

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 1: THE C-TRANSFORM AND THE CROSS-ENERGY

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ABSTRACT. Let $X_N(s) = \sum_{k \leq N} k^{-s}$ and let $C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma}$ be the cross-energy of consecutive partial sums. This is the first part of a series of seventeen papers on C_2 and its links with the classical reformulations of the Riemann hypothesis (RH). With the C-transform $C_N(f) = \sum_{n \leq N} f(n) - \int_1^N f$, $\zeta(s) = C(x^{-s}) - 1/(1-s)$ in the strip, and a zero is a point of exact balance of X_N and its continuous envelope; the ratio of the two cannot see the critical line, and RH is the convergence of the C-transform of the primes in $\sigma > \frac{1}{2}$. Theorem A: at every point of the strip $\lim C_2(N, s)$ is $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$, a trace of the pole, and the zeros enter only through the wave $\zeta(s)N^{-s}$. The dynamics of $C_2 + iS_2$ is an exact rotation, contraction and forcing. For large $|\gamma|$, at a zero on the line $C_2(\cdot, \rho)$ changes sign once beyond $|\gamma|^{3/2}$, at $\rho(1-\rho) + \frac{1}{12} + \dots$, a universal series with rational coefficients. The same anatomy holds for every Dirichlet series with a simple pole and regular coefficient sums; the Mellin transform of C_2 is explicit and a zero makes its pole removable; and away from the zeros the sign changes on the line stop by an explicit bound. An appendix collects the formulas used throughout the series. We do not prove the Riemann hypothesis.

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1. INTRODUCTION

Let $X_N(s) = \sum_{k \leq N} k^{-s}$ be the partial sums of the Dirichlet series of ζ , with $s = \sigma + it$. The *cross-energy* of consecutive partial sums is

$$C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} (X_{N-1}(s) N^{-\bar{s}}) = 2N^{-\sigma} \sum_{n < N} n^{-\sigma} \cos \left(t \log \frac{N}{n} \right).$$

It records how the newest term N^{-s} interacts with all the previous ones. This is the first part of a series on C_2 . It contains the foundations: the C-transform, which writes ζ in the critical strip as the balance of the partial sums and their continuous envelope, and the exact structure and asymptotics of C_2 at every point of the strip. The later parts relate C_2 to the zeros near the critical line, to Hardy's function, to the mean square of the error term of the prime number theorem, to Jensen's formula and to Weil's explicit formula, and they test every statement on the Davenport–Heilbronn function f , which has a functional equation of the same type as ζ but zeros off the critical line, and on an Epstein zeta function, which shares with ζ every structural feature that C_2 sees and still has a real zero off the line.

Throughout, $\rho = \beta + i\gamma$ is a nontrivial zero of ζ , $\Theta = \sup \beta$, $\lambda = s(1-s)$, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, and

$$C_B = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots,$$

the sum taken in symmetric order [Da]; under RH, $\rho(1-\rho) = |\rho|^2 = \frac{1}{4} + \gamma^2$. We write $Y_N(s) = N^{1-s}/(1-s)$ for the continuous envelope of X_N and $X_N^C = X_N - \frac{1}{2}N^{-s}$.

1.1. The series. The series replaces the author's earlier preprints [C1]–[C8] and the consolidated paper *The cross-energy of Dirichlet partial sums and the zeros of the Riemann zeta function* (versions 1–4, [doi:10.5281/zenodo.23243641](https://doi.org/10.5281/zenodo.23243641)). Version 4 of that paper has been divided into Parts 1, 2, 5, 10 and 16, and the material is ordered by object, in five blocks (Table 1): the foundations (Parts 1–4), the primes and Euler products (Parts 5–9), positivity criteria (Parts 10–12), the de Bruijn–Newman constant (Parts 13–15), and limits and controls (Parts 16–17). Each part is self-contained up to the results it cites; a result proved in another part is cited as [CEk, Theorem $x.y$].

paper	object
<i>I. Foundations</i>	
1 this paper	the C-transform: ζ as the balance of X_N and its continuous envelope; Theorem A, the trichotomy, the crossing law and the general anatomy of C_2
2 [CE2]	the Fresnel transition law at a zero, the monotonicity windows, and C_2 on the critical line: Hardy's Z , Riemann–Siegel and the mollified cross-energy
3 [CE3]	the reflected cross-energy $C_2^{\text{FR}}(N, s) = X_N(s)X_N(1-s) - X_{N-1}(s)X_{N-1}(1-s) - \frac{1}{N}$, which pairs s with $1-s$
4 [CE4]	zeros of truncations of ζ : spirals, sections and the circles of the primes
<i>II. Primes and Euler products</i>	
5 [CE5]	the mean energy of the prime number theorem error is $\sum_{\rho} \lim C_2(N, \rho)$; Li, the explicit formula, Nyman–Beurling and the arithmetic cross-energy
6 [CE6]	second-order asymptotics and limiting fluctuations of the prime energy
7 [CE7]	multiplicative resonance and large values of Dirichlet polynomials
8 [CE8]	the Euler product inside the critical strip
9 [CE9]	zeros of the prime zeta function in its half-plane of absolute convergence
<i>III. Positivity criteria</i>	
10 [CE10]	the Jensen formula of the C_2 wave (Balazard–Saias–Yor), the energy slope $\partial_{\sigma} \log \xi $ (Lagarias) and Weil's functional
11 [CE11]	the Euler–Caceres bound: the Jensen moment of ζ against a smoothed Euler product
12 [CE12]	Weil positivity at the windows of the zeros
<i>IV. The de Bruijn–Newman constant</i>	
13 [CE13]	the bound $\Lambda \leq 0.174$ for the de Bruijn–Newman constant, by optimised mollifiers
14 [CE14]	the constant in the $C/\log X$ law for triangle-inequality upper bounds on Λ
15 [CE15]	how long an off-line zero survives the heat flow: $\Lambda \geq \delta^2/(70 \log \gamma \log \log \gamma)$
<i>V. Limits and controls</i>	
16 [CE16]	structural limits: the Epstein obstruction, superpositions, identities blind to s and the pole barrier
17 [CE17]	controls: Davenport–Heilbronn, Beurling systems and function fields

TABLE 1. The seventeen parts of the series.

1.2. Main results. The additive representation (Section 2). With the C-transform $C_N(f) = \sum_{n \leq N} f(n) - \int_1^N f(x) dx$, one has $C_N(x^{-s}) = X_N(s) - Y_N(s) + \frac{1}{1-s}$ exactly, hence $\zeta(s) = C(x^{-s}) - \frac{1}{1-s}$ for $\sigma > 0$, $s \neq 1$: in the strip ζ is the additive balance $X - Y$ of the discrete sum and its continuous envelope, and a zero is a point where the balance is reached, $X_N(\rho) - Y_N(\rho) \rightarrow 0$ (Proposition 2.3). The multiplicative ratio X_N/Y_N tends to 1 at every point and cannot see the critical line (Proposition 2.4); the energies $|X_N^C|^2$ and $|Y_N|^2$ agree to all orders at every zero, and the coefficient of the first correction equals $-B_2$ exactly when $\beta = \frac{1}{2}$ (Proposition 2.5). For general coefficients the C-transform of Λ converges exactly in $\sigma > \Theta$, so RH is the statement that the C-transform of the primes converges in $\sigma > \frac{1}{2}$ (Proposition 2.6), and the difference of the transforms of Λ and 1 is $1 - \zeta - \zeta'/\zeta$, in which the pole cancels identically (Proposition 2.8).

The cross-energy (Section 3). Theorem 3.2 (Theorem A) gives, for $0 < \sigma < 1$,

$$X_{N-1}(s) N^{-\bar{s}} = \frac{N^{1-2\sigma}}{1-s} + \zeta(s) N^{-\sigma} e^{it \log N} - \frac{1}{2} N^{-2\sigma} + O(|s| N^{-2\sigma-1}),$$

so that $\lim_N C_2(N, s)$ is $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$, at every point of the strip: the trichotomy is the trace of the pole at $s = 1$, and the zeros enter only through the

wave $\zeta(s)N^{-\sigma+it}$. The exact dynamics of $W_N = C_2 + iS_2$ is a rotation, a contraction and a forcing (Proposition 3.1). At a zero on the line with $|\gamma|$ large, $C_2(\cdot, \rho)$ crosses the axis once beyond $|\gamma|^{3/2}$, at $n^* = \lambda + \frac{1}{12} + \frac{1}{180\lambda} + \dots$, a universal series in λ with rational coefficients (Propositions 3.6 and 3.8). The same anatomy holds for every Dirichlet series with a simple pole and coefficient sums $cx + O(x^\theta)$ (Theorem 3.11). The Mellin transform of C_2 is $\zeta(2\sigma + q - 1)$ times two simple fractions, and a zero makes the pole removable with value $\zeta'(\rho)$ (Theorem 3.12). At points that are not zeros the sign changes of $C_2(\cdot, \frac{1}{2} + it)$ stop by $4|\zeta|^2\lambda^2(1 + O(1/\lambda))$ (Proposition 3.13).

The other parts of the series, in one sentence each: the transition at the critical line and Hardy's function are in Part 2; the energy of the primes ([CE5, Theorem B]) in Part 5; the Jensen formula of the wave and the energy slope in Part 10; the structural limits, with the Epstein obstruction, in Part 16. Appendix A collects the formulas used throughout the series.

1.3. The classical reformulations in cross-energy form. Table 2 lists the classical criteria and theorems that appear in the series, with the form they take there and the part in which they are treated. None of the reformulations is a proof of RH; each is equivalent to RH or implied by it, or it is a classical theorem restated, and the text says which.

1.4. What is new and what is classical. The C-transform representation, Theorem A and its consequences, the crossing law and its all-orders series, the exact dynamics, the general anatomy, the Mellin transform and the sign-change bound are, as far as we know, new in this form; they are elementary consequences of Euler–Maclaurin summation and we give complete proofs. Euler–Maclaurin summation, Backlund's bound and Landau's theorem are classical and cited as such. We do not prove the Riemann hypothesis.

classical statement	form in the series	where
Landau: abscissa of $\sum(\Lambda(n) - 1)n^{-s}$ is Θ	the C-transform of the primes converges exactly in $\sigma > \Theta$	Prop. 2.6, 2.8
Hardy's Z , Riemann–Siegel, Turing	signed radius of the helix; phase-locking extracts $2Z$; winding counts zeros	Sect. 3; [CE2, Sect. 3]
Hardy's theorem	infinitely many crossings of the oriented helix	[CE2, Thm. 3.6]
Gonek's $J_{-1}(T)$	total monotonicity window of C_2	[CE2, Prop. 2.20]
Balazard–Saias–Yor	C_2 –Jensen formula; $\mathcal{J} = 0$	[CE10, Cor. 2.8]
Cramér; Montgomery	mean energy of $E(u)$ equals $\sum_{\rho} \lim C_2(n, \rho)$	[CE5, Thm. B, Cor. 2.3]
Li's criterion	$\lambda_1 = C_B/2 = \frac{1}{2} \sum_{\rho} \lim C_2(n, \rho)$ under RH	[CE5, Prop. 2.4]
Nyman–Beurling, Báez-Duarte, BBLS	Gram data and $d_N^2 \log N \rightarrow C_B$ in C_2 coordinates	[CE5, Prop. 2.14]
Explicit formula, Schoenfeld	cross-energy coordinates of $1/\rho$	[CE5, Thm. 2.7]
$\psi(x) - x = O(x^{1/2+\varepsilon})$	$\sum_{n \leq N} C_2^{\Lambda}(n, \frac{1}{2}) = 4N + O(\sqrt{N})$	[CE5, Thm. 2.23]
Lagarias, Sondow–Dumitrescu	energy slope $S > 0$; disc lemma; exceptional set	[CE10, Sect. 3]
de la Vallée Poussin; Bohr–Landau	zero-free region and density in slope and energy form	[CE10, Prop. 3.6, 3.9]
Weil's criterion	S is Weil's functional at a Poisson kernel	[CE10, Thm. 3.13]
Pólya, Jensen polynomials, de Bruijn–Newman	the spectral variable λ ; extrema of Z and Lehmer pairs; $\Lambda \leq 0.174$	Rem. 3.9, [CE2, Sect. 3], [CE13]
Levinson, Conrey, Selberg	mollified cross-energy	[CE2, Sect. 3]
Davenport–Heilbronn, Epstein	controls: each criterion fails in the predicted way	[CE16, Sect. 2]
Gonek–Montgomery truncations	reflected cross-energy	[CE3]
Gonek–Hughes–Keating	Euler–Caceres bound	[CE11]

TABLE 2. Classical statements and their form in terms of the cross-energy, by part of the series.

2. THE ADDITIVE REPRESENTATION AND THE C-TRANSFORM

Everything in this paper rests on one classical identity, read additively. In the critical strip the partial sum $X_N(s)$ and its continuous envelope $Y_N(s) = N^{1-s}/(1-s)$ both diverge, and only their difference converges, to $\zeta(s)$. In the shorthand of this series, $\zeta = X - C$: the zeta function is the discrete sum minus its continuous component, and the C-transform is the operation that performs this subtraction. A zero of ζ is the point where the two components balance additively. This section sets up the representation, shows why the multiplicative comparison X_N/Y_N cannot see the critical line, computes the X - Y energies to all orders, and extends the C-transform to Dirichlet coefficients, where for $a = \Lambda$ its abscissa of convergence is the supremum Θ of the real parts of the zeros. The cross-energy of Section 3 is the increment of the discrete energy $|X_N|^2$; it is the bridge between this additive picture and the zeros.

2.1. Discrete sum and continuous envelope. For $N \geq 1$ and $s = \sigma + it \neq 1$ put

$$X_N(s) = \sum_{n \leq N} n^{-s}, \quad X_N^C(s) = X_N(s) - \frac{1}{2}N^{-s}, \quad (2.1)$$

$$Y_N(s) = Y_N^C(s) = \frac{N^{1-s}}{1-s}, \quad I_N(s) = \int_1^N x^{-s} dx = Y_N(s) - \frac{1}{1-s}.$$

X_N^C is the trapezoidal sum, with weight $\frac{1}{2}$ at the endpoint. For a locally integrable f on $[1, \infty)$ the C-transform is

$$C_N(f) = \sum_{n \leq N} f(n) - \int_1^N f(x) dx, \quad C(f) = \lim_{N \rightarrow \infty} C_N(f) \quad (2.2)$$

whenever the limit exists. For $f(x) = 1/x$ it gives Euler's constant, $C_N(x^{-1}) = H_N - \log N \rightarrow \gamma_0$.

Lemma 2.1 (the C-transform of x^{-s}). For every $N \geq 1$ and $s \neq 1$,

$$C_N(x^{-s}) = X_N(s) - Y_N(s) + \frac{1}{1-s}. \quad (2.3)$$

Proof. $C_N(x^{-s}) = X_N(s) - I_N(s)$ and $I_N(s) = Y_N(s) - 1/(1-s)$. $\square$

Status: proved.

We use Euler–Maclaurin summation in the following form throughout. Write $(s)_k = s(s+1) \cdots (s+k-1)$, $c_k = B_{2k}/(2k)!$, let $\zeta(s, x)$ be the Hurwitz zeta function and $\|B_j\|_\infty$ the maximum of the Bernoulli polynomial $|B_j|$ on $[0, 1]$.

Lemma 2.2 (Euler–Maclaurin for the tail). Let $x > 0$, $s \neq 1$ and $m \geq 1$ with $\sigma > -2m$. Then

$$\zeta(s, x) = \frac{x^{1-s}}{s-1} + \frac{1}{2}x^{-s} + \sum_{j=1}^m c_j (s)_{2j-1} x^{-s-2j+1} + R_m(s, x), \quad (2.4)$$

$$|R_m(s, x)| \leq \frac{\|B_{2m+1}\|_\infty}{(2m+1)!} \frac{|(s)_{2m+1}|}{\sigma + 2m} x^{-\sigma-2m},$$

and for every integer $N \geq 1$, $X_{N-1}(s) = \zeta(s) - \zeta(s, N)$.

Proof. For $\sigma > 1$, Euler–Maclaurin summation of $f(u) = u^{-s}$ over $u = x, x+1, \dots$ gives (2.4) with $R_m = -\frac{1}{(2m+1)!} \int_x^\infty \tilde{B}_{2m+1}(u) f^{(2m+1)}(u) du$, $\tilde{B}_{2m+1}$ the periodic Bernoulli function, and $f^{(2m+1)}(u) = -(s)_{2m+1} u^{-s-2m-1}$; this integral converges absolutely and is holomorphic in $\sigma > -2m$, which gives the continuation and the bound. The identity $X_{N-1} = \zeta(s) - \zeta(s, N)$ holds for $\sigma > 1$ and extends by analytic continuation. $\square$

Status: classical [T, §2.1], [Ed, Ch. 6].

Proposition 2.3 (the additive representation). Let $\sigma > 0$, $s \neq 1$.

(i) For every $K \geq 1$,

$$X_N^C(s) - Y_N(s) = \zeta(s) - \sum_{k=1}^K c_k(s) 2k-1 N^{-s-2k+1} + O_{s,K}(N^{-\sigma-2K-1}), \quad (2.5)$$

uniformly for s in compact subsets of $\{\sigma > 0\}$; in particular $D_N(s) := X_N^C(s) - Y_N(s) = \zeta(s) + O_s(N^{-\sigma-1})$ and $X_N(s) - Y_N(s) \rightarrow \zeta(s)$.

(ii) The C-transform of x^{-s} is the regular part of ζ at its pole:

$$\zeta(s) = C(x^{-s}) - \frac{1}{1-s} = C(x^{-s}) + \frac{1}{s-1}, \quad (2.6)$$

the function $C(x^{-s})$ is holomorphic in $\sigma > 0$, and its value at $s = 1$ is $C(x^{-1}) = \gamma_0$, the constant term of the Laurent expansion $\zeta(s) = \frac{1}{s-1} + \gamma_0 + O(s-1)$.

(iii) $\zeta(\rho) = 0$ if and only if $X_N(\rho) - Y_N(\rho) \rightarrow 0$, if and only if $C(x^{-\rho}) = 1/(1-\rho)$; at a zero $\rho = \beta + i\gamma$, $D_N(\rho) = O_\rho(N^{-\beta-1})$.

Proof. (i) By Lemma 2.2 with $x = N$, $X_N^C = X_{N-1} + \frac{1}{2}N^{-s} = \zeta(s) - \zeta(s, N) + \frac{1}{2}N^{-s}$, and (2.4) with $m = K + 1$ gives (2.5); the constants are uniform on compact sets. (ii) By Lemma 2.1 and (i), $C_N(x^{-s}) \rightarrow \zeta(s) + \frac{1}{1-s}$. Each $C_N(x^{-s})$ is entire in s (I_N is entire), and the convergence is uniform on compact subsets of $\sigma > 0$, including neighbourhoods of $s = 1$, so the limit is holomorphic there; at $s = 1$ it is $\lim(H_N - \log N) = \gamma_0$. (iii) follows from (i) and (ii). $\square$

Status: proved (classical in substance).

Identity (2.6) is first-order Euler–Maclaurin summation; what matters here is the reading. The two components diverge separately for $\sigma < 1$, with $|Y_N(s)| = N^{1-\sigma}/|1-s|$, and a nontrivial zero is an *additive asymptotic balance* between the discrete sum and its continuous envelope, $X_N^C(\rho) - Y_N(\rho) = O(N^{-\beta-1})$. The continuous component Y_N is the trace of the pole of ζ at $s = 1$; it is the source of the trichotomy of the cross-energy in Section 3, and a Dirichlet series without a pole, such as the Davenport–Heilbronn function f , has no continuous component and no trichotomy.

2.2. What the ratio cannot see. Proposition 2.4 (the ratio and the energies at every point). Let $0 < \sigma < 1$ and $\lambda = s(1-s)$.

