Riemann Zeta function – Nine propositions

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Abstract:

The Riemann Zeta function or Euler–Riemann Zeta function, ζ(s), is a function of a complex variable z that analytically continues the sum of the Dirichlet series:

$$\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$$

The Riemann zeta function is a meromorphic function on the whole complex z-plane, which is holomorphic everywhere except for a simple pole at z = 1 with residue 1. One of the most important advance in the study of Prime numbers was the paper by Bernhard Riemann in November 1859 called "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (On the number of primes less than a given quantity). In this paper, Riemann gave a formula for the number of primes less than x in terms the integral of 1/log(x), and also provided insights into the roots (zeros) of the zeta function, formulating a conjecture about the location of the zeros of $\zeta(z)$ in the critical line Re(z)=1/2.

The Riemann Zeta function is one of the most studied and well known mathematical functions in history. In this paper, we will formulate nine new propositions to advance in the knowledge of the Riemann Zeta function.

1. Nomenclature and conventions

a. ζ(z)=$\lim_{n\to\infty}\sum_{k=1}^{n} k^{-z}$ is the Zeta function of Riemann in the complex plane

b. z*: any non-trivial solution of the Zeta function verifying that ζ(z*)=0+i0. By default, a reference to zero of ζ(z) will refer to a non-trivial zero of ζ(z).

c. b(n) is the n^{th} zero of the Riemann function in the critical line x=1/2 in C. e.g. b1=14.134725…

d. α=Re(z) is the real part of z

e. ß=Im(z) is the imaginary part of z

f. If z=α+iß, we define Modulus(z)= $|z|^2 = \alpha^2+ß^2$

g. For notation simplification, all modulus of complex functions in this paper, such as $|\zeta(z)|^2$, $|X(z)|^2$ and $|Y(z)|^2$, with z=α+iß, that will be represented in the form of infinite series when $n \to \infty$, must be understood as functions in R over the variables α=Re(z), ß=Im(z) and n.

2. Constants used in this paper

2.1. Glaisher-Kinkelin constant

$$A = e^{\frac{1}{12}-\zeta'(-1)} = 1.28242712\ldots$$

2.2. Euler-Mascheroni constant:

$$\gamma = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{1}{k} - \int\frac{dm}{m}\right) = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{1}{k} - \ln(m)\right) = 0.57721566490\ldots$$

2.3. Stieltjes constants:

$$\gamma_n = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{(\ln k)^n}{k} - \int\frac{(\ln m)^n}{m}\,dk\right) = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1}\right)$$

3. The Riemann Zeta function ζ(s) in R

As defined in literature (Sondow et al, Ellinor et al, Andrews)

3.1. $\zeta(\mathbf{k}) = \sum_{j=1}^{\infty} \mathbf{j}^{-\mathbf{k}}$ converges for $\mathbf{k} \neq 1$

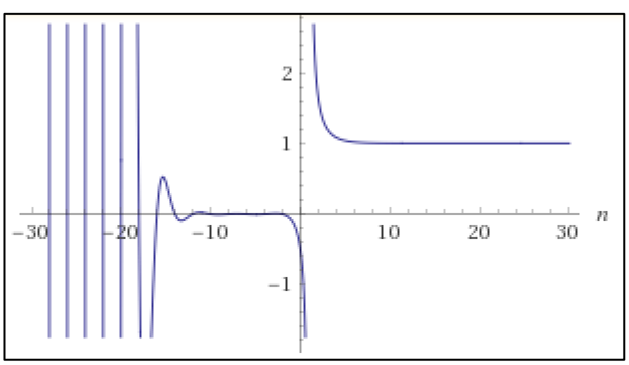

Figure 1. Riemann Zeta function in R

3.2. Euler Product Formula that ties $\zeta(k)$ with the distribution of prime numbers

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \prod_{p\ \mathrm{prime}} \frac{1}{1-p^{-s}}$$

Example for k=2

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{1}{1-2^{-2}} x \frac{1}{1-3^{-2}} x \frac{1}{1-5^{-2}} x \frac{1}{1-7^{-2}} x \ldots$$

3.3. Integral definition:

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{1}{e^x - 1} x^s \frac{dx}{x}$$

Where $\Gamma(s)$, is the Gamma function

3.4. Analytical continuation for:

Re(s)>0: [Dirichlet]

$$\zeta(s) = \frac{1}{s-1} \sum_{k=1}^{\infty} (\frac{n}{(n+1)^s} - \frac{n-s}{n\hat{}s}$$

0<Re(s)<1:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}$$

-k<Re(s) [Kopp, Konrad. 1945]:

$$\zeta(s) = \frac{1}{s-1} \sum_{k=1}^{\infty} \frac{k(k+1)}{2} (\frac{2k+3+s}{(k+1)^{s+2}} - \frac{2k-1-s}{k^{s+2}})$$

3.5. Laurent series at s=1:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=}^{\infty} \frac{(-1)^k \gamma_n}{k!} (s-1)^k)$$

where $\gamma_n$are the Stieltjes constants.

3.6. Hurwitz function $\zeta(k, z)$:

$$\zeta(k, z) = \sum_{j=0}^{\infty} (j+z)^{-k} = \sum_{j=z}^{\infty} j^{-k} \qquad \text{converges for } k > 1$$

3.7. Generalized Harmonic Function $H_n^{(k)}$:

$$H_n^{(k)} = \sum_{j=1}^{n} j^{-k} = \left(\frac{1}{1^k} + \frac{1}{2^k} + \cdots + \frac{1}{n^k}\right) \qquad \text{converges for } k > 1$$

3.8. ζ(s) converges for s>1 to the following values:

s	ζ(s)	Known ζ(s) representations over π
2	1.6449179	$\pi^2/6$
4	1.0823232	$\pi^4/90$
6	1.0173431	$\pi^6/945$
8	1.0040774	$\pi^8/9450$

Table 1. Values of ζ(s)

What happens with the odd values of s? Do they have also a representation in a close form?

$$\zeta(2\mathrm{n}+1)=\frac{\pi^{2n+1}}{k}$$

We are going to propose two different approaches to answer the question.

3.8.1. Solution 1: A close form for $\boldsymbol{\zeta(2n+1)=\frac{a}{b}\,C^{2n+1}}$

The problem is proposed as an optimization problem to minimize the function:

$$\sum_{n=1}^{\infty}\left|\zeta(2n+1)-\frac{a}{b}\,C^{2n+1}\right|^2$$

This function is the aggregated quadratic error of the approximations of $\zeta(2n+1)$ to $\left(\frac{a}{b}\,C^{2n+1}\right)$.

The result of this calculation is:

C=3.0677724318720092274489185949583964933828786704320291807 73...

And the values for $\zeta(2n+1)$:

(2n+1)	a	b	Error $\zeta(2n+1)-\frac{a}{b}C^{2n+1}$
ζ (3)	21635641	519653864	$2.9\,x\,10^{-16}$
ζ (5)	18604295	4875065106	$3.8\,x\,10^{-16}$
ζ (7)	9001873	22828841506	$3.8\,x\,10^{-16}$
ζ (9)	759823	18249431499	$3.6\,x\,10^{-16}$
ζ (11)	1	226381	0
ζ (13)	13555	28889967723	$6.6x\,10^{-16}$

Table 2: Values of $\zeta(2n+1)$ as a function of the C-constant

As we can see, C can be defined as:

[1] $\boldsymbol{C=[\zeta(11)*226381]^{\frac{1}{11}}}$ **[Caceres Proposition 1]**

3.8.2. Solution 2: An approximation for the values of $\zeta(s)$ in R

We can calculate that:

[2] $\lim_{s\to\infty}\left(\frac{\zeta(s)}{\zeta(s)+1}\right)^{1/s} = 1$

And:

[3] $\lim_{s\to\infty}\left(\frac{\zeta(s)}{\zeta(s)-1}\right)^{1/s} = 2$

Based on this expression, we can say that for s sufficiently large, we can represent $\zeta(s)$ as a multiple of π^s:

[4] $\zeta(s) = \frac{\pi^s}{K_s}$ with $K_s = (2^s - 1) * \frac{\pi^s}{2^s}$

with a very good approximation given by:

[5] $K_s^* = int\left((2^s - 1) * \frac{\pi^s}{2^s}\right) - 1$ where int(k) is the integer part of k.

The error between the K_s^* calculated and K_s actual is very small for s>4.

