

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 12: WEIL POSITIVITY AT THE WINDOWS OF THE ZEROS

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ABSTRACT. By Weil’s criterion the Riemann hypothesis (RH) holds if and only if a quadratic form in the pole, the Gamma factor and the prime powers is nonnegative. On test functions supported in a window of length L only the prime powers up to e^L enter, and the form is a rank-two pole term plus the energy of $\hat{\varphi}$ against the prime partial sum on the critical line. We study the smallest value $\lambda_{\min}(L)$ of the form on such windows. In a fixed finite basis it fits $10^{-6.3L^2}$; in numerically stabilized bases the computed upper bound is positive and empirically close to $10^{-(5.42e^L - 12.6)}$ for $2 \leq L \leq 5$. Its size is a property of the density of the zeros: regular points with that density give the same values, and curves over finite fields give the constant $2 \log p$. The primes create the positivity: at $L = 1.25$ the primes 2, 3 turn a negative form positive and move the zeros of the minimiser onto the first zeros of ζ , and in every window tested the form becomes negative when the most recent prime power is removed. For the Davenport–Heilbronn function f the form is negative beyond a threshold $L^* \approx 3.425$ ($L^* \leq 3.43$ certified in interval arithmetic), once the coefficients up to $n \approx 31$ enter; beyond it a real zero of its minimiser locks onto the off-line ordinate. At large height, two windows at the ends of the support detect an isolated zero $\frac{1}{2} + \delta + i\gamma$, $0 < \delta < \frac{1}{2}$, with the prime powers up to $(C \log \gamma)^{1/\delta}$ (proved for γ large under an isolation hypothesis; explicit for $\delta \geq 0.1$ up to 10^{1000}); a related two-window positivity holds for ζ at 2400 sampled heights up to $3 \cdot 10^6$ and fails for a random multiplicative model. Prime translations are isometries of the form; what is missing over $\mathbb{Q}$ is a finite set of lags closed under them. We do not prove the Riemann hypothesis.

1. INTRODUCTION

1.1. Weil’s criterion on windows. Let $\varphi \in C_c^\infty(\mathbb{R})$, $\tilde{\varphi}(u) = \overline{\varphi(-u)}$, $g = \varphi * \tilde{\varphi}$ and $\hat{\varphi}(r) = \int \varphi(u) e^{iru} du$. The Riemann–Weil explicit formula expresses the sum $\sum_\rho \hat{\varphi}(\gamma_\rho) \overline{\hat{\varphi}(\bar{\gamma}_\rho)}$ over the non-trivial zeros ρ , $\gamma_\rho = -i(\rho - \frac{1}{2})$, as a pole term, an archimedean term and a sum over the prime powers; call the latter side $W(\varphi)$. Under RH every γ_ρ is real and $W(\varphi) = \sum_\rho |\hat{\varphi}(\gamma_\rho)|^2 \geq 0$, and Weil proved the converse [We2], [Bo]. The criterion, and the identification of the energy slope $\partial_\sigma \log |\xi|$ with W at a Poisson kernel together with its exact truncation, are in [CE10, Sect. 3]; we use them freely.

If φ is supported in an interval of length L , then g is supported in $[-L, L]$ and only the prime powers $n \leq e^L$ enter. This paper is about the smallest value of $W(\varphi)/\|\varphi\|_2^2$ over such φ ,

$$\lambda_{\min}(L) = \inf \left\{ \frac{W(\varphi)}{\|\varphi\|_2^2} : \text{supp } \varphi \subset [-\frac{L}{2}, \frac{L}{2}] \right\},$$

which is a finite computation in the primes up to e^L for each L , and nonnegative for every L if and only if RH holds (Proposition 1). We compute it for ζ and for the Davenport–Heilbronn function f , which has a functional equation of the same type, no Euler product and zeros off the critical line [DH], [CE17]. Throughout, $X_N(s) = \sum_{k \leq N} k^{-s}$ and $C_2(n, s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma}$ are as in [CE1], and $C_B = 2 + \gamma_0 - \log 4\pi$.

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1.2. Results. The paper collects the computations on Weil's form that were scattered through version 3 of [CP] and the preprint [C7], with the corrections recorded below.

- (1) *Structure* (Sect. 2). $W(\varphi)$ is a rank-two pole term plus $\frac{1}{2\pi} \int |\hat{\varphi}|^2 \Psi_L$, where Ψ_L is built from the Gamma factor and the prime partial sum $\sum_{n \leq e^L} \Lambda(n) n^{-1/2-ir}$; the prime part is the cross-energy of φ with its translates by $\log n$; $\lambda_{\min}(L)$ is independent of the position of the window and nonincreasing in L (Proposition 1; proved, elementary).
- (2) *The margin* (Sect. 3). In a fixed sine basis $\lambda_{\min}(L) \approx 10^{-6.3L^2}$, computed from the primes and from the zeros; in larger bases with 50 to 120 digits the values are smaller and fit $10^{-5.0L^{2.53}}$ on $1.25 \leq L \leq 3$, so the law $10^{-6.3L^2}$ belongs to the basis and is only an upper bound for the true margin. In bases large enough for the values to stabilize numerically, with the matrix in closed form and up to 1050 digits, the margin is smaller still, positive, and follows $\log_{10} \lambda_{\min} \approx -(5.42e^L - 12.6)$ for $2 \leq L \leq 5$, about $e^{-12.5X}$ in the length $X = e^L$ of the prime sum (numerical). The computations of [CE11, Sect. 13.6] on fixed intervals agree.
- (3) *Why the margin is small* (Sect. 4). The collapse is a property of the density of the zeros, not of the primes: it occurs for regular points with the density of the zeros of ζ , follows $\log_{10} \lambda_{\min} \approx -6.5L^2/c$ when the density is multiplied by c , and is absent for curves over $\mathbb{F}_p$, where the margin is the constant $2 \log p$ [CE10, Prop. 3.15]. In a basis of Dirichlet polynomials it is a counting effect, the same for f (numerical; the law $-6.5L^2/c$ has a heuristic explanation that we do not prove).
- (4) *Prime powers and positivity* (Sect. 5). Without the primes the form is negative from $L \approx 0.75$ on; at $L = 1.25$ the primes 2, 3 make it positive and move the zeros of the minimiser to 14.134, 21.022, 25.012. Truncating the prime sum below its natural length makes the computed form negative in every window tested, so that in these windows the computed positivity has no reserve (numerical; that each prime power preserves positivity for all test functions is not proved). Read as an induction in the length through the Levinson recursion of the Toeplitz spline matrix, it fails for f at a single step and, in fine spline spaces, without warning (numerical, high precision).
- (5) *The Davenport–Heilbronn threshold* (Sect. 6). For f the form is negative for $L > L^*$. The upper bound $L^* \leq 3.43$ is a computer-assisted proof in interval arithmetic: an explicit test function on a window of length 3.43 has $W_f < 0$ (Proposition 4). The lower bound is numerical: the computed smallest eigenvalue at $L = 3.42$ is positive (60 digits, up to 500 basis functions), which does not certify positivity, so $L^* \approx 3.425$ is a numerical estimate. The value 3.6 given in [C7] is where the negative eigenvalue first exceeds double precision. The form separates ζ from f once the coefficients up to $n \approx e^{3.44} \approx 31$ enter, and the threshold depends mainly on the height of the off-line pair and little on its distance to the line; the second off-line pair of f , at height 114.16, is seen from $L \approx 3.72$ on (numerical).
- (6) *The minimiser as an approximate spectrum* (Sect. 7). Connes' observation that the zeros of the minimiser reproduce the zeros of ζ [Cn26] holds equally for f up to L^* ; beyond it the approximation degrades at once and a real zero of the minimiser locks onto the ordinate of the off-line pair, to within 0.009 at $L = 4$ (numerical, 60 to 380 digits).
- (7) *Two windows at large height* (Sect. 8). A family of two windows at the ends of the support reduces the form to a 2×2 matrix (Lemma 5), detects an isolated zero $\frac{1}{2} + \delta + i\gamma$, $0 < \delta < \frac{1}{2}$, with the prime powers up to $(C \log \gamma)^{1/\delta}$ (Proposition 6: an explicit inequality, proved; the asymptotic statement for $\gamma \geq \gamma_1(\delta, A)$ under an isolation radius $(\log \gamma)^A$, $A > 1/(10\delta)$, proved; its evaluation for $\delta \geq 0.1$ at all heights $10^6 \leq \gamma \leq 10^{1000}$, numerical with a margin of 1%). A related family, with $w = L/4$ and the windows $L \approx 11$ –13 at which pairs are detected in the measurements, is positive for ζ at 2400 sampled heights in $[2 \cdot 10^3, 3 \cdot 10^6]$ with a margin of 6 to 15%, while a random multiplicative model violates it at 1.5% of the

heights (numerical). RH is verified far beyond these heights [PT], so this excludes nothing new.

- (8) *Transport* (Sect. 9). Prime translations are isometries of Weil's form (Lemma 7; elementary); the margin depends on the difference set of the support, and a single translate of a window loses eleven orders of magnitude of margin whether the shift is $\log 2$ or not (numerical).

1.3. What is new, and what is not. Weil's criterion and the explicit formula are classical [We2], [Bo], [IK, Thm. 5.12]. Positivity on short windows was proved by Yoshida [Yo]; Connes and Consani explained it for the archimedean place [CC], studied the very small eigenvalues of the form on a fixed interval through prolate spheroidal functions [CC2], and with Moscovici related the zeros to a prolate wave operator [CCM]. Connes observed that the minimiser on short windows reproduces the zeros of ζ to many digits [Cn26], Groskin pushed this to several hundred digits [Gk], and certified lower bounds for the form beyond the unit window have been announced [Liu]. In the continuous setting, Suzuki attached to ζ a function $g = -\Psi$, built from the prime powers and the Gamma factor, and proved that RH holds if and only if g is a screw function in the sense of Kreĭn on $\mathbb{R}$, a positivity statement of Weil's type, and, unconditionally, that it is a screw function on short intervals [Su, Thm. 1.2 and Sect. 4]; Kim and coauthors computed numerically the spectrum of an operator form of Weil's quadratic form proposed by Suzuki [Ki+]. Proposition 1 and Lemmas 5, 7 are elementary. What is new here is the numerical description of the margin and of who pays for it: the density explanation of its size, the role of each prime power, the control by the Davenport–Heilbronn function with a sharp threshold, the behaviour of the Connes minimiser for a function that violates RH, and Proposition 6, which shows that at large height an isolated off-line zero is detected by prime powers up to a power of $\log \gamma$. None of these statements proves positivity on any window beyond what is known, and positivity on a fixed window is compatible with off-line zeros, as f shows for $L < L^*$.