- (i) $\frac{X_N^C(s)}{Y_N(s)} = 1 + (1-s)\zeta(s)N^{s-1} - \frac{\lambda}{12N^2} + O_s(N^{-4})$ and $\frac{X_N(s)}{Y_N(s)} = \frac{X_N^C(s)}{Y_N(s)} + \frac{1-s}{2N}$; in particular $X_N(s)/Y_N(s) \rightarrow 1$ at every point of the open strip, whether or not $\sigma = \frac{1}{2}$ and whether or not $\zeta(s) = 0$.
- (ii) At a zero ρ , $\lim_N 2N(X_N(\rho)/Y_N(\rho) - 1) = 1 - \rho$ and $\lim_N N^2(X_N^C(\rho)/Y_N(\rho) - 1) = -\lambda(\rho)/12$.
- (iii) $|Y_N(s)|^2 = \frac{N^{2-2\sigma}}{(1-\sigma)^2 + t^2}$ and $\frac{d}{dN}|Y_N(s)|^2 = A_s N^{1-2\sigma}$ with $A_s = \frac{2(1-\sigma)}{|1-s|^2}$, which is constant in N if and only if $\sigma = \frac{1}{2}$, with value $1/(\frac{1}{4} + t^2)$. At every zero, on or off the line,

$$|X_N^C(\rho)|^2 - |Y_N(\rho)|^2 = O_\rho(N^{-2\beta}), \quad \frac{|X_N^C(\rho)|^2}{|Y_N(\rho)|^2} = 1 + O_\rho(N^{-2}). \quad (2.7)$$

Proof. Divide (2.5) by $Y_N = N^{1-s}/(1-s)$: the term $\zeta(s)$ gives $(1-s)\zeta(s)N^{s-1}$, of modulus $|1-s||\zeta(s)|N^{\sigma-1} \rightarrow 0$, and $-c_1 s N^{-s-1}$ gives $-s(1-s)/(12N^2)$; the next term is $O_s(N^{-4})$. Then $X_N = X_N^C + \frac{1}{2}N^{-s}$ and $\frac{1}{2}N^{-s}/Y_N = (1-s)/(2N)$. This proves (i) and (ii). (iii) The first two formulas are direct. At a zero $D_N(\rho) = O(N^{-\beta-1})$, so $|X_N^C|^2 - |Y_N|^2 = 2\operatorname{Re}(Y_N \overline{D_N}) + |D_N|^2 = O(N^{1-\beta}N^{-\beta-1})$, and $|Y_N|^2 \asymp N^{2-2\beta}$. $\square$

Status: proved.

The multiplicative comparison of the two components therefore carries no information about the critical line: the ratio tends to 1 at every point of the strip, and the two energies agree to relative

order N^{-2} at every zero, whatever its abscissa. This disposes of the *multiplicative three-regime mechanism* of the author's earliest version of this approach, which sought the line in the zeroth-order ratio X_N/Y_N ; that mechanism is withdrawn, as already stated in [C1]. For the same reason a second assertion of the author's earlier preprints is not valid: the additive balance $D_N(\rho) \rightarrow 0$ does not force $|X_N^C(\rho)|^2$ and $|Y_N(\rho)|^2$ to be linear in N ; by (2.7) they agree at every zero, and they are linear only when $\beta = \frac{1}{2}$. The information survives in the subleading terms. The first-order defect in (ii) recovers $1 - \rho$ only through the endpoint term $\frac{1}{2}N^{-s}$, which is not information about ζ ; with the endpoint removed, the second-order defect recovers $\lambda(\rho)$, and the normalized energy gap of Proposition 2.5 recovers β (Figure 3, left column).

2.3. The X–Y energies to all orders. With $D_N = X_N^C - Y_N$,

$$|X_N^C|^2 - |Y_N|^2 = 2 \operatorname{Re}(Y_N \overline{D_N}) + |D_N|^2, \quad (2.8)$$

$$D_N(s) = \zeta(s) - \sum_{k \leq K} c_k(s)_{2k-1} N^{-s-2k+1} + O_{s,K}(N^{-\sigma-2K-1}).$$

Proposition 2.5 (the X–Y energies to all orders, and two zeros).

(i) At every zero $\rho = \beta + i\gamma$ of ζ , trivial or not,

$$N^{2\beta}(|X_N^C(\rho)|^2 - |Y_N(\rho)|^2) = \kappa(\rho) + O_\rho(N^{-2}), \quad (2.9)$$

$$\kappa(\rho) = -\frac{1}{6} \frac{\operatorname{Re} \lambda(\rho)}{|1 - \rho|^2} = -\frac{1}{6} - \frac{(1 - \beta)(2\beta - 1)}{6|1 - \rho|^2}.$$

For a nontrivial zero, $\kappa(\rho) = -B_2 = -\frac{1}{6}$ if and only if $\beta = \frac{1}{2}$, with $\kappa < -\frac{1}{6}$ for $\beta > \frac{1}{2}$ and $\kappa > -\frac{1}{6}$ for $\beta < \frac{1}{2}$. At the trivial zeros $\kappa(-2k) = k/(3(2k + 1))$.

(ii) On the critical line $|Y_N|^2 = N/\lambda$ exactly, $\lambda = \frac{1}{4} + t^2$. If $\zeta(\frac{1}{2} + it) = 0$, then for every $K \geq 1$

$$|X_N^C|^2 = \frac{N}{\lambda} + \sum_{m=1}^K Q_m(\lambda) N^{1-2m} + O_K(N^{-2K-1}), \quad (2.10)$$

$$Q_m = -2c_m \operatorname{Re}(s+1)_{2m-2} + \sum_{k+l=m} c_k c_l \operatorname{Re}[(s)_{2k-1} \overline{(s)_{2l-1}}],$$

and each Q_m is a polynomial of degree $m - 1$ in λ with rational coefficients: $Q_1 = -\frac{1}{6}$, $Q_2 = \frac{1}{90} + \frac{\lambda}{240}$, $Q_3 = -\frac{1}{210} + \frac{19\lambda}{7560} + \frac{\lambda^2}{6048}$, $Q_4 = \frac{1}{210} - \frac{1439\lambda}{302400} + \frac{17\lambda^2}{151200} + \frac{\lambda^3}{172800}$. If $\zeta(\frac{1}{2} + it) \neq 0$ and $t \neq 0$, then $\limsup_N N^{-1/2} ||X_N^C|^2 - N/\lambda| = 2|\zeta(s)|/|1 - s|$. Hence $\lim_N N(|X_N^C|^2 - N/\lambda)$ exists if and only if $\zeta(\frac{1}{2} + it) = 0$, and it is then $-\frac{1}{6}$.

(iii) On the line the diagonal part of $|X_N^C|^2$ is $H_{N-1} + \frac{1}{4N}$, the same for every t . For two zeros $\rho_j = \frac{1}{2} + i\gamma_j$, $\lambda_j = \frac{1}{4} + \gamma_j^2$,

$$|X_N^C(\rho_1)|^2 - |X_N^C(\rho_2)|^2 = N \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) + \frac{\lambda_1 - \lambda_2}{240 N^3} + O(N^{-5}), \quad (2.11)$$

and $N^2[X_N^C/Y_N(\rho_1) - X_N^C/Y_N(\rho_2)] \rightarrow -(\lambda_1 - \lambda_2)/12$, while the same quantity is of exact order $N^{3/2}$ along a subsequence if one of the two points is not a zero.

Proof. (i) At a zero $D_N(\rho) = -\frac{\rho}{12}N^{-\rho-1} + O(N^{-\beta-3})$, so $Y_N \overline{D_N} = -\frac{1}{12}N^{-2\beta} \bar{\rho}/(1 - \rho) + O(N^{-2\beta-2})$ and $|D_N|^2 = O(N^{-2\beta-2})$; since $\bar{\rho}/(1 - \rho) = \bar{\rho}(1 - \bar{\rho})/|1 - \rho|^2 = \overline{\lambda(\rho)}/|1 - \rho|^2$ this gives κ , and $\operatorname{Re} \lambda - |1 - \rho|^2 = (1 - \beta)(2\beta - 1)$ gives the second form. For $\rho = -2k$, $\lambda = -2k(2k + 1)$ and $|1 - \rho|^2 = (2k + 1)^2$. (At $\rho = -2$ everything is exact: $X_N^C(-2) = \frac{N^3}{3} + \frac{N}{6}$, $Y_N(-2) = \frac{N^3}{3}$, and $N^{-4}(|X_N^C|^2 - |Y_N|^2) = \frac{1}{9} + \frac{1}{36N^2}$.) (ii) If $\zeta(s) = 0$, (2.8) gives $|X_N^C|^2 = |Y_N|^2 - 2 \operatorname{Re}[Y_N \sum_k c_k(s)_{2k-1} N^{-s-2k+1}] + |\sum_k c_k(s)_{2k-1} N^{-s-2k+1}|^2$. On the line $s + \bar{s} = 1$, so $Y_N N^{-\bar{s}-2k+1} = N^{1-2k}/(1 - s)$, every oscillating factor disappears, and $\overline{(s)_{2k-1}}/(1 - s) = \overline{(s + 1)_{2k-2}}$

because $\bar{s} = 1 - s$; collecting powers of N gives Q_m . A polynomial with real coefficients evaluated at $s = \frac{1}{2} + it$ has a real part even in t , so Q_m is a polynomial in $t^2 = \lambda - \frac{1}{4}$, of degree $m - 1$. If $\zeta(s) \neq 0$, $N(|X_N^C|^2 - N/\lambda) = 2N \operatorname{Re}(Y_N \overline{\zeta(s)}) + O(N) = 2N^{3/2} |\zeta(s)| |1 - s|^{-1} \cos(\varphi - t \log N) + O(N)$ with φ fixed, and $t \log N$ is dense modulo 2π . (iii) The coefficients of X_N^C are 1 for $n < N$ and $\frac{1}{2}$ for $n = N$, so the diagonal is $H_{N-1} + \frac{1}{4N}$. Subtracting (2.10) at two zeros gives (2.11), since Q_1 is constant and $Q_2(\lambda_1) - Q_2(\lambda_2) = (\lambda_1 - \lambda_2)/240$. The ratio statement follows from Proposition 2.4(i). $\square$

Status: proved; the coefficients Q_m are exact (`anc/xy_energy/xy_energy.py`); the checks below are numerical.

Up to the terms with $k = N$, the off-diagonal part of the left side of (2.11) is

$$-4 \sum_{j < k < N} (jk)^{-1/2} \sin(m\ell) \sin\left(\frac{h}{2}\ell\right),$$

with $m = \frac{1}{2}(\gamma_1 + \gamma_2)$, $h = \gamma_1 - \gamma_2$, $\ell = \log(k/j)$: the mean-height and sine structure of the vertical displacement between two zeros. Identity (2.11) is the all-orders form of the relation between two zeros of [C20], which obtained only $\lim_N |X_N^C(\rho)|^2/N = 1/\lambda$; the corresponding statement for the accumulated cross-energy, which corrects an earlier two-zero claim of [E24] (see [Er]), is Proposition 3.14. On the critical line the X - Y energies see a zero only through λ , to all orders, and two zeros interact only through $\lambda_1 - \lambda_2$, exactly as the crossing point of the cross-energy does (Proposition 3.8). Off the line the first correction $\kappa(\rho)$ would differ from $-B_2$; this reads the abscissa of a point already known to be a zero and does not constrain it. Numerically, at the first five zeros $N(|X_N^C|^2 - N/\lambda)$ is -0.16666666 , -0.16666665 , -0.16666664 , -0.16666663 , -0.16666662 for $N = 10^4$; the quantity $N^5(|X_N^C|^2 - N/\lambda + \frac{1}{6N} - Q_2 N^{-3})$ at $N = 100$ is 7.11946 and 33.48647 at ρ_1, ρ_2 , against $Q_3 + Q_4 N^{-2} = 7.11941$ and 33.48655 ; the two-zero residual times N^3 is -1.0089 at $N = 10^3$, against $(\lambda_1 - \lambda_2)/240 = -1.0089$; at ρ_1 and $N = 1600$, $N^2(X_N^C/Y_N - 1)$ is $-16.6701 + 6 \cdot 10^{-6}i$, against $-\lambda/12 = -16.6700$; at $\frac{1}{2} + 17.2i$, which is not a zero, $N(|X_N^C|^2 - N/\lambda)$ is $272, 3251, 305240$ for $N = 10^2, 10^3, 10^4$.

2.4. The C-transform of a coefficient sequence. For coefficients $a(n)$ put

$$C_N(s; a) = \sum_{n \leq N} a(n) n^{-s} - \epsilon_a \int_1^N x^{-s} dx, \quad (2.12)$$

with $\epsilon_a = 1$ if $\sum a(n) n^{-s}$ has a simple pole of residue 1 at $s = 1$ and $\epsilon_a = 0$ otherwise. For $a = 1$ this is $C_N(x^{-s})$, which tends to $\zeta(s) + 1/(1 - s)$ for $\sigma > 0$, $s \neq 1$, with error $\frac{1}{2} N^{-s} + O(N^{-\sigma-1})$. For $a = \Lambda$ the transform at $s = 0$ is the error term of the prime number theorem,

$$C_N(0; \Lambda) = \psi(N) - N + 1. \quad (2.13)$$

Let $\Theta = \sup \beta$ over the nontrivial zeros.

Proposition 2.6 (the C-transform of the primes). (i) For $\sigma > \Theta$, $C_N(s; \Lambda)$ converges, to $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$. (ii) If $C_N(s_0; \Lambda)$ converges at a point with $\sigma_0 = \operatorname{Re} s_0 > 0$, then ζ has no zeros with $\beta > \sigma_0$. Hence in the half-plane $\sigma > 0$ the abscissa of convergence of the C-transform of Λ is Θ , and RH holds if and only if the C-transform of the primes converges in $\sigma > \frac{1}{2}$, as the C-transform of 1 converges in $\sigma > 0$.

Proof. (i) By Stieltjes integration, $C_N(s; \Lambda) = \int_1^N x^{-s} d(\psi(x) - x) = (\psi(N) - N)N^{-s} + 1 + s \int_1^N (\psi(x) - x) x^{-s-1} dx$, and $\psi(x) - x = O_\epsilon(x^{\Theta+\epsilon})$ [In]; so the limit exists for $\sigma > \Theta$, uniformly on compact sets, and it equals $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ for $\sigma > 1$, hence in $\sigma > \Theta$ by analytic continuation. (ii) $C_N(s; \Lambda) - C_N(s; 1) = \sum_{n \leq N} (\Lambda(n) - 1) n^{-s}$ is the partial sum of a Dirichlet series, and $C_N(s_0; 1)$ converges by Proposition 2.3. So the Dirichlet series $\sum_n (\Lambda(n) - 1) n^{-s}$ converges at s_0 , hence

converges and is holomorphic in $\sigma > \sigma_0$; it equals $-\frac{\zeta'}{\zeta}(s) - \zeta(s)$ for $\sigma > 1$, and this function has a pole at every zero of ζ . $\square$

Status: proved (classical in substance [In]).

Remark 2.7 (the transform of $\log \zeta$). The same argument applies to the multiplicative transform

$$\sum_{p \leq N} -\log(1 - p^{-s}) - \int_2^N \frac{-\log(1 - x^{-s})}{\log x} dx,$$

the C-transform of $\log \zeta$ with the density $1/\log x$ of the primes in place of the density 1 of the integers: it converges for $\sigma > \Theta$ by $\pi(x) - \text{li}(x) = O(x^{\Theta+\epsilon})$, and its convergence at a point with $\sigma_0 > \frac{1}{2}$ would make $\log \zeta(s)$ minus the continuation of the integral holomorphic in $\sigma > \sigma_0$, which a zero there forbids. Status: proved in outline.

Numerically, at $t = 85.7$ with $N \leq 10^7$, the exponent of $|C_N(s; a) - \lim_N C_N(s; a)|$ fitted on dyadic blocks from 2^{12} is $-\sigma$ to three decimals for $a = 1$, and within 0.02 of $\frac{1}{2} - \sigma$ for $a = \Lambda$, for $0.55 \leq \sigma \leq 1.1$. For the coefficients $c(n)$ of $-f'/f$, with f the Davenport–Heilbronn function ($\epsilon = 0$) and $N \leq 4 \cdot 10^6$, it is $0.8085 - \sigma$ to three decimals ($+0.209$ at $\sigma = 0.6$, $+0.009$ at $\sigma = 0.8$, -0.092 at $\sigma = 0.9$): the wave of the zero $0.808517 + 85.699348i$ (Figure 1). This is the measured exponent near one height, not the abscissa of $\sum c(n)n^{-s}$ itself, which is at least 1 because f has zeros in $\sigma > 1$ [DH]. The measurement does not extend: all zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ are on the line [PT], so an off-line zero is seen at a fixed $s_0 = \frac{1}{2} + i\gamma_0$ with amplitude $N^{\beta-1/2}/|\rho - s_0|$ and $|\rho - s_0| \geq 3 \cdot 10^{12} - |\gamma_0|$, and its wave exceeds the level $\log N$ only for $N \gtrsim 10^{29}$ if $\beta = 0.99$, $N \gtrsim 10^{150}$ if $\beta = 0.6$, and $N \gtrsim 10^{1600}$ if $\beta = 0.51$. Status: numerical (`anc/ctransform/C_transform_paper.py`).

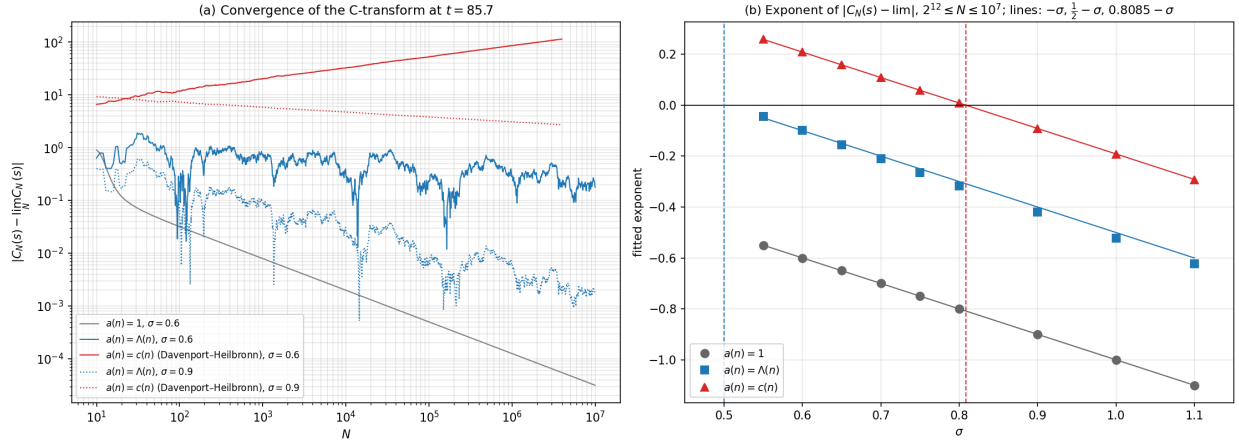


FIGURE 1. The C-transform $C_N(s; a)$. (a) Distance to the limit at $t = 85.7$ for $a = 1$, $a = \Lambda$ and the coefficients $c(n)$ of $-f'/f$ (f the Davenport–Heilbronn function), at $\sigma = 0.6$ and 0.9 . (b) Fitted exponent against σ , with the lines $-\sigma$, $\frac{1}{2} - \sigma$ and $0.8085 - \sigma$; the abscissae $\frac{1}{2}$ and 0.8085 are marked.

2.5. Primes minus integers. Put $X_N^\Lambda(s) = \sum_{n \leq N} \Lambda(n)n^{-s}$ and, as for the integers in Section 3, $W_N^\Lambda = C_2^\Lambda + iS_2^\Lambda = 2N^{-s}X_{N-1}^\Lambda$ (no factor $\Lambda(N)$). Since Λ has mean 1, the prime sum carries the same continuous component Y_N as the integers, and in the difference it cancels identically.