Some calculated values of K_s^* calculated and K_s actual:

s	Calculated	Actual
2	6.0	6.0
3	26.0	25.8
4	90.0	90.0
5	295.0	295.1
6	945.0	945.0
7	2,995.0	2,995.3
8	9,450.0	9,450.0
9	29,749.0	29,749.4
10	93,555.0	93,555.0
11	294,059.0	294,058.7

Table 3. Values of K_s^* calculated and K_s actual

3.9. We can use 3.8.2 to propose the following approximation for $\zeta(s)$ [Caceres, Pedro. 2017]:

[6] $\mathbf{CZ(s) \approx \frac{1}{1-\pi^{-s}-2^{-s}}}$ **[Caceres Proposition 2]**

	s=3	s=4	s=10	s=14
ζ(s)Actual	1.20206	1.0823	1.000994	1.0000612
ζ(s)Approx	1.18659	1.0784	1.000988	1.0000611

Table 4: Values of the approximation CZ(s)

Graphically:

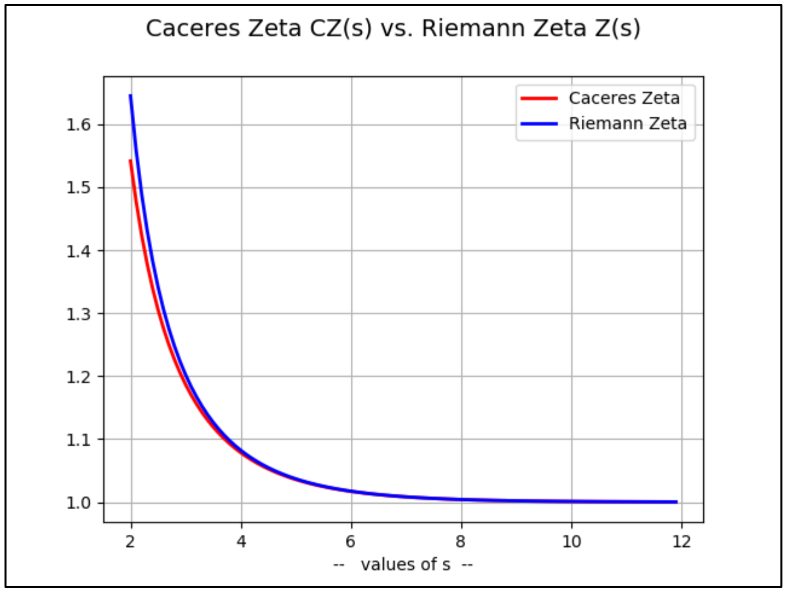

Figure 2. Caceres' approximation for the Riemann Zeta function in R

3.10. PrimeZeta function

One could ask what would change in Euler's expression if we did the infinite sum also on the prime numbers instead of all the natural numbers? As the number of primes is a fraction of the naturals, this infinite sum should be smaller. The PrimeZeta (PZ(s)) function can be represented by:

[7] $$\zeta^{p}\ (s) = \ \sum_{p\ prime}^{\infty} \frac{1}{p^{s}}$$

PrimeZeta(s) has a complex behavior in the interval [0,1) and it has a pole in every s=1/k where k is any square free integer (2,3,5,6,7,8,10,11,13...) OEIS A005117

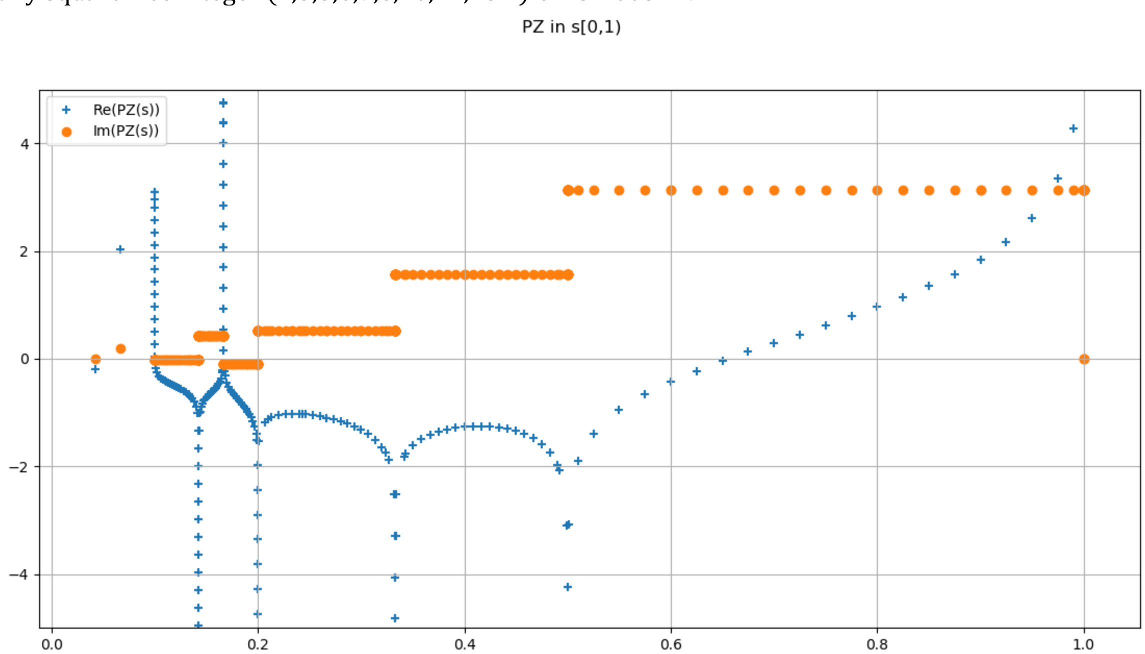

Figure 3: Representation of PrimeZeta(s) for s ∈ (0,1)

PrimeZeta(s) representation for s>1 shows how this function is very close to the values of ln(ζ(s)) and ln(CZ(s)). In the following chart we represent e^(PrimeZeta(s)), ζ(s), and CZ(s).

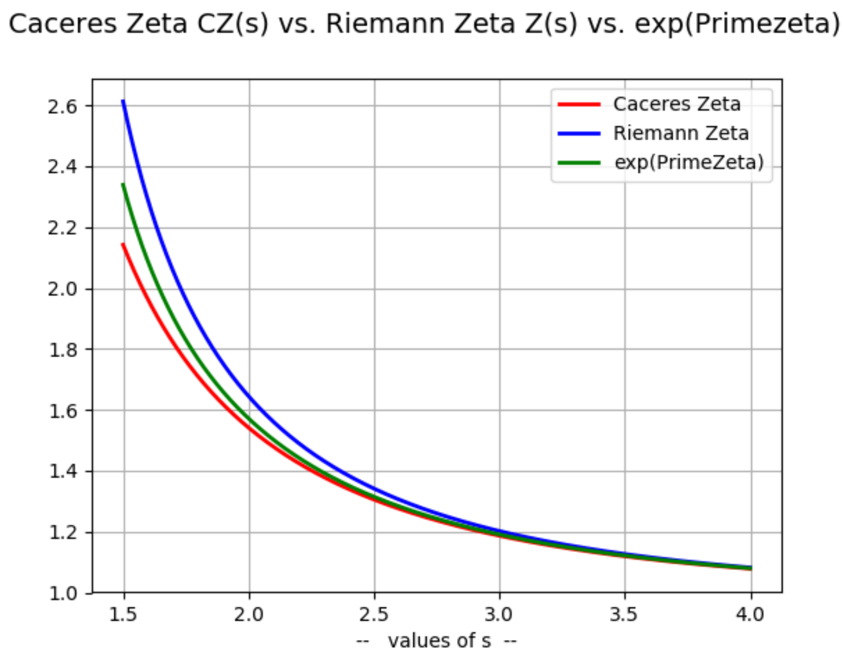

Figure 4: Comparison of Caceres Zeta, Riemann Zeta and exp(PrimeZeta)

One interesting finding is the way this partial sum over primes evolves as s increases.

Let's call:

[8] $$\theta(s) = \frac{\zeta(s)}{\zeta^p(s)}$$

with ζ(s) the regular zeta function as a sum over n naturals and $\zeta^p(s)$ the zeta function as a sum over the first n primes, then we can obtain:

[9] $$\lim_{s\to\infty} \theta(s)/\theta(s-1) = 2$$

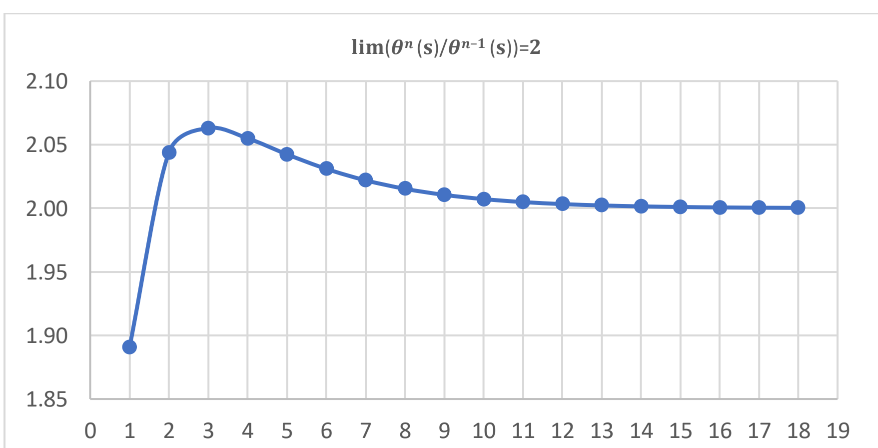

Figure 5: Representation of $\lim_{s\to\infty} \theta(s)/\theta(s-1)$

We know that:

[10] $$\lim_{s\to\infty}\frac{\zeta(s)}{\zeta(s-1)} = 1$$

So, we can conclude that:

[11] $$\lim_{s\to\infty}\frac{\zeta^p(s)}{\zeta^p(s-1)} = \frac{1}{2}$$

The sum over all primes of $\zeta^p(s)$ decreases by 50% for each unit increment of s.

4. A set of Constants involving $\boldsymbol{\zeta}(\mathbf{s})$

4.1. Lemma 1:

$$\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = 1$$

Proof:

$$\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = \lim_{n,m\to\infty}\sum_{j=2}^{n}\sum_{k=2}^{m} j^{-k} =$$

$$= \lim_{n,m\to\infty}\sum_{j=2}^{n}\frac{j^{-m-1}(j^m - j)}{j-1} = \lim_{n\to\infty}\sum_{j=2}^{n}\frac{j^{-1}}{j-1} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 1$$

4.2. Lemma 2:

$$\sum_{j=2}^{\infty}(\zeta(j)-1) = 1$$

Proof:

$$\sum_{j=2}^{\infty}(\zeta(j)-1) = \sum_{j=2}^{\infty}(\sum_{k=1}^{\infty} j^{-k} - 1) = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty} j^{-k} = 1$$

Per lemma 1

4.3. Lemma 3:

$$\lim_{k\to\infty}\zeta(k) = 1$$

Proof:

$$\zeta(k) = \sum_{j=1}^{\infty} j^{-k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + \cdots$$

$$\lim_{k\to\infty}\zeta(k) = \lim_{k\to\infty} 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + \cdots = 1$$

4.4. Theorem 1.0: [Caceres Proposition 3]

The infinite sums ∑_(j = 1)^∞▒[ζ(u ∗ k ± n) − ζ(v ∗ k ± m)] converge to a value in the interval (-1 ,1) for all $u \geq 1, v \geq 1, n, m \in N$ such that $(u * k \pm n) > 1$ and $(v * k \pm m) > 1$.