Corrections. The preprint [C7] gave $L^* \approx 3.6$ for f and presented $10^{-6.3L^2}$ as the margin itself. Both are corrected here: $3.42 < L^* \leq 3.43$ (Section 6; the upper bound certified, the lower bound numerical), and $10^{-6.3L^2}$ holds only in a fixed finite basis; so does the fit $10^{-5.0L^{2.53}}$ in the larger bases of Section 3, which the converged bases of Section 3.5 replace by $10^{-(5.42e^L - 12.6)}$.

1.4. Organization. Section 2 sets up the form and its discretizations. Sections 3–5 treat ζ : the margin, its density explanation and the role of the primes. Section 6 treats f , and Section 7 the minimiser as an approximate spectrum. Section 8 is about large heights, Section 9 about prime translations, and Section 10 discusses what the computations say and lists open problems.

2. WEIL'S FORM ON A WINDOW

2.1. The form. For ζ the explicit formula [CE10, (3.4)] gives, for φ supported in an interval of length L ,

$$(1) \quad \begin{aligned} W(\varphi) = & \hat{g}\left(\frac{i}{2}\right) + \hat{g}\left(-\frac{i}{2}\right) - g(0) \log \pi + \frac{1}{2\pi} \int \hat{g}(r) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) dr \\ & - \sum_{n \leq e^L} \frac{\Lambda(n)}{\sqrt{n}} (g(\log n) + g(-\log n)), \end{aligned}$$

with $\hat{g}(z) = \hat{\varphi}(z) \overline{\hat{\varphi}(\bar{z})}$, equal to $|\hat{\varphi}(r)|^2$ for real r , and $\psi = \Gamma'/\Gamma$. For the Davenport–Heilbronn function the completed function $(5/\pi)^{s/2} \Gamma(\frac{s+1}{2}) f(s)$ is entire and symmetric under $s \mapsto 1-s$; its form W_f has no pole term, the constant is $+g(0) \log(5/\pi)$, the gamma term is $\operatorname{Re} \psi(\frac{3}{4} + \frac{ir}{2})$, and $\Lambda(n)$ is replaced by the coefficients $b(n)$ of $-f'/f$, which change sign and live mostly on composite

n [CE17]. With $\hat{g}(r) = e^{-(r/a)^2}$, $2 \leq a \leq 5$, the two sides of each explicit formula agree to 10^{-14} for ζ (682 zeros) and to 10^{-16} for f [CE10, Sect. 3].

Domain. For an interval I let $\mathcal{V}(I)$ be the space of $\varphi \in L^2(\mathbb{R})$ supported in $\bar{I}$ with $\|\varphi\|_{\mathcal{V}}^2 = \int |\hat{\varphi}(r)|^2 \log(2 + |r|) dr < \infty$. For such φ , $\hat{\varphi}$ is entire, $g = \varphi * \tilde{\varphi}$ is continuous, and $\operatorname{Re} \psi(\frac{1}{4} + \frac{ir}{2}) = \log \frac{|r|}{2} + O(r^{-2})$, so every term of (1) is defined and continuous for $\|\cdot\|_{\mathcal{V}}$; and C_c^∞ functions supported in the interior of I are dense in $\mathcal{V}(I)$ (smooth by mollification after a slight contraction of the support). Every function of bounded variation supported in $\bar{I}$ lies in $\mathcal{V}(I)$, since then $\hat{\varphi}(r) = O(1/|r|)$; this covers the sine, cosine and spline bases used below, the functions φ_L below, indicators of intervals, and the two-window functions of Section 8. For such φ the zero side is also available: $|\hat{\varphi}(z)| \ll (1 + |z|)^{-1}$ in the strip $|\operatorname{Im} z| \leq \frac{1}{2}$, so $\sum_\rho |\hat{g}(\gamma_\rho)| < \infty$, and approximating φ by $\varphi * \eta_\varepsilon$ with a mollifier η supported in $[-1, 1]$, the explicit side converges by continuity in $\mathcal{V}$ and the zero side by dominated convergence (since $|\hat{\eta}(\varepsilon z)| \leq e^{\varepsilon/2}$ in the strip), so $W(\varphi) = \sum_\rho \hat{g}(\gamma_\rho)$ holds for every such φ . By density and continuity, the infimum of $W(\varphi)/\|\varphi\|_2^2$ is the same over C_c^∞ functions supported in I and over $\mathcal{V}(I)$.

For $x \in \mathbb{R}$ let $T_x \varphi(u) = \varphi(u - x)$.

Proposition 1 (structure of the form). *Let φ be supported in an interval of length L , and put*

$$\Psi_L(r) = \operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) - \log \pi - 2 \operatorname{Re} \sum_{n \leq e^L} \frac{\Lambda(n)}{n^{1/2+ir}}.$$

- (i) $W(\varphi) = 2 \operatorname{Re} [\hat{\varphi}(\frac{i}{2}) \overline{\hat{\varphi}(-\frac{i}{2})}] + \frac{1}{2\pi} \int |\hat{\varphi}(r)|^2 \Psi_L(r) dr$.
- (ii) *The pole term $P(\varphi) = 2 \operatorname{Re} [\hat{\varphi}(\frac{i}{2}) \overline{\hat{\varphi}(-\frac{i}{2})}]$ is a Hermitian form of rank at most two, with at most one positive and one negative direction, and it vanishes on the subspace $\hat{\varphi}(\pm \frac{i}{2}) = 0$ of codimension at most two.*
- (iii) *The prime part of W is $-2 \sum_{n \leq e^L} \Lambda(n) n^{-1/2} \operatorname{Re} \langle \varphi, T_{\log n} \varphi \rangle$, the cross-energy of φ with its translates by $\log n$.*
- (iv) $W(T_x \varphi) = W(\varphi)$ for every real x . Hence the infimum of $W(\varphi)/\|\varphi\|_2^2$ over φ supported in an interval I depends only on the length L of I ; it is the number $\lambda_{\min}(L)$ above, and it is nonincreasing in L .
- (v) *RH holds if and only if $\lambda_{\min}(L) \geq 0$ for every $L > 0$. For every finite-dimensional subspace $V_0 \subset \mathcal{V}(-\frac{L}{2}, \frac{L}{2})$, the smallest eigenvalue of W on V_0 with respect to the L^2 inner product is at least $\lambda_{\min}(L)$, and it does not increase when V_0 is enlarged.*

The same holds for W_f , with no pole term and with

$$\Psi_L^f(r) = \operatorname{Re} \psi\left(\frac{3}{4} + \frac{ir}{2}\right) + \log \frac{5}{\pi} - 2 \operatorname{Re} \sum_{n \leq e^L} \frac{b(n)}{n^{1/2+ir}};$$

in (v), RH is replaced by the statement that all zeros of f lie on the critical line, which is false.

Proof. (i) $\hat{g}(\frac{i}{2}) + \hat{g}(-\frac{i}{2}) = \hat{\varphi}(\frac{i}{2}) \overline{\hat{\varphi}(-\frac{i}{2})} + \hat{\varphi}(-\frac{i}{2}) \overline{\hat{\varphi}(\frac{i}{2})}$. For the other terms, $g(u) = \frac{1}{2\pi} \int |\hat{\varphi}(r)|^2 e^{-iru} dr$, so $g(0) = \frac{1}{2\pi} \int |\hat{\varphi}|^2$ and $g(\log n) + g(-\log n) = \frac{1}{2\pi} \int |\hat{\varphi}(r)|^2 2 \cos(r \log n) dr$, while $2 \operatorname{Re} n^{-ir} = 2 \cos(r \log n)$. (ii) With $\ell_\pm(\varphi) = \hat{\varphi}(\pm \frac{i}{2})$ one has $P = \frac{1}{2}(|\ell_+ + \ell_-|^2 - |\ell_+ - \ell_-|^2)$. (iii) $g(x) = \int \varphi(y) \overline{\varphi(y-x)} dy = \langle \varphi, T_x \varphi \rangle$ and $g(-x) = \overline{g(x)}$. (iv) $T_x \varphi(z) = e^{izx} \hat{\varphi}(z)$, so $\hat{g}$ is unchanged on the real line and $\hat{g}(\pm \frac{i}{2}) = \hat{\varphi}(\pm \frac{i}{2}) \overline{\hat{\varphi}(\mp \frac{i}{2})}$ is multiplied by $e^{\mp x/2} e^{\pm x/2} = 1$; equivalently, $g = \varphi * \tilde{\varphi}$ is unchanged. An interval of length L is a translate of $[-\frac{L}{2}, \frac{L}{2}]$, which is contained in $[-\frac{L'}{2}, \frac{L'}{2}]$ for $L' > L$. (v) Every $\varphi \in C_c^\infty$ is supported in some window, so the first statement is Weil's criterion [We2], [Bo] together with (iv). The infimum over C_c^∞ functions equals the infimum over the form domain $\mathcal{V}$, which contains the trigonometric and spline bases used below (see *Domain* above); the rest is the Rayleigh–Ritz principle. In particular a positive smallest eigenvalue on a

finite-dimensional space is only an upper bound for $\lambda_{\min}(L)$ and certifies nothing about positivity on the window. $\square$

Status: proved (elementary, given Weil's criterion).

Part (iii) is the sense in which Weil's form is a cross-energy: the primes enter only through the overlaps of φ with its translates by $\log n$, weighted by $\Lambda(n)n^{-1/2}$, exactly as the cross-energy $C_2(n, s) = 2 \operatorname{Re}(X_{n-1}(s)\overline{n^{-s}})$ of [CE1] is the overlap of the last term with the previous partial sum. Under RH, Proposition 1(i) says that the energy of $\hat{\varphi}$ against Ψ_L exceeds the pole term; in the minimising directions the two nearly cancel.