Proposition 2.8 (primes minus integers). Let

$$F(s) = 1 - \zeta(s) - \frac{\zeta'}{\zeta}(s) = \sum_{n \geq 2} \frac{\Lambda(n) - 1}{n^s} \quad (\sigma > 1). \quad (2.14)$$

Then, for $\sigma > \Theta$,

$$\begin{aligned} F(s) &= 1 + \lim_{N \rightarrow \infty} [C_N(s; \Lambda) - C_N(s; 1)] = 1 + \lim_{N \rightarrow \infty} [X_N^\Lambda(s) - X_N(s)] \\ &= 1 + \lim_{N \rightarrow \infty} \frac{1}{2} N^{\bar{s}} (W_N^\Lambda(s) - W_N(s)), \end{aligned} \quad (2.15)$$

the three limits exist at no point with $\sigma < \Theta$, and the poles of ζ and $-\zeta'/\zeta$ at $s = 1$ cancel, with $F(1) = 1 - 2\gamma_0$. Moreover, exactly,

$$C_2^\Lambda(N, s) - C_2(N, s) = 2 \operatorname{Re} \left[N^{-\bar{s}} \left(F(s) - 1 - \sum_{n \geq N} \frac{\Lambda(n) - 1}{n^s} \right) \right] \quad (\sigma > \Theta). \quad (2.16)$$

Proof. The three brackets in (2.15) are the same partial sum $\sum_{n \leq N} (\Lambda(n) - 1)n^{-s}$, up to the term $n = N$ in the last one, since $\frac{1}{2} N^{\bar{s}} W_N = X_{N-1}$. Convergence for $\sigma > \Theta$ and divergence for $\sigma < \Theta$ are Proposition 2.6. Near $s = 1$, $\zeta = \frac{1}{s-1} + \gamma_0 + O(s-1)$ and $-\zeta'/\zeta = \frac{1}{s-1} - \gamma_0 + O(s-1)$. (2.16) is $X_{N-1}^\Lambda - X_{N-1} = F(s) - 1 - \sum_{n \geq N} (\Lambda(n) - 1)n^{-s}$ multiplied by $2N^{-\bar{s}}$. $\square$

Status: classical identities, in the notation of this paper.

By the explicit formula the tail in (2.16) is a superposition of waves $N^{\rho-s}/(\rho-s)$, one for each zero, so the cross-energies of the primes and of the integers agree asymptotically, and they differ by $2 \operatorname{Re}[N^{-\bar{s}}(F(s) - 1)]$ plus a wave of size $N^{\beta-2\sigma}$ for each zero: F measures the difference between the primes and the integers, read on the cross-energy.

The rotation of the cross-energy difference. Put $\Delta W_N = W_N^\Lambda - W_N = \Delta C_2 + i\Delta S_2$, with $\Delta C_2 = C_2^\Lambda - C_2$ and $\Delta S_2 = S_2^\Lambda - S_2$. Since $X_{N-1}^\Lambda - X_{N-1} = \frac{1}{2} N^{\bar{s}} \Delta W_N$ and $N^{\bar{s}} = N^\sigma e^{-it \log N}$, taking real parts gives, exactly for every $N \geq 2$ and every s ,

$$2 \sum_{n < N} \frac{\Lambda(n) - 1}{n^\sigma} \cos(t \log n) = N^\sigma [\Delta C_2(N, s) \cos(t \log N) + \Delta S_2(N, s) \sin(t \log N)], \quad (2.17)$$

the term $n = 1$ being -1 . The cosine series of the primes minus the integers is the difference of the two complex cross-energies, rotated by the angle $t \log N$ and scaled by N^σ ; the rotation mixes the two components, so S_2 is needed as well as C_2 . Taking moduli instead, $N^\sigma |\Delta W_N(s)| = 2|R_{N-1}(s)|$ with R_N as in Proposition 2.9. Hence, for every fixed real t , RH holds if and only if

$$\sup_{N \geq 2} N^\sigma |W_N^\Lambda(\sigma + it) - W_N(\sigma + it)| < \infty \quad \text{for every } \sigma > \frac{1}{2}.$$

Indeed, under RH the series $\sum_n (\Lambda(n) - 1)n^{-s}$ converges for $\sigma > \frac{1}{2}$ (Propositions 2.3 and 2.6), so its partial sums $R_{N-1}(s)$ are bounded; conversely, bounded partial sums at $s_0 = \sigma_0 + it$ make the Dirichlet series converge for $\sigma > \sigma_0$, and Proposition 2.6(ii) excludes the zeros with $\beta > \sigma_0$. In words, the cross-energy of the primes follows that of the integers with a lag that decays like $N^{-\sigma}$: under RH, by (2.16), $\Delta W_N = 2N^{-\bar{s}}(F(s) - 1) + O_\varepsilon(N^{1/2-2\sigma+\varepsilon})$, while a zero with $\beta > \sigma$ makes $N^\sigma |\Delta W_N|$ as large as $N^{\beta-\sigma-\varepsilon}$ along a sequence of N . This is Proposition 2.6 written in the coordinates of the cross-energy, not a step towards it. Status: proved (classical in substance).

Proposition 2.9 (resonance of prime polynomials). Let $R_N(s) = \sum_{n \leq N} (\Lambda(n) - 1)n^{-s} = C_N(s; \Lambda) - C_N(s; 1)$. For a single real γ_0 , RH holds if and only if $R_N(\frac{1}{2} + i\gamma_0) = O_\varepsilon(N^\varepsilon)$ for every $\varepsilon > 0$; under RH, $R_N(\frac{1}{2} + i\gamma_0) = O((1 + |\gamma_0|) \log^3 N)$.

Proof. Under RH, $\psi(x) - \lfloor x \rfloor = O(x^{1/2} \log^2 x)$, and partial summation gives the bound. Conversely, the bound at $s_0 = \frac{1}{2} + i\gamma_0$ makes the Dirichlet series $\sum_n (\Lambda(n) - 1)n^{-s} = -\zeta'(s) - \zeta(s)$ convergent in $\sigma > \frac{1}{2} + \varepsilon$ for every ε , and Proposition 2.6(ii) applies. $\square$

Status: classical [In].

R_N is the prime sum with the continuous component removed, and what remains is a superposition of waves $N^{\rho-s}/(\rho-s)$, of the same form $N^{\beta-\sigma} \cos((\gamma-t) \log N + \phi)$ as the wave $N^{1-\sigma} \cos(t \log N + \phi)$ of the pole. One frequency γ_0 controls every height, but only in the limit: at

finite N a zero at height γ is seen at γ_0 with amplitude $N^{\beta-1/2}/|\rho - \frac{1}{2} - i\gamma_0|$. The computation therefore measures the resonance exponent

$$E_T(N) = \frac{\log \max_{10 \leq \gamma \leq T} |R_N(\frac{1}{2} + i\gamma)|}{\log N}, \quad T = 100,$$

and the same quantity for $L(s, \chi)$ mod 5 (coefficients $\chi(n)\Lambda(n)$, no pole) and for the Davenport–Heilbronn function (coefficients $c(n)$ of $-f'/f$, obtained by Dirichlet inversion and checked against $-f'/f$ at $s = 2$ to 10^{-9}). For ζ and $L(s, \chi)$ with $N \leq 2 \cdot 10^7$ the maximum follows $\log N$ (16.4 at $N = 2 \cdot 10^7$, where $\log N = 16.8$) and $E_T(N)$ follows $\log \log N / \log N$. For f with $N \leq 4 \cdot 10^6$ the maximum sits at $\gamma = 85.7$ from $N \approx 100$ on, and $E_T(4 \cdot 10^6) = 0.385$, against $0.3085 + \log(1/0.3085)/\log N = 0.386$ predicted by its zero $0.808517 + 85.699348i$ (Figure 2b). With a Hann window of width 6 in $u = \log n$, the peak at an ordinate grows like $3e^{(\beta-1/2)u_0}$ in the centre u_0 of the window: the fitted slopes are below 0.003 and 0.01 in absolute value for the 21 isolated zeros of ζ and the 19 of $L(s, \chi)$ below height 100, and 0.303 for f at 85.70, against $\beta - \frac{1}{2} = 0.3085$ (Figure 2a). The anatomy of that zero of f in its approximate functional equation, as an interference of two Euler products carried by the coefficients at which multiplicativity fails (Figure 2c), belongs to the Euler product inside the strip and is treated in [CE8]. What this leaves is the question whether a Dirichlet polynomial with multiplicative coefficients can resonate at one frequency with amplitude $N^{\beta-1/2}$, $\beta > \frac{1}{2}$; by Proposition 2.9 it is equivalent to RH, and it is not answered here. Status: numerical (`anc/resonance/`, double precision for the sums).

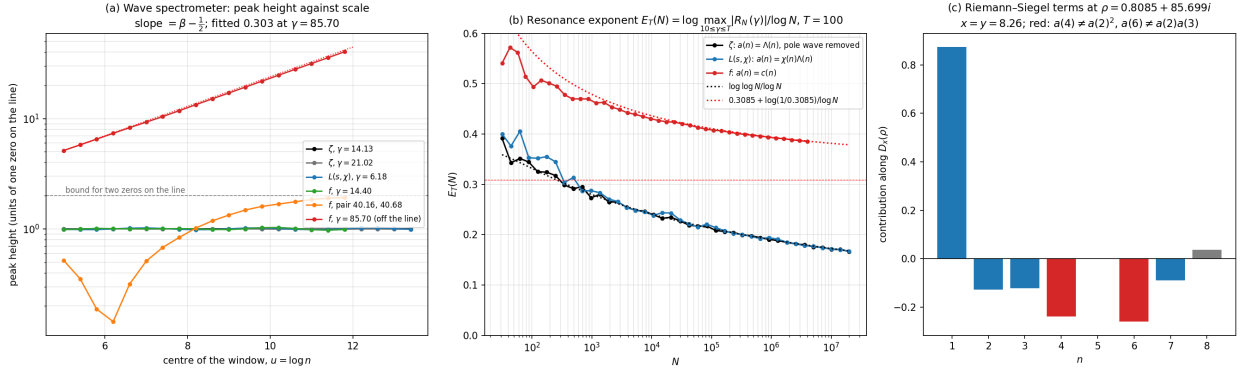


FIGURE 2. Resonance of prime polynomials, $R_N = C_N(s; \Lambda) - C_N(s; 1)$. (a) Height of the spectral peak at an ordinate against the centre of a Hann window of width 6 in $\log n$, in units of the height of one zero on the line; dotted: slope 0.3085. (b) The resonance exponent $E_T(N)$, $T = 100$, for ζ , $L(s, \chi)$ mod 5 and the Davenport–Heilbronn function f , with the predictions $\log \log N / \log N$ and $0.3085 + \log(1/0.3085)/\log N$. (c) Contributions of the terms of the approximate functional equation of f at its zero $0.8085 + 85.699i$, projected on the direction of the first half (see [CE8]).

2.6. Two factorizations. In the critical strip ζ has the additive factorization of this section, $\zeta = \lim_N [X_N^C - Y_N]$, equivalently $\zeta(s) = C(x^{-s}) - \frac{1}{1-s}$, built from the equally spaced integers and the pole; the cross-energy reads it through $X_{N-1} = \frac{1}{2}N^s W_N$ (Section 3), and on the line $\zeta(s) = \lim_N N^{1-s} [\frac{1}{2}W_N - \frac{1}{1-s}] + \frac{1}{2}N^{-s}$. The multiplicative factorization $\zeta = E_g^* Z_g$, a smoothed continuation of the Euler product times a factor carrying the zeros, belongs to [CE11] and [CE8]. Figure 3 collects both, the bridge C_2 between them, and the equivalent forms of RH, together with the two controls: the Davenport–Heilbronn function has the additive side and the symmetry without an Euler product, a Beurling system has the multiplicative side without the integers $1, 2, 3, \dots$ [CE17], and both have zeros off the line.

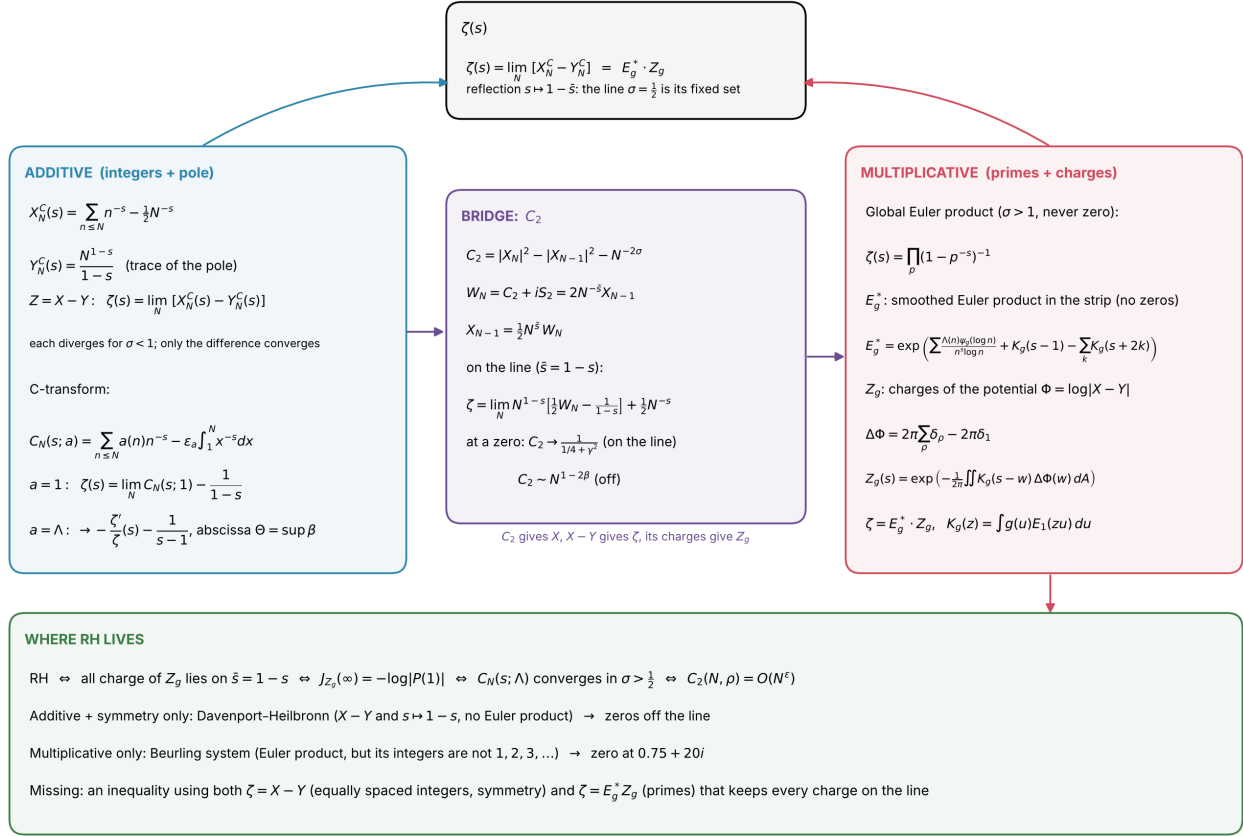
Two exact factorizations of ζ in the critical strip, and a bridge

FIGURE 3. The additive factorization $\zeta = \lim_N [X_N^C - Y_N]$ and the C-transform (left), the multiplicative factorization $\zeta = E_g^* Z_g$ of [CE11, CE8] (right), the cross-energy C_2 as the bridge between them (centre), and the equivalent forms of RH with the two controls (bottom).

3. THE CROSS-ENERGY

For $s = \sigma + it$ and $N \geq 1$ the cross-energy is the increment of the discrete energy $|X_N(s)|^2$ of Section 2 that is due to the interaction of the new term N^{-s} with the accumulated sum:

$$\begin{aligned} C_2(N, s) &= |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} (X_{N-1}(s) N^{-\bar{s}}) \\ &= 2N^{-\sigma} \sum_{j < N} j^{-\sigma} \cos \left(t \log \frac{N}{j} \right). \end{aligned} \quad (3.1)$$

Its imaginary companion and the complex wave are

$$S_2(N, s) = 2 \operatorname{Im} (X_{N-1}(s) N^{-\bar{s}}), \quad W_N(s) = C_2(N, s) + iS_2(N, s) = 2X_{N-1}(s) N^{-\bar{s}}, \quad (3.2)$$

so that $X_{N-1}(s) = \frac{1}{2} N^{\frac{s}{2}} W_N(s)$: the wave carries the partial sum, and through Proposition 2.3 the value $\zeta(s)$. Since $C_2(N, \sigma, -t) = C_2(N, \sigma, t)$ we may take $t \geq 0$ when convenient. In (3.1) C_2 is the projection of the weighted sum $\sum_{j < N} j^{-\sigma} e^{-it \log j}$ onto a direction that rotates logarithmically with N . Two extensions to real arguments are used. The *Hurwitz extension* $C_2(x, s) = 2 \operatorname{Re} [(\zeta(s) - \zeta(s, x)) x^{-\bar{s}}]$, $x \geq 1$ real, agrees with (3.1) at integers by Lemma 2.2 and is smooth in x ; it is used for the crossing law. The *step extension* $\tilde{C}_2(x, s) = 2x^{-\sigma} \sum_{j < x} j^{-\sigma} \cos(t \log(x/j))$ is used for the Mellin transform.

3.1. Exact structure. Proposition 3.1 (exact structure). Let $s = \sigma + it$ and $N \geq 1$.

(i) (Energy identity.) (3.1) holds, and with $C_2(1, s) = 0$,

$$|X_N(s)|^2 = \sum_{n \leq N} C_2(n, s) + X_N(2\sigma). \quad (3.3)$$

(ii) (Geometry of a crossing.) If $C_2(N_0, s) = 0$, then $X_{N_0-1}(s)$ and N_0^{-s} are orthogonal in $\mathbb{C} \cong \mathbb{R}^2$,

$$|X_{N_0}(s)|^2 = |X_{N_0-1}(s)|^2 + N_0^{-2\sigma}, \quad \arg X_{N_0-1}(s) \equiv \frac{\pi}{2} - t \log N_0 \pmod{\pi}. \quad (3.4)$$

(iii) (Rotation, contraction and forcing.) With $r_N = (N/(N+1))^\sigma$ and $\delta_N = t \log(1 + 1/N)$,

$$W_{N+1} = r_N e^{i\delta_N} W_N + 2[N(N+1)]^{-\sigma} e^{i\delta_N}, \quad (3.5)$$

that is, in real coordinates,

$$\begin{pmatrix} C_2(N+1) \\ S_2(N+1) \end{pmatrix} = r_N \begin{pmatrix} \cos \delta_N & -\sin \delta_N \\ \sin \delta_N & \cos \delta_N \end{pmatrix} \begin{pmatrix} C_2(N) \\ S_2(N) \end{pmatrix} + g_N \begin{pmatrix} \cos \delta_N \\ \sin \delta_N \end{pmatrix}, \quad g_N = 2[N(N+1)]^{-\sigma}.$$

The propagator telescopes, $\prod_{m=k}^{N-1} r_m e^{i\delta_m} = (k/N)^\sigma e^{it \log(N/k)}$, and iterating (3.5) from $W_1 = 0$ returns (3.2): the system is exactly solvable.

(iv) (A recurrence for C_2 alone.) If $t \neq 0$ and $N > |t|/\pi$, then

$$C_2(N+2) = \alpha_N C_2(N+1) - \beta_N C_2(N) + F_N, \quad (3.6)$$

$$\alpha_N = \left(\frac{N+1}{N+2}\right)^\sigma \frac{\sin(t \log \frac{N+2}{N})}{\sin(t \log \frac{N+1}{N})}, \quad \beta_N = \left(\frac{N}{N+2}\right)^\sigma \frac{\sin(t \log \frac{N+2}{N+1})}{\sin(t \log \frac{N+1}{N})},$$

$$F_N = g_{N+1} \cos \delta_{N+1} - r_{N+1} g_N \frac{\sin \delta_{N+1}}{\sin \delta_N}.$$

Proof. (i) Expand $|X_N|^2 = |X_{N-1} + N^{-s}|^2$; summing gives (3.3), since $\sum_{n \leq N} n^{-2\sigma} = X_N(2\sigma)$. (ii) $C_2(N_0) = 0$ says that $X_{N_0-1} N_0^{-\bar{s}} = X_{N_0-1} N_0^{-\sigma} e^{it \log N_0}$ is purely imaginary. (iii) $W_{N+1} = 2(N+1)^{-\bar{s}}(X_{N-1} + N^{-s})$ and $(N+1)^{-\bar{s}} = N^{-\bar{s}}(N/(N+1))^\sigma e^{it \log((N+1)/N)}$; the telescoping is immediate. (iv) The first row of the real form gives $S_2(N)$ in terms of $C_2(N), C_2(N+1)$ when $\sin \delta_N \neq 0$, which holds for $N > |t|/\pi$ since $0 < |\delta_N| < |t|/N < \pi$; substituting into the second row gives $S_2(N+1) = [r_N C_2(N) + g_N - \cos \delta_N C_2(N+1)] / \sin \delta_N$, and inserting this into the first row at step $N+1$, with $\sin \delta_N \cos \delta_{N+1} + \cos \delta_N \sin \delta_{N+1} = \sin(t \log \frac{N+2}{N})$, gives (3.6). $\square$

Status: proved; the identities are checked to $2 \cdot 10^{-14}$ at $s = 0.6 + 7.3i$, $N \leq 4000$ (`anc/c2core/exact_identities_check.py`).