Proof:

$$\sum_{j=1}^{\infty}[\zeta(uj \pm n) - \zeta(vj \pm m)] =$$

$$= \zeta(u \pm n) - \zeta(v \pm m) + \zeta(2u \pm n) - \zeta(2v \pm m) + \cdots = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{j^{uk\pm n}} - \frac{1}{j^{vk\pm m}}\right)$$

The largest differences in value between the parameters $uk \pm n$ and $vk \pm m$ occurs when $u = 1, n = 1,$ and $v =$ infinity.

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{j^{uk\pm n}} - \frac{1}{j^{vk\pm m}}\right) \leq \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{j^{k+1}}\right) - 1 = \sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j^{k+1}} = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{j^{k}} = 1$$

As per lemmas 1,2,3.

Following the same logic, we can also say that:

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{1}{j^{uk\pm n}} - \frac{1}{j^{vk\pm m}}\right) \geq -1$$

Therefore, the infinite sums $\sum_{j=1}^{\infty}[\zeta(uk \pm n) - \zeta(vk \pm m)]$ converge to a value in the interval (-1, 1) for all $u, v, n, m \in N$ such that $(u * k \pm n) > 1$ and $(v * k \pm m) > 1$ for all $j \in N$

Similarly, we can propose and formulate the following Theorem.

4.5. Theorem 2.0:

The infinite sums $\sum_{j=1}^{\infty}[\zeta(u * k \pm n) - \zeta(v * k \pm m)]$ converge to a value in the interval (-∞,∞) for all $u \geq 1, v \geq 1, n, m \in R$ such that $(u * k \pm n) > 1$ and $(v * k \pm m) > 1$

Example:

$$\sum_{j=1}^{\infty}\zeta(1.1 * j + 2) - \zeta(1.2 * j + 0.1) = -3.14132 \ldots$$

$$\sum_{j=1}^{\infty}\zeta(1.001 * j) - \zeta(1.1 * j + 1) = 999.733 \ldots$$

4.6. These theorems let us define a set of infinite constants of the type:

$$\mathbf{CZ}^{(\mathbf{q})}_{\mathbf{u,v,n,m}} = \sum_{\mathbf{j=q}}^{\infty} [\boldsymbol{\zeta}(\mathbf{uj} \pm \mathbf{n}) - \boldsymbol{\zeta}(\mathbf{vj} \pm \mathbf{m})] = \text{constant} \qquad \text{[Caceres Proposition 4]}$$

(By default, when q=1, q will be omitted)

Some of the CZ constants we will use through the paper are:

$$CZ_{2,0,2,1} = \sum_{j=1}^{\infty} [\zeta(2j) - \zeta(2j+1)] = 0.5$$

$$CZ_{4,0,4,-2} = \sum_{j=1}^{\infty} [\zeta(4j) - \zeta(4j-2))] = -0.5766744746\ldots$$

$$CZ_{4,1,4,-1} = \sum_{j=1}^{\infty} [\zeta(4j+1) - \zeta(4j-1))] = -0.171865985524\ldots$$

The following table shows some values of the $CZ_{u,v,1,0}$ constants:

n=1, m=0	v=2	v=3	v=4	v=5	v=200
u=2	-0.500000	0.028310	0.163337	0.212046	0.250000
u=3	-0.658193	-0.129882	0.005144	0.053853	0.091800
u=4	-0.719330	-0.182622	-0.047596	0.001113	0.039067
u=5	-0.732147	-0.203836	-0.068810	-0.020101	0.017853
u=200	-0.750000	-0.221689	-0.086663	-0.037954	0.000000

Table 5: Values of constants of the form $CZ_{u,v,m,n}$

These constants appear in sums and products of infinite Dirichlet-like functions.

4.6.1. Example for f(z)= $[\mathbf{re^{i\theta}}]^{-\mathbf{k}}$ **with z=**$\mathbf{re^{i\theta}}$

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty}[re^{i\theta}]^{-k} = \lim_{n\to\infty} \frac{(\psi^{(0)}(n+1-e^{-i\boldsymbol{\theta}}) - \psi^{(0)}(n+1) - \psi^{(0)}(2-e^{-i\boldsymbol{\theta}}) - \gamma + 1}{e^{i\boldsymbol{\theta}}})$$

Values for some θ:

Θ	*Sum Matrix* $[re^{i\theta}]^{-k}$	Numeric result
π/2	$i\left(-1+\gamma+\psi^{(0)}(2+i)\right)$	$CZ_{4,0,4,-2} - CZ_{4,1,4,-1}\, i$
Π	½	$CZ_{2,0,2,1}$
-π/2	$-i\left(-1+\gamma+\psi^{(0)}(2-i)\right)$	$CZ_{4,0,4,-2} + CZ_{4,1,4,-1}\, i$

Table 6

4.6.2. Example for f(z)= $1+\mathbf{re^{i\theta}}]^{-k}$, with z=re$^{i\theta}$

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty}[1+re^{i\theta}]^{-k} = \lim_{n\to\infty} \frac{(-\psi^{(0)}(n+1-e^{-i\theta})+\psi^{(0)}(n+1)+\psi^{(0)}(2+e^{-i\theta})+\gamma-1}{e^{i\theta}})$$

Θ	*Sum Matrix* $[1+re^{i\theta}]^{-k}$
0	$CZ_{2,0,2,1}$=1/2
π/2	$-i\left(-1+\gamma+\psi^{(0)}(2-i)\right)=CZ_{4,0,4,-2}+CZ_{4,1,4,-1}\,i$
Π	$2*CZ_{2,0,2,1}=1$
-π/2	$i\left(-1+\gamma+\psi^{(0)}(2+i)\right)=CZ_{4,0,4,-2}-CZ_{4,1,4,-1}\,i$

Table 7

4.6.3. Example for f(z)= $1-\mathbf{re^{i\theta}}]^{-k}$, with z=re$^{i\theta}$

$$\sum_{r=2}^{\infty}\sum_{k=2}^{\infty}[1-re^{i\theta}]^{-k} = \lim_{n\to\infty} \frac{(\psi^{(0)}(n+1-e^{-i\theta})-\psi^{(0)}(n+1)-\psi^{(0)}(2-e^{-i\theta})-\gamma+1}{e^{i\theta}})$$

$$= \frac{\left(\psi^{(0)}(e^{-i\theta}-2\right)-\gamma+1}{e^{i\theta}} \quad \textit{when } e^{i\theta}\neq 0$$

Θ	*Sum Matrix* $[1-re^{i\theta}]^{-k}$
0	$2*CZ_{2,0,2,1}=1$
π/2	$i\left(-1+\gamma+\psi^{(0)}(2+i)\right)=CZ_{4,0,4,-2}-CZ_{4,1,4,-1}\,i$
Π	$CZ_{2,0,2,1}=1/2$
-π/2	-i $\left(-1+\gamma+\psi^{(0)}(2-i)\right)=CZ_{4,0,4,-2}+CZ_{4,1,4,-1}\,i$

Table 8

6. Function in $C_2(x,a,b)$ →R with zeros at a=1/2 and b=Im(z) such that ζ(z)=0.

Let's define the function $C_2(x,a,b)$ in R such that:

[12] $$C_2(x,a,b)=\ 2*x^{-a}*\left(\sum_{j=1}^{x-1} j^{-a}*\cos\left(b*(\ln\left(\frac{x}{j}\right))\right)\right)$$

With the following wave representation:

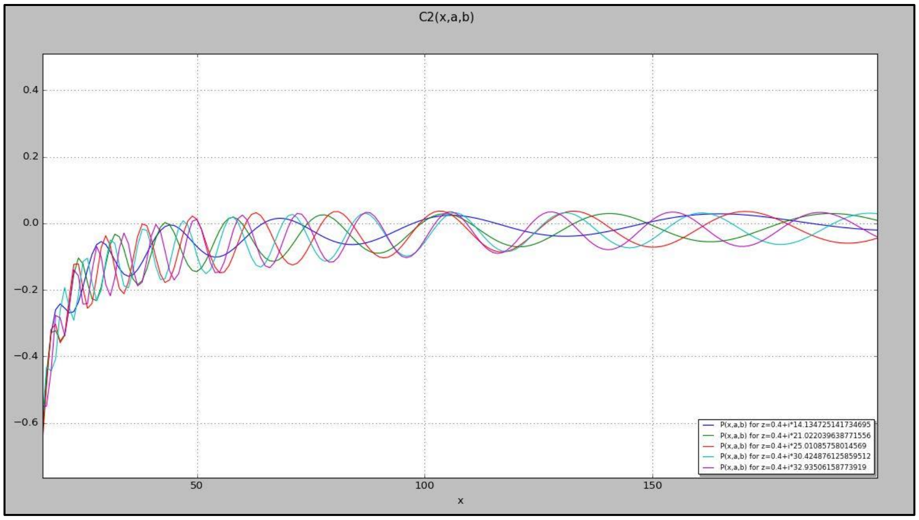

Figure 6. $C_2(x,a,b)$ *for a=0.4 and several b*

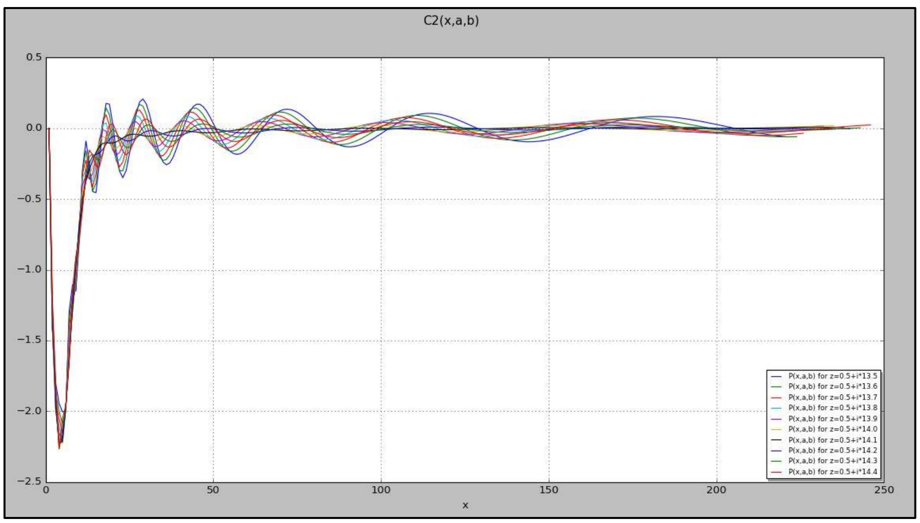

Figure 7. $C_2(x,a,b)$ *for a=0.5 and several b*

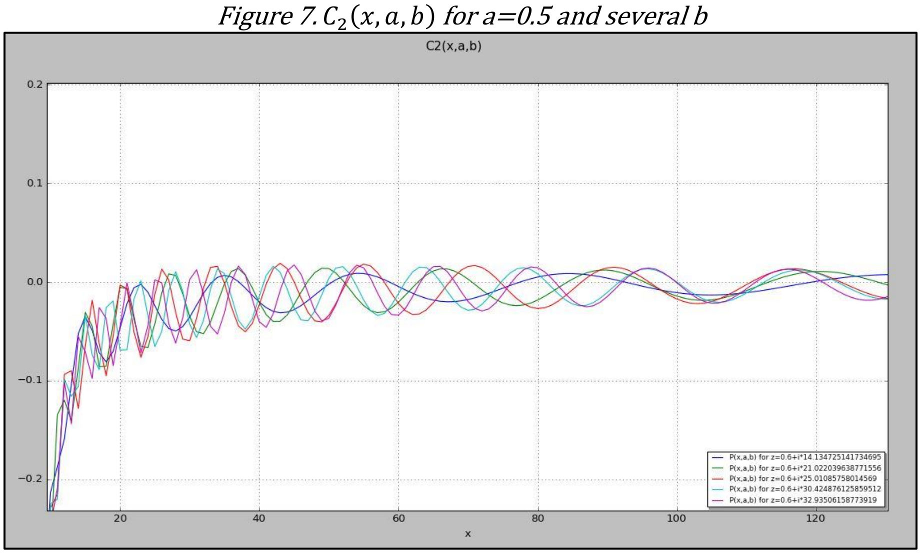

Figure 8. $C_2(x,a,b)$ *for a=0.6 and several b*

As a wave, $C_2(x,a,b)$ can have one or more zeros. For $C_2(x,a,b)$ to have only one zero, it must cross the axis y=0 only once, which means that the wave collapses to a polynomial line. A numeric method has been created and coded to find the values of (x,a,b) such that $C_2(x,a,b)$=0. The following table shows an example of those calculated values:

Alfa	Beta	Number of Zeros	Zero at X=
0.4	14.1	5	
0.4	14.2	5	
0.4	14.3	5	
0.4	14.4	5	
0.5	14.07	5	
0.5	14.08	5	
0.5	14.09	5	
0.5	14.1	4	
0.5	14.11	4	
0.5	14.12	3	
0.5	14.13	1	200
0.5	14.14	3	
0.5	20.97	11	
0.5	20.98	11	
0.5	20.99	11	
0.5	21	9	
0.5	21.01	5	
0.5	21.02	1	442
0.5	21.03	3	
0.5	24.96	16	
0.5	24.97	16	
0.5	24.98	15	
0.5	24.99	11	
0.5	25	7	
0.5	25.01	1	626
0.5	25.02	6	
0.5	25.03	10	

Table 9. Number of Zeros of $C_2(x,a,b)$ for different values of a,b

The calculations for $a \in (0,1)$ and $b \in [1,100]$ only found single zeros for $C_2(x,a,b)$ for values of $a = 0.5$ as shown in the following table that summarizes the single zeros found in those intervals:

Values (x,a,b) | C2(x,a,b)=0 SINGLE		
x*	a*	b*
200.1000	0.5000	14.1368
442.2000	0.5000	21.0226
625.8000	0.5000	25.0110
926.0000	0.5000	30.4261
1085.0000	0.5000	32.9355
1413.0000	0.5000	37.5866
1674.6000	0.5000	40.9188
1877.5000	0.5000	43.3272
2304.8000	0.5000	48.0057

Table 10. Showing only the first single Zeros of $C_2(x,a,b)$

It can be observed that:

$$if\ \ C_2(x,a,b) = 0 \rightarrow$$

$$a = 1/2$$

$$b = \ Im(z) \qquad with\ \zeta(z) = 0$$

$$\text{(a, b) are the Non} - \text{Trivial Zeros of } \zeta(\text{z}) \text{ in the critical line.}$$

$$x = b^2 + (1-a)^2$$

And the calculated values of $\lim_{x\to\infty} C_2(x,a,b)$ for the values of (a,b) from Table 10 are:

Values (x,a,b) | C2)x,a,b=0			Limit (C2(x,a,b))
x	a	b	when x->∞
200.1000	0.5000	14.1368	0.0050
442.2000	0.5000	21.0226	0.0023
625.8000	0.5000	25.0110	0.0016
926.0000	0.5000	30.4261	0.0011
1085.0000	0.5000	32.9355	0.0009
1413.0000	0.5000	37.5866	0.0007
1674.6000	0.5000	40.9188	0.0006
1877.5000	0.5000	43.3272	0.0005

Table 11. Limit of $C_2(x,a,b)$ for b in Table 10 and x->∞

Values (x,a,b) | C2(x,a,b)=0			Limit (C2(x,a,b))	
x	a	b	when x->∞	Known Zero
200.1000	0.5000	14.1368	0.0050	14.1347
442.2000	0.5000	21.0226	0.0023	21.0220
625.8000	0.5000	25.0110	0.0016	25.0109
926.0000	0.5000	30.4261	0.0011	30.4249
1085.0000	0.5000	32.9355	0.0009	32.9351
1413.0000	0.5000	37.5866	0.0007	37.5862
1674.6000	0.5000	40.9188	0.0006	40.9187
1877.5000	0.5000	43.3272	0.0005	43.3271
2304.8000	0.5000	48.0057	0.0004	48.0052
2477.7000	0.5000	49.7740	0.0004	49.7738

Table 12. Comparing "b" calculated with known zeros of ζ(z)

[Caceres Proposition 5] $\boldsymbol{C_2(x,a,b)}$ **has the following special properties for all (a,b) such that ζ(a+bi)=0.**

if $\quad S = \frac{1}{b^2+1/4}$

$$\boldsymbol{C_2(x,a,b) = 0\ when\ x = \frac{1}{S},\ a = \frac{1}{2},\ \ b = Im(z^*)\ \text{with z}^*\ \text{a non} - \text{trivial zero of}\ \zeta(z)}$$

$$\boldsymbol{\lim_{n\to\infty} C_2(x,1/2,b) = S}$$

Graphically:

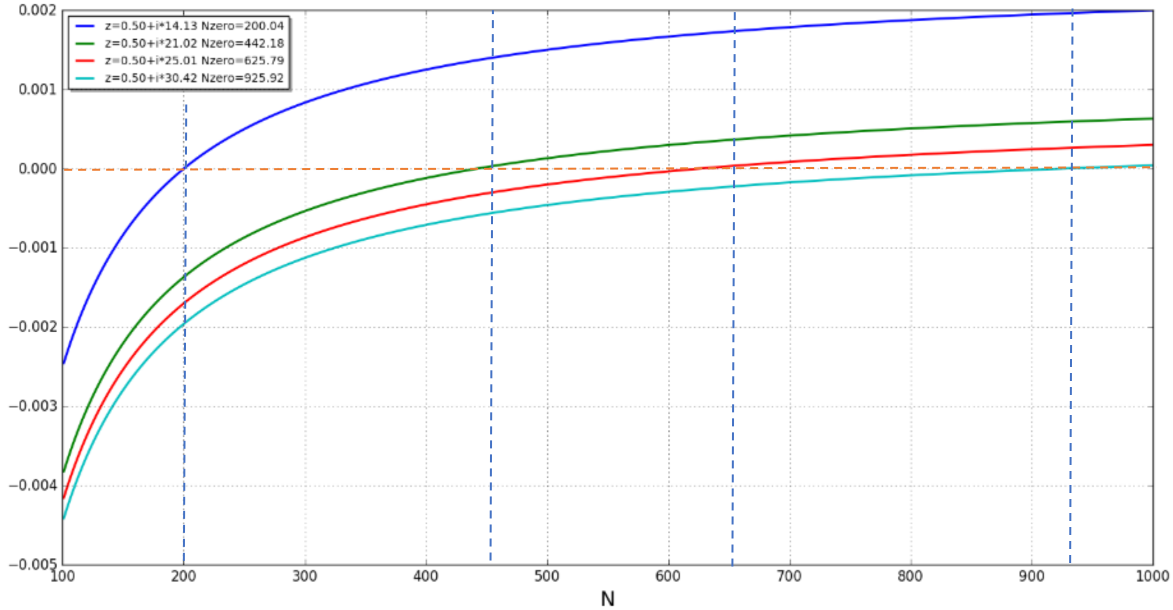

Figure 9. $C_2(x, 1/2, b)$ *such that ζ(1/2+bi)=0*

7. An analytic continuation for *ζ(z), z∈ C, Re(z)>0*

Let's define the C-transformation of a function f(x) as:

[13] $C_n\{f\} = \sum_{k=1}^{n} f(k) - \int_0^n f(k)dk$

And let's call $C\{f\} = \lim_{n\to\infty} C_n\{f\}$ the C-transformation values.