Remark 2 (positivity by construction). The weight $1/(\frac{1}{4} + \gamma^2)$, which is the limit of $C_2(n, \frac{1}{2} + i\gamma)$ [CE1], is $\hat{G}(\gamma)$ for $G(u) = e^{-|u|/2}$. With this G the explicit formula becomes Cramér's identity for the autocorrelation of the normalized error $E(u) = e^{-u/2}(\psi(e^u) - e^u + \dots)$ of the prime number theorem: under RH and with simple zeros, $\lim_U \frac{1}{U} \int_0^U |(\varphi * E)(u)|^2 du = \sum_{\rho} |\hat{\varphi}(\gamma_{\rho})|^2 / (\frac{1}{4} + \gamma_{\rho}^2)$ [CE5, Prop. 2.1]. Weil positivity with the weight $1/(\frac{1}{4} + \gamma^2)$ therefore holds by construction as soon as the prime signal E has finite energy, and the sign problem becomes a size problem: the growth rate of $\int_0^U E^2$ is $2\Theta - 1$, $\Theta = \sup \operatorname{Re} \rho$, and the mean energy is $\sum_{\rho} \lim_n C_2(n, \rho)$, finite exactly under RH [CE5, Thm. 2.2 and Thm. B]. The kernel G is not admissible in (1) ($\hat{G}$ has poles at $\pm \frac{i}{2}$), which is why the windows of this paper are needed. Status: classical [Cr], with the C_2 reading of [CE5].

The energy slope $S(\sigma, t) = \partial_{\sigma} \log |\xi(\sigma + it)|$ is W at the Poisson kernel $\delta/(\delta^2 + (r - t)^2)$, $\delta = \sigma - \frac{1}{2}$, and its truncation $W_L(\sigma, t) = W(\varphi_L)$ with $\varphi_L(u) = \sqrt{2\delta} e^{-\delta u - it u} \mathbf{1}_{[0, L]}(u)$ is an exact finite expression in the prime powers up to e^L , within $(1 \pm e^{-\delta L})^2$ of $2S$ under RH [CE10, Thm. 3.13 and Cor. 3.14]. That is one admissible family on each window; $\lambda_{\min}(L)$ is the infimum over all of them.

2.2. Discretizations. We use four bases. Each value reported was computed by two routes where possible: from the explicit side (1) (pole, archimedean term, primes) and from the zero side $\sum_{\rho} \hat{g}(\gamma_{\rho})$ over the 682 zeros with $|\gamma| < 600$.

(a) *Sine basis, double precision.* $\varphi_m(u) = \sin(m\pi(u + \frac{L}{2})/L)$, $1 \leq m \leq M = \lfloor 10 + 12L \rfloor$ ($\lfloor 12 + 30L \rfloor$ for f , to resolve frequencies up to about 100). The transforms $\hat{\varphi}_m$ are explicit; the matrix of (1) is assembled from the rank-two pole part and the integral of $\hat{\varphi}_m \hat{\varphi}_n \Psi_L$ over $|r| \leq 800$ with step 0.004, and the smallest generalized eigenvalue with respect to the Gram matrix $\frac{L}{2}I$ is taken. On the zero side we use the smallest singular value of the matrix $(\hat{\varphi}_m(\gamma_{\rho}))$, squared and divided by $\frac{L}{2}$, which resolves values far below the floor of the explicit side (PartVII/weil.py, PartVII/lamcurve.py).

(b) *Cosine basis, high precision.* Even functions $\cos(m\pi x/L)$, m odd, K functions, with all entries in mpmath at 50 to 120 digits and the archimedean term by Gauss–Legendre quadrature (weil_hp/connes_hp.py, weil_hp/step.py). The same matrix has a closed form, which allows $K \leq 2000$ and up to 1050 digits (Section 3.5; weil_cf/).

(c) *Cubic B-splines.* Splines of mesh 0.02 or 0.03, for which the matrix of the form is Toeplitz; its entries were checked against the zero side to 10^{-10} [CE16, Prop. 2.7] (weil/tz8.py, weil/run8.py, and in mpmath weil_by_prime/prime_by_prime.py).

(d) *Sines and cosines on several windows*, with 50 digits, for Section 9; script transport/hp_transport.py.

The form commutes with the reflection $u \mapsto -u$, since the explicit side of (1) is invariant under $g(u) \mapsto g(-u)$; bases (a) and (d) split into even and odd parts, and the smallest eigenvalue is attained in the even part in all cases computed. By Proposition 1(v) every value below is an upper bound for $\lambda_{\min}(L)$. Three statements must be kept apart: (A) $\lambda_{\min}^{(K)}(L) > 0$, the smallest eigenvalue in a K -dimensional space, a computation; (B) $\lambda_{\min}(L) > 0$, strict positivity of the form on the window; (C) $\lambda_{\min}(L) \geq 0$ for every L , which is RH. (A) gives only an upper bound for the number

in (B) and does not imply (B); a computed negative value, if certified, does show that (B) fails. Every positive value reported in this paper, in the text and in the tables, is of type (A).

3. THE MARGIN ON WINDOWS

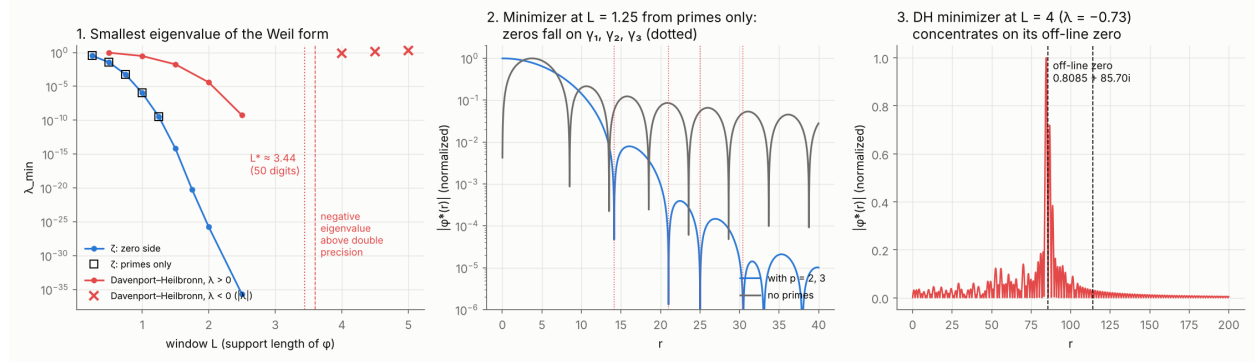


FIGURE 1. Left: $\lambda_{\min}(L)$ in the sine basis for ζ (zero side: line; primes only: squares) and for the Davenport–Heilbronn function f (positive values: line; $|\lambda_{\min}|$ where negative: crosses); dotted: the threshold $L^* \approx 3.44$ found with 50 digits ($3.42 < L^* \leq 3.43$ in the larger bases of Section 6); dashed: $L = 3.6$, where the negative eigenvalue first exceeds double precision. Centre: the minimiser at $L = 1.25$ with and without the primes 2, 3; dotted lines at the zeros of ζ . Right: the minimiser for f at $L = 4$ and its first two off-line zeros.

3.1. The sine basis. Table 1 gives $\lambda_{\min}(L)$ in basis (a), from the primes and from the zeros (Figure 1, left). The two routes agree to three or four digits wherever the explicit side is above its double-precision floor of about 10^{-11} . The data fit

$$(2) \quad \log_{10} \lambda_{\min}(L) \approx -6.3L^2 \quad (\text{sine basis, } M = \lfloor 10 + 12L \rfloor).$$

L	from the primes	from the zeros
0.25	$3.305 \cdot 10^{-1}$	$3.305 \cdot 10^{-1}$
0.50	$3.719 \cdot 10^{-2}$	$3.718 \cdot 10^{-2}$
0.75	$5.798 \cdot 10^{-4}$	$5.798 \cdot 10^{-4}$
1.00	$1.079 \cdot 10^{-6}$	$1.079 \cdot 10^{-6}$
1.25	$3.47 \cdot 10^{-10}$	$3.40 \cdot 10^{-10}$
1.50	—	$6.3 \cdot 10^{-15}$
1.75	—	$5.7 \cdot 10^{-21}$
2.00	—	$1.7 \cdot 10^{-26}$
2.50	—	$1.4 \cdot 10^{-36}$

TABLE 1. Smallest eigenvalue of Weil’s form for ζ in the sine basis on windows of length L . Dashes: below the double-precision floor of the explicit side. The zero side omits the zeros with $|\gamma| > 600$, whose contribution is negligible here. The value at $L = 2.5$ is within two orders of magnitude of the floor of the singular-value computation (about $3 \cdot 10^{-38}$ at $L \geq 3$); a rerun gives $1.9 \cdot 10^{-36}$.

The reason is visible in the minimiser. The transform $\hat{\varphi}^*$ of the optimal function vanishes near the zeros of ζ : at the first 2 for $L = 1$, the first 8 (up to $\gamma = 43.33$) for $L = 1.5$, and the first 11 for $L = 2$. A function of exponential type $L/2$ can be placed near more and more zeros as

L grows, and so can make $\sum_{\rho} |\hat{\varphi}(\gamma_{\rho})|^2$ very small. Status: numerical (`PartVII/lamcurve.py`, `PartVII/fig7.py`).

3.2. The law belongs to the basis. The law (2) is a property of the basis (a), and by Proposition 1(v) enlarging the basis can only lower the smallest eigenvalue. In basis (b) with $K = 42, 51, 60$ functions and 70 to 120 digits the explicit side gives

$$\lambda = 8.0 \cdot 10^{-30}, \quad 3.9 \cdot 10^{-54}, \quad 6.5 \cdot 10^{-82} \quad \text{at } L = 2, 2.5, 3,$$

and $3.0 \cdot 10^{-10}$ at $L = 1.25$ with 24 functions and 30 digits. These values fit

$$(3) \quad \log_{10} \lambda \approx -5.0 L^{2.53} \quad (1.25 \leq L \leq 3)$$

(Figure 5a). The law $10^{-6.3L^2}$ is therefore only an upper bound for the true margin. So is (3): the values in these bases have not stabilized in K , and in larger bases the margin is much smaller, about $10^{-(5.42e^L - 12.6)}$ (Section 3.5). Status: numerical (`weil_hp/`, states `state_zeta*.json`, figure `weil_hp/fig_noche_en.py`).

