for ζ it rotates through both signs, between -0.95 and $+0.95$. The ratio r of consecutive forcings gives the Prony estimate $s + \log r / \log 2$, which for the Beurling system scatters around its zero (real parts 0.56 to 0.93, heights 19.81 to 20.20, mean $0.76 + 20.00i$) and for ζ does not converge (`scale_recursions/lyap-prony.py`).
- (b) *Type I against type II.* Vaughan's identity with $U = V = M^{1/3}$ splits $A(M)$ exactly into a type I part P , which uses only the data at scales $\leq M^{1/3}$ together with the exact integers, and a type II part $T_3 = \sum_{m, n > M^{1/3}} a_m b_n (mn)^{-s}$. At $s = 0.6$ and $M = 2^{19}$, $|P| = 20.86$, $|T_3| = 20.76$ and $|A| = 0.103$; at $s = 0.6 + 20i$, $|P| = 1.92$, $|T_3| = 1.87$ and $|A| = 0.171$ (`scale_recursions/vaughan_split.py`; the author's notes give $|P| = 20.84$, with the integer sums in P replaced by their Euler–Maclaurin main terms). The cancellation that would have to be proved is between the two types, not within either; bounding the blocks one at a time gives only the exponents $+0.385$ (with the internal cancellation of each block) and $+0.427$ (absolute values), against the trivial $1 - \sigma = 0.400$ (author's notes, $s = 0.6 + 20i$). The blocks $|A(M)|$ themselves stay between 0.04 and 0.42 for $M = 2^{10}, \dots, 2^{23}$.
- (c) *Möbius times divisor.* From $\Lambda - 1 = \mu * (\log - \tau)$, $A(M) = \sum_d \mu(d) d^{-s} \Gamma(M/d)$ with $\Gamma(y) = \sum_{y < b \leq 2y} (\log b - \tau(b)) b^{-s}$. The divisor problem splits Γ into a smooth main term and an error $O(y^{\theta_D - \sigma + \varepsilon})$; summed over $d \leq M^{1/2}$ by the triangle inequality the error contributes at most $M^{(1+\theta_D)/2 - \sigma + \varepsilon}$, which with Huxley's $\theta_D = 131/416$ [Hu03] is $M^{0.057 + \varepsilon}$ at $\sigma = 0.6$, below the trivial exponent 0.400. That part is provable. What remains are long Möbius sums: the main term, which carries $\sum_{d \leq 2M} \mu(d)/d$ (measured exponent $+0.329$ at $s = 0.6 + 20i$), and the range $d > M^{1/2}$ (-0.248 ; the divisor error with $d \leq M^{1/2}$ measures -0.246 with its cancellation and -0.019 by the triangle inequality; author's notes). So a power saving for the primes follows from the divisor problem, which is solved, and a power saving for long Möbius sums, which is the open core.
- (d) *The Möbius recursion.* The identity $\sum_{k \leq x} M(x/k) = 1$ holds exactly in both systems up to 10^6 (2 269 791 generalized integers for the Beurling system), and the scale operator is

multiplication by ζ , respectively by ζ_P , so the recursion has the same form in a system where M grows: by itself it cannot force decay.

- (e) *Periodicity and multiplicativity.* The same recursion $\sum_{n \leq x} a(n)B(x/n) = 1$, $b = a^{-1}$, in four systems gives the exponents of the twisted dyadic increments of $\sum b(v)v^{-s}$ in Table 2 (`scale_recursions/sierra.py`, `mobius_res.py`). The Davenport–Heilbronn function reproduces $0.8085 - \sigma$ to 0.01, the Beurling system $0.75 - \sigma$ to 0.04, and ζ and $L(s, \chi_5)$ lie 0.03 to 0.06 above $\frac{1}{2} - \sigma$ (logarithmic factors at this range). Only the cell that has both properties behaves as RH predicts, and by Hilberdink’s theorem [Hil] that cell contains a single system.

system	periodic	multiplicative	$\sigma = 0.6$	0.7	0.8	0.9
ζ ($t = 20$)	yes	yes	−0.05	−0.15	−0.24	−0.34
$L(s, \chi_5)$ ($t = 20$)	yes	yes	−0.07	−0.17	−0.27	−0.36
Davenport–Heilbronn ($t = 85.7$)	yes	no	+0.22	+0.12	+0.02	−0.09
Beurling, zero $0.75 + 20i$ ($t = 20$)	no	yes	+0.11	+0.01	−0.09	−0.18

TABLE 2. Exponents of the twisted dyadic increments of $\sum_{v \leq x} b(v)v^{-s}$, b the Dirichlet inverse of the coefficients a of each system, up to $x = 10^6$; “periodic” refers to $A(x) - cx$, A the summatory function of a . The values predicted by the rightmost zero are $\frac{1}{2} - \sigma$, $\frac{1}{2} - \sigma$, $0.8085 - \sigma$ and $0.75 - \sigma$.