The C-values for $f(x) = \frac{1}{x^z}$ for z≠1, are equal to $\zeta(z)$:

[14] $C_n\{f\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{\mathrm{k}^{\mathrm{z}}} - \int_0^n \frac{dk}{\mathrm{k}^{\mathrm{z}}}$

Applying the exponential expression to the power of *k*∈ R to a complex number *z*∈ C:

[15 $k^{-z} = k^{\alpha}\,[\cos(ß * \ln(k)) + i\,(\sin(ß * \ln(k))]$

And:

[16] $\int_1^n \frac{1}{k^z}\,\mathrm{dk} = \frac{1}{(1-\alpha)-iß}(n^{(1-\alpha)-ßi})$

Or:

[17] $\int n^{-z}\,\mathrm{dn} = (n^{(1-\alpha)}\,[\cos(ß * ln(n) - i\,sin(ß * ln(n)]) * \frac{[(1-\alpha)+iß]}{[(1-\alpha)^2+ß^2]}$

We can now express the real and imaginary components of $C_n\{f\}$ as:

[18] $Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}$ (cos(ß $*$ ln(k))-

$- \frac{1}{[(1-\alpha)^2+ß^2]} (n^{(1-\alpha)}$ [(1-α)*cos(ß*ln(n))+ß* sin(ß*ln(n))])

[19] $Im(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}$ (sin(ß $*$ ln(k))+

$+ \frac{1}{[(1-\alpha)^2+ß^2]} (n^{(1-\alpha)}$ [ß*cos(ß*ln(n))-(1-α)*sin(ß*ln(n))])

The values of $C_n\{f\}$ are very close to the values of $\zeta(z)$ when $Re(z) \geq 2$ as we can see in the following table:

α	ß	$C_n\{f\}$	$\zeta(z)$	$\lvert C_n\{f\} - \zeta(z)\rvert$
2	0	1.644934068	1.654934067	$< 10^{-8}$
2	1	1.150355702 + 0.437530865 i	1.150355703+0.437530866 i	$< 10^{-8}$
3	0	1.202056903	1.202056903	$< 10^{-9}$
3	1	1.107214408+0.148290867 i	1.107214408+0.148290867 i	$< 10^{-9}$

Table 13. Values of $C_n\{f(n) = k^{-z}\}$ for Re(z)>1

The error $C_n\{f\}$-$\zeta(z)$ grows significantly in the critical strip for 0<Re(z)<1. The following table contains some examples:

α	ß	$C_n\{f\}$	$\zeta(z)$	$\lvert C_n\{f\} - \zeta(z)\rvert$
0.0	0	-1.0	-0.5	0.5
0.2	2	0.399824505+0.322650799 i	0.360103 + 0.266246 i	> 0.05
0.7	0	-2.777900606	-2.7783884455	$> 10^{-4}$
0.8	1	0.374712821-0.886329153 i	0.37487366-0.8864126 i	$< 10^{-4}$

Table 14. Values of $C_n\{f(n) = k^{-z}\}$ for 0≤Re(z)<1

To understand better the value of the difference $C_n\left\{\frac{1}{k^z}\right\}$-$\zeta(z)$, let's plot it for $\alpha \in [0,1)$ and ß $= 0$:

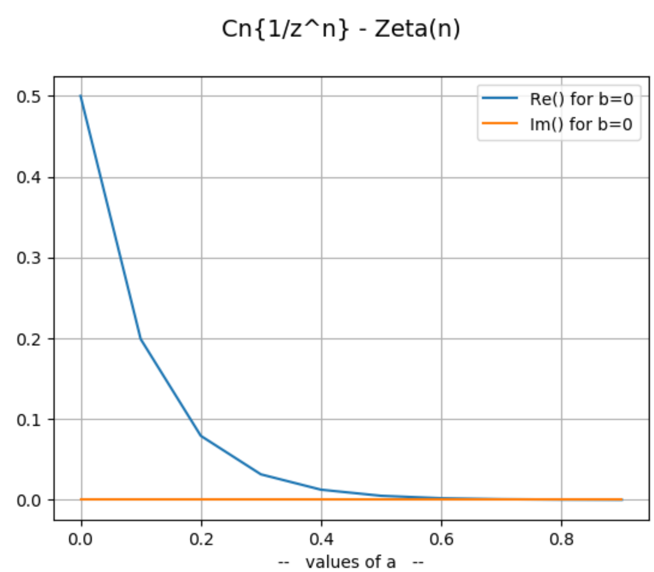

Figure 10

Similar exponential charts occur for all values of $a \in [0,1)$ for any given value of b.

Let's plot now $C_n\left\{\frac{1}{z^n}\right\}$-$\zeta(z)$, for ß ∈ [0,10) and $\alpha = 0$:

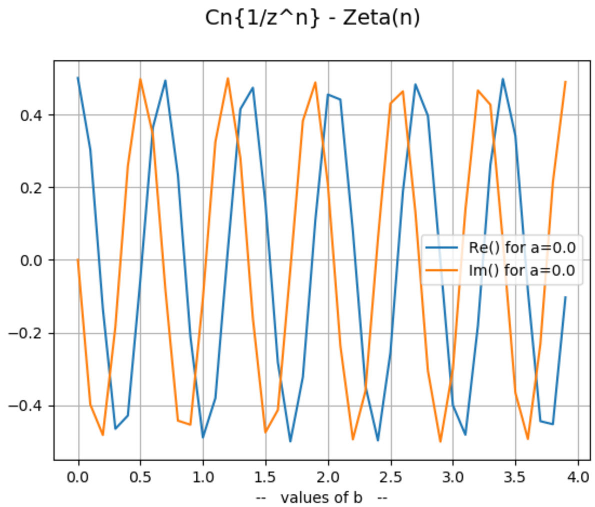

Figure 11

Similar sine charts occur for all values of ß ∈ [0,1) for any given value of α.

Based on these two distributions of errors, we are proposing corrections to $C_n\left\{\frac{1}{k^z}\right\}$ to eliminate the errors. This will allow us to define $C_n^*\left\{\frac{1}{k^z}\right\}$ also as an analytic continuation for $\zeta(z)$ in 0≤Re(z)<1. The numeric research conducted has determined that the best approximation for the errors are:

[20] $\mathrm{Re}[C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)] = \frac{1}{2}\, n^{-a} * \cos(ß * \ln(\mathrm{n})) + \mathrm{O}(\frac{1}{\mathrm{n}})$

[21] $\mathrm{Im}[C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)] = \frac{1}{2}\, n^{-a} * \sin(ß * \ln(\mathrm{n})) + \mathrm{O}(\frac{1}{\mathrm{n}})$

With O(1/n)->0 when n->∞.

Now, we can define the analytic continuation for $\zeta(z)$ in 0≤Re(z)<1 as the following function:

7.1. [Caceres Proposition 6: An analytic continuation for $\zeta(z)$ for Re(z)>0]

[22] **ζ(z) = X(z) - Y(z)**, where:

[23] $\mathbf{X(z)} = \;(\sum_{k=1}^{n} k^{-\alpha}\,(\cos(ß * \ln(k)) + \frac{1}{2}\, n^{-\alpha}\cos(ß\ln(n)) +$
$+\, i * (\sum_{k=1}^{n} k^{-\alpha}\,(\sin(ß * \ln(k)) + \frac{1}{2}\, n^{-\alpha}\,\sin(ß\ln(n)))$

[24] $\mathbf{Y}(\mathbf{z}) = \mathrm{n}^{(1-\alpha)}\, \frac{1}{[(1-\alpha)^2 + ß^2]}\, [\big((1-\alpha) * \cos(ß\ln(\mathrm{n})) + ß * \sin(ß\ln(\mathrm{n}))\big) +$
$+\, \mathrm{i}\;\,(ß * \cos(ß\ln(\mathrm{n})) - (1-\alpha) * \sin(ß\ln(\mathrm{n})))]$

Let's compare some examples of the values of X(z)-Y(z) vs. ζ(z) for Re(z)>0:

z= 0 +j* 0
Zeta(z) = -0.5 + i* 0.0 X(z)-Y(z) = -0.5 +i* 0.0 ---> Error = 0.0 +i* 0.0
z= 0.2 +j* 2
Zeta(z) = 0.360102590022591 + i* -0.266246199765574 X(z)-Y(z) = 0.360102741838091 +i* -0.266246128959438 ---> Error= -1.51815500282204e-7 +i* -7.08061368426272e-8
z= 0.4 +j* 0
Zeta(z) = -1.13479778386698 + i* 0.0 X(z)-Y(z) = -1.1347977871726 +i* 0.0 ---> Error= 3.30561999994927e-9 +i* 0.0
z= 0.8 +j* 7
Zeta(z) = 1.02504872765481 + i* 0.33812156136926 X(z)-Y(z) = 1.02504872814145 +i* 0.338121561040815 ---> Error= -4.86640283625661e-10 +i* 3.28445437514091e-10

Table 15

The highest error for α∈ [0,1), ß ∈ [0,100], n=10^6 is $8x10^{-6}$.