3.3. Comparison with the fixed-interval computation of Part 11. Weil's form on fixed intervals was computed independently in [CE11, Sect. 13.6, Table 13], with the archimedean term in position space, the basis $\sin(j\pi(u + \frac{\ell}{2})/\ell)$, $j \leq J$, for $J = 40$ (80 digits) and $J = 60$ (130 digits), ℓ up to 5, even and odd functions separately, and with and without the prime term. We summarize the points of contact. (1) The values agree with Table 1 and (3) where the bases are comparable: $3.03 \cdot 10^{-10}$ at $\ell = 1.25$ for even functions and $J = 60$. (2) They decrease with J for $\ell \geq 2$, down to $4.8 \cdot 10^{-87}$ at $\ell = 5$ with $J = 60$, and are far from converged: at $\ell = 3$, $J = 60$ gives $9.2 \cdot 10^{-58}$ for even functions, against $6.5 \cdot 10^{-82}$ with the 60 even functions of basis (b) (the sine basis with $J = 60$ contains only 30 even functions), as Proposition 1(v) requires. (3) The odd functions have margins larger by two to three orders of magnitude. (4) Without the prime term, the pole and the Gamma factor change sign at $\ell \approx 0.745$ for odd and 0.82 for even functions, consistent with the value 0.755 for the full sine basis in Section 5; at $\ell = \log 2$, where no prime enters, the margin is $1.4 \cdot 10^{-3}$. As emphasized there, positivity on a finite-dimensional subspace at fixed ℓ is not Weil's criterion, and these numbers are not evidence for RH beyond what is known: the form on these windows is governed by low zeros, which are known to lie on the line [PT]. Status: numerical [CE11].

3.4. What the margin means. Positivity on a window of length L is decided by a cancellation between the pole term and $\frac{1}{2\pi} \int |\hat{\varphi}|^2 \Psi_L$ whose depth grows at least like $6.3L^2$ decimal digits, and in converged bases like $5.42e^L$ digits (Section 3.5). This explains why certified lower bounds are so small: a computer-assisted certificate must resolve this cancellation in interval arithmetic [Liu], and the theorems on positivity concern short windows [Yo], [CC]. The near-null directions are consistent with the very small eigenvalues found by Connes and Consani on fixed intervals [CC2]; we have not compared eigenvectors.

3.5. Numerically stabilized bases. In basis (b) every entry of the matrix has a closed form. With $a = m\pi/L$, $b = n\pi/L$, $m \neq n$ odd, the autocorrelation of $\cos ax$ and $\cos bx$ on the window is $g(u) = \pm(a \sin bu - b \sin au)/(a^2 - b^2)$ for $0 \leq u \leq L$, so the archimedean term needs only $\psi(\frac{1}{4} + \frac{ia}{2})$ and $\psi'(\frac{1}{4} + \frac{ia}{2})$ at the K frequencies of the basis (the tails beyond $u = L$ are geometric series in e^{-2L}), and the prime part only the sums $P(a) = \sum_{n \leq e^L} \Lambda(n) n^{-1/2} \sin(a \log n)$. The off-diagonal entries are $\mp 2(aF(b) - bF(a))/(a^2 - b^2)$ plus the rank-one pole term, where F is P plus the archimedean part: a Loewner matrix, of displacement rank four with respect to $\text{diag}(a^2)$, whose LDL^T factorization a generalized Schur algorithm with diagonal pivoting computes in $O(K^2)$ operations; the smallest eigenvalue then follows by inverse iteration. The closed form agrees with the quadrature of Section 2.2 to $2 \cdot 10^{-35}$ at 40 digits and reproduces the values above to eight digits.

The values above are not stabilized in K (Table 7 in Appendix A). At $L = 3$ the smallest eigenvalue falls from $6.5 \cdot 10^{-82}$ ($K = 60$) to $9.5 \cdot 10^{-97}$ ($K = 120$), and then only to $4.7 \cdot 10^{-97}$ at $K = 260$. In general it falls fast until the largest frequency $\Omega = (2K - 1)\pi/L$ of the basis reaches about $14e^L$, and by less than a factor 3 when K is doubled beyond that. In bases with $\Omega \geq 14e^L$, checked at every L with two basis sizes and, for $L = 3.5, 4, 4.5, 5$, with two precisions, the computed smallest eigenvalue has stabilized in the sense that doubling K changes it by less than a factor 3. This stabilization is observed, not proved, and by Proposition 1(v) every value is an upper bound for $\lambda_{\min}(L)$, whose positivity it does not certify; with this caveat,

$$\log_{10} \lambda_{\min}(L) \approx -(5.42 e^L - 12.6) \quad (2 \leq L \leq 5),$$

within one unit for $L \geq 2.5$ and two at $L = 2$: -29.2 at $L = 2$, -96.3 at $L = 3$, -282.6 at $L = 4$, -474.9 at $L = 4.5$ and -792.5 at $L = 5$. No power law fits: the best, $-3.6L^{3.14}$, misses by 225 units at $L = 5$. In terms of the length $X = e^L$ of the prime sum the margin is about $e^{-12.5X}$. The margin is positive in every computation, as it must be under RH. The cut-off $14e^L$ is close to the height R of Remark 3, $\log(R/2\pi e) \approx L$, that is $R \approx 17e^L$, up to which a function of exponential type $L/2$ can follow the zeros; the sine basis (a), whose largest frequency grows only linearly in L , and the bases of (3) stop below that height, which is why they give the slower laws (2) and (3). This reading is heuristic. Status: numerical, 60 to 1050 digits, $K \leq 2000$; scripts `weil_cf/weil_cf.py`, `check_cf.py`, `run_min.py`, `kscan.py`, `margin_small_L.py`, `margin_summary.py` (output `margin_summary.txt`).

4. WHY THE MARGIN IS SMALL: THE DENSITY OF THE ZEROS

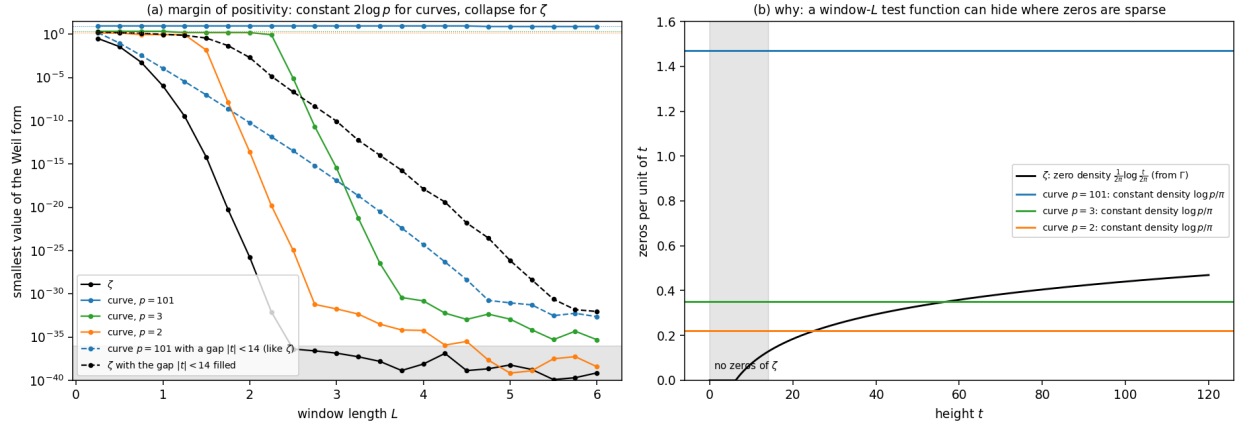


FIGURE 2. (a) Smallest value of Weil's form on windows of length L , from the zeros: ζ , curves over $\mathbb{F}_p$ (constant value $2 \log p$), the curve for $p = 101$ with its zeros below height 14 removed, and ζ with that gap filled. Grey: floor of the computation. (b) Density of the zeros: constant for the curves, $\frac{1}{2\pi} \log \frac{t}{2\pi}$ for ζ , with no zeros below 14.13.

4.1. Curves over finite fields. The collapse of $\lambda_{\min}(L)$ is already present for $L < \log 2$, before any prime enters (Table 1), so it is not caused by the primes. A comparison with curves over finite fields locates its cause. Let E be an elliptic curve over $\mathbb{F}_p$; its zeros are $\frac{1}{2} + i\gamma$ with γ in two arithmetic progressions of difference $2\pi/\log p$, and for test functions supported in an interval of length $L < \log p$ one has $\sum_{\gamma} |\hat{\varphi}(\gamma)|^2 = 2 \log p \|\varphi\|_2^2$ [CE10, Prop. 3.15], by Poisson summation. So the margin of the curve is the constant $2 \log p$, independent of $L < \log p$ and of the number of points; numerically it is $9.23 = 2 \log 101$ for $p = 101$ up to $L = 4$, and $2.20, 1.39$ for $p = 3, 2$ up to

$L \approx \log p$ (Figure 2a). The terms that enter at $L \geq \log p$ are the closed points of E , which play the role of the primes.

The zeros of E have constant density $\log p/\pi$; those of ζ have density $\frac{1}{2\pi} \log \frac{t}{2\pi}$, which comes from the Gamma factor, and none below height 14.13 (Figure 2b). A function supported in a window of length L can be small on a set of points only where the points are sparser than $L/2\pi$, and it hides its mass there. Two experiments confirm that this is what happens. Removing the zeros of the curve for $p = 101$ with $|t| < 14$ makes its smallest value collapse like that of ζ : $1.2 \cdot 10^{-4}$ at $L = 1$ and $5.9 \cdot 10^{-11}$ at $L = 2$. Adding these zeros to those of ζ raises the value of ζ at $L = 2$ from $1.7 \cdot 10^{-26}$ to $2.0 \cdot 10^{-3}$. Status: numerical (`weil_margin/m.py`, `m2.py`, `fig_margin.py`).

4.2. Only the density matters. The positions of the zeros of ζ play no role beyond their density. Let $N_0(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8}$ be the smooth counting function and define regular points by $N_0(\gamma_k) = k - \frac{1}{2}$. For these points the smallest value is $1.3 \cdot 10^{-6}$, $6.2 \cdot 10^{-15}$, $1.4 \cdot 10^{-26}$ at $L = 1, 1.5, 2$, against $1.1 \cdot 10^{-6}$, $6.3 \cdot 10^{-15}$, $1.7 \cdot 10^{-26}$ for the zeros of ζ . Random displacements of 25% of the local spacing change nothing, and a Poisson process with the same density gives smaller values, because of its larger gaps (Figure 3a). Multiplying the density of the regular points by c gives $\log_{10} \lambda_{\min} \approx -11.9L^2$, $-6.6L^2$, $-3.2L^2$ for $c = \frac{1}{2}, 1, 2$, that is,

$$(4) \quad \log_{10} \lambda_{\min}(L) \approx -6.5 L^2/c \quad (L \leq 2, \text{ sine basis})$$

(Figure 3b). Status: numerical (`weil_margin/syn.py`, `fig2.py`).