The contraction is governed by σ , the rotation by t , and the forcing by the arrival of each new term. Since $\delta_N = t/N + O(tN^{-2})$ and $r_N = 1 - \sigma/N + O(N^{-2})$, the recurrence (3.6) approaches a second-order differential regime for large N . In (3.3) the diagonal energy is the partial sum of ζ itself at the real point 2σ : it converges to $\zeta(2\sigma)$ for $\sigma > \frac{1}{2}$, grows like $N^{1-2\sigma}/(1-2\sigma)$ for $\sigma < \frac{1}{2}$, and on the critical line, and only there, falls on the pole of ζ : $X_N(1) = H_N = \log N + \gamma_0 + O(N^{-1})$. These identities are consequences of $X_N - X_{N-1} = N^{-s}$ alone and hold equally at a hypothetical zero off the line, where $X_N(\rho) - Y_N(\rho) = \frac{1}{2}N^{-\rho} - \frac{\rho}{12}N^{-\rho-1} + O(N^{-\beta-3}) \rightarrow 0$ for every $\beta \in (0, 1)$ (Proposition 2.3); by themselves they cannot exclude such a zero.

3.2. The anatomy and the trichotomy. Taking real parts in Lemma 2.2, every Euler–Maclaurin term of $\zeta(s, n)$ acquires the real factor $n^{-\bar{s}}n^{-s} = n^{-2\sigma}$ and loses its phase. Put $A_s = 2 \operatorname{Re} \frac{1}{1-s} = \frac{2(1-\sigma)}{|1-s|^2}$; on the critical line $A_s = 1/\lambda$ with $\lambda = s(1-s) = \frac{1}{4} + t^2$.

Theorem 3.2 (Theorem A: anatomy and trichotomy). Let $0 < \sigma < 1$. For real $n \geq 1$ (Hurwitz extension) and every $m \geq 1$,

$$C_2(n, s) = \underbrace{A_s n^{1-2\sigma}}_{\text{structural mode}} + \underbrace{2n^{-\sigma} \operatorname{Re} [\zeta(s) n^{it}]}_{\text{wave of } \zeta} - n^{-2\sigma} - 2 \sum_{j=1}^m c_j \operatorname{Re} [(s)_{2j-1}] n^{-2\sigma-2j+1} + E_m, \quad (3.7)$$

with $|E_m| \leq \frac{2\|B_{2m+1}\|_\infty}{(2m+1)!} \frac{|(s)_{2m+1}|}{\sigma+2m} n^{-2\sigma-2m}$. In particular

$$W_n(s) = \frac{2n^{1-2\sigma}}{1-s} + 2\zeta(s)n^{-\bar{s}} - n^{-2\sigma} + O(|s|n^{-2\sigma-1}), \quad (3.8)$$

$$C_2(n, s) = A_s n^{1-2\sigma} + 2|\zeta(s)|n^{-\sigma} \cos(t \log n + \arg \zeta(s)) - n^{-2\sigma} + O(|s|n^{-2\sigma-1}),$$

with an absolute implied constant for $n \geq (|s| + 2)^2$, and at every point of the strip, whether or not $\zeta(s) = 0$,

$$\lim_{n \rightarrow \infty} C_2(n, s) = \begin{cases} +\infty, & 0 < \sigma < \frac{1}{2}, \\ \frac{1}{\frac{1}{4} + t^2}, & \sigma = \frac{1}{2}, \\ 0, & \frac{1}{2} < \sigma < 1. \end{cases} \quad (3.9)$$

In particular $\liminf_n C_2(n, s) > 0$ if and only if $\sigma \leq \frac{1}{2}$.

Proof. By Lemma 2.2, $X_{n-1}(s) = \zeta(s) - \zeta(s, n)$, and $\zeta(s, n)n^{-\bar{s}} = n^{-2\sigma} n^s \zeta(s, n)$ with

$$n^s \zeta(s, n) = \frac{n}{s-1} + \frac{1}{2} + \sum_{j=1}^m c_j (s)_{2j-1} n^{1-2j} + n^s R_m(s, n).$$

Multiply by $-2n^{-2\sigma}$, take real parts and use $-2 \operatorname{Re} \frac{1}{s-1} = A_s$; this is (3.7). For (3.8) take $m = 1$: the $j = 1$ term is $-\frac{\sigma}{6} n^{-2\sigma-1}$, and $|(s)_3| \leq |s|(|s| + 1)(|s| + 2) \leq |s|n$ when $n \geq (|s| + 2)^2$, so $|E_1| \ll |s|n^{-2\sigma-1}$; the same computation without the real part gives W_n . The exponents in (3.8) are ordered, $1 - 2\sigma > -\sigma > -2\sigma$, so the structural mode dominates at every point, and its limit is (3.9). $\square$

Status: proved.

The limit sees the line, not the zeros. The limit (3.9) is decided by the structural mode, which depends only on s : at $s = 0.7 + 20i$, where $\zeta(s) \neq 0$, $C_2 \rightarrow 0$; at $0.3 + 20i$, $C_2 \rightarrow \infty$; at $\frac{1}{2} + 20i$, which is not a zero, $C_2(N, s)$ equals 0.02357, 0.01827, -0.00452 , 0.00303 for $N = 10^3, \dots, 10^6$ and tends to $1/400.25 = 0.002498$, while at the zero $\frac{1}{2} + 14.134725i$ it equals 0.003999, 0.004899, 0.004989, 0.004998 and tends to 0.004999. A zero changes only the wave, of lower order. The structural mode is the tail $n^{1-s}/(1-s)$ of the harmonic series, the trace of the pole of ζ , that is, the continuous component Y_n of Section 2. The Davenport–Heilbronn function f has coefficients of mean zero, no pole and partial sums that converge throughout the strip, so its analogue of C_2 tends to 0 for every $\sigma > 0$: it has no trichotomy. The Epstein zeta function of Part 16 [CE16] keeps the pole and the trichotomy, by Theorem 3.11 below.

At a zero the spiral collapses. Geometrically, $\frac{1}{2}W_n$ winds around the centre $n^{1-2\sigma}/(1-s)$ with radius $|\zeta(s)|n^{-\sigma}$ and angle $t \log n$; on the critical line $\sqrt{n}(\frac{1}{2}W_n - \frac{1}{1-s})$ is a helix of radius $|\zeta(\frac{1}{2} + it)|$ (Figure 4). At a zero the helix collapses and $\arg W_n \rightarrow \arg \frac{1}{1-\rho}$. RH is the statement that every collapse occurs on a vertical axis, $\sigma = \frac{1}{2}$. The two properties come from different terms of (3.8), and the expansion alone does not link them: at a hypothetical zero with $\beta > \frac{1}{2}$ the sequence $W_n(\rho)$ converges to 0 along the tilted axis without any inconsistency. For $t = 17, 20, 14.1347$ and 21.0220 the expansion agrees with the computed W_n to within $4 \cdot 10^{-4}$ for $100 < n \leq 2 \cdot 10^4$. The orientation of the helix, whose signed radius is Hardy’s function $Z(t)$, is the subject of Part 2 [CE2].

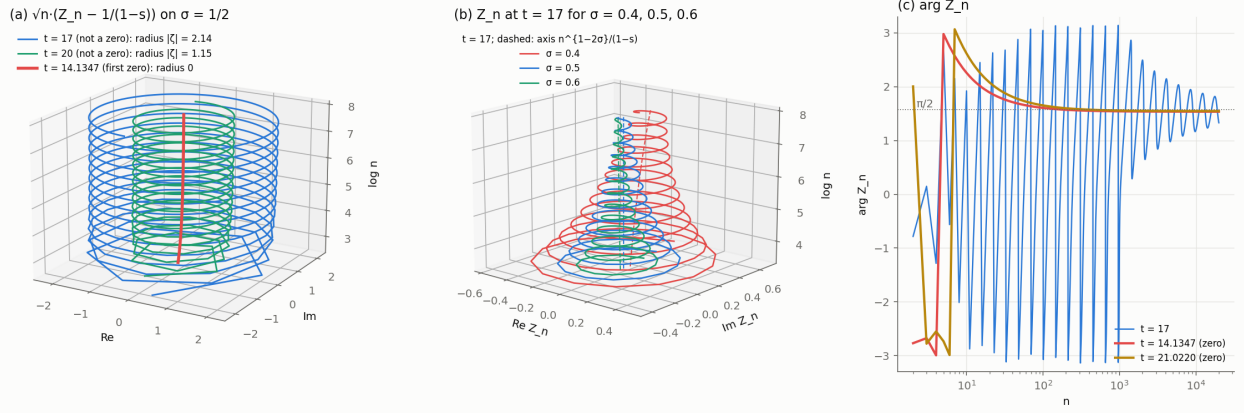


FIGURE 4. The wave $Z_n = \frac{1}{2}W_n = X_{n-1}(s)\overline{n}^{-s}$ (Theorem 3.2). (a) On the critical line $\sqrt{n}(Z_n - \frac{1}{1-s})$ is a helix of radius $|\zeta(s)|$, which degenerates to the axis at a zero. (b) At $t = 17$ the axis $n^{1-2\sigma}/(1-s)$ (dashed) is vertical only for $\sigma = \frac{1}{2}$. (c) $\arg Z_n$ converges to $\arctan(2\gamma)$ at zeros and rotates otherwise.

The complex wave shows the three cases at once (Figure 5). By (3.8), W_n is a centre $2n^{1-2\sigma}/(1-s)$, a rotating term of radius $2|\zeta(s)|n^{-\sigma}$, and a drift. At a zero the rotating term vanishes and W_n approaches the centre $2/(1-\rho)$ along a smooth curve, crossing the imaginary axis ($C_2 = 0$) at $n^* \approx \gamma^2 + \frac{1}{3}$ (Proposition 3.6). On the critical line away from a zero, W_n is a spiral of radius $2|\zeta(s)|n^{-1/2}$ around a fixed centre. Off the line the centre moves toward 0 and the radius decays like $n^{-\sigma}$. Only for $\sigma = \frac{1}{2}$ is the centre fixed; this is a property of the pole and not of RH, and for the Davenport–Heilbronn function, whose series has no pole, the centre is 0 on and off the line. What distinguishes a zero is the rate of collapse: $|W_n(\rho) - 2n^{1-2\beta}/(1-\rho)|$ decays like $n^{-2\beta}$, and RH states that every collapse has rate n^{-1} . For f , with $W_f(n, s) = 2X_{n-1}^f(s)n^{-\bar{s}}$ built from its partial sums, the collapse at a zero is a shrinking pentagon, from the period 5 of the coefficients, both at the zero $\frac{1}{2} + 43.0813i$ on the line and at the zero $0.808517 + 85.699348i$ off it (Figure 6); the two pictures look alike, and the distance to the centre decays with slopes -1.000 and $-1.617 = -2\beta$ in $\log n$. Status: the description follows from (3.8); the figures are numerical (`anc/spiral/figs_paper.py`, `anc/wave/w.py`, `anc/wave/w2.py`).

Proposition 3.3 (the wave of ζ detects and counts zeros). Let $0 < \sigma < 1$.

- (i) With $\mathcal{R}_N(s) = N^\sigma [C_2(N, s) - A_s N^{1-2\sigma} + N^{-2\sigma} + \frac{\sigma}{6} N^{-2\sigma-1}]$,

$$\mathcal{R}_N(s) = 2|\zeta(s)| \cos(t \log N + \arg \zeta(s)) + O_s(N^{-\sigma-3}), \quad (3.10)$$

so for $t \neq 0$, $\limsup_N \mathcal{R}_N = 2|\zeta(s)|$, $\liminf_N \mathcal{R}_N = -2|\zeta(s)|$, and $\zeta(s) = 0$ if and only if $\mathcal{R}_N(s) \rightarrow 0$.

- (ii) Put $F_n(s) = W_n(s) - \frac{2n^{1-2\sigma}}{1-s} + n^{-2\sigma} + \frac{\sigma}{6}n^{-2\sigma-1}$, so that $F_n(s) = 2\zeta(s)n^{-\bar{s}} + O(|s|^3 n^{-2\sigma-3})$. Let R be a closed rectangle in $0 < \sigma < 1$ with $\zeta \neq 0$ on ∂R . Then for all sufficiently large n the winding number of F_n along ∂R equals the number of zeros of ζ in R , counted with multiplicity.

Proof. (i) is (3.7) with $m = 2$, and $t \log N$ is dense modulo 2π because its increments tend to 0. (ii) The expansion of F_n is (3.7) with $m = 2$ written for W_n , uniformly on ∂R . With $\mu = \min_{\partial R} |\zeta| > 0$, the error is smaller than $|2\zeta(s)n^{-\bar{s}}| \geq 2\mu n^{-\sigma}$ on ∂R once n is large, so F_n and $2\zeta(s)n^{-\bar{s}}$ have the same winding number. The factor $n^{-\bar{s}} = e^{-\bar{s} \log n}$ has the continuous logarithm $-\bar{s} \log n$ on the plane, so its winding along a closed curve is 0, and the winding of ζ is the number of zeros by the argument principle. $\square$

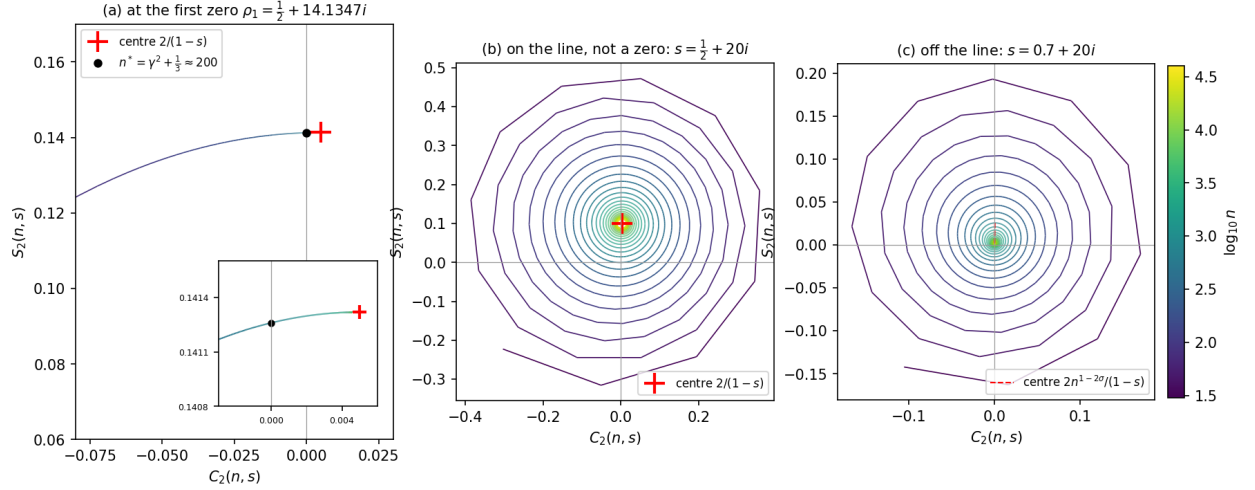


FIGURE 5. The complex wave $W_n(s) = C_2(n, s) + iS_2(n, s)$ for $3 \leq n \leq 40000$, coloured by $\log_{10} n$. (a) At $\rho_1 = \frac{1}{2} + 14.1347i$: no rotation; W_n reaches the centre $2/(1 - \rho_1)$ (cross) and crosses $C_2 = 0$ at $n^* \approx 200$ (dot; inset). (b) At $s = \frac{1}{2} + 20i$: a spiral of radius $2|\zeta(s)|n^{-1/2}$ around the fixed centre $2/(1 - s)$. (c) At $s = 0.7 + 20i$: the centre $2n^{1-2\sigma}/(1 - s)$ (dashed) moves toward 0.

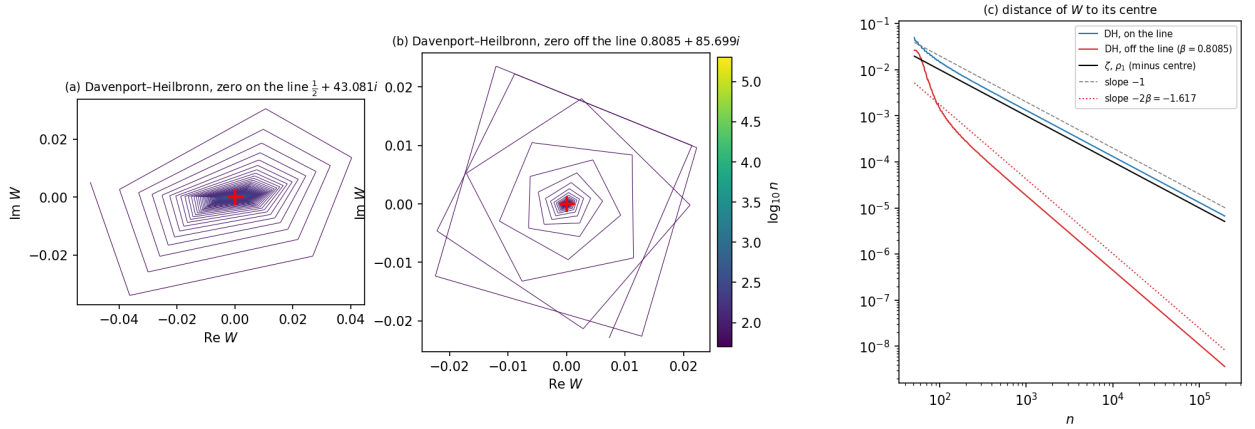


FIGURE 6. The wave W_f of the Davenport-Heilbronn function. (a) At the zero $\frac{1}{2} + 43.0813i$ on the line. (b) At the zero $0.808517 + 85.699348i$ off the line. (c) Distance to the centre (maximum over blocks of five consecutive n) for these two zeros and for ζ at ρ_1 , with slopes -1 and -2β .

Status: proved; the counts below are numerical (`anc/wave/wind.py`, `anc/wave/figc.py`).

F_n is not holomorphic in s , but its winding number is that of ζ : the argument principle can be read on the cross-energy, from the partial sums alone. With $n = 2000$ and the rectangles $0.1 < \sigma < 0.9$, $10 < t < T$, the winding number equals the number of zeros for all 21 values of T tested in $[13, 60.5]$, and is an integer to 10^{-14} (Figure 7a). For the Davenport–Heilbronn function the same argument applies to W_f itself, since its partial sums converge with error $O(n^{-\sigma})$; the winding with $n = 20000$ on cells of a grid over $[0.1, 0.95] \times [80, 90]$ locates its seven zeros there, five on the line and the two at $0.191483 + 85.699348i$ and $0.808517 + 85.699348i$ off it (Figure 7b), and with $n = 10^5$ the winding on the whole rectangle is 7, as for f . The count finds every zero in the region it is applied to, on or off the line, and therefore decides RH only up to the height computed.

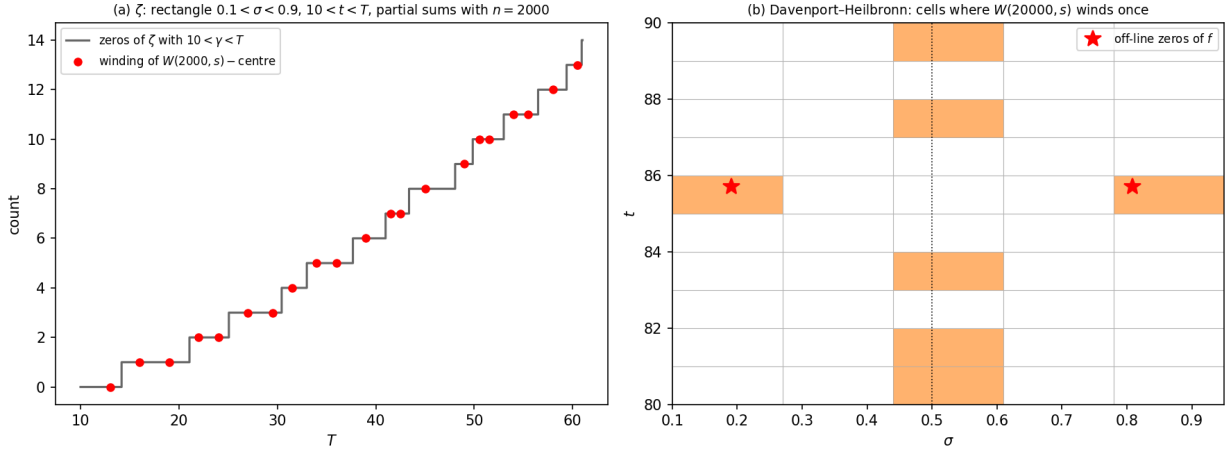


FIGURE 7. Counting zeros with the wave. (a) Winding number of F_{2000} on $0.1 < \sigma < 0.9$, $10 < t < T$ (dots) and the number of zeros of ζ with $10 < \gamma < T$. (b) Cells of $[0.1, 0.95] \times [80, 90]$ on which $W_f(20000, \cdot)$ winds once, for the Davenport–Heilbronn function; stars: its two off-line zeros.