- (f) *Möbius against the sawtooth.* In $\sum_{n \leq x} \mu(n)\{x/n\} = x m(x) - 1$, $m(x) = \sum_{n \leq x} \mu(n)/n$, the fast half $n \leq \sqrt{x}$, where $\{x/n\}$ oscillates, is trivially at most $\sqrt{x}$ and has measured exponent 0.256 up to 10^7 ; the slow half, where $\{x/n\}$ is piecewise linear, has exponent 0.485, as the total (0.484), and the two halves are uncorrelated (correlation -0.04). The 2×2 table is visible in the fast half (exponents 0.26, 0.31, 0.43 for ζ , Davenport–Heilbronn and Beurling up to 10^6), but for ζ that half is free. The hyperbola identity $\sum_{k \leq r} M(x/k) = 1 - x m(r) + \sum_{n \leq r} \mu(n)\{x/n\} + r M(r)$, $r = \lfloor \sqrt{x} \rfloor$, is exact (checked to 10^7), but with $m(r) = M(r)/r - \int_r^\infty M(u)u^{-2}du$ it brings back M at every scale above $\sqrt{x}$: it is circular (`scale_recursions/mitades.py`, `hiperbola.py`).

All the variants end in long Möbius sums or in bilinear forms of type II with both factors at least $x^{1/3}$, the front on which the best known result is the Vinogradov–Korobov region. Status: (a), (b), (d), (e) and (f) are numerical (`anc/scale_recursions/`), except the block-inheritance exponents in (b) and the measured exponents in (c), which are from the author’s working notes; the reduction in (c) is proved in outline; the conclusion is heuristic.

2.6. The Davenport–Heilbronn function as a C_2 control. The Davenport–Heilbronn function $f = c_1 L(s, \chi) + c_2 L(s, \bar{\chi})$ (χ a complex character modulo 5) has a functional equation of the same type as ζ and zeros off the line, the first at $0.808517 + 85.699348i$ [DH]. Each criterion of this paper was tested on it; Table 3 summarizes the results, with the corrections of Part 10 [CE10] and of [CE12] applied.

The row of the Euler product is the only one in which f behaves like ζ : convergence of a logarithmic Dirichlet series is controlled by the zeros to its right, whether or not there is an Euler product. In every other row the difference comes from the coefficients. Those of $\log \zeta$ are $\Lambda(n)/\log n \geq 0$ and live on prime powers; those of $\log f$ change sign and live mostly on composite integers. The pole is the second difference: the trichotomy of C_2 and its crossing at a zero are traces of the pole of ζ at $s = 1$, and f , whose series has no pole, has neither. The Epstein function Z_Q of Theorem 2.2 removes this second difference and keeps the first, which is why it is the sharper control for the statements of this paper about C_2 . The full anatomy of an off-line zero of f , the Beurling controls and the function-field controls are in [CE17]; the Beurling system of [BDR] type, which shares every

quantity	ζ	Davenport–Heilbronn f
limit of C_2	trichotomy at $\sigma = \frac{1}{2}$	0 for all $\sigma > 0$ (no pole)
sign changes of the signed helix radius, $0 < t < 400$	202 of 202 zeros	287 of 303 zeros
convergence of a truncated zero [CE4]	rate $n^{-1/2}$	rates $n^{-\beta}$, $n^{-(1-\beta)}$ at the off-line pair (measured $n^{-0.809}$, $n^{-0.191}$)
explicit spiral of a truncated zero [CE4]	rate $N^{-3/2}$	$M^{-2.81}$, $M^{-2.19}$
zeros on the circles of the primes [CE4]	bias toward π	spikes of both signs
prime-signal energy	$\approx C_B$ up to 10^8	grows, rate $\approx 2\beta - 1$
local energy in Gaussian windows	inside $C_B \pm B(H)$	grows from 0.22 to 2.06
energy slope ([CE10, Theorems 3.2, 3.7])	$\min_t S \approx 0.39(\sigma - \frac{1}{2})$	one failure interval per off-line zero
truncated functional W_L ([CE10, Theorem 3.13])	inside the RH band [CE10, (3.5)]	negative outside the disc
Weil matrices on windows [CE12]	positive where computed; margin $\approx 10^{-(5.42e^L - 12.6)}$, $L \leq 5$	negative for $L > L^*$, $3.42 < L^* \leq 3.43$
two-window Weil form [CE12]	$ Y /D \leq 0.94$ up to $3 \cdot 10^6$	—
Euler product [CE8]	converges right of the zeros	converges right of the zeros
Euler helix, pairs $\frac{1}{2} \pm \delta$ [CE8]	$\delta_{\min} \approx 4 \cdot 10^{-5}$ at $x = 10^8$	—
extrema of Z without crossing	none for $t < 10^5$	8, at the off-line pairs