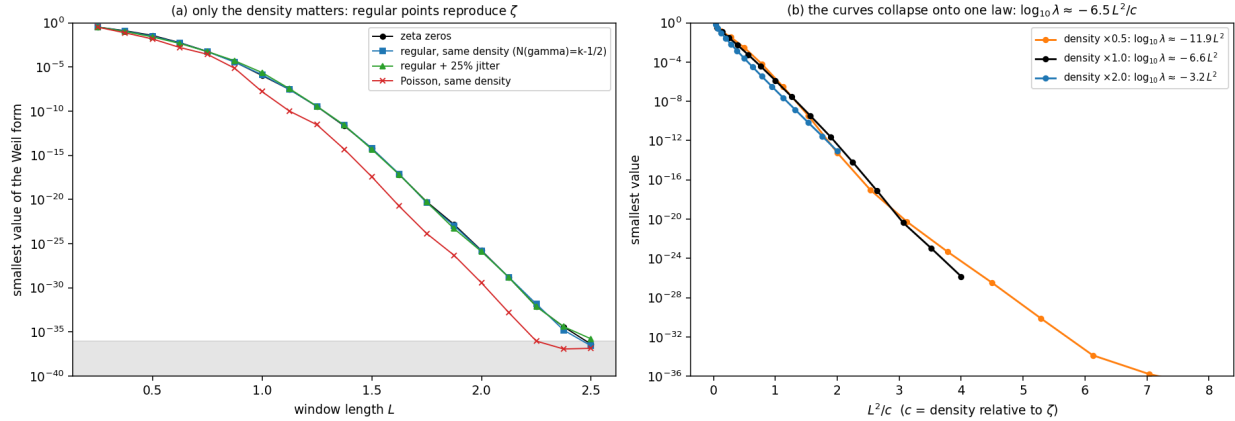


FIGURE 3. (a) Smallest value of Weil's form, from the points, for the zeros of ζ , for regular points with the same density, for these points displaced at random by 25% of the spacing, and for a Poisson process with the same density. (b) Regular point sets with c times the density of the zeros of ζ , against L^2/c .

Remark 3 (a heuristic for the law). A function of exponential type $L/2$ has at most about LR/π zeros in $[-R, R]$, and the zeros of ζ in $[-R, R]$ number about $c \frac{R}{\pi} \log \frac{R}{2\pi e}$. The minimiser can therefore follow them up to the height R with $\log R \approx L/c$. If its residual beyond that height decays like a power $R^{-\alpha L}$, then $\log \lambda_{\min} \approx -\alpha L^2/c$, and the data give $\alpha \approx 6.5 \log 10 \approx 15$. Status: heuristic; we have not proved it.

The consequence is that the size of the margin carries no arithmetic information. It is the same for points with the density of the zeros and no relation to the primes, and no argument based on its size alone can distinguish the zeros of ζ from such points.

4.3. Dirichlet polynomials as test functions. A second basis shows the same collapse as a counting effect. The angles $\gamma \log p$ of the zeros do not see the real parts [CE5], and the natural test functions for the joint behaviour of the zeros at the primes are $\varphi = \sum_{n \leq X} a_n \delta_{\log n}$, smoothed by a Gaussian. With $w(r) = e^{-(r/T)^2}$ put

$$A_{mn} = \sum_{\rho} w(\gamma_{\rho}) \left(\frac{m}{n} \right)^{i\gamma_{\rho}}, \quad 1 \leq m, n \leq X.$$

Under RH, $\sum_{m,n} a_m \bar{a}_n A_{mn} = \sum_{\rho} w(\gamma_{\rho}) |\sum_n a_n n^{i\gamma_{\rho}}|^2 \geq 0$, and (1) with $\hat{g}(r) = w(r)(m/n)^{ir}$ expresses each entry through the prime powers k with $|\log k - \log \frac{m}{n}| \lesssim 1/T$, the pole and the archimedean term; for f the same holds with its explicit formula. With $T = 60$ the entries computed from the primes (or from the coefficients $b(n)$) and from the zeros agree to six digits or better. The smallest eigenvalue is in Table 2.

X	16	32	64	128	256	384	512
ζ	0.28	$1.9 \cdot 10^{-3}$	$2.6 \cdot 10^{-7}$	$7 \cdot 10^{-15}$	0*	0*	0*
Davenport–Heilbronn	18.9	8.0	0.11	$3 \cdot 10^{-11}$	$-3.8 \cdot 10^{-4}$	-1.6	-5.1

TABLE 2. Smallest eigenvalue of the matrix $(A_{mn})_{m,n \leq X}$, $T = 60$. 0*: below 10^{-13} in modulus, the rounding level.

For f the form becomes negative between $X = 192$ and $X = 256$; at $X = 512$ the off-line zeros contribute -9.1 to the minimal value and the zeros on the line $+4.0$. For ζ the margin collapses as on windows. At $X = 128$ the minimiser, normalized as $c_n = a_n \sqrt{n}/a_1$, alternates between 1 at odd n and -1.14 at even n : the polynomial $\sum c_n n^{-s}$ is of the type of the alternating series $\eta(s) = (1 - 2^{1-s})\zeta(s)$, has modulus below 10^{-4} at the first three zeros, and lies between 0.2 and 2.3 at the other points tested. This vector is not typical. The eigenvalues below $10^{-9} \lambda_{\max}$ span a space of dimension 11 for ζ and 6 for f at $X = 128$, and 87 and 74 at $X = 256$; on average 78% ($X = 128$) and 88% ($X = 256$) of the mass of these vectors lies on $\frac{X}{2} < n \leq X$ for ζ , and 100% and 93% for f ; after division by ζ (Möbius inversion), respectively by f , at most 4% of the mass lies on $n \leq 10$. The collapse is therefore a counting effect, present in the same form for f : Dirichlet polynomials on the top octave have more free coefficients than there are zeros of non-negligible weight, and can nearly vanish at all of them. It is not produced by multiples of ζ by short Euler factors and does not involve the Euler product. In this basis only the sign of the minimum separates ζ from f , and it is decided by the off-line zeros. Status: numerical (joint/joint.py, big.py, minz.py, null2.py, nullsp.py).

5. PRIME POWERS AND POSITIVITY

5.1. The first primes. The statements of this section concern the spectrum of the form in the finite-dimensional spaces tested and its minimiser. The term $-2\Lambda(n)n^{-1/2} \operatorname{Re}\langle \varphi, T_{\log n} \varphi \rangle$ of a single prime power can have either sign, so no prime power contributes positively in every direction; what the computations show is that the successive inclusion of prime-power terms restores positivity in the tested finite-dimensional spaces.

At $L = 1.25$ only $n \leq e^{1.25} = 3.49$ enters, that is, the primes 2 and 3. Table 3 compares the form with and without these two terms (Figure 1, centre).

Without the primes the form is not positive: the archimedean term alone behaves like the explicit formula of a function whose “zeros” lie off the line. The two smallest primes make the form positive and move the zeros of the minimiser onto the first three zeros of ζ , to three decimals. At $L = 1$ only $p = 2$ enters, and the minimiser still vanishes at 14.136 and 21.058. Status: numerical (PartVII/primes_only.py).

	$\lambda_{\min}$	zeros of $\hat{\varphi}^*$ in $(0, 26)$
pole + archimedean term only	-0.571	8.504, 13.492, 18.516, 23.562
with $p = 2, 3$	$3.47 \cdot 10^{-10}$	14.134, 21.022, 25.012
zeros of ζ		14.135, 21.022, 25.011

TABLE 3. Weil’s form at $L = 1.25$ in the sine basis, with and without the primes 2, 3, and the real zeros of the transform of the minimiser.

In linear inequalities of the type $3 + 4 \cos \theta + \cos 2\theta \geq 0$ the pole destroys the positivity supplied by $\Lambda(n) \geq 0$, because a trigonometric polynomial that is nonnegative at every prime has a positive mean, and the mean sits at the pole [CE16, Sect. 2]. In the quadratic form the pole is a rank-two term (Proposition 1(ii)) and the positivity is produced by the primes, weighted by $\Lambda(n)n^{-1/2}$.

5.2. The prime powers one at a time. On a window of length L the prime powers enter one at a time, at $L = \log n$. Let $\lambda_N(L)$ be the smallest eigenvalue of the form in which only the prime powers $n \leq N$ are kept; for $N \geq e^L$ it is $\lambda_{\min}(L)$. We computed $\lambda_N(L)$ in basis (a) in double precision on a grid of step 0.02 in L , and in basis (c) with mesh 0.02 in mpmath with 36 to 79 digits for $1 \leq L \leq 2.8$ (Figure 4).

- With the pole and archimedean terms only, the form is positive up to $L = 0.755$. This is consistent with [CC], which treats $L \leq \log 2$, and with the sign changes at 0.745 (odd) and 0.82 (even) of [CE11, Sect. 13.6].
- On the subspace $\hat{\varphi}(\pm \frac{i}{2}) = 0$, where the pole term vanishes, it is positive up to $L = 1.207$, beyond $\log 3$; adding the prime 2 alone lowers this to 1.146.
- In every window tested the form becomes negative when the most recent prime power is removed, however short the time since it entered: at $L = 2.0$, removing 7 (which entered at $\log 7 = 1.946$) gives -0.0031 ; at $L = 2.6$, removing 13 (2.565) gives $-2.5 \cdot 10^{-4}$; at $L = 2.8$, removing 16 (2.773) gives $-1.4 \cdot 10^{-8}$.
- The deficit grows roughly like δ^6 in $\delta = L - \log q$, with fitted exponents between 4.8 and 7.2 (Figure 4b).

Since $\lambda_{\min}(L) \lesssim 10^{-6.3L^2}$, from the fitted upper bounds of Section 3, a truncated form survives only for a length of about 10^{-L^2} after the next prime power enters: positivity is maintained prime power by prime power, with no reserve. Two statements must be kept apart here. What the computations support is that, in the windows and discretizations tested, removing the most recent prime power destroys positivity. What is not proved, and would be a statement of the strength of RH, is that adding each prime power preserves positivity for all test functions and all lengths. The two directions of a computation are not symmetric: a reliably computed negative eigenvalue certifies that the form is indefinite, while a positive one certifies nothing. The spline values below 10^{-14} in Figure 4a are a discretization floor of order h^8 , not the true $\lambda_{\min}$, which Table 1 gives. Status: numerical (`weil_by_prime/run.py`, `cross.py`, `prime_by_prime.py`, `fig2.py`).