A control with a planted zero. The wave of Proposition 3.3 is decided by the value $\zeta(s)$ and not by σ : the comparison “ $|X_N|^2$ is a wave and $|Y_N|^2$ a polynomial, so they can agree only on the line” uses neither the Euler product nor the functional equation, and it is defeated by the simplest function with the same continuous component and a zero off the line. Fix s_0 with $0 < \sigma_0 < 1$, $\sigma_0 \neq \frac{1}{2}$, and put $f_0(s) = \zeta(s) - \zeta(s_0)$, whose partial sums are $X_N(s) - \zeta(s_0)$ (the first coefficient is changed) and whose continuous component is the same Y_N . By (2.5), $X_N^C(s_0) - \zeta(s_0) = Y_N(s_0) - \frac{s_0}{12} N^{-s_0-1} + O(N^{-\sigma_0-3})$, so

$$\frac{|X_N^C(s_0) - \zeta(s_0)|^2}{|Y_N(s_0)|^2} - 1 = -\frac{\operatorname{Re} \lambda(s_0)}{6N^2} + O(N^{-4}),$$

exactly as at a zero of ζ (Proposition 2.4(iii)): there is no wave. At $s_0 = 0.3 + 20i$ and $s_0 = 0.7 + 20i$ with $N = 2 \cdot 10^4$ both sides equal $-1.6675 \cdot 10^{-7}$ (with X_N in place of X_N^C the endpoint term adds $(1 - \sigma_0)/N$). The cross-energy of f_0 at s_0 therefore has the anatomy of an off-line zero, with the trichotomy of Theorem 3.2 deciding its limit, although f_0 has no Euler product and no functional equation. An argument that sees only the wave and the envelope cannot exclude off-line zeros: its conclusion “the wave vanishes only on the line” is RH itself. Status: the expansion is proved; the numbers are numerical (`anc/valley/misc_checks.py`).

Proposition 3.4 (generic consequences of the structural mode). Let $0 < \sigma < 1$, $t \neq 0$.

- (i) $\lim_N \log |C_2(N, s)| / \log N = 1 - 2\sigma$, so $\sigma = \frac{1}{2}(1 - \lim_N \log |C_2(N, s)| / \log N)$ at every point.

- (ii) (The logarithmic spiral of X_N .) $\Delta \log |X_N(s)| = \frac{1-\sigma}{N} + O_s(N^{\sigma-2})$ and $\Delta \arg X_N(s) = -\frac{t}{N} + O_s(N^{\sigma-2})$, where Δ is the step $N \rightarrow N+1$. The partial sums follow a logarithmic spiral of pitch $\Delta \arg X_N / \Delta \log |X_N| \rightarrow -t/(1-\sigma)$. At the reflected point $1-\bar{s}$ the pitch is $-t/\sigma$, and the two pitches coincide if and only if $\sigma = \frac{1}{2}$.
- (iii) (Eventual monotonicity off the line.) If $\sigma \neq \frac{1}{2}$, then

$$\Delta C_2(N, s) = A_s(1-2\sigma)N^{-2\sigma}(1+O(N^{-1})) + O(|\zeta(s)||s|N^{-\sigma-1}) + O_s(N^{-2\sigma-1}), \quad A_s = \frac{2(1-\sigma)}{|1-s|^2},$$

so $\Delta C_2(N, s)$ has eventually the sign of $1-2\sigma$, whether or not s is a zero: $C_2(\cdot, s)$ is eventually increasing for $\sigma < \frac{1}{2}$ and eventually decreasing for $\sigma > \frac{1}{2}$.

Proof. (i) follows from (3.9) for $\sigma \neq \frac{1}{2}$, the structural mode being eventually dominant and positive, and from $C_2 \rightarrow 1/\lambda > 0$ for $\sigma = \frac{1}{2}$. (ii) By Proposition 2.4(i), $\log X_N = (1-s) \log N - \log(1-s) + \log(1+u_N)$ with $u_N = (1-s)\zeta(s)N^{s-1} + O_s(N^{-1})$ and $\Delta u_N = O_s(N^{\sigma-2})$; and $\Delta[(1-s) \log N] = (1-s)/N + O(N^{-2})$. Take real and imaginary parts. (iii) Difference the three terms of (3.7): $\Delta(A_s N^{1-2\sigma}) = A_s(1-2\sigma)N^{-2\sigma}(1+O(N^{-1}))$, $\Delta(2N^{-\sigma} \operatorname{Re}[\zeta(s)N^{it}]) = O(|\zeta(s)||s|N^{-\sigma-1})$, and the remaining terms change by $O_s(N^{-2\sigma-1})$. Since $-2\sigma > -\sigma-1$ for $\sigma < 1$, the structural difference dominates. $\square$

Status: proved. The crossover in (iii) occurs where $N^{1-\sigma} \approx |\zeta(s)||s|/(A_s|1-2\sigma|)$, which can be very large: for $s = 0.7 + 20i$ it is near $N \approx 10^{16}$. For small t it is observable. On $10^4 \leq N \leq 2 \cdot 10^6$, $\Delta C_2(N, s)$ has no sign change at ρ_1 (increasing), 33 sign changes at $\frac{1}{2} + 20i$, which is not a zero and where the wave keeps ΔC_2 oscillating (the last in this range at $N = 1\,710\,576$), none at $0.3 + i$ (increasing), and two at $0.8 + i$, the last at $N = 122\,511$, after which it is decreasing (`anc/c2sq/c2_squared_linear.py`, with [C1]). Monotonicity therefore carries no information about zeros off the critical line: at a hypothetical zero with $\sigma > \frac{1}{2}$, C_2 would be eventually decreasing to 0, as at every other point with that σ . Numerically, at $t = 7.3$ the pitch measured over $10^5 \leq N \leq 10^6$ and $10^6 \leq N \leq 4 \cdot 10^6$ is -10.433 and -10.435 for $\sigma = 0.3$ (prediction -10.429) and -14.68 and -14.80 for $\sigma = \frac{1}{2}$ (prediction -14.60); for $\sigma = 0.7$ the correction, of relative size $|1-s||\zeta(s)|N^{\sigma-1}/(1-\sigma)$, is still of order one at $N = 4 \cdot 10^6$ (`anc/c2core/exact_identities_check.py`). The equality of the pitches at s and $1-\bar{s}$ is a property of the pole term and characterizes the line, not the zeros.

3.3. Limits at zeros and reformulations of RH. Corollary 3.5 (limits at zeros, monotonicity and boundedness). Let $\zeta(\rho) = 0$ with $\rho = \beta + i\gamma$, $0 < \beta < 1$.

- (i) $C_2(n, \rho) = A_\rho n^{1-2\beta} - n^{-2\beta} + O(|\rho|n^{-2\beta-1})$, $A_\rho = \frac{2(1-\beta)}{|1-\rho|^2}$, so $\lim_n C_2(n, \rho)$ is $1/(\frac{1}{4} + \gamma^2)$,

0 or $+\infty$ according as $\beta = \frac{1}{2}$, $\beta > \frac{1}{2}$ or $\beta < \frac{1}{2}$. On the critical line this is the straight-line law

$$|X_n(\rho)|^2 = \frac{n + \frac{1}{2}}{\frac{1}{4} + \gamma^2} + O_\rho(n^{-1}) : \quad (3.11)$$

the slope is $\lim C_2$, and the line passes through $n = -\frac{1}{2}$. At a hypothetical zero with $\beta \neq \frac{1}{2}$, $|X_n(\rho)|^2 \sim n^{2-2\beta}/|1-\rho|^2$ is strictly concave ($\beta > \frac{1}{2}$) or convex ($\beta < \frac{1}{2}$).

- (ii) If $\beta = \frac{1}{2}$, then for every integer $n \geq 1$

$$C_2(n, \rho) = \frac{1}{\gamma^2 + \frac{1}{4}} - \frac{1}{n} - \frac{1}{12n^2} + \frac{\operatorname{Re}(\rho)_3}{360n^4} + R_n, \quad |R_n| \leq 2\kappa_B |(\rho)_5| n^{-5}, \quad (3.12)$$

with $\kappa_B = \|B_5\|_\infty/540 < 4.6 \cdot 10^{-5}$. Consequently $C_2(n+1, \rho) > C_2(n, \rho)$ for every n with $n^4/(n+1) > K(\rho) := |(\rho)_3|/90 + 4\kappa_B |(\rho)_5|$, so $C_2(\cdot, \rho)$ is eventually strictly increasing, bounded, and converges to $1/(\gamma^2 + \frac{1}{4})$ from below; the threshold is of order $(4\kappa_B)^{1/3}|\gamma|^{5/3} \approx 0.057|\gamma|^{5/3}$.

- (iii) The following are equivalent: RH; $\sup_n C_2(n, \rho) < \infty$ for every zero ρ ; $\lim_n C_2(n, \rho)$ is finite and positive for every zero; $\lim_n C_2(n, \rho) = 1/(\frac{1}{4} + \gamma^2)$ for every zero; $\liminf_n C_2(n, \rho) > 0$ for every zero; $C_2(n, \rho) = O_{\rho, \varepsilon}(n^\varepsilon)$ for every zero and every $\varepsilon > 0$.
- (iv) Unconditionally: (a) $\lim_n C_2(n, \rho) = 1/(\gamma^2 + \frac{1}{4})$ for every zero with $0 < \gamma \leq 3 \cdot 10^{12}$; (b) the same holds for a proportion greater than $\frac{5}{12}$ of all zeros, counted by height; (c) for every $\delta > 0$ the number of zeros with $0 < \gamma \leq T$ at which $C_2(n, \rho) \neq O(n^{2\delta})$ is $O_\delta(T) = o(N(T))$; (d) at every zero, $C_2(n, \rho) \leq (A_\rho + o(1)) n^{1-2/(R_0 \log \gamma)}$ with $R_0 = 5.573412$.

Proof. (i) is Theorem 3.2 with $\zeta(\rho) = 0$. For (3.11), Proposition 2.5(ii) gives $|X_n^C|^2 = n/\lambda + O(n^{-1})$, and $|X_n|^2 = |X_n^C|^2 + \operatorname{Re}(X_n^C \overline{n^{-\rho}}) + \frac{1}{4n}$ with $X_n^C \overline{n^{-\rho}} = Y_n n^{-\bar{\rho}} + O(n^{-2}) = \frac{1}{1-\rho} + O(n^{-2})$ and $\operatorname{Re} \frac{1}{1-\rho} = \frac{1}{2\lambda}$; the shape off the line follows from $X_n = Y_n + O(1)$. (ii) is (3.7) with $m = 2$ at $\sigma = \frac{1}{2}$ and $\zeta(\rho) = 0$: $-2c_1 \operatorname{Re} \rho = -\frac{1}{12}$, $-2c_2 = \frac{1}{360}$, and $|E_2| \leq \frac{2\|B_5\|_\infty}{120 \cdot 4.5} |(\rho)_5| n^{-5}$. Then $C_2(n+1, \rho) - C_2(n, \rho) \geq \frac{1}{n(n+1)} - \frac{|(\rho)_3|}{360} (n^{-4} - (n+1)^{-4}) - 4\kappa_B |(\rho)_5| n^{-5}$, and $n^{-4} - (n+1)^{-4} \leq 4n^{-5}$. (iii) If $\beta < \frac{1}{2}$ the limit is infinite by (i); if $\beta > \frac{1}{2}$, then $1 - \bar{\rho}$ is a zero with real part $1 - \beta < \frac{1}{2}$. By (i), the second and the last condition hold at ρ exactly when $\beta \geq \frac{1}{2}$, the fifth exactly when $\beta \leq \frac{1}{2}$, and the third and fourth exactly when $\beta = \frac{1}{2}$; since $\rho \mapsto 1 - \bar{\rho}$ maps β to $1 - \beta$, each of them holds at every zero exactly when every zero has $\beta = \frac{1}{2}$. (iv) By (i) the limit is $1/(\gamma^2 + \frac{1}{4})$ whenever $\beta = \frac{1}{2}$. (a) All zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ lie on the line [PT]. (b) More than five twelfths of the zeros lie on the line [PRZZ]. (c) By (i), $C_2(n, \rho) \neq O(n^{2\delta})$ requires $\beta < \frac{1}{2} - \delta$, and then $1 - \bar{\rho}$ has real part $> \frac{1}{2} + \delta$; the number of such zeros up to height T is $O_\delta(T)$ by the theorem of Bohr and Landau [T, §9.15], while $N(T) \sim \frac{T}{2\pi} \log T$. (d) There are no zeros with $\sigma \geq 1 - 1/(R_0 \log |t|)$, $|t| \geq 2$ [MoT]; applied to $1 - \bar{\rho}$ this gives $\beta \geq 1/(R_0 \log \gamma)$, and (i) gives the bound. $\square$

Status: proved; (iii) is a reformulation of RH.

For the 341 zeros with $\gamma < 600$, $C_2(20000, \rho)$ agrees with $1/(\gamma^2 + \frac{1}{4}) - \frac{1}{N} - \frac{1}{12N^2}$ to $1.4 \cdot 10^{-12}$, and $|X_n(\rho_1)|^2(\frac{1}{4} + \gamma_1^2) - n$ equals 0.516662, 0.501667, 0.500167, 0.500017 for $n = 10^3, \dots, 10^6$, converging to $\frac{1}{2}$ at rate $1/n$ as (3.11) says; at the non-zero point $\frac{1}{2} + 20i$ the same quantity is $-839.8, +3928.0, +3973.9, -43868.8$. The last decrease of $C_2(\cdot, \rho)$ occurs at $n \leq 0.952\gamma/2\pi$ for every one of the 341 zeros (median $0.932\gamma/2\pi$), the threshold of the approximate functional equation, while the proven threshold in (ii) is $n_0 = 6, 39, 513, 2413$ for $\gamma = 14.13, 49.77, 236.52, 599.55$; Part 2 [CE2] proves monotonicity down to that threshold and locates its onset by a Fresnel transition law. Part (iv) is the unconditional part of (iii): boundedness is proved for all zeros up to height $3 \cdot 10^{12}$ and for a positive proportion of all zeros, an exponent 2δ for almost all of them, and for every zero only the exponent $1 - 2/(R_0 \log \gamma)$, barely below 1. The step from these statements to every zero is RH. Part (iii) is a reformulation and not a constraint. By (3.9), $\liminf_n C_2(n, s) > 0$ is equivalent to $\sigma \leq \frac{1}{2}$ at every point s , zero or not, so the condition $\liminf_n C_2(n, \rho) > 0$ at every zero, called the “non-collapse lemma” in [C1], is not a lemma but RH itself; positivity alone is not the relevant notion, since $C_2(n, \rho) > 0$ eventually at every hypothetical zero, including those with $\beta > \frac{1}{2}$. At a zero, $C_2(n, \rho) = -2 \operatorname{Re}[n^{-\bar{\rho}} \zeta(\rho, n)]$ depends only on the position of ρ through the Hurwitz function, and a proof of RH along these lines would need an independent reason for the boundedness of $C_2(\cdot, \rho)$ at the zeros.

3.4. The crossing law. Proposition 3.6 (crossing law). Use the Hurwitz extension, and for $s = \sigma + it$ put

$$g_s(x) = -\operatorname{Re}[x^s \zeta(s, x)], \quad \nu(s) = \frac{1}{A_s} = \frac{|1-s|^2}{2(1-\sigma)}, \quad c(\sigma) = (1-\sigma) \left(\frac{1-\sigma^2}{30} - \frac{\sigma^2}{18} \right), \quad (3.13)$$

so that $C_2(x, s) = 2x^{-\sigma} \operatorname{Re}[\zeta(s)x^{it}] + 2x^{-2\sigma} g_s(x)$. There is an absolute constant t_0 such that for $0 < \sigma < 1$ and $|t| \geq t_0$:

- (i) g_s has exactly one zero $n^*(s)$ in $x \geq |t|^{3/2}$; it lies in $[0.99\nu, 1.01\nu]$, and g_s changes sign there from negative to positive;
- (ii) uniformly in σ ,

$$n^*(s) = \nu(s) + \frac{\sigma}{6} + \frac{c(\sigma)}{t^2} + O(t^{-4}). \quad (3.14)$$

If $\zeta(\rho) = 0$, then $C_2(x, \rho) = 2x^{-2\beta}g_\rho(x)$ for every real $x \geq 1$. Hence, for every zero with $|\gamma| \geq t_0$, $C_2(\cdot, \rho)$ has exactly one sign change in $x \geq |\gamma|^{3/2}$, upward, at $n^*(\rho)$, and

$$n^*(\rho) - \left(\gamma^2 + \frac{1}{3}\right) = (2\beta - 1) \left(\frac{\gamma^2}{2(1-\beta)} - \frac{1}{6}\right) + \frac{c(\beta)}{\gamma^2} + O(\gamma^{-4}). \quad (3.15)$$

In particular $n^*(\rho) = \gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} + O(\gamma^{-4})$ if $\beta = \frac{1}{2}$, so that $n^*(\rho) \cdot \lim_n C_2(n, \rho) = 1 + \frac{1}{12(\gamma^2+1/4)} + O(\gamma^{-4})$; for $\beta \neq \frac{1}{2}$ the deviation is $(2\beta - 1)\gamma^2/(2(1 - \beta)) + O(1)$, and for an off-line pair $\rho, 1 - \bar{\rho}$ the two deviations add up to $(2\beta - 1)^2\gamma^2/(2\beta(1 - \beta)) + O(\gamma^{-2}) \geq 0$.

Proof. Lemma 2.2 with $m = 2$ and $\operatorname{Re} \frac{1}{1-s} = \frac{1}{2\nu}$ give

$$g_s(x) = \frac{x}{2\nu} - \frac{1}{2} - \frac{\sigma}{12x} + \frac{\operatorname{Re}(s)_3}{720x^3} + E(x), \quad |E(x)| \leq \kappa' |(s)_5| x^{-4}, \quad \kappa' = \frac{\|B_5\|_\infty}{480} < 5.1 \cdot 10^{-5}.$$

For $x \geq |t|^{3/2}$ the last three terms are bounded by $\frac{1}{12}|t|^{-3/2} + (|t|+3)^3|t|^{-9/2}/720 + \kappa'(|t|+5)^5|t|^{-6}$, which is less than 0.002 for $|t| \geq 16$. Hence $g_s < 0$ for $|t|^{3/2} \leq x \leq 0.99\nu$ and $g_s > 0$ for $x \geq 1.01\nu$ (note $\nu \geq t^2/2$). On $[0.99\nu, 1.01\nu]$, $g'_s(x) = \frac{1}{2\nu} + \frac{\sigma}{12x^2} + O(|t|^3x^{-4} + |t|^6x^{-5})$, and since $x \asymp \nu \geq t^2/2$ the error is $O(t^{-2})$ times $\frac{1}{2\nu}$; so g_s is increasing there and has one zero, which proves (i). For (ii) take $m = 4$, so that the remainder contributes $O(|t|^9x^{-8}) = O(t^{-7})$ near $x = \nu$ and the Bernoulli terms with $j = 3, 4$ contribute $O(t^{-6})$. Write $n^* = \nu + \frac{\sigma}{6} + \varepsilon$. Expanding $\frac{\sigma}{12x} = \frac{\sigma}{12\nu} \left(1 - \frac{\sigma}{6\nu} + O\left(\frac{1+|\varepsilon|}{\nu^2}\right)\right)$ and using $\operatorname{Re}(s)_3 = -3(\sigma+1)t^2 + O(1)$, the equation $g_s(n^*) = 0$ becomes

$$\frac{\varepsilon}{2\nu} + \frac{\sigma^2}{72\nu^2} - \frac{(\sigma+1)t^2}{240\nu^3} = O(t^{-6}),$$

so $\varepsilon = -\frac{\sigma^2}{36\nu} + \frac{(\sigma+1)t^2}{120\nu^2} + O(t^{-4})$, and $\nu^{-1} = 2(1-\sigma)t^{-2} + O(t^{-4})$ gives $\varepsilon = c(\sigma)t^{-2} + O(t^{-4})$. All constants are independent of $\sigma \in (0, 1)$ because $\nu \geq t^2/2$. At a zero the wave term vanishes. Finally $\nu(\rho) - \gamma^2 - \frac{1}{4} = (2\beta - 1) \left(\frac{\gamma^2}{2(1-\beta)} - \frac{1}{4}\right)$ and $\frac{\beta}{6} - \frac{1}{12} = \frac{2\beta-1}{12}$, which gives (3.15); $c(\frac{1}{2}) = \frac{1}{180}$. On the line $\lim C_2 = 1/\lambda$ and $n^*/\lambda = 1 + \frac{1}{12\lambda} + O(\lambda^{-2})$. Adding (3.15) for β and $1 - \beta$ gives the statement on pairs. $\square$

Status: proved.