TABLE 3. ζ and the Davenport–Heilbronn function under the C_2 criteria. The threshold $3.42 < L^* \leq 3.43$ replaces the value 3.6 of earlier versions, which is where the negative eigenvalue first exceeds double precision; the margin law $10^{-6.3L^2}$ of earlier versions holds only in a fixed finite basis; in converged bases the margin of ζ follows $10^{-(5.42e^L - 12.6)}$ [CE12].

zero of ζ and has an additional real zero at 0.6, and its detection by the reflected cross-energy, are in [CE3].

2.7. What does not hold. The following assertions of the author’s earlier preprints, or of earlier versions of the consolidated paper, are not valid, or not proved, and are not used here.

1. $C_2(N, \sigma, t) = 0$ does not imply $\sigma = \frac{1}{2}$. Finite crossings of the cross-energy occur off the critical line. The simplest example is $C_2(2, s) = 2 \operatorname{Re} 2^{-\bar{s}} = 2^{1-\sigma} \cos(t \log 2)$, which vanishes at $t = \pi/(2 \log 2)$ for every σ . At zeros, the crossing law of Part 1 [CE1] is a statement about the Hurwitz tail at an arbitrary point s : a zero ρ with $\beta \neq \frac{1}{2}$ and $|\gamma|$ large, if it existed, would have its own crossing at $n^*(\rho) = \gamma^2 + \frac{1}{3} + (2\beta - 1)(\frac{\gamma^2}{2(1-\beta)} - \frac{1}{6}) + O(\gamma^{-2})$.
2. The vanishing of $X_N^C - Y_N^C$ does not force $|X_N^C|^2$ and $|Y_N^C|^2$ to be linear in N . The continuous energy $|Y_N^C(s)|^2 = N^{2-2\sigma}/|1-s|^2$ is linear in N if and only if $\sigma = \frac{1}{2}$, but at every zero, whatever its abscissa, $X_N^C(\rho) - Y_N^C(\rho) \rightarrow 0$ and $|X_N^C(\rho)|^2 - |Y_N^C(\rho)|^2 = O_\rho(N^{-2\beta})$: the two energies agree at every zero, and their comparison does not locate it.
3. The multiplicative three-regime mechanism is withdrawn. The ratio $X_N(s)/Y_N(s)$ tends to 1 at every point of the strip, zeros included, so it carries no information on the zeros; only the additive discrepancy does ([CE1, Proposition 2.4]).
4. The crossing point at a zero on the line is $n^* = \gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} + O(\gamma^{-4})$, not $\gamma^2 + \frac{1}{4}$ as stated in [C1].
5. Uniqueness of the crossing. The claim of exactly one sign change of $C_2(\cdot, \rho)$ for $N \geq K|\gamma|$ (Part IV of the earlier series) rests on a sketch; what is proved here is uniqueness for

$n \geq |\gamma|^{3/2}$ at every zero on the line with $|\gamma| \geq t_0$ ([CE1, Proposition 3.6]) and, for $|\gamma| \geq t_0$ with t_0 not effective, on the whole range beyond the transition $N_{\text{last}} = |\gamma|/2\pi + \kappa\sqrt{|\gamma|} + O(1)$ ([CE2, Corollary 2.13], which inherits one lemma verified numerically); for small $|\gamma|$ the range between $|\gamma|/2\pi$ and $|\gamma|^{3/2}$ is numerical.

6. *The transition-law prediction “confirmed on 18 zeros up to $t = 20000$ ”* in [C1] is exact for 16 of the 18 zeros and off by one for the other 2.
7. *Weil’s form for the Davenport–Heilbronn function* becomes negative at $3.42 < L^* \leq 3.43$, not 3.6, and the law $10^{-6.3L^2}$ for the margin of ζ holds only in a fixed finite basis; in bases large enough to converge the margin follows $\log_{10} \lambda_{\min} \approx -(5.42e^L - 12.6)$ on $2 \leq L \leq 5$ [CE12]. (An intermediate fit $10^{-5.0L^{2.53}}$ on $1.25 \leq L \leq 3$ was not converged in the basis size.)

None of these points affects the exact identities or the theorems of the series. We do not prove the Riemann hypothesis.