Positivity as an induction in the length. In the spline basis (c), and more generally for the translates $B(u - kh)$, $k \geq 0$, of a B-spline B of order q and mesh h ($q = 4$: cubic splines), the matrix of the form is Toeplitz, so its leading principal sections obey the Levinson–Durbin (Schur) recursion $e_K = e_{K-1}(1 - \kappa_K^2)$, where e_K is the ratio of consecutive leading minors and κ_K the K -th reflection coefficient: given $t_0 = W(B) > 0$, the form is positive on the first $K + 1$ translates, which fill a window of length $(K + q)h$, if and only if $|\kappa_j| < 1$ for $1 \leq j \leq K$. RH is equivalent to $|\kappa_j(h)| < 1$ for all j and all h in a sequence $h \rightarrow 0$: under RH the form is strictly positive on nonzero functions, since $W(\varphi) = \sum_{\rho} |\hat{\varphi}(\gamma_{\rho})|^2$ and a nonzero function of exponential type cannot vanish at all the ordinates, which are too dense; conversely, the spline spaces of mesh $h \rightarrow 0$ are dense in $\mathcal{V}$, and Proposition 1(v) applies. Positivity is thus an induction in the length of the window, one translate

at a time. In the decomposition $W_{L+h} = W_L + R_L$ suggested by this induction, positivity on the larger space is positivity on the smaller one together with positivity of the Schur complement of the new translate, which is $e_K = e_{K-1}(1 - \kappa_K^2)$. This complement has no sign of its own: positivity of the previous section does not imply positivity of the next, and the condition $|\kappa_K| < 1$ has to be checked at each step. A double-precision computation in the author's working notes (cubic splines, $h = 0.03$) suggested that the induction gives no warning. We have redone it in mpmath with 60 to 220 digits, with the archimedean part of the symbol in closed form (Hurwitz and Lerch values) and the symbols checked against the double-precision ones to 10^{-15} , for $q = 4, 6, 8, 10, 12, 16$ and h from 0.03 down to 0.0025 (Table 9 in Appendix A).

For ζ there is no break up to $L = 4.5$, with $1 - \kappa_K^2 \geq 0.66$ at every step ($q = 4, h = 0.03$, and $q = 8, h = 0.01$, at 150 and 220 digits); the apparent break at $L \approx 4.26$ in double precision was rounding. For f the break comes at a single step, at which $|\kappa_K|$ jumps above 1. In the coarse cubic spaces $1 - \kappa_K^2$ dips to 0.08 to 0.13 at the step before; in the finer spaces ($q \geq 6, h \leq 0.005$) it stays above 0.36 up to the break, so the recursion gives no warning. Its position is a property of the spline space, not of the precision (the same K and the same κ_K with 80 and 120 digits), and it moves towards L^* as the space is refined: from 3.75 for cubic splines with $h = 0.03$ (the double-precision run labelled this window by $Kh = 3.66$ instead of $(K + 3)h$) to 3.57 and 3.54 for cubic splines with $h = 0.01$ and 0.005, 3.46, 3.445 and 3.4325 for $q = 8$ with $h = 0.01, 0.005$ and 0.0025, next to $3.42 < L^* \leq 3.43$ (Section 6). In the finest space ($K = 1365$ at the break) $1 - \kappa_K^2 \geq 0.59$ at every step before the break, 0.65 at the last one, and $\kappa = 1.32$ at the break. (At fixed mesh, very high orders break later, at 3.50 and 3.53 for $q = 12$ and 16 with $h = 0.01$: their translates vanish to high order at the ends of the window, which shortens it.) An induction step therefore needs global information at each step, which is the absence of reserve of the preceding paragraphs in another form. Status: numerical, 60 to 220 digits, script `weil_cf/levinson_hp.py`, outputs `weil_cf/runs/lev_*.json`, summary `lev_summary.txt`.

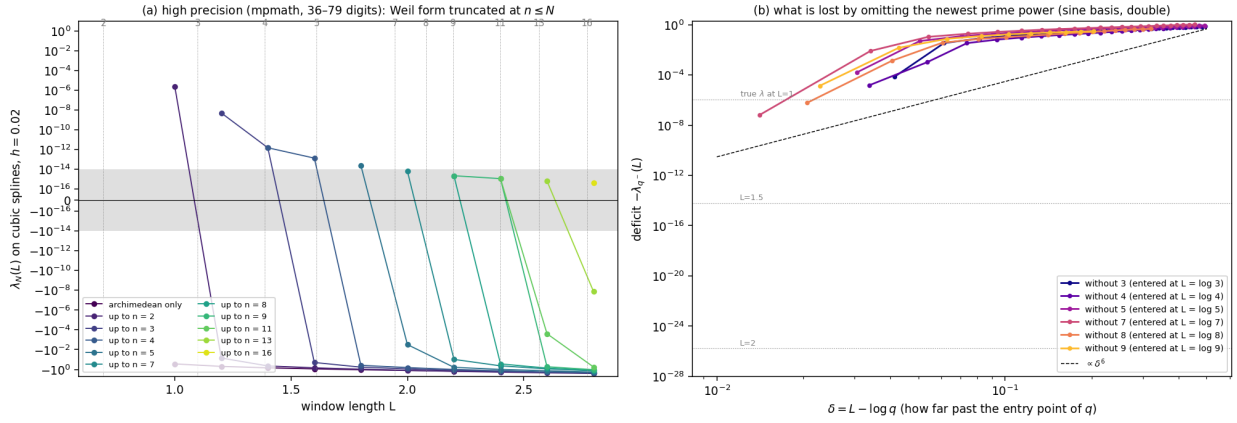


FIGURE 4. Weil positivity prime power by prime power. (a) $\lambda_N(L)$ for the form truncated at $n \leq N$, on cubic splines of mesh 0.02 in high precision; dotted lines at $L = \log q$; grey band: discretization floor. (b) The deficit $-\lambda_{q-}(L)$ caused by omitting the most recent prime power q , against $\delta = L - \log q$ (sine basis); dotted lines: $\lambda_{\min}$ at $L = 1, 1.5, 2$.

5.3. Who pays for positivity. In the direction of the minimiser the form splits into the pole term, the archimedean term (with the constant), and one term for each n . For ζ at $L = 3$, in basis (b) with 120 digits, these are $+1.5413$, -1.4722 and -0.0691 ; the last consists of -0.0682 from $n = 2$, $-8.5 \cdot 10^{-4}$ from $n = 3$, and terms decreasing to $-4.8 \cdot 10^{-65}$ at $n = 19$, with total

$6.5 \cdot 10^{-82}$. At $L = 2$ the split is $+1.4694, -1.4186, -0.0509$. Every prime power term is negative, because $\Lambda(n) \geq 0$ and the extremal g is positive at every $\log n$: positivity is a coherent balance of the pole against the archimedean term and all prime powers together. For f , which has no pole, the archimedean term is $+0.0072$ at $L = 2$ and -0.0297 at $L = 3.42$, and the coefficients carry the rest with both signs: at $L = 3.42$, -0.160 from $n = 2$, $+0.084$ from $n = 3$, $+0.120$ from $n = 4$, -0.013 from $n = 6$, and further terms at composite n (12, 14, 18, ...) that exist only because f has no Euler product. Positivity for f rests on cancellation between coefficients of opposite sign, and it fails at L^* . Status: numerical (`weil_hp/balance.py`).

6. THE DAVENPORT–HEILBRONN THRESHOLD

6.1. The threshold. For f the same construction gives a form that is positive up to a threshold L^* and negative beyond it. In double precision, in basis (a), the negative eigenvalue first appears at $L \approx 3.6$ (Table 4, left). With 50 digits in basis (b) the threshold is earlier (Table 4, right): at $L = 3.40$ the value is the same with 67 and with 91 basis functions, and the table gives $3.42 < L^* \leq 3.45$. The numerically stabilized bases of Section 3.5 (60 digits) sharpen this. At $L = 3.42$ the smallest eigenvalue decreases with K but stays positive, $1.5 \cdot 10^{-29}$, $3.7 \cdot 10^{-30}$, $1.3 \cdot 10^{-30}$, $1.15 \cdot 10^{-30}$ for $K = 67, 150, 330, 500$, while at $L = 3.43$ it is $-7.05 \cdot 10^{-30}$ for $K = 330$; so

$$3.42 < L^* \leq 3.43, \quad L^* \approx 3.425.$$

where the lower bound and the estimate are numerical: positivity of the computed eigenvalues at $L = 3.42$ is an upper-bound statement for $\lambda_{\min}(3.42)$ and does not certify it. The upper bound is a theorem.

Proposition 4 (a certified upper bound for the threshold). *Let $L = 3.43$. There is an explicit function*

$$\varphi(x) = \sum_{i < 200} v_i \cos \frac{(2i+1)\pi x}{L} \quad (|x| \leq \frac{L}{2}),$$

with dyadic rational coefficients v_i , such that $W_f(\varphi) < 0$; in fact

$$W_f(\varphi)/\|\varphi\|_2^2 \in [-6.854847504 \cdot 10^{-30} \pm 3 \cdot 10^{-36}].$$

Consequently $\lambda_{\min}(L) < 0$ for f for every $L \geq 3.43$, that is, $L^* \leq 3.43$.

Proof. The coefficients are the eigenvector of the most negative eigenvalue of the 60-digit computation, rounded to dyadic rationals. The value of the form is $v^T W v$ for the closed-form matrix of Section 3.5, whose entries are exact expressions in $\psi(\frac{3}{4} + \frac{ia}{2})$, $\psi'(\frac{3}{4} + \frac{ia}{2})$, the coefficients $b(n)$, $n \leq 30$, and series in e^{-2L} . We evaluate every entry in the ball arithmetic of Arb at 240 bits, with the digamma and trigamma functions from Arb, κ and $b(n)$ as balls, and the remainder of each e^{-2L} series added to the radius. The entries have radius at most 10^{-67} , and $v^T W v$ and $\|v\|^2$ are evaluated in the same arithmetic. Monotonicity in L is Proposition 1(iv). $\square$

Status: computer-assisted proof (`weil_cf/certify_dh_upper.py`, about 20 seconds). The closed form of the matrix is an exact identity (derived in the docstring of `weil_cf.py`); it agrees with the Gauss–Legendre quadrature of the archimedean term to $2 \cdot 10^{-35}$ and with the zero side of the explicit formula.

Between 3.43 and 3.6 the negative eigenvalue lies below double precision; this is why [C7] reported 3.6.