We can calculate the modulus of this complex expressions for X(z) and Y(z).

7.2. Modulus $|Y(z)|^2$

[25] $$|Y(z)|^2 = [(n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [(1-\alpha) * cos(ß * ln(n)) + ß * sin(ß * ln(n))])2 + (n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2+ß^2]} [ß * cos(ß * ln(n)) - (1-\alpha) * sin(ß * ln(n))])2]$$

[26] $$|\boldsymbol{Y(z)}|^2 = \boldsymbol{n^{2(1-\alpha)}} * \frac{\mathbf{1}}{[ß^2+(1-\alpha)^2]}$$

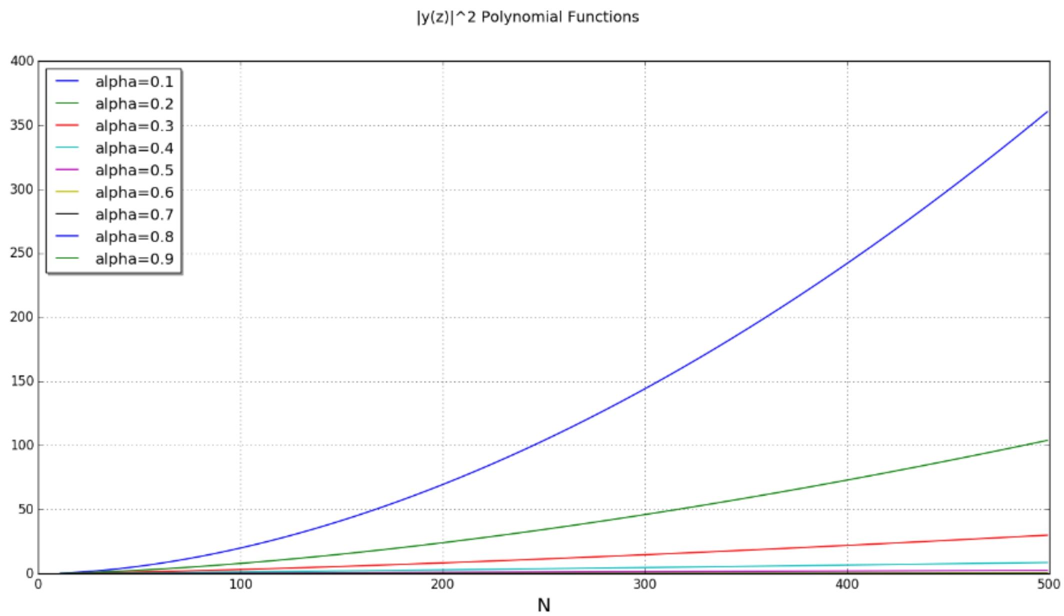

Figure 12: $|Y(z)|^2$ *has a polynomial representation*

7.2.1. Lemma: $|Y(z)|^2$ is a straight line if and only if α=1/2

The slope of $|y(z)|^2$ with respect to n is given by:

[27] $slope(|y(z)|^2) = d(|y(z)|^2)/dn$

Which equals to:

[28] $slope(|y(z)|^2) = 2(1-\alpha)\, n^{1-2\alpha} * \frac{1}{[ß^2+(1-\alpha)^2]}$

$|y(z)|^2$ can only be a line when the slope is constant, which can only happen if and only if (1-2α)=0 , therefore:

$|y(z)|^2$ is a line if and only if α=1/2

7.2.2. Conclusion: On the function $|Y(z)|^2$ for Re(z)=1/2 and $n \to \infty$:

We have calculated that for all $z \in C$:

- ⇨ the slope $|Y(z)|^2$ is constant if and only if α=1/2
- ⇨ and slope $|Y(z)|^2 = \frac{1}{[ß^2+1/4]}$ for with $z = \frac{1}{2} + iß$

7.3. Modulus $|X(z)|^2$

[29] $|X(z)|^2 = (\frac{1}{2} n^{-\alpha} \cos(ß \ln(n)) + \sum k^{-\alpha} cos(ß\, ln(n))^2 + (\frac{1}{2} n^{-\alpha} \sin(ß \ln(n)) + \sum k^{-\alpha} sin(ß\, ln(n))^2$

[30] $|X(z)|^2 = \frac{1}{4} n^{-2\alpha}(\cos^2(ß \ln(n)) + \sin^2(ß \ln(n))) + \sum k^{-\alpha} cos(ß\, ln(n))^2 + \sum k^{-\alpha} sin(ß\, ln(n))^2 +$
$n^{-\alpha}[\cos(ß \ln(n)) * \sum k^{-\alpha} cos(ß\, ln(k))] + n^{-\alpha}[\sin(ß \ln(n)) * \sum k^{-\alpha} sin(ß\, ln(k))] =$

[31] $|X(z)|^2 = \frac{1}{4} n^{-2\alpha} + \sum_{k=1}^{n} \sum_{1}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\, (ln\left(\frac{k}{j}\right)\right) +$
$n^{-\alpha}[\cos(ß \ln(n)) * \sum k^{-\alpha} cos(ß\, ln(k))] + n^{-\alpha}[\sin(ß \ln(n)) * \sum k^{-\alpha} sin(ß\, ln(k))] =$

[32] $|X(z)|^2 = \frac{1}{4} n^{-2\alpha} + \sum_{k=1}^{n} \sum_{1}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\, (ln\left(\frac{k}{j}\right)\right) +$
$n^{-\alpha} * \sum k^{-\alpha} \left[cos\left(ß\, ln\left(\frac{k}{n}\right) + \cos(ß * \ln(kn))\right)\right] +$
$n^{-\alpha} * \sum k^{-\alpha} \left[cos\left(ß\, ln\left(\frac{k}{n}\right) - \cos(ß * \ln(kn))\right)\right) =$

[33] $\boldsymbol{|X(z)|^2 = \frac{1}{4} n^{-2\alpha} + \sum_{k=1}^{n} \sum_{1}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\, (ln\left(\frac{k}{j}\right)\right) + 2\, n^{-\alpha} * \sum k^{-\alpha} \cos(ß * \ln\left(\frac{k}{n}\right))}$

Let's represent $|X(z)|^2$ graphically. We will prove that:

- $|x(n)|^2$ is a wave that converges when $n \to \infty$ and α>1 (Fig. 13)
- $|x(n)|^2$ is a wave that does not converge when $n \to \infty$ and α<1 (Fig. 14)
- $|x(n)|^2$ is a wave that collapses to a line when $n \to \infty$ and α=1/2 and ß=Im($\zeta(z^*)$) (Fig. 15)