3. DISCUSSION

The controls show which property of ζ each statement of the series uses.

1. *The Epstein obstruction is the sharpest limit.* Every property that C_2 sees holds for the Epstein function of $m^2 + 100n^2$, as does the C-transform representation. These properties are nonnegative coefficients, a simple pole, a functional equation and coefficient sums with a power saving. The Epstein function still has a real zero at $0.8133678\dots$, at which its cross-energy collapses (Theorem 2.2). An argument for RH must therefore use a property that the Epstein function lacks. The natural one is the Euler product. Section 2 shows that the convolution identities that express the Euler product in terms of partial sums are blind to s (Proposition 2.6), and that twists, superpositions and scale recursions do not supply it either.
2. *The two structures and their meeting point.* The Davenport–Heilbronn function shows that the functional equation alone does not suffice, and the Epstein function and the Beurling systems of [BDR] (see [CE17]) show that a pole, positivity and regular integers do not suffice either. What remains is the coupling of the functional equation with the Euler product. The cross-energy carries the first through the pole and the second only through the primes in its arithmetic version. Its reflection C_2^{FR} , which pairs s with $1 - s$ in a single partial-sum identity, carries both at once and is studied in [CE3]. The Euler–Caceres bound of [CE11] measures the Jensen moment of ζ against a smoothed Euler product.

3.1. Open problems.

- Which property of the Euler product, absent for the Epstein function of $m^2 + 100n^2$, can be expressed in terms of the partial sums X_N alone?

We do not prove the Riemann hypothesis.

APPENDIX A. NUMERICAL METHODS AND DATA

All computations use double precision in Python (NumPy, SciPy), except where a number of digits is stated; those use mpmath or Python’s `decimal` arithmetic. The zeros of ζ with $0 < \gamma < 600$ (341 ordinates) and the first 1000 ordinates were computed once and are reused throughout; where a statement depends on the precision of an ordinate ([CE2, Remark 2.12]), the ordinates are refined to 10^{-12} . ζ is evaluated by Euler–Maclaurin summation with Bernoulli corrections, and the Davenport–Heilbronn function f through Hurwitz zeta functions; its eight off-line zeros with $\gamma < 400$ are those listed in [CE17]. $\Lambda(n)$ is computed by sieve up to 10^8 (up to 10^9 for ψ). Every value in the tables was obtained by the script cited in its Status line, and every script prints the numbers quoted in the text.

- *Section 2.* The Epstein zeta function of $m^2 + 100n^2$ from the incomplete-gamma expansion, its real zero, the collapse table and the coefficients b_Q (`epstein/`); the superposition $\xi + \varepsilon\Lambda(\cdot, \chi_5)$ (`quantum/`); the convolution identities and polarization (`euler_identities/`, `polarization/`); the scale recursions, the Vaughan split, the coherence and Prony estimates and the 2×2 table (`scale_recursions/`, with the Beurling primes and generalized integers built as in [CE17]). The block-inheritance exponents and the Möbius-times-divisor exponents quoted after Remark 2.10 are from the author’s working notes.

How this work was produced. The cross-energy C_2 and the research program are the author’s, beginning with his preprints of 2017 and the series [C1]–[C8]. The author used Claude (Anthropic), an AI assistant, as a research and writing aid: to derive and check proofs, to write and run the Python computations, to test claims against the Davenport–Heilbronn control and against counterexamples, to search the literature, and to draft and edit the text and figures. In preparing this series, the material of the consolidated paper (version 3), of the earlier series [C1]–[C8] and of the author’s manuscript on the asymptotic balance of partial sums was inventoried result by result, regrouped into the seventeen parts of the series (Table 1 of [CE1]), and checked again; the corrections found in the process are stated where they apply and listed in Section 2. Several claims of the author’s earlier preprints were found to be incorrect and are not used. The author directed the work, checked the results and is responsible for the content.

Data availability. The scripts are included in the ancillary directory `anc/` of the source package, one subdirectory per topic as listed above. The directory `anc/lean/` contains a draft Lean 4/mathlib formalization of the pair identity and the strict part of [CE10, Theorem 3.2] (disc lemma), and of the finite core of Proposition 2.8 (a nonnegative trigonometric polynomial has positive mean, and the weighted prime sums are nonnegative with constant part $a_0 \sum w_p$); the prime number theorem asymptotic of Proposition 2.8 is not formalized. The files are a draft: they have not been compiled or checked by Lean, and no result of this paper depends on them (`anc/lean/README.md`). The earlier preprints [C1]–[C8] and the versions 1–4 of the consolidated paper are deposited on Zenodo under the DOIs listed in the references.

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