At $L = 3.8$ the pair of the off-line zero $\rho_f = 0.8085 + 85.6993i$ contributes -0.786 in the direction of the minimiser, and the other seven off-line zeros below height 400 less than 0.006 together. At $L = 4$ the minimiser concentrates at $r \approx 84.5$ (Figure 1, right), the pair $\rho_f, 1 - \bar{\rho}_f$ contributes $\hat{g}(\gamma_{\rho_f}) = -0.486$ in the double-precision basis, and the next off-line zero, $0.6508 + 114.16i$, contributes $+0.002$. An independent discretization by cubic B-splines of mesh 0.03 (basis

double precision, sine basis				50 digits, cosine basis	
L	$\lambda_{\min}$	L	$\lambda_{\min}$	L	$\lambda_{\min}$
0.5	0.946	3.6	$-1.8 \cdot 10^{-9}$	3.40	$+1.2 \cdot 10^{-28}$
1.0	0.307	3.7	-0.017	3.42	$+1.5 \cdot 10^{-29}$
1.5	$1.8 \cdot 10^{-2}$	3.8	-0.226	3.45	$-7.2 \cdot 10^{-30}$
2.0	$4.0 \cdot 10^{-5}$	3.9	-0.440	3.50	$-1.7 \cdot 10^{-23}$
2.5	$5.4 \cdot 10^{-10}$	4.0	-0.730	3.55	$-2.3 \cdot 10^{-18}$
3.0	≈ 0	5.0	-1.83	3.60	$-1.7 \cdot 10^{-7}$
				3.80	-0.228

TABLE 4. Smallest eigenvalue of Weil’s form for the Davenport–Heilbronn function f . Left: double precision, sine basis with $M = \lfloor 12 + 30L \rfloor$; values at $L \approx 3$ are at the rounding level. Right: 50 digits, 67 to 74 even cosine functions.

(c)) becomes negative only at the window of length 3.75 (122 splines): in high precision the smallest eigenvalue is $-1.7 \cdot 10^{-8}$, $-2.2 \cdot 10^{-4}$ and -0.30 for 122, 125 and 130 splines, that is at $L = 3.75$, 3.84 and 3.99, with the eigenvector peaked at $r = 84.5$. (The double-precision values $-5 \cdot 10^{-7}$ at 3.66 and -0.006 at 3.75 printed by `weil/eig8.py` are these divided by the mesh, at lengths labelled Kh instead of $(K + 3)h$.) The delay with respect to L^* is a property of this spline space, which resolves the negative direction, an oscillation at $r \approx 85$, only on finer meshes or with splines of higher order (Section 5, Table 9). The same spline space gives positive values for ζ up to $L = 4.5$ in high precision, and in the converged cosine bases the form of ζ is positive up to $L = 5$ (Section 3.5).

In this sense Weil positivity separates ζ from f once the coefficients up to $n \approx e^{3.43} \approx 31$ are included; up to that length the form of f behaves like that of ζ , with a positive margin that collapses (Figure 5a), although more slowly than for ζ : $3.6 \cdot 10^{-5}$, $3.0 \cdot 10^{-18}$, $9.4 \cdot 10^{-29}$ at $L = 2$, 3, 3.4 in numerically stabilized bases, against $6.3 \cdot 10^{-30}$, $4.7 \cdot 10^{-97}$ and, by the fit of Section 3.5, about 10^{-150} . This agrees with Section 4: the zeros of f are denser and begin at 5.09. Beyond L^* the negative eigenvalue is $-8.6 \cdot 10^{-30}$, $-2.4 \cdot 10^{-23}$, $-3.1 \cdot 10^{-12}$, $-2.7 \cdot 10^{-7}$ at $L = 3.45$, 3.5, 3.55, 3.6 ($K = 200$; the entries of Table 4 at 3.55 and 3.6 are limited by their smaller bases) and reaches -0.70 at $L = 4$ (Table 8). Status: numerical (`PartVII/dhcurve.py`, `weil_hp/` with the states `state_dh*.json`, `weil/run8.py`, `weil/eig8.py`; in numerically stabilized bases, 60 to 90 digits, `weil_cf/run_min.py`, `dh_margin.py`, `levinson_hp.py`, outputs `weil_cf/runs/min_dh*.json`, `dh_margin.json`).

6.2. Sensitivity to an off-line pair. The threshold depends on the height of the off-line zero and only weakly on its distance to the line. To see this we remove from the form of f its eight off-line pairs below height 400, replacing each by a double zero on the line. This leaves a positive form, with smallest eigenvalue $2.5 \cdot 10^{-30}$ at $L = 3.45$ and $1.6 \cdot 10^{-36}$ at 3.6. We then split one double zero at height γ into $\frac{1}{2} \pm \delta + i\gamma$; the change of the form is a finite sum over zeros and is computed exactly. The results are in Table 5 and Figure 5b,c. At $L = 3.8$ and $\gamma = 85.7$ the smallest eigenvalue is negative down to $\delta = 10^{-4}$ and scales roughly like $-\delta^{11}$ until it meets the margin of the clean form.

$\gamma = 85.7$					$\delta = 0.3$						
δ	0.3	0.1	0.03	0.01	γ	10	20	40	60	85.7	100
L^*	3.45	3.55	3.60	≤ 3.80	L^*	2.5	3.0	3.0	3.4	3.45	> 3.8

TABLE 5. First window length, on the computed grid, at which the clean form of f with a split pair $\frac{1}{2} \pm \delta + i\gamma$ becomes negative.

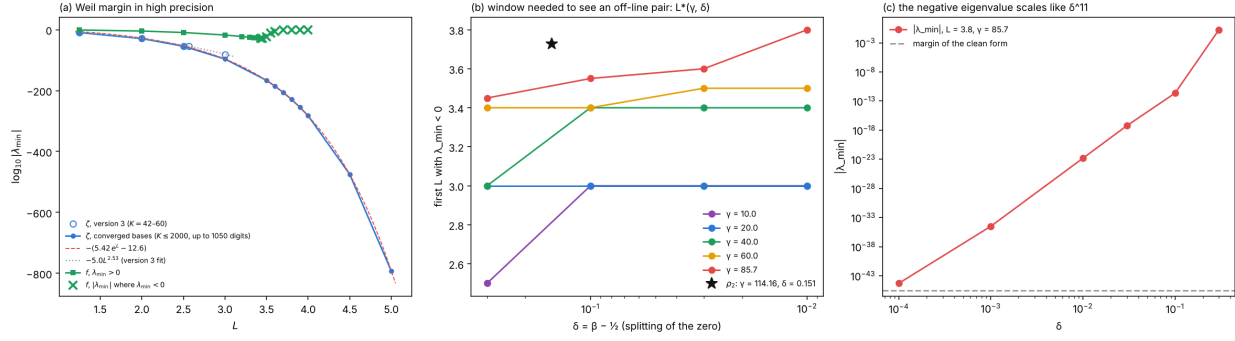


FIGURE 5. Weil’s form in high precision. (a) $\log_{10} |\lambda_{\min}|$ for ζ in the bases of Section 3 (42 to 60 functions, 70 to 120 digits; open circles) and in the converged bases of Section 3.5 (dots; up to 2000 functions and 1050 digits), with the fit $-5.0L^{2.53}$ of the former and the fit $-(5.42e^L - 12.6)$ of the latter (dashed); for the Davenport–Heilbronn function f in converged bases (60 digits; crosses: $|\lambda_{\min}|$ where negative). (b) First window length at which the clean form of f , with a double zero at height γ split to $\frac{1}{2} \pm \delta + i\gamma$, becomes negative; star: the second off-line pair ρ_2 of f (Section 6). (c) Smallest eigenvalue at $L = 3.8$, $\gamma = 85.7$, against δ ; dashed: margin of the clean form.

So $L^* \approx L_0(\gamma) + b \log(1/\delta)$ with a small b : an off-line pair at height γ makes the form negative once the coefficients up to a power of $1/\delta$ with a small exponent enter, against $x \approx \gamma^{1/\delta}$ for the Gaussian-window energy of the prime signal [CE5, Cor. 2.12(c)]. The price is precision, since the margin to be resolved decreases superexponentially in L . This is a statement about sensitivity, not a certificate: positivity on a fixed window is compatible with off-line zeros, as f shows for $L < L^*$. Status: numerical (`weil_hp/clean_scan.py`, `delta_scan.py`, `weil_probe.py`).

The second off-line pair. The second off-line zero of f is $\rho_2 = 0.65083 + 114.16334i$ ($\delta = 0.1508$). Since the form of f is already negative from L^* on because of $\rho_1 = 0.80852 + 85.69935i$, we use the same method: in the form of f we replace the quartet of ρ_1 by a double zero on the line at the same height, which leaves ρ_2 as the lowest off-line zero, and compute in the converged bases of Section 3.5 (the off-line zeros to 115 digits; 80 to 140 digits; $K = 200$ to 700, $\Omega \geq 330$). This form becomes negative at

$$3.72 < L_2^* \leq 3.725 \quad (K = 400, 550, 700; 100 \text{ and } 140 \text{ digits}),$$

With $K = 200$ and 300 the bracket is $(3.725, 3.73]$; it moves down by one step of the grid as the margin shrinks in the larger bases. The negative direction is centred at the height of ρ_2 : its transform has a real zero at 114.155, at distance 0.008 from the ordinate, as for ρ_1 in Section 7. Moving every off-line pair below height 400 except ρ_2 onto the line gives the same bracket with $K = 200$, and the clean form, with all eight pairs moved, stays positive up to $L = 4$; in the form of f itself a second negative eigenvalue appears between $L = 3.7$ and 3.8 (Table 8). So ρ_2 is seen once the coefficients up to $n \approx e^{3.72} \approx 41$ enter, close to the rule $L^* \approx \log(\gamma/2\pi) + 0.8 = 3.70$ found at large heights in Section 8. The threshold is not a function of the height alone: with $K = 200$ the split pair at $\gamma = 100$, $\delta = 0.3$ of Table 5 still gives a positive form at $L = 3.8$, and the entry 3.45 at $\gamma = 85.7$, $\delta = 0.3$ becomes 3.43, as for f . Status: numerical, 80 to 140 digits, scripts `weil_cf/pair2_scan.py`, `pair2_summary.py`, `dh_zeros.py` (outputs `weil_cf/runs/pair2-*.jsonl`, `dh_zeros_hp2.json`).

7. THE MINIMISER AS AN APPROXIMATE SPECTRUM

Connes observed that the zeros of the Fourier transform of the minimiser of Weil’s form on short windows reproduce the zeros of ζ with remarkable accuracy: with the primes up to 13 the first 50

zeros agree to between $2.6 \cdot 10^{-55}$ and 10^{-3} [Cn26], and Groskin has pushed this to several hundred digits [Gk]; whether the approximation converges as $L \rightarrow \infty$ is open. We repeated the construction in the even part of basis (a), for ζ and for f , in double precision, and at two points in basis (b) in high precision.