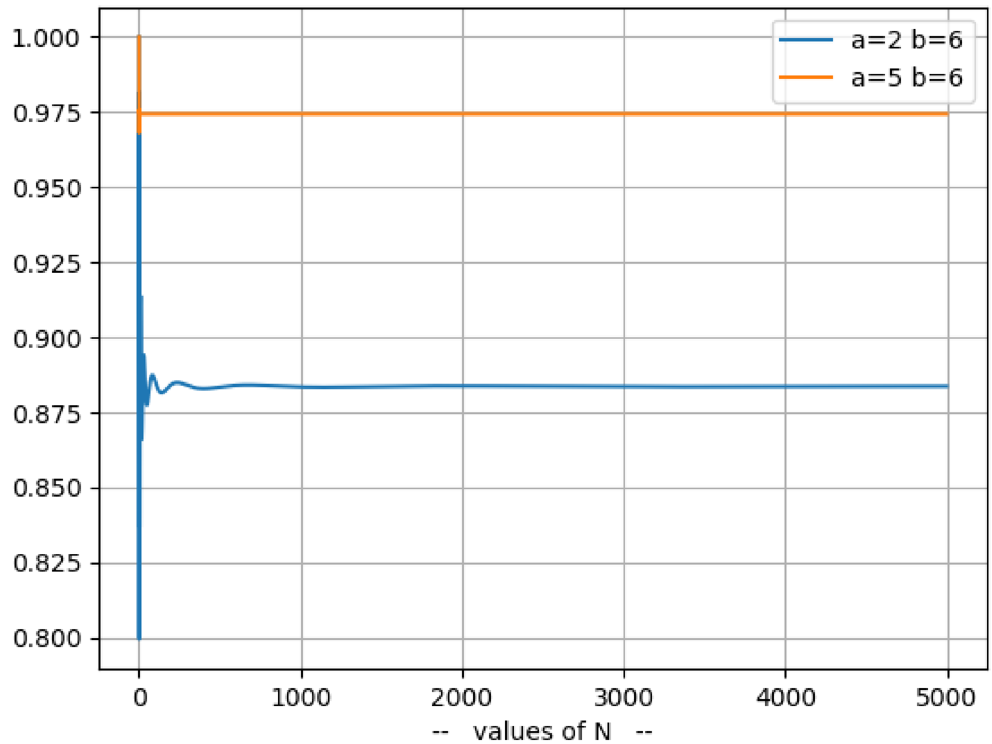

Figure 13: $|X(z)|^2$ *for* $\alpha>1$

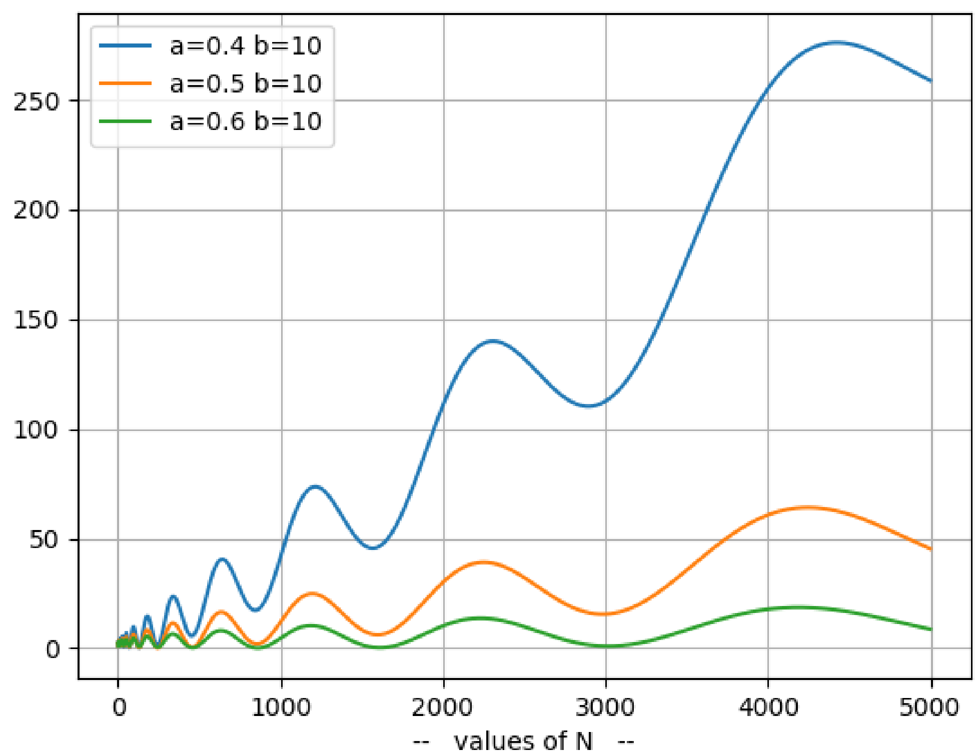

Figure 14: $|X(z)|^2$ *for* $\alpha<1$

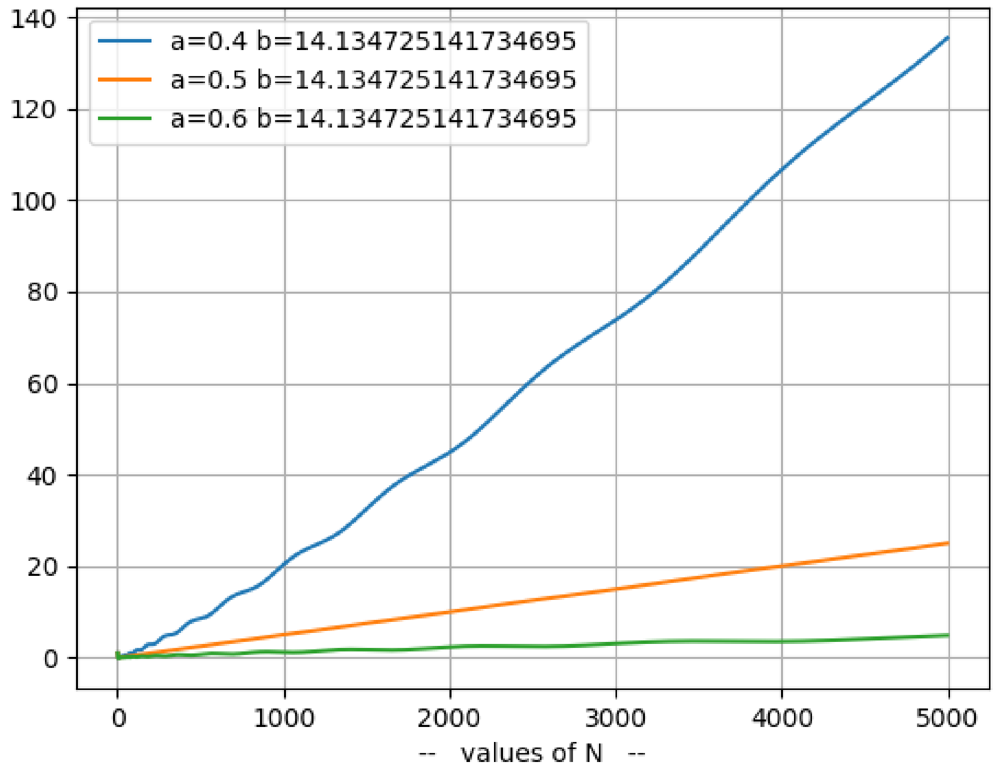

Figure 15: For a=0.5, b=b1, the wave collapses to a line

7.3.1. Lemma: $|X(n)|^2$ converges when $n \to \infty$ and α>1 to $|\zeta(\alpha, ß)|^2$

This Lemma provides the limit of $|x(z)|^2$ outside the critical strip [0,1]

[34] $$\lim_{n\to\infty} |X(z)|^2 = \sum_{k=1}^{n} \sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\left(ln\left(\frac{k}{j}\right)\right)\right)$$

As we can see in some examples in the following table where z=α+iß:

α	ß	$\lim_{n\to\infty} \|x(z)\|^2$	$\|\zeta(\alpha, ß)\|^2$
1.0	7	1.074711506185445	1.074756
1.0	10	1.4413521753699579	1.441430
2.5	7	1.0093487944300192	1.009349
2.5	10	1.0507402208589398	1.050740

Table 16

[35] $$\lim_{n\to\infty} |X(z)|^2 = |\zeta(\alpha, ß)|^2 = \zeta(\alpha + ßi) * \zeta(\alpha - ßi) \quad for\ \alpha > 1$$

And also, in the following figure 16:

|X(z)|² with α and ß variable

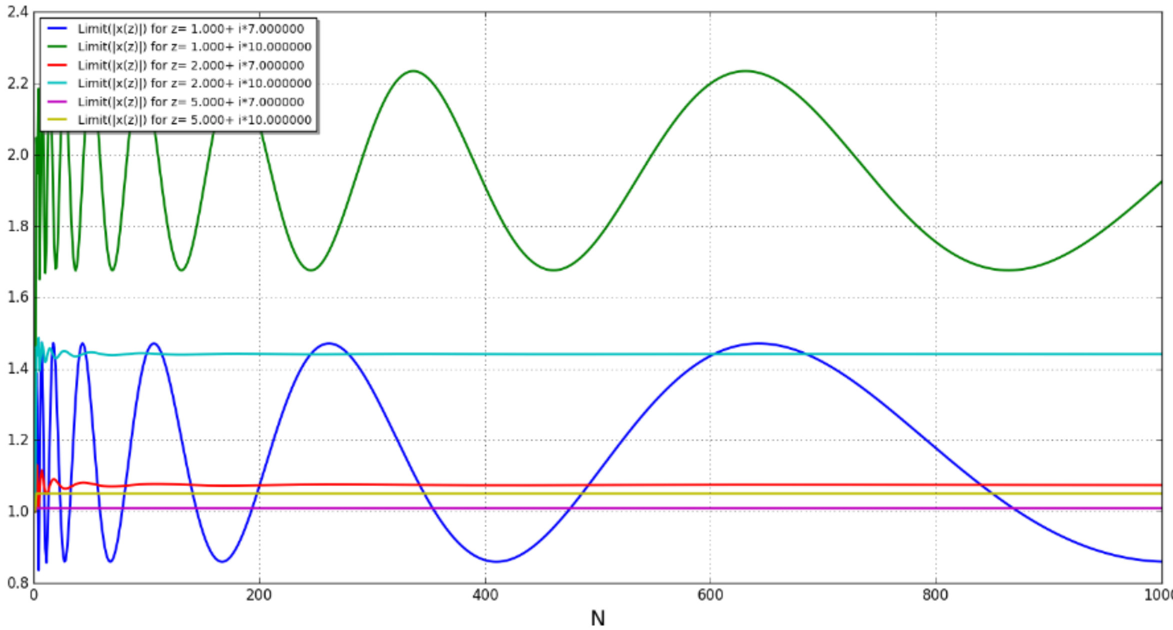

Figure 16. $|X(n)|^2$ converges when $n \to \infty$ and α>1

The graphs for α=1 do not converge while all other graphs for α>1 they all converge to a $|\zeta(\alpha, ß)|^2$. We will use this observation to prove later that there are no zero values of ζ(z) for z with Re(z)=α>1.

7.3.2. Lemma: $|X(z)|^2$ diverges when $\boldsymbol{n} \to \infty$ for α≤1

$|x(z)|^2$ diverges to ∞ when $\boldsymbol{n} \to \infty$ for α<1 because:

[36] $$|\cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)| < 1$$

And:

[37] $$\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha}\ diverges\ for\ \alpha < 1$$

Therefore:

[38] $$\lim_{n\to\infty} |X(z)|^2 \ = \sum_{k=1}^{n}\sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\,(ln\left(\frac{k}{j}\right)\right)\ diverges\ for\ \alpha < 1$$

Now let's evaluate if the function $|X(z)|^2$admits a polynomial representation inside the critical strip [0,1]

7.3.3. Lemma: $|X(z)|^2$ does not collapse to any polynomial function $|\mathbf{X(z)}|^2 = \mathbf{C * n^t\ for\ t > 1}$

We will prove it with a reduction to absurd.