The constant $\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$ is the sum of the leading term and of the first Bernoulli term, $\frac{1}{12} = \frac{B_2}{2} = -\zeta(-1)$; the location and the height of the cross-energy at a zero on the line are the same number $\lambda = \frac{1}{4} + \gamma^2$, once as a position and once as a reciprocal. (The crossing was first observed near $\gamma^2 + \frac{1}{4}$; the correct point is $\gamma^2 + \frac{1}{3}$ [C1].) For the first eight zeros of ζ the root of $C_2(\cdot, \frac{1}{2} + i\gamma)$ agrees with $\gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2}$ to within $3 \cdot 10^{-7}$, against $3 \cdot 10^{-5}$ for $\gamma^2 + \frac{1}{3}$ (at $\gamma_1 = 14.1347$ the root is 200.1238157 and $\gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} = 200.1238159$); with integer n , $C_2(n, \rho_1)$ is negative for $n \leq 200$ and positive for $n \geq 201$. The range $x \geq |t|^{3/2}$ in (i) comes from the use of a fixed number of Bernoulli terms. Numerically the sign change is unique from about $x = |\gamma|/2\pi$ on: for $\gamma = 37.59, 79.34$ and 165.54 the function $C_2(\cdot, \rho)$ has 9, 28 and 98 sign changes in $1 < x < |\gamma|/2\pi$, the last at 4.02, 9.72 and 22.07, and then only the one at $n^*(\rho)$; below that threshold $X_{n-1}(\rho)$ is a short partial sum and oscillates, and the transition between the two regimes is the Fresnel law of Part 2 [CE2].

Proposition 3.6 holds for all zeros at once because it is a statement about the Hurwitz tail at an arbitrary point. For $s = 0.7 + 20i, 0.6 + 30i, 0.8 + 15i$ and $0.3 + 40i$, which are not zeros,

the root of g_s agrees with $\nu + \frac{\sigma}{6} + c(\sigma)t^{-2}$ to within $3 \cdot 10^{-8}$. The zero condition enters only by removing the wave of Theorem 3.2. The law therefore characterizes the critical line, on which $n^*(s) - t^2 - \frac{1}{3} \rightarrow 0$, and not the set of zeros: the location of the crossing determines β once ρ is known, but it does not exclude $\beta \neq \frac{1}{2}$, and a crossing $C_2(N, s) = 0$ does not imply $\sigma = \frac{1}{2}$ (an assertion of the author's earlier preprints that is not valid). The function $t \mapsto C_2(t^2 + \frac{1}{3}, \frac{1}{2} + it)$ vanishes at every zero on the critical line (within $9 \cdot 10^{-5}$ on a grid of step 0.002), but it has 110 sign changes on $10 \leq t \leq 50$, against 10 zeros, because of the wave. The crossing is a trace of the pole: for the Davenport–Heilbronn function there is no structural mode, and its analogue of C_2 has no such sign change. Status: numerical (`anc/crossing/nstar.py`, `anc/crossing/uniform.py`).

Remark 3.7 (the shape at a hypothetical off-line zero). At a zero with $\beta \neq \frac{1}{2}$, Theorem 3.2 gives $C_2(n, \rho) = A_\rho n^{1-2\beta} - n^{-2\beta} - \frac{\beta}{6} n^{-2\beta-1} + O_\rho(n^{-2\beta-3})$. The trajectory crosses upward at $n^*(\rho) = \nu(\rho) + \frac{\beta}{6} + \dots$ (Proposition 3.6). For $\beta > \frac{1}{2}$, the leading form $n^{-2\beta}(A_\rho n - 1)$ has a maximum at

$$N_m \sim \frac{2\beta}{A_\rho(2\beta-1)}, \quad \frac{N_m}{n^*(\rho)} \sim \frac{2\beta}{2\beta-1} = 1 + \frac{1}{2\beta-1}, \quad C_2(N_m, \rho) \sim \frac{N_m^{-2\beta}}{2\beta-1},$$

after which it decays to 0^+ ; as $\beta \downarrow \frac{1}{2}$ the maximum escapes to infinity, and at $\beta = \frac{1}{2}$ it is replaced by the asymptote $1/\lambda$. For $\beta < \frac{1}{2}$ the trajectory crosses upward and grows without bound. These features describe what an off-line zero would look like; they do not exclude one. [CE2, Proposition 2.15] proves the location and uniqueness of the maximum. Status: proved in outline here (leading order); see [CE2, Proposition 2.15].

Proposition 3.8 (the crossing law to all orders). At a zero $\rho = \frac{1}{2} + i\gamma$, with $\lambda = \frac{1}{4} + \gamma^2$, for every $m \geq 1$

$$C_2(x, \rho) = \frac{1}{\lambda} - \frac{1}{x} - 2 \sum_{j=1}^m \frac{B_{2j}}{(2j)!} P_j(\lambda) x^{-2j} + O_m(\lambda^m x^{-2m-2}), \quad P_j(\lambda) = \operatorname{Re}(\rho)_{2j-1}, \quad (3.16)$$

each P_j is a polynomial of degree $j-1$ in λ with rational coefficients ($P_1 = \frac{1}{2}$, $P_2 = 3 - \frac{9}{2}\lambda$, $P_3 = 60 - 125\lambda + \frac{25}{2}\lambda^2$), and

$$n^*(\rho) = \lambda + \frac{1}{12} + \frac{1}{180\lambda} - \frac{53}{5040\lambda^2} - \frac{683}{129600\lambda^3} + \frac{232349}{39916800\lambda^4} + O(\lambda^{-5}). \quad (3.17)$$

Proof. (3.16) is (3.7) with $\zeta(\rho) = 0$, $\sigma = \frac{1}{2}$ and $A_\rho = 1/\lambda$, with the remainder of Lemma 2.2 at order $m+1$. $\operatorname{Re}(\rho)_{2j-1}$ is the real part of a polynomial with real coefficients at $\frac{1}{2} + i\gamma$, hence a polynomial in $\gamma^2 = \lambda - \frac{1}{4}$, of degree $j-1$ since the top-degree term $(i\gamma)^{2j-1}$ is imaginary. Near $x = \lambda$ the j -th term is $O(\lambda^{-j-1})$, so $C_2(n^*, \rho) = 0$ can be solved term by term in $1/\lambda$ in exact rational arithmetic, and Proposition 3.6(i) identifies the root. $\square$

Status: proved; the coefficients are exact (`anc/crossing/nstar_series.py`).

On the critical line the crossing point depends on the zero only through λ : $n^*(\rho) = \mathcal{N}(\lambda)$ for a universal asymptotic series whose coefficients are fixed by the Bernoulli numbers. Off the line the coefficients involve β separately, as in (3.15). At the first four zeros the remainder after the term $1/(180\lambda)$, multiplied by λ^2 , is -0.01054 , -0.01053 , -0.01052 , -0.01052 , against $-\frac{53}{5040} = -0.010516$; at the first three, the next remainder multiplied by λ^3 is -0.00524 , -0.00526 , -0.00525 , against $-\frac{683}{129600} = -0.005270$. The same exact computation continues (3.17) with

$$+ \frac{1779759901}{217945728000 \lambda^5} - \frac{13216630789}{2615348736000 \lambda^6} - \frac{152069053220813}{8002967132160000 \lambda^7} + O(\lambda^{-8}),$$

and gives the coefficients of (3.16) at a zero as polynomials in λ up to x^{-8} : $-\frac{1}{12}$; $\frac{1}{120} - \frac{\lambda}{80}$; $-\frac{1}{252} + \frac{25\lambda}{3024} - \frac{5\lambda^2}{6048}$; $\frac{1}{240} - \frac{49\lambda}{4800} + \frac{91\lambda^2}{43200} - \frac{7\lambda^3}{172800}$ (`anc/crossing/nstar_series.py`, output in `nstar_series_output.txt`).

Remark 3.9 (the spectral variable). $\lambda = s(1-s)$ is the variable in which the Hilbert–Pólya picture places the zeros as eigenvalues: it is real for every zero on the line, while an off-line zero has $\text{Im } \lambda = \gamma(1-2\beta) \neq 0$; the crossing point $n^*(\rho)$ is real in both cases. Since $\xi(s) = \xi(1-s)$ there is an entire function Ψ with $\xi(s) = \Psi(s(1-s))$, and with Riemann’s kernel $\Phi_R(u) = 2 \sum_{n \geq 1} (2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}) e^{-\pi n^2 e^{2u}}$,

$$\Psi(\lambda) = \sum_{k \geq 0} \frac{(-1)^k b_k}{(2k)!} \left(\lambda - \frac{1}{4} \right)^k, \quad b_k = \int_{-\infty}^{\infty} \Phi_R(u) u^{2k} du > 0.$$

By (3.17) the numbers $n^* - \frac{1}{12}$ are, up to $O(\lambda^{-1})$, the zeros of an entire function defined by positive moments, without reference to the zeros. We computed $b_0, \dots, b_{400}$ by the trapezoidal rule (step $\frac{1}{400}$, 260 digits); $\Psi(\frac{1}{4}) = 0.4971207781883141 = \xi(\frac{1}{2})$, and the truncation after $K = 50, 100, 200, 400$ terms has its first 3, 10, 24 and 25 zeros (all those with $\lambda < 8000$) equal to $\frac{1}{4} + \gamma^2$ within 10^{-8} . RH is equivalent to Ψ having only real zeros, and in Pólya’s formulation [Po] to the hyperbolicity of all the Jensen polynomials $J^{d,n}(X) = \sum_{j \leq d} \binom{d}{j} \hat{\gamma}(n+j) X^j$, $\hat{\gamma}(j) = j! b_j / (2j)!$, proved for each fixed d and all sufficiently large n [GORZ]; all $J^{d,n}$ with $d \leq 150$ and $n \in \{0, 1, 2, 5, 10\}$ are hyperbolic. The crossing of the cross-energy measures the spectral variable with an explicit correction; this is a reading of a classical reformulation, not a new constraint. Status: the identity is classical; the computations are numerical (`anc/spectral/`).

3.5. The general anatomy. The anatomy of Theorem 3.2 uses only the size of the coefficient sums. Let (a_n) be real, $F(s) = \sum a_n n^{-s}$, $X_x^a(s) = \sum_{n \leq x} a_n n^{-s}$, and

$$K^a(x, s) = 2 \operatorname{Re} [x^{-\bar{s}} X_x^a(s)] = 2x^{-\sigma} \sum_{n \leq x} a_n n^{-\sigma} \cos \left(t \log \frac{x}{n} \right); \quad (3.18)$$

for $a_n \equiv 1$ and integer $x = N$, $K(N, s) = C_2(N, s) + 2N^{-2\sigma}$, which has the same limits.

Hypothesis (H). $\sum_{n \leq x} a_n = cx + P(x)$ with $c > 0$ and $P(x) = O(x^\theta)$ for some $0 \leq \theta < 1$.

Lemma 3.10. Under (H), for fixed s with $\theta < \sigma < 1$,

$$X_x^a(s) = \frac{cx^{1-s}}{1-s} + F(s) + O_s(x^{\theta-\sigma}), \quad F(s) = \frac{c}{s-1} + c + s \int_1^\infty P(u) u^{-s-1} du, \quad (3.19)$$

the last formula giving the continuation of F to $\sigma > \theta$, $s \neq 1$.

Proof. Stieltjes integration with $\sum_{n < 1} a_n = 0$, hence $P(1^-) = -c$, gives $X_x^a(s) = c \frac{x^{1-s}-1}{1-s} + P(x)x^{-s} + c + s \int_1^x P(u) u^{-s-1} du$. Split $\int_1^x = \int_1^\infty - \int_x^\infty$; for $\sigma > 1$ the formula for F reproduces the Dirichlet series. $\square$

Status: proved. For ζ : $c = 1$, $P(x) = [x] - x$, $\theta = 0$.

Theorem 3.11 (general anatomy and trichotomy). Under (H), for fixed s with $\theta < \sigma < 1$,

$$K^a(x, s) = \frac{2c(1-\sigma)}{(1-\sigma)^2 + t^2} x^{1-2\sigma} + 2x^{-\sigma} \operatorname{Re} [F(s)x^{it}] + O_s(x^{\theta-2\sigma}). \quad (3.20)$$

If moreover $\theta < \frac{1}{2}$, then since $1-2\sigma > -\sigma > \theta-2\sigma$ the limit of $K^a(x, s)$ is $+\infty$, $c/(\frac{1}{4} + t^2)$ or 0 according as $\sigma < \frac{1}{2}$, $\sigma = \frac{1}{2}$ or $\sigma > \frac{1}{2}$, whether or not $F(s) = 0$.

Proof. Multiply (3.19) by $2x^{-\bar{s}}$ and take real parts. $\square$

Status: proved.

The structural mode, the trichotomy and every consequence of Theorem 3.2 that depends only on the structural mode hold for any Dirichlet series with a simple pole and coefficient sums $cx + O(x^\theta)$, $\theta < \frac{1}{2}$, and its zeros enter only through the wave $F(s)x^{-\sigma+it}$. Part 16 [CE16] applies this to an Epstein zeta function with nonnegative coefficients, a pole, a functional equation and a real zero off the line, at which the whole anatomy holds and the limit of K^a is 0: no argument that uses only

these structures can prove the conditions of Corollary 3.5(iii) for ζ . The same construction for the functional reflection of a Beurling system is in [CE3].

3.6. The Mellin transform. Theorem 3.12 (Mellin transform of the cross-energy). For $0 < \sigma < 1$ and $\operatorname{Re} q > 2 - 2\sigma$,

$$\int_1^\infty \tilde{C}_2(x, s) x^{-q} dx = \zeta(2\sigma + q - 1) \left[\frac{1}{q - (1 - \sigma + it)} + \frac{1}{q - (1 - \sigma - it)} \right]. \quad (3.21)$$

The right side is meromorphic in $q \in \mathbb{C}$. Its poles are $q = 2 - 2\sigma$, with residue A_s , from the pole of ζ (the structural mode $A_s x^{1-2\sigma}$), and $q = 1 - \sigma \pm it$, with residues $\zeta(\sigma \pm it)$ (the wave $x^{-\sigma \pm it}$). In particular, for $t \neq 0$, $\zeta(s) = 0$ if and only if the singularity at $q = 1 - \sigma + it$ is removable, and at a zero the continued transform takes the value

$$\mathcal{M}[\tilde{C}_2](1 - \sigma + it) = \zeta'(\sigma + it). \quad (3.22)$$

Proof. Write $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and exchange sum and integral, which is justified by absolute convergence for $\operatorname{Re} q > 2 - 2\sigma$:

$$\begin{aligned} \int_1^\infty \tilde{C}_2(x, s) x^{-q} dx &= \sum_{j \geq 1} j^{-\sigma} \int_j^\infty x^{-\sigma-q} [(x/j)^{it} + (x/j)^{-it}] dx \\ &= \sum_{j \geq 1} j^{1-2\sigma-q} \left[\frac{1}{\sigma + q - 1 - it} + \frac{1}{\sigma + q - 1 + it} \right]. \end{aligned}$$

The sum is $\zeta(2\sigma + q - 1)$. At $q = 2 - 2\sigma$ the residue is $\frac{1}{1-\sigma-it} + \frac{1}{1-\sigma+it} = A_s$. At $q_0 = 1 - \sigma + it$ the first term has residue $\zeta(2\sigma + q_0 - 1) = \zeta(s)$; if $\zeta(s) = 0$ it tends to $\zeta'(s)$ as $q \rightarrow q_0$, and the second term tends to $\zeta(s)/(2it) = 0$. $\square$

Status: proved; (3.21) is checked to ten digits at $s = 0.6 + 7.3i$, $q = 2.5$ (`anc/c2core/exact_identities_check.py`).

The Mellin picture restates Theorem 3.2: the structural mode is the real pole and is present for every s , while a zero removes only the complex pole, and the derivative $\zeta'(\rho)$ is what remains there.

3.7. The cross-energy at a point that is not a zero. On the critical line away from a zero the wave $2|\zeta(s)|n^{-1/2} \cos(t \log n + \arg \zeta(s))$ of Theorem 3.2 oscillates around the level $1/\lambda - 1/n$, and $C_2(\cdot, s)$ changes sign as long as the wave exceeds that level. The last sign change measures $|\zeta(s)|$.

Proposition 3.13 (sign changes at non-zeros). There are absolute constants $t_1, c > 0$ such that, for $s = \frac{1}{2} + it$ with $|t| \geq t_1$ and $\zeta(s) \neq 0$, every real $x \geq 2\lambda$ with $C_2(x, s) \leq 0$ satisfies

$$x \leq 4|\zeta(\tfrac{1}{2} + it)|^2 \lambda^2 \left(1 + \frac{c}{\lambda}\right) + 2\lambda, \quad \lambda = \tfrac{1}{4} + t^2. \quad (3.23)$$

In particular every sign change of $C_2(\cdot, s)$ in $x \geq 2\lambda$, at integers or at real points, lies below the right side, and there is none if $8\lambda|\zeta(s)|^2 < 1 - c/\lambda$.

Proof. By (3.7) with $m = 2$ at $\sigma = \frac{1}{2}$,

$$C_2(x, s) = \frac{1}{\lambda} - \frac{1}{x} + 2x^{-1/2} \operatorname{Re} [\zeta(s)x^{it}] - \frac{1}{12x^2} + \frac{\operatorname{Re}(s)_3}{360x^4} + E_2, \quad |E_2| \leq 2\kappa|(s)_5|x^{-5}.$$

For $x \geq 2\lambda$ we have $\frac{1}{\lambda} - \frac{1}{x} \geq \frac{1}{2\lambda}$, while $\frac{1}{12x^2} \leq \frac{1}{48\lambda^2}$, $|(s)_3|x^{-4} \ll \lambda^{3/2}\lambda^{-4}$ and $|(s)_5|x^{-5} \ll \lambda^{5/2}\lambda^{-5}$; so the last three terms are at most $\frac{c_1}{\lambda^2} \leq \frac{2c_1}{\lambda} \left(\frac{1}{\lambda} - \frac{1}{x}\right)$ for an absolute c_1 . Hence $C_2(x, s) \leq 0$ implies $(\frac{1}{\lambda} - \frac{1}{x})\sqrt{x} \leq 2Z_1$ with $Z_1 = |\zeta(s)|/(1 - 2c_1/\lambda)$. With $u = \sqrt{x}$ this reads $u^2 - 2Z_1\lambda u - \lambda \leq 0$, so $u \leq Z_1\lambda + \sqrt{Z_1^2\lambda^2 + \lambda}$ and $x \leq (Z_1\lambda + \sqrt{Z_1^2\lambda^2 + \lambda})^2 \leq 4Z_1^2\lambda^2 + 2\lambda$. Since $u \mapsto (u^2 - \lambda)/(\lambda u)$ is increasing and equals $1/\sqrt{2\lambda}$ at $u = \sqrt{2\lambda}$, such an x exists only if $1/\sqrt{2\lambda} \leq 2Z_1$. $\square$

Status: proved. This sharpens the bound $x \leq (16 + o(1))|\zeta(\frac{1}{2} + it)|^2 \lambda^2$ of [C1, Prop. 4.6] by a factor 4, and turns the heuristic location of the last sign change in version 3 of the consolidated paper into an upper bound.