Let's assume that $|X(z)|^2 = C * n^t\ for\ t > 1$ where C and t integers C>0 and t>0

If $|X(z)|^2 = C * n^t$ then:

[39] $\lim_{n\to\infty} |X(z)|^2/n^t = C$

But:

[40] $\lim_{n\to\infty} |\mathrm{X(z)}|^2 / n^t \ = \frac{1}{n^t} * \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2\alpha} + \frac{1}{n^t} * \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)$

And:

[41] $\frac{1}{n^t} * \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2\alpha} = 0 \quad for\ t > 1$

[42] $\frac{1}{n^t} * \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right) = \ 0\, for\ t > 1$

So, C must be 0 which is an absurd.

7.3.4. Lemma: |X(z)|² collapses to a straight-line $|\boldsymbol{x}(\boldsymbol{z})|^2 = \boldsymbol{Cn} \quad \boldsymbol{if}\ \boldsymbol{Re}(\boldsymbol{z}) = \mathbf{1/2}$

The proposition says that the following limit exists only for Re(z) = 1/2

[43] $\lim_{n\to\infty} (|\mathrm{X(z)}|^2 / \, n) = S$

And we know the expression:

[44] $\lim_{n\to\infty} (|X(z)|^2 \,/\, n) = \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\,(ln\left(\frac{k}{j}\right)\right))$

7.3.4.1. For α>1/2 , we can see that $\lim_{n\to\infty} (|\mathrm{x(z)}|^2 / \, n) = 0$:

[45] $\lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha}) = 0$ because 2α>1 and the series is convergent

[46] $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\,(ln\left(\frac{k}{j}\right)\right)) \ < \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} (k^{-\alpha} * j^{-\alpha}) < \ \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha})$

So:

[47] $\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)\right) = 0$

7.3.4.2. For α<1/2, we can see that $\lim_{n\to\infty} (|\mathrm{X(z)}|^2/n) = \infty$ as:

[48] $\lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-2\alpha}) < \lim_{n\to\infty} \frac{1}{n} \left(n * \frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = \ 0$

And:

[49] $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)) > \lim_{n\to\infty} (\frac{1}{n} * \ n^2 * \frac{1}{n^{2\alpha}}) = \infty$

Where we replaced the summations by the number of elements in the matrix (n x n) times the smallest value in each row (1/n) and 1>(2-1-2α)>0 when α<1/2

7.3.4.3. Let's calculate the limit for α=1/2.

When α=1/2, we can express (|X(z)|²/n) as:

[50] $\lim_{n\to\infty} (|X(z)|^2 /n) =$

$$= \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} k^{-1} + \sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right))$$
$$= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} k^{-1}\right) + \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)) =$$

$$= 0 + \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)) =$$
$$= \lim_{n\to\infty} \frac{2n}{n} (\sum_{j=1}^{n-1} n^{-1/2} * j^{-1/2} * \cos\left(ß\left(\ln\left(\frac{n}{j}\right)\right)\right)) =$$

$$= \lim_{n\to\infty} 2(n^{-\frac{1}{2}} \sum_{j=1}^{n-1} * j^{-\frac{1}{2}} * \cos\left(ß\left(\ln\left(\frac{n}{j}\right)\right)\right)) =$$

Using the integral approximation of the infinite series

$$= 2 * \lim_{n\to\infty} \frac{2 * \sqrt{n} * \cos\left(ß * \ln\left(\frac{n}{n}\right)\right) - 2 * ß * \sin(ß * \ln(\frac{n}{n})}{4 * ß^2 + 1} * n^{-\frac{1}{2}}$$
$$= 2 * \frac{2 * \sqrt{n}}{4 * ß^2 + 1} n^{-\frac{1}{2}} = 2 * \frac{2}{4 * ß^2 + 1} = \frac{1}{ß^2 + 1/4}$$

So, if $\lim_{n\to\infty} (|X(z)|^2 /n)$ exists will be equal to:

[51] $\lim_{n\to\infty} (|X(z)|^2 /n) = \frac{1}{ß^2+1/4}$ if z=1/2+iß

And this limit can only exist when $|X(z)|^2$ is monotonous which means that the curve will cross the x axis only once.

[52] $|X(z)|^2 = \left(\sum_{k=1}^{n} \sum_{j=k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)\right) = 2 * n^{-a} * \left(\sum_{j=1}^{n-1} j^{-a} * \cos\left(b * (\ln\left(\frac{x}{j}\right))\right)\right)$

From Proposition 5 we obtain:

[53] $|X(z)|^2 = C_2(n, \alpha, ß)$ with z=α+i*ß

And from table 10 we know that $|X(z)|^2$ crosses the x axis once and is therefore monotonous for values of ß equal to the non-trivial zeros of ζ(z).

7.4. [Caceres Proposition 7]. Proof of Riemann Hypothesis.

Theorem 3: If z* is a non-trivial zero of ζ(z) then Re(z*)=1/2

Proof:

- We defined two functions X(z) and Y(z) such that X(z)-Y(z) is the analytic continuation for ζ(z) on C for Re(z)>0
- We proved that $|X(z)|^2$ is a wave function that has only one polynomial representation in the form of a straight line if and only if Re(z)= ½ and for certain values of Im(z)=ß* the imaginary part of the non-trivial zeros of ζ(z) that we calculated using this condition.
- $|X(z^*)|^2 = \frac{n}{[ß^{*2}+1/4]}$ when $n \to \infty$ for z*=1/2+iß*
- We showed that $|Y(z)|^2$ is always a polynomial line.
- We proved that $|Y(z)|^2$ is only straight line if and only if Re(z)= ½
- $|Y(z)|^2 = \frac{n}{[ß^2+1/4]}$ when $n \to \infty$ for all z=1/2+iß
- If z=z* is a zero of ζ(z) and $n \to \infty$ ➔
- $\zeta(z^*) = 0 + i0$ ➔
- $|\zeta(z)|^2$ must be 0 ➔
- All z* non-trivial solution of ζ(z) must have $|X(z^*)|^2 = |Y(z^*)|^2$ when $n \to \infty$
- We proved that of all possible representations of $|x(z^*)|^2$ and $|y(z^*)|^2$ the only one in common for both functions is a representation as a straight line when Re(z)=1/2 when $n \to \infty$
- Therefore, all z* non-trivial solution of ζ(z) must have Re(z*) = ½ and we can also state that any zero of ζ(z) with z=α+iß meet these two conditions:

(condition 1) α=1/2

(condition 2) If $S = \frac{1}{[ß^2+1/4]}$ then for n=1/S ->
$\left(\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)\right) = 0$
This is a finite sum, with n∈ $[1, \frac{1}{S}]$.

when $n \to \infty$, where the size of n will determine the degree of accuracy of the solution. **Q.E.D.**

We can also derive the following propositions from ζ(z)=X(z)-Y(z).

7.5. Corollary: There are no zeros of ζ(z) when Re(z)>1

From Figure 16:

- we proved that $|x(z)|^2$ converges to a given value for α=Re(z)>1, which means that
- $|x(z)|^2$ tends to a horizontal line with slope =0 as $n \to \infty$ when α>1 .
- We know that all zeros of ζ(z) must make $|x(z)|^2$ a straight line with slope $\frac{1}{[ß^2+(1-\alpha)^2]}$.
- Therefore, this contradiction proves that there can't be any zeros of the ζ(z) function for α>1

8. **[Caceres Proposition 8]. A linearization of the Harmonic series using zeros of ζ(z).**

The precedent formulations describe also a way to approximate the Harmonic function to a straight line with slope $\frac{1}{[ß^2+(1-\alpha)^2]}$ where α=1/2 and ß=R(n):

[54] $$Hn = \frac{n}{[ß^2+(1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos(ß\,(ln\left(\frac{k}{j}\right)) \qquad \text{when } n \to \infty$$

We can see this graphically for $ß_1$=14.134725… with O(n) given by:

[55] $$O(n) = \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos(ß\,(ln\left(\frac{k}{j}\right)$$

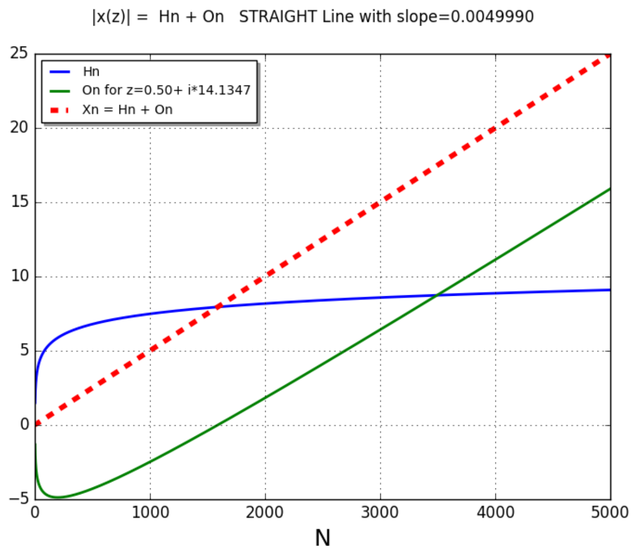

Figure 17. H_n and $|x(z)|^2$

9. **[Caceres Proposition 9] All ß zeros of ζ(z) are related algebraically.**

The fact that the same H_n can be expressed in an infinite number of ways as a function of ß for every ß imaginary part of a non-trivial solution of ζ(z), provides an algorithm to calculate all non-trivial zeros from any known zero through the expression. If $ß_1$ and $ß_2$ are imaginary part of a non-trivial solution of ζ(z), z=α+ßi, where α=1/2, from Proposition 8 we obtain:

[56] $$\frac{n}{[ß_2^2+(1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos(ß_2\,(ln\left(\frac{k}{j}\right)) =$$
$$\frac{n}{[ß_1^2+(1-\alpha)^2]} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos(ß_1\,(ln\left(\frac{k}{j}\right)) =$$

when $n \to \infty$, where the size of n will determine the degree of accuracy of the solution.

{__end__}

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