At a general point of the strip the same comparison of the structural mode $A_s n^{1-2\sigma}$ with the wave $2|\zeta(s)|n^{-\sigma}$ predicts the last sign change near

$$n_{\text{last}}(s) \approx \left(\frac{|\zeta(s)| |1-s|^2}{1-\sigma} \right)^{1/(1-\sigma)}, \quad (3.24)$$

which is $4|\zeta(s)|^2 \lambda^2$ on the critical line: (3.23) says that on the line this prediction is an upper bound, up to 2λ and a relative error $O(\lambda^{-1})$; off the line (3.24) is a heuristic. At a zero the wave is absent and only the crossing n^* of Proposition 3.6 remains: the crossing law is the limit of (3.24) as $|\zeta(s)| \rightarrow 0$. Equivalently, $d \log |C_2| / d \log n \rightarrow 1 - 2\sigma$ at every point carries only σ , and its poles, the sign changes, carry $|\zeta(s)|$.

At $s = \frac{1}{2} + 20i$, where $|\zeta(s)| = 1.1478424$, $4|\zeta|^2 \lambda^2 = 844\,281$, the right side of (3.23) is about 845\,082, and the bound $16|\zeta|^2 \lambda^2$ of [C1] is 3\,377\,125. We recounted the sign changes with the partial sums for $N < 3000$ and, for $3000 \leq N \leq 10^7$, with the splitting $C_2(N, s) = 2N^{-1/2} \operatorname{Re}[\zeta(s)N^{it}] - 2N^{-1} \operatorname{Re}[N^s \zeta(s, N)]$, in which the second term has no oscillating factor (the two evaluations agree to $4 \cdot 10^{-15}$, and $|C_2| \geq 1.7 \cdot 10^{-8}$ at both neighbours of every sign change). The integer sequence $C_2(N, \frac{1}{2} + 20i)$, $2 \leq N \leq 10^7$, has 78 sign changes, 44 of them with $N \geq 2\lambda = 800.5$, the last between $N = 682\,701$ and $682\,702$, and none afterwards, as (3.23) guarantees beyond 845\,082. The Hurwitz extension $C_2(x, s)$ on real $x \in [2, 10^7]$ has 82 sign changes, 15 of them below $x = 20$, where the real extension oscillates between consecutive integers, and the last at $x = 682\,699$. The two counts quoted earlier, 79 in [C1] and 82 in version 3 of the consolidated paper, correspond to these two conventions; the count for the integer sequence is 78. At nine points with $\sigma \in \{0.5, 0.6, 0.7\}$ and $t \in \{20, 30, 40\}$, the last sign change of the real extension lies between 0.75 and 0.96 times $n_{\text{last}}(s)$, for values from $6.8 \cdot 10^5$ to $3.4 \cdot 10^{12}$ (Figure 8). Status: numerical (`anc/c2core/sign_changes_recount.py`, `anc/c2core/fig_sign_changes.py`).

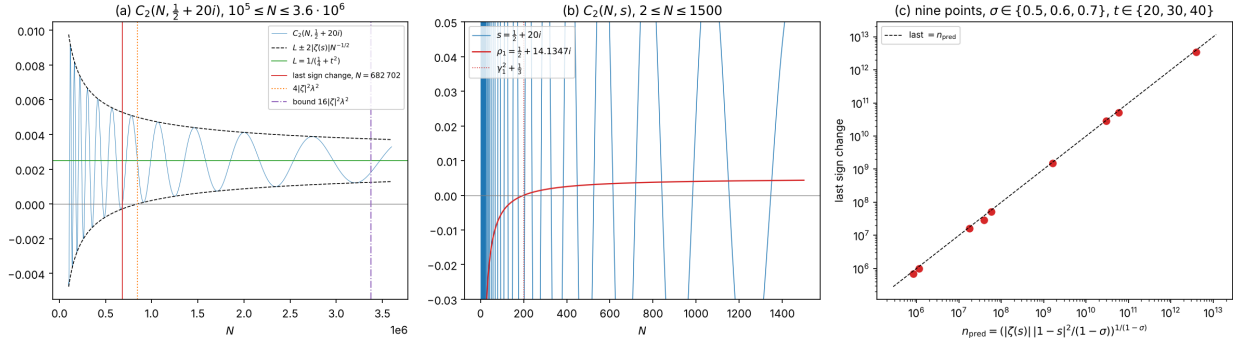


FIGURE 8. Sign changes of the cross-energy at points that are not zeros (linear axes in (a) and (b)). (a) $C_2(N, \frac{1}{2} + 20i)$ for $10^5 \leq N \leq 3.6 \cdot 10^6$, with the band $1/\lambda \pm 2|\zeta(s)|N^{-1/2}$; the last sign change at $N = 682\,702$, the prediction $4|\zeta(s)|^2 \lambda^2$ (dotted), which by Proposition 3.13 is the upper bound up to 2λ and a factor $1 + O(\lambda^{-1})$, and the bound $16|\zeta(s)|^2 \lambda^2$ of [C1] (dash-dotted). (b) $C_2(N, s)$ for $2 \leq N \leq 1500$ at $\frac{1}{2} + 20i$ and at ρ_1 , which crosses once, at $\gamma_1^2 + \frac{1}{3}$. (c) Last sign change of the Hurwitz extension against $n_{\text{last}}(s)$ of (3.24) at nine points of the strip (logarithmic axes).

Proposition 3.14 (the integrated detector). Let $s = \frac{1}{2} + it$, $t \neq 0$, $L = 1/\lambda = 1/(\frac{1}{4} + t^2)$, and let

$$\mathcal{H}(t, n) = \sum_{m \leq n} [L - C_2(m, s)] = nL - |X_n(s)|^2 + H_n \quad (3.25)$$

be the accumulated deficit of the cross-energy with respect to its limit (the second form is (3.3)).

- (i) (Half-limit law.) If $\zeta(s) = 0$, then $\mathcal{H}(t, n) = H_n - \frac{L}{2} + O_t(n^{-1}) = \log n + \gamma_0 - \frac{L}{2} + O_t(n^{-1})$; equivalently $\sum_{m \geq 1} [L - \frac{1}{m} - C_2(m, s)] = -\frac{L}{2}$.
- (ii) If $\zeta(s) \neq 0$, then

$$\mathcal{H}(t, n) - H_n = -\frac{L}{2} - |\zeta(s)|^2 - 2\sqrt{nL} |\zeta(s)| \cos(t \log n + \vartheta) + O_t(n^{-1/2}) \quad (3.26)$$

for a phase $\vartheta = \vartheta(t)$.

Consequently $\zeta(\frac{1}{2} + it) = 0$ if and only if $\mathcal{H}(t, n) - H_n$ converges, if and only if it is bounded, and at a zero

$$\lim_{n \rightarrow \infty} [\mathcal{H}(t, n) - \log n] = \underbrace{\gamma_0}_{\text{constant term of } \zeta \text{ at } s=1} - \underbrace{\frac{1}{2}L}_{\text{signature of the zero}}. \quad (3.27)$$

Proof. (i) is (3.11): $|X_n|^2 = nL + \frac{L}{2} + O(n^{-1})$. (ii) By (2.5), $X_n = Y_n + \zeta(s) + \frac{1}{2}n^{-s} + O_t(n^{-3/2})$. Here $|Y_n|^2 = nL$, $2 \operatorname{Re}(Y_n \cdot \frac{1}{2}n^{-\bar{s}}) = \operatorname{Re} \frac{1}{1-s} = \frac{L}{2}$, and $2 \operatorname{Re}(Y_n \bar{\zeta(s)})$ has modulus $2\sqrt{n} |\zeta(s)|/|1-s| = 2\sqrt{nL} |\zeta(s)|$ and phase $-t \log n$ up to a constant; the remaining terms are $|\zeta|^2 + O_t(n^{-1/2})$. The equivalences follow because $t \log n$ is dense modulo 2π . (3.27) is (i) with $H_n = \log n + \gamma_0 + O(n^{-1})$. $\square$

Status: proved.

In (3.27) the divergence $\log n$ and the constant $\gamma_0 = C(x^{-1})$ (Proposition 2.3(ii)) come from the pole of ζ , through the diagonal energy $X_n(1) = H_n$ of (3.3), which on the line falls on the pole; $-\frac{1}{2}L$ is the trace of the zero. The quantity $\mathcal{H}$ is the observable of the author's earlier work [E24], where it was claimed that $|\mathcal{H}(\gamma_1, n) - \mathcal{H}(\gamma_2, n)| < \varepsilon$ for any two zeros and all large n . By (i) the difference converges to $\frac{1}{2}[L(\gamma_2) - L(\gamma_1)]$, which is not zero; it is small, -0.001369 for the first two zeros, which is why it appeared to vanish numerically [Er], [C1]. Numerically, $\mathcal{H}(\gamma_1, n) - H_n = -0.002583, -0.002508, -0.002500, -0.002500$ for $n = 10^3, \dots, 10^6$, against $-L(\gamma_1)/2 = -0.002499$, and $\mathcal{H}(\gamma_1, n) - \mathcal{H}(\gamma_2, n) = -0.001369$ for every one of these n ; at the non-zero point $t = 20$, $\mathcal{H} - H_n$ takes the values $+2.10, -9.81, -9.93, +109.60$, as (3.26) predicts. The statement concerns each zero individually; it follows from the X - Y balance and says nothing about how the zeros are distributed.

3.8. The cross-energy at its own crossing. Let $\theta(t) = \operatorname{Im} \log \Gamma(\frac{1}{4} + \frac{it}{2}) - \frac{t}{2} \log \pi$ and $Z(t) = e^{i\theta(t)} \zeta(\frac{1}{2} + it)$, Hardy's function, which is real (Part 2 [CE2]).

Proposition 3.15 (the cross-energy at its own crossing). Let $0 < \sigma < 1$, $|t| \geq t_0$, let $n^*(s)$ be the crossing point of Proposition 3.6, and put $K(s) = C_2(n^*(s), s)$ (Hurwitz extension). Then:

- (i) $K(s) = 2n^*(s)^{-\sigma} \operatorname{Re} [\zeta(s) n^*(s)^{it}]$;
- (ii) on the critical line $K(\frac{1}{2} + it) \sqrt{n^*}/2 = Z(t) \cos \varphi(t)$, with $\varphi(t) = t \log n^*(\frac{1}{2} + it) - \theta(t)$;
- (iii) $\varphi'(t) = \frac{3}{2} \log t + O(1)$, so the zeros of K on the critical line up to height T are the zeros of Z together with $\varphi(T)/\pi + O(1) = \frac{3}{2\pi} T \log T + O(T)$ further zeros, asymptotically three for each zero of ζ ;
- (iv) the zeros of ζ in the strip with $|t| \geq t_0$ are the common zeros of K and of

$$K^\perp(s) = 2n^*(s)^{-\sigma} \operatorname{Im} [\zeta(s) n^*(s)^{it}],$$

and at a zero of multiplicity m the phase of $K + iK^\perp$ turns by $2\pi m$ along a small circle.

Proof. By (3.13), $C_2(x, s) = 2x^{-\sigma} \operatorname{Re}[\zeta(s)x^{it}] + 2x^{-2\sigma} g_s(x)$, and $g_s(n^*(s)) = 0$; this is (i). On the line $\zeta(\frac{1}{2} + it) = e^{-i\theta(t)} Z(t)$, which gives (ii). By Proposition 3.6, $n^*(\frac{1}{2} + it) = t^2 + O(1)$, and n^* is smooth in s by the implicit function theorem ($g'_s > 0$ at the root) with $\frac{d}{dt} n^*(\frac{1}{2} + it) = 2t(1 + O(t^{-2}))$; so $\frac{d}{dt} [t \log n^*] = 2 \log t + 2 + O(t^{-1})$, while $\theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O(t^{-2})$. Thus $\varphi'(t) = \frac{3}{2} \log t + O(1) > 0$ for large t , $\cos \varphi$ has $\varphi(T)/\pi + O(1)$ zeros up to T , and $N(T) \sim \frac{T}{2\pi} \log T$ gives the ratio 3. For (iv), $K + iK^\perp = 2n^{*- \sigma} n^{*it} \zeta(s)$, and $n^*(s)$ is a positive smooth function, so the factor in front of ζ is smooth and does not vanish, and the index of $K + iK^\perp$ at a zero equals that of ζ . $\square$

Status: proved; the figures are numerical (`anc/crossing_K/k.py`, `anc/crossing_K/fig.py`).

K can be read from the partial sums: at $s = \frac{1}{2} + 20i$, $n^* = 400.33$, and $C_2(400, s) = 0.0849$ from $X_{399}(s)$, against $K(s) = 0.0861$; the difference is the smooth part at the integer point. Identity (ii) holds numerically to 10^{-13} , and on $10 < t < 100$ there are 278 zeros of K on the line against 29 zeros of ζ (Figure 9). In the strip the sign of K forms bands, the nodal lines of K , and by (iv) each zero of ζ is a dislocation of this pattern, where a band begins or ends. In this picture RH says that all dislocations lie on $\sigma = \frac{1}{2}$. The function K is real but not the real part of a holomorphic function, because $n^*(s)$ is real; this is why $K = 0$ alone does not locate the zeros.

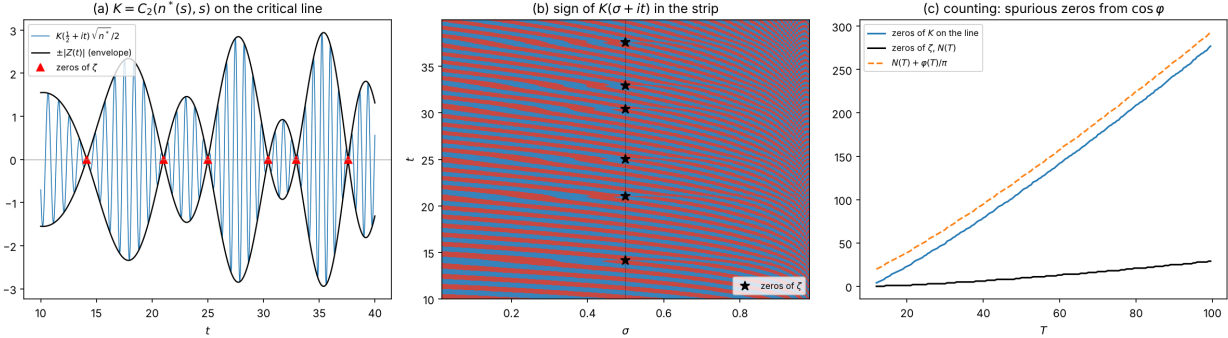


FIGURE 9. The cross-energy at its own crossing, $K(s) = C_2(n^*(s), s)$. (a) $K(\frac{1}{2} + it)\sqrt{n^*}/2$ and the envelope $\pm|Z(t)|$; triangles: zeros of ζ . (b) Sign of $K(\sigma + it)$ on $0 < \sigma < 1$, $10 < t < 40$; stars: zeros of ζ , at the dislocations of the band pattern. (c) Number of zeros of K on the critical line up to T , of Z , and $N(T) + \varphi(T)/\pi$.

3.9. What this section does not contain. The transition of $C_2(\cdot, \rho)$ at the threshold $n \approx \gamma/2\pi$, the Fresnel law for the onset of monotonicity and the monotonicity windows are in Part 2 [CE2], together with Hardy's function, the orientation of the helix, the extrema of Z , the Riemann–Siegel formula as a cross-energy sum, the detector of zeros on the critical line and the mollified cross-energy. The product $C_2(n, \rho)C_2(n, 1 - \bar{\rho})$ at a reflected pair, the functional reflection C_2^{FR} , its Wronskian and the dual range of Weil's form are the subject of [CE3]. The spiral of the partial sums at a zero, the zeros of the sections X_N and the circles of the primes are the subject of [CE4].

4. DISCUSSION

Two features of the cross-energy, established in this part, govern the whole series.

1. *The cross-energy sees the pole first.* The trichotomy of Theorem A, the fixed centre of the wave on the critical line, the crossing law and the anatomy of Theorem 3.11 are traces of the pole at $s = 1$. They hold for every Dirichlet series with a simple pole and regular coefficient sums, including the Epstein function of $m^2 + 100n^2$, which has a real zero off the line ([CE16, Theorem 2.2]). They locate the critical line, and they do not locate the zeros.
2. *The zeros enter through one wave.* By Theorem 3.2 the zeros enter C_2 only through the term $\zeta(s)N^{-\sigma+it}$. At a zero the wave vanishes, and what is left is the structural mode, whose collapse rate is $n^{-2\beta}$. The C-transform says the same thing additively: $\zeta(s)$ is the limit of $X_N - Y_N$, and a zero is a point of exact balance (Proposition 2.3). RH is therefore a statement about the rate of collapse at every zero: the collapse must be n^{-1} . No finite computation of C_2 can establish such a rate uniformly in the height.

4.1. Open problems.

- The sign changes of $C_2(\cdot, \frac{1}{2} + it)$ at a point that is not a zero stop by $4|\zeta|^2\lambda^2(1 + O(1/\lambda))$ (Proposition 3.13). Is the bound sharp, and does the off-line heuristic (3.24) hold?

We do not prove the Riemann hypothesis.

APPENDIX A. COMPENDIUM OF FORMULAS

This appendix collects the formulas used in the paper and some related ones, grouped by structure. Each group states its status: an exact identity, an asymptotic result, a criterion equivalent to RH, or a statement that would imply RH and is not proved. Throughout, $s = \sigma + it$, $\rho = \beta + i\gamma$ is a nontrivial zero, $\rho^* = 1 - \bar{\rho}$ is its reflection, and $X_N(s) = \sum_{n \leq N} n^{-s}$.

A.1. Partial sums. *Status: exact identities and classical asymptotics.*

$$X_N(s) - X_{N-1}(s) = N^{-s}, \quad -X'_N(s) = \sum_{n \leq N} \frac{\log n}{n^s}. \quad (\text{A.1})$$

For fixed $s \neq 1$ with $0 < \sigma < 1$, Euler–Maclaurin summation gives

$$X_N(s) = \frac{N^{1-s}}{1-s} + \zeta(s) + \frac{1}{2}N^{-s} + O_s(N^{-\sigma-1}), \quad (\text{A.2})$$

and at a zero

$$\begin{aligned} X_N(\rho) &= \frac{N^{1-\rho}}{1-\rho} + \frac{1}{2}N^{-\rho} + O_\rho(N^{-\beta-1}), \\ -X'_N(\rho) &= N^{1-\rho} \left[\frac{\log N}{1-\rho} - \frac{1}{(1-\rho)^2} \right] - \zeta'(\rho) + O_\rho(N^{-\beta} \log N). \end{aligned} \quad (\text{A.3})$$

The $X - Y$ decomposition. Put $X_N^C(s) = X_N(s) - \frac{1}{2}N^{-s}$ and $Y_N^C(s) = N^{1-s}/(1-s)$. By (A.2), for $0 < \sigma < 1$,

$$\zeta(s) = \lim_{N \rightarrow \infty} [X_N^C(s) - Y_N^C(s)], \quad D_N(s) := X_N^C(s) - Y_N^C(s) = \zeta(s) + O_s(N^{-\sigma-1}). \quad (\text{A.4})$$

The two components diverge separately for $\sigma < 1$; only their difference converges. The continuous component has

$$|Y_N^C(s)|^2 = \frac{N^{2-2\sigma}}{(1-\sigma)^2 + t^2}, \quad \frac{d}{dN} |Y_N^C(s)|^2 = \frac{2(1-\sigma)N^{1-2\sigma}}{(1-\sigma)^2 + t^2}, \quad (\text{A.5})$$

which is linear in N if and only if $\sigma = \frac{1}{2}$, with slope $1/(t^2 + \frac{1}{4})$. At a zero, $D_N(\rho) = O_\rho(N^{-\beta-1})$, hence

$$|X_N^C(\rho)|^2 - |Y_N^C(\rho)|^2 = O_\rho(N^{-2\beta}), \quad \frac{|X_N^C(\rho)|^2}{|Y_N^C(\rho)|^2} = 1 + O_\rho(N^{-2}). \quad (\text{A.6})$$

The two energies agree to leading order at every zero, on or off the critical line; their comparison therefore does not locate the zeros. More precisely, the first Bernoulli term gives $X_N^C(\rho)/Y_N^C(\rho) = 1 - \rho(1-\rho)/(12N^2) + O_\rho(N^{-4})$, so that on the critical line $|X_N^C(\rho)|^2/|Y_N^C(\rho)|^2 = 1 - \lambda/(6N^2) + O(N^{-4})$ with $\lambda = \frac{1}{4} + \gamma^2$; at ρ_1 and $N = 1600$, $N^2(X_N^C/Y_N^C - 1) = -16.6701 + 6 \cdot 10^{-6}i$ against $-\lambda/12 = -16.6700$. The normalised gap reads the abscissa of a zero (Section 2): for every zero ρ , trivial or not,

$$N^{2\beta}(|X_N^C(\rho)|^2 - |Y_N^C(\rho)|^2) \rightarrow \kappa(\rho) = -\frac{1}{6} \frac{\operatorname{Re} \lambda(\rho)}{|1-\rho|^2} = -\frac{1}{6} - \frac{(1-\beta)(2\beta-1)}{6|1-\rho|^2}, \quad (\text{A.6}')$$

so $\kappa(\rho) = -B_2$ if and only if $\beta = \frac{1}{2}$, and $\kappa(-2k) = k/(3(2k+1))$. On the critical line, at a zero and to all orders in N ,

$$|X_N^C(\rho)|^2 = \frac{N}{\lambda} - \frac{1}{6N} + \left(\frac{1}{90} + \frac{\lambda}{240} \right) \frac{1}{N^3} + \left(-\frac{1}{210} + \frac{19\lambda}{7560} + \frac{\lambda^2}{6048} \right) \frac{1}{N^5} + \cdots, \quad (\text{A.6}'')$$

every coefficient a rational polynomial in λ ; for two zeros $|X_N^C(\rho_1)|^2 - |X_N^C(\rho_2)|^2 = N(\lambda_1^{-1} - \lambda_2^{-1}) + (\lambda_1 - \lambda_2)/(240N^3) + O(N^{-5})$, and $N(|X_N^C(s)|^2 - N/\lambda)$ converges on the line if and only if $\zeta(s) = 0$.

A.2. The cross-energy. *Status: exact identities; asymptotics as in Theorem 3.2 and Corollary 3.5.*

$$C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} [N^{-\bar{s}} X_{N-1}(s)],$$

$$C_2(N, s) = 2N^{-\sigma} \sum_{n < N} n^{-\sigma} \cos \left(t \log \frac{N}{n} \right). \quad (\text{A.7})$$

From (A.2),

$$C_2(N, s) = A_s N^{1-2\sigma} + 2N^{-\sigma} \operatorname{Re} [\zeta(s) e^{it \log N}] - N^{-2\sigma} + O_s(N^{-2\sigma-1}), \quad (\text{A.8})$$

with $A_s = 2(1 - \sigma)/((1 - \sigma)^2 + t^2)$. At a zero the oscillating term is absent:

$$C_2(N, \rho) = A_\rho N^{1-2\beta} - N^{-2\beta} + O_\rho(N^{-2\beta-1}), \quad (\text{A.9})$$

and on the critical line Corollary 3.5 refines this to $C_2(N, \rho) = \frac{1}{\gamma^2+1/4} - \frac{1}{N} - \frac{1}{12N^2} + \frac{\operatorname{Re}(\rho)_3}{360N^4} + R_N$. Consequently

$$\lim_{N \rightarrow \infty} C_2(N, \rho) = \begin{cases} +\infty, & \beta < \frac{1}{2}, \\ \frac{1}{\gamma^2 + 1/4}, & \beta = \frac{1}{2}, \\ 0, & \beta > \frac{1}{2}, \end{cases} \quad \lim_{N \rightarrow \infty} \frac{\log C_2(N, \rho)}{\log N} = 1 - 2\beta, \quad (\text{A.10})$$

the second for $0 < \beta < 1$, and

$$\lim_{N \rightarrow \infty} N^{2\beta-1} C_2(N, \rho) = A_\rho = N^{2\beta-1} \frac{d}{dN} |Y_N^C(\rho)|^2, \quad (\text{A.11})$$

so that at a zero C_2 is asymptotically the growth rate of the continuous energy (A.5). The two leading terms of (A.9) alone would vanish at $N \approx \gamma^2 + \frac{1}{4}$; the Bernoulli term moves the crossing, and the crossing point of Proposition 3.6 is $n^* = \gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} + O(\gamma^{-4})$.

Criterion (Corollary 3.5(iii)). RH holds if and only if $0 < \lim_N C_2(N, \rho) < \infty$ for every ρ , if and only if $C_2(N, \rho) = O_{\rho, \varepsilon}(N^\varepsilon)$ for every ρ and $\varepsilon > 0$ (applied to ρ and ρ^* the latter forces $\beta = \frac{1}{2}$). Equivalent to RH, not proved.

A.3. Functional equation and derivatives. *Status: exact identities; they hold equally for hypothetical off-line zeros.* With $h(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})$ and $\xi(s) = h(s)\zeta(s)$,

$$\xi(s) = \xi(1-s), \quad \xi^{(k)}(s) = (-1)^k \xi^{(k)}(1-s), \quad \xi'(\rho^*) = -\overline{\xi'(\rho)}, \quad \xi''(\rho^*) = \overline{\xi''(\rho)}. \quad (\text{A.12})$$

At a simple zero, $\xi'(\rho) = h(\rho)\zeta'(\rho)$, $\xi''(\rho) = 2h'(\rho)\zeta'(\rho) + h(\rho)\zeta''(\rho)$, and

$$\frac{\xi''(\rho)}{\xi'(\rho)} = 2 \frac{h'(\rho)}{h(\rho)} + \frac{\zeta''(\rho)}{\zeta'(\rho)}, \quad \frac{\xi''(\rho^*)}{\xi'(\rho^*)} = -\overline{\left(\frac{\xi''(\rho)}{\xi'(\rho)} \right)}. \quad (\text{A.13})$$

A.4. Primes in the partial sums. *Status: exact identities, except (A.17), which is asymptotic.*

$$\log n = \sum_{d|n} \Lambda(d), \quad -\frac{\zeta'}{\zeta}(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} \quad (\sigma > 1), \quad (\text{A.14})$$

$$-X'_N(s) = \sum_{d \leq N} \frac{\Lambda(d)}{d^s} X_{[N/d]}(s), \quad (\text{A.15})$$

$$\frac{\partial C_2(N, s)}{\partial \sigma} = -(\log N) C_2(N, s) - 2 \operatorname{Re} \left[N^{-\bar{s}} \sum_{d < N} \frac{\Lambda(d)}{d^s} X_{[(N-1)/d]}(s) \right]. \quad (\text{A.16})$$

At a zero, inserting (A.3) in (A.15) and using $\sum_{d \leq N} \Lambda(d)/d = \log N - \gamma_0 + o(1)$,

$$-X'_N(\rho) - \frac{N^{1-\rho}}{1-\rho} \sum_{d \leq N} \frac{\Lambda(d)}{d} = N^{1-\rho} \left[\frac{\gamma_0}{1-\rho} - \frac{1}{(1-\rho)^2} + o(1) \right]. \quad (\text{A.17})$$

The $o(1)$ comes from the error in Mertens' formula and the replacement of $\lfloor N/d \rfloor$ by N/d ; it dominates the constant $-\zeta'(\rho)$ of (A.3), which is therefore not visible at this precision.

A.5. The prime error and its energy. *Status: classical identities and criteria; (A.22) is exact; the constant term is in [CE6].* With $\psi(x) = \sum_{n \leq x} \Lambda(n)$ and $x > 1$ not a prime power,

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}), \quad -\frac{x^{\rho}}{\rho} - \frac{x^{\bar{\rho}}}{\bar{\rho}} = -\frac{2x^{\beta}}{|\rho|} \cos(\gamma \log x - \arg \rho), \quad (\text{A.18})$$

the sum over zeros taken in symmetric order. In the variable $u = \log x$ the normalized error of the paper is $E(u) = e^{-u/2}(\psi(e^u) - e^u + \log 2\pi + \frac{1}{2} \log(1 - e^{-2u}))$ (Part 5 [CE5]), and a pair of zeros contributes an oscillation of amplitude $2e^{(\beta-1/2)u}/|\rho|$ and frequency γ . The Mellin transform of $\psi(x) - x$ is

$$\int_1^{\infty} (\psi(x) - x)x^{-s-1} dx = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} \quad (\sigma > 1), \quad (\text{A.19})$$

and RH holds if and only if this function continues holomorphically to $\sigma > \frac{1}{2}$, if and only if $\psi(x) - x = O_{\varepsilon}(x^{1/2+\varepsilon})$ for every $\varepsilon > 0$.

Energy. Put

$$I(X) = \int_1^X \frac{(\psi(x) - x)^2}{x^2} dx = \int_0^{\log X} E(u)^2 du, \quad Q(X) = \frac{I(X)}{\log X}. \quad (\text{A.20})$$

By [CE5, Theorem 2.2] and [CE5, Theorem B], RH holds if and only if $\sup_{X \geq 2} Q(X) < \infty$, and then $Q(X) \rightarrow \sum_{\rho} m_{\rho}^2 |\rho|^{-2}$, which equals $C_B = 2 + \gamma_0 - \log 4\pi$ when the zeros are simple. If $I(X) = O(\log X)$, then $\int_1^{\infty} |\psi(x) - x|x^{-\sigma-1} dx < \infty$ for $\sigma > \frac{1}{2}$ by the Cauchy-Schwarz inequality on dyadic intervals, which excludes zeros with $\beta > \frac{1}{2}$. Expanding the square,

$$\int_1^X \frac{\psi(x)^2}{x^2} dx = \sum_{m, n \leq X} \Lambda(m)\Lambda(n) \left(\frac{1}{\max(m, n)} - \frac{1}{X} \right), \quad (\text{A.21})$$

$$I(X) = \sum_{m, n \leq X} \Lambda(m)\Lambda(n) \left(\frac{1}{\max(m, n)} - \frac{1}{X} \right) - 2 \sum_{n \leq X} \Lambda(n) \log \frac{X}{n} + X - 1. \quad (\text{A.22})$$

The double sum splits into the diagonal $D(X) = \sum_{n \leq X} \Lambda(n)^2 (\frac{1}{n} - \frac{1}{X}) \sim \frac{1}{2}(\log X)^2$ and the cross terms $T(X) = 2 \sum_{n \leq X} \Lambda(n)\psi(n^-)(\frac{1}{n} - \frac{1}{X})$. Identity (A.22) expresses the energy through prime powers alone, without reference to ζ or its zeros; the terms on the right are of size X , and the statement $I(X) = O(\log X)$ asks for their cancellation down to $O(\log X)$. At $X = 10^7$, $D(X) = 113.21$, $T(X) = 9999877.64$, $2 \sum_{n \leq X} \Lambda(n) \log \frac{X}{n} = 19999986.59$ and $X - 1 = 9999999$, and they cancel to $I(10^7) = 3.2558$: about six and a half digits (`anc/valley/misc/checks.py`). It is a classical reformulation of the boundedness in [CE5, Theorem B], not a new constraint.

The constant term. Under RH, simple zeros and an additional spacing condition on the zeros, $I(X) = C_B \log X + K + \Phi(\log X) + o(1)$ with Φ the almost periodic interference of pairs of zeros; under RH alone K has a closed form without zeros and $K = 2.512479790006 \dots$ (certified in interval arithmetic). The formula for K , the hypotheses it needs, the limiting distribution of the fluctuations and the numerics from the primes up to 10^9 are in [CE6].

A.6. The two factorizations. *Status: a summary of exact identities of the paper; the last line of the figure is the open step.* In the critical strip ζ has an additive factorization, $\zeta = \lim_N [X_N^C - Y_N^C]$ (A.4), equivalently the C-transform of 1, and a multiplicative one, $\zeta = E_g^* Z_g$, the smoothed continuation of the Euler product after Gonek, Hughes and Keating [GHK] ([CE11], [CE8]). The cross-energy links them: $X_{N-1} = \frac{1}{2} N^s W_N$ with $W_N = C_2 + iS_2$, and the charges of $\Phi = \log |X^C - Y^C|$ give Z_g [CE11, Prop. 25 and Rem. 26]. Figure 3 collects the formulas and the equivalent forms of RH, together with the two controls: the Davenport–Heilbronn function has the additive side and the symmetry without an Euler product, and the Beurling systems of [CE11, CE17] have the multiplicative side without the integers $1, 2, 3, \dots$; both have zeros off the line.

The map of these factorizations, with the bridge C_2 between them and the equivalent forms of RH, is Figure 3.

Primes minus integers. The function $F(s) = 1 - \zeta(s) - \frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 2} (\Lambda(n) - 1)n^{-s}$ ($\sigma > 1$) is the difference of the C-transforms of Λ and 1, and its three forms, in which the pole cancels identically, are Proposition 2.8; their existence in $\sigma > \frac{1}{2}$ is RH.

A.7. Finite parts at a zero. *Status: exact identities from the Hadamard product and the functional equation; they hold for hypothetical off-line zeros.* With $F = 1 - \zeta - \zeta'/\zeta$, at a simple zero $F(s) = -1/(s - \rho) + c_0(\rho) + O(s - \rho)$ with $c_0(\rho) = 1 - \zeta''(\rho)/(2\zeta'(\rho))$. With $G(s) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2} \log \pi + \frac{1}{2} \psi(\frac{s}{2})$ and $\xi'/\xi(s) = \sum_{\rho} 1/(s - \rho)$ in symmetric order,

$$c_0(\rho) = 1 + G(\rho) - \sum_{\rho' \neq \rho} \frac{1}{\rho - \rho'}. \quad (\text{A.23})$$

If $\beta = \frac{1}{2}$ the real parts of the sum cancel in pairs, so $\text{Re } c_0(\rho) = 1 - \frac{1}{2} \log \pi + \frac{1}{2} \text{Re } \psi(\frac{1}{4} + \frac{i\gamma}{2})$ whether or not RH holds; for an off-line zero the reflection alone contributes $-1/(2\delta)$ to $\text{Re } c_0(\rho) - 1 - \text{Re } G(\rho)$. Numerically the identity holds to $8 \cdot 10^{-8}$ at the first 100 zeros of ζ and to $3 \cdot 10^{-8}$ at 122 on-line zeros of f (with its own gamma factor), and the four off-line zeros of f below height 200 lie below by 1.88, 3.46, 6.81, 2.51, against $1/(2\delta) = 1.62, 3.32, 6.72, 2.23$. On the line the imaginary part is

$$\text{Im } c_0(\rho) = \sum_{\gamma' \neq \gamma} \frac{1}{\gamma - \gamma'} + \text{Im } G(\frac{1}{2} + i\gamma) = \frac{Z''(\gamma)}{2Z'(\gamma)}, \quad (\text{A.24})$$

the force of the other zeros along the line, which drives the de Bruijn–Newman flow [Pm]; it correlates with $1/g_- - 1/g_+$, $g_{\pm}$ the gaps to the neighbours, with coefficient 0.98 for ζ and 0.96 for f .

A graded form of [CE10, Theorem 3.2]. For $0 < \eta \leq \frac{1}{2}$, ζ has no zeros in $\sigma > 1 - \eta$ if and only if $S(\sigma, t) = \text{Re } \xi'/\xi(s) > 0$ there, if and only if

$$\text{Re } F(s) < \Phi(s) := 1 - \text{Re } \zeta(s) + \text{Re } G(s) \quad (\sigma > 1 - \eta), \quad (\text{A.25})$$

since $S = \Phi - \text{Re } F$ and $S = \sum_{\rho} (\sigma - \beta)/|s - \rho|^2$. On the line $\text{Re } F = \Phi$; the right side is explicit, of size $\frac{1}{2} \log \frac{t}{2\pi}$, and the left side is the sum over the primes minus the integers.

Jensen moment and the Euler–Caceres bound. The Jensen moment of ζ/E_X^* at a centre s_0 with $\text{Re } s_0 > \frac{1}{2}$, its law $\kappa_m/\log X$ under RH, the explosion when $\Theta > \frac{1}{2}$ is attained, and the resulting criteria are in [CE11].

A.8. Statements that would imply RH. *Status: each is equivalent to RH or implies it; none is proved.* For every zero ρ and every $\varepsilon > 0$:

$$C_2(N, \rho) = O_{\rho, \varepsilon}(N^{\varepsilon}), \quad X_N(\rho) = O_{\rho, \varepsilon}(N^{1/2+\varepsilon}), \quad (\text{A.26})$$

and, for the primes,

$$\psi(x) - x = O_{\varepsilon}(x^{1/2+\varepsilon}), \quad I(X) = O(\log X), \quad \sum_{\rho} \frac{1}{|\rho|^2} = 2 + \gamma_0 - \log 4\pi. \quad (\text{A.27})$$

The last is the vanishing of the non-holomorphic gap $\mathcal{Q}$ of [CE5, Proposition 2.5]. The two energy structures of the paper, the cross-energy $C_2(N, \rho)$ of the integers and the energy $I(X)$ of the prime powers, are linked by [CE5, Theorem B]; what is missing is an inequality, independent of the location of the zeros, that excludes the growth $N^{1-2\beta}$ of $C_2(N, \rho)$ for $\beta < \frac{1}{2}$.

A.9. Statements not included as results. The assertions of the author’s earlier preprints that are not valid in general are listed, with counterexamples, in Part 16 [CE16] (“What does not hold”).

APPENDIX B. NUMERICAL METHODS AND DATA

All computations use double precision in Python (NumPy, SciPy), except where a number of digits is stated; those use mpmath or Python’s `decimal` arithmetic. The zeros of ζ with $0 < \gamma < 600$ (341 ordinates) and the first 1000 ordinates were computed once and are reused throughout; where a statement depends on the precision of an ordinate ([CE2, Remark 2.12]), the ordinates are refined to 10^{-12} . ζ is evaluated by Euler–Maclaurin summation with Bernoulli corrections, and the Davenport–Heilbronn function f through Hurwitz zeta functions; its eight off-line zeros with $\gamma < 400$ are those listed in [CE17]. $\Lambda(n)$ is computed by sieve up to 10^8 (up to 10^9 for ψ). Every value in the tables was obtained by the script cited in its Status line, and every script prints the numbers quoted in the text.

- *Section 2.* The C-transform of 1, Λ and the coefficients of $-f'/f$ at $t = 85.7$, with exponents fitted on dyadic blocks from 2^{12} (`ctransform/`); the resonance exponents (`resonance/`); the X – Y energies (`xy_energy/`).
- *Section 3.* The exact identities of Proposition 3.1 and the Mellin transform, checked to 10^{-10} (`c2core/exact_identities_check.py`); the sign changes at non-zeros, recounted on the integers $2 \leq N \leq 10^7$ with partial sums below 3000 and the exact split into ζ plus a smooth Hurwitz part above (`c2core/sign_changes_recount.py`); the wave and its winding (`wave/`); the crossing law and its all-orders series in exact rational arithmetic (`crossing/`, `crossing_K/`); the spiral of X_N (`spiral/`); the spectral variable (`spectral/`); the sign changes of ΔC_2 for $10^4 \leq N \leq 2 \cdot 10^6$ (`c2sq/`); the control $\zeta - \zeta(s_0)$ with a planted zero (`valley/misc_checks.py`).
- *Appendix A.* The cancellation budget of (A.22) at $X = 10^7$ (`valley/misc_checks.py`).

How this work was produced. The cross-energy C_2 and the research program are the author’s, beginning with his preprints of 2017 and the series [C1]–[C8]. The author used Claude (Anthropic), an AI assistant, as a research and writing aid: to derive and check proofs, to write and run the Python computations, to test claims against the Davenport–Heilbronn control and against counterexamples, to search the literature, and to draft and edit the text and figures. In preparing this series, the material of the consolidated paper (version 3), of the earlier series [C1]–[C8] and of the author’s manuscript on the asymptotic balance of partial sums was inventoried result by result, regrouped into the seventeen parts of the series (Table 1), and checked again; the corrections found in the process are stated where they apply and listed in Part 16 [CE16]. Several claims of the author’s earlier preprints were found to be incorrect and are not used. The author directed the work, checked the results and is responsible for the content.

Data availability. The scripts are included in the ancillary directory `anc/` of the source package, one subdirectory per topic as listed above. The earlier preprints [C1]–[C8] and the versions 1–4 of the consolidated paper are deposited on Zenodo under the DOIs listed in the references.

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