For ζ the median error of the first ten zeros decreases steadily, from 0.8 at $L = 1$ to the double-precision floor 10^{-12} for $L \geq 2.4$; at $L = \log 13$ with 100 digits the first zero is reproduced to $1.0 \cdot 10^{-51}$ and the tenth to $3.4 \cdot 10^{-33}$. For f the construction works equally well while its form is positive: at $L = 3.4$, with 50 digits, the first zero of f on the line is reproduced to $2.8 \cdot 10^{-27}$ and the tenth to $7 \cdot 10^{-16}$, although f violates RH. The off-line pair ρ_f is projected onto the line: at $L = 3.4$ the minimiser has two real zeros, 85.569 and 85.852, on either side of the ordinate 85.6993. In both cases the error of the first zero is about 10 to 10^4 times $\lambda_{\min}(L)$. In larger bases ($K \geq 200$, 60 digits) the accuracy for f is the same up to $L = 3.405$ ($1.2 \cdot 10^{-27}$ for the first zero), but from $L = 3.41$ on, just before L^* , the lowest eigenvector is the mode that becomes negative at L^* , which reproduces the first ten zeros of f only to about 10^{-15} .

Beyond the threshold the picture changes abruptly. As the negative eigenvalue grows from 10^{-30} at $L = 3.45$ to 10^{-7} at 3.6 and 0.2 at 3.8, the median error of the first ten zeros of f rises from $3 \cdot 10^{-13}$ at $L = 3.55$ to $3 \cdot 10^{-5}$ at 3.65, $8 \cdot 10^{-3}$ at 3.70 and about 10^{-1} for $L \geq 3.8$, and the zero of the minimiser next to ρ_f converges to its ordinate, 85.7092 at $L = 4.0$ and 85.6993 at $L = 4.25$: the minimiser becomes the mode of the off-line pair (Figure 6). In double precision the degradation appears only where the negative eigenvalue exceeds the rounding level, near $L = 3.6$.

High precision beyond the threshold. Table 8 in Appendix A gives the minimiser at $L = 3.6, 3.7, 3.8, 3.9, 4.0$ in converged bases: for ζ with 260 to 380 digits and $K = 330$ to 550, and for f with 60 digits and $K = 200$ (at $L = 3.6$ and 4.0 also $K = 300$, and at $L = 3.6$ also 90 digits, with the same values to the digits shown). For ζ the construction keeps improving: the first zero is reproduced to $7.9 \cdot 10^{-182}$ at $L = 3.6$ and to $3.3 \cdot 10^{-279}$ at $L = 4$, about $10^4 \lambda_{\min}(L)$ in both cases, and all 29 zeros below height 100 to $1.4 \cdot 10^{-229}$ at $L = 4$. For f the negative eigenvalue grows from $-7.1 \cdot 10^{-30}$ at $L = 3.43$ to $-2.7 \cdot 10^{-7}$, -0.026 , -0.26 , -0.45 , -0.70 at $L = 3.6, \dots, 4.0$; a second negative eigenvalue, due to ρ_2 (Section 6), appears between 3.7 and 3.8. The median error of the first ten zeros of f is $8 \cdot 10^{-16}$ at $L = 3.43$, $5 \cdot 10^{-8}$ at 3.6, $9 \cdot 10^{-3}$ at 3.7 and 0.03 to 0.09 for $3.8 \leq L \leq 4$. The minimiser does not acquire zeros at the off-line points: a search for complex zeros of $\hat{\varphi}$ started at $\gamma_{\rho_f} = 85.6993 - 0.3085i$ (and at $114.163 - 0.151i$ for ρ_2) converges to a real zero or not at all. What locks is a real zero, which moves towards the ordinate: 85.7312 at $L = 3.43$, 85.7305 at 3.6, 85.7289 at 3.7, 85.7203 at 3.8, 85.7145 at 3.9 and 85.7083 at 4.0, at distance 0.009 from 85.6993. A real zero at the ordinate makes $\hat{\varphi}(\gamma_{\rho_f}) \approx -0.3085i \hat{\varphi}'(85.6993)$ purely imaginary, so the contribution $4 \operatorname{Re} \hat{\varphi}(\gamma_{\rho_f})^2$ of ρ_f , $1 - \bar{\rho}_f$ and their conjugates is negative; per unit norm it is $-5.8 \cdot 10^{-6}$ at $L = 3.6$, -0.25 at 3.7, -0.80 at 3.8 and -0.94 at 4.0. The minimiser thus locks onto the ordinate of the off-line pair, not onto ρ_f and its reflection. Status: numerical, 60 to 380 digits; scripts `weil_cf/run_min.py`, `connes_analysis.py` (output `connes_summary.txt`), reference zeros `zeta.zeros_hp.py`, `dh.zeros.py`; figure `weil_cf/connes_fig_v4.py`.

Two conclusions follow. The accuracy of the construction on short windows does not by itself say anything about RH, since a function that violates RH reproduces its zeros on the line just as well up to and slightly beyond L^* . And an off-line zero shows up as a positivity threshold at a finite L , beyond which all the approximations degrade at once and one zero of the minimiser locks onto the ordinate of the pair. For ζ , RH is the statement that no such threshold occurs. Status: numerical (`connes/connes.py`, `connes_fig.py`, `connes_hp.py`, `step.py`, `analiza.py`, `analiza_dh.py`; figure `connes_fig-en.py`).

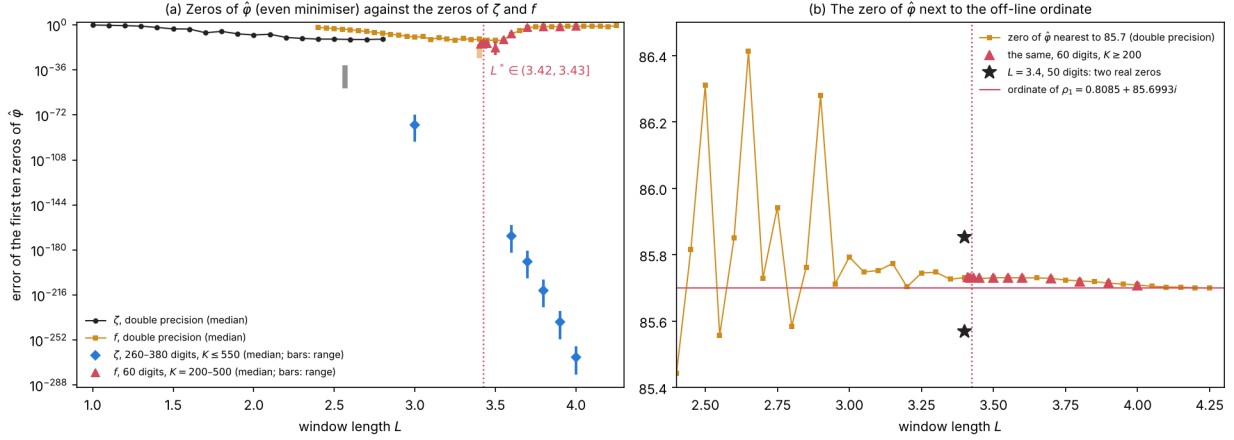


FIGURE 6. The minimiser of Weil's form as an approximate spectrum. (a) Error of the first ten zeros of $\hat{\varphi}$, for the even minimiser on a window of length L , against the zeros of ζ and of the Davenport–Heilbronn function f : median in double precision (lines); range in the high-precision runs of Section 7 (grey bars: ζ , $L = \log 13$, 100 digits; f , $L = 3.4$, 50 digits) and of Table 8 (markers: median; bars: range; ζ with 260 to 380 digits, f with 60 digits). Dotted: $L^* \approx 3.425$. (b) The real zero of $\hat{\varphi}$ nearest to 85.7 for f , in double precision and with 60 digits; beyond the threshold it converges to the ordinate of the off-line zero ρ_1 . Stars: at $L = 3.4$, with 50 digits, the minimiser has two real zeros on either side of that ordinate.

8. TWO WINDOWS AT LARGE HEIGHT

The thresholds of Section 6 are at small heights, where an off-line pair is detected only once the window resolves the neighbouring zeros. At large heights a simple family suffices. Let $w > 0$, $\varphi_0(x) = \cos^2(\pi x/w)$ for $|x| \leq \frac{w}{2}$ and 0 otherwise, $K = \varphi_0 * \varphi_0$ (supported in $[-w, w]$), $L > 2w$, $L' = L - w$, $c = L'/2$, and for $a, b \in \mathbb{C}$ and $\gamma_0 > 0$

$$\varphi_{a,b}(x) = (a\varphi_0(x-c) + b\varphi_0(x+c))e^{-i\gamma_0 x}, \quad \text{supp } \varphi_{a,b} \subset [-\frac{L}{2}, \frac{L}{2}].$$

Lemma 5 (two windows). $W(\varphi_{a,b}) = (|a|^2 + |b|^2)D - 2\text{Re}(a\bar{b}Y)$, where $D = W(\varphi_0 e^{-i\gamma_0 x})$ and

$$Y = \sum_n \frac{\Lambda(n)}{\sqrt{n}} K(\log n - L') n^{-i\gamma_0} + R,$$

the sum running over the prime powers with $|\log n - L'| < w$, and R the contribution of the pole and archimedean terms to the cross term. The smallest value of W on the family, normalized by $|a|^2 + |b|^2$, is $D - |Y|$; in particular the form is indefinite on the family if and only if $|Y| > D$. Moreover D is the archimedean term at height γ_0 , of size $\|\varphi_0\|_2^2 \log \frac{\gamma_0}{2\pi}$ up to $O(1)$, minus the prime powers $n \leq e^w$, plus a pole term $O(e^{w/2}\gamma_0^{-6})$, and $R = O(e^{-(L'-w)/2} + e^{L/2}\gamma_0^{-6})$.

Proof. Let B be the Hermitian form with $B(\varphi, \varphi) = W(\varphi)$, and $f_1 = \varphi_0(x-c)e^{-i\gamma_0 x}$, $f_2 = \varphi_0(x+c)e^{-i\gamma_0 x}$. By Proposition 1(iv), $B(f_1, f_1) = B(f_2, f_2) = D$, since f_1 and f_2 are translates of $\varphi_0 e^{-i\gamma_0 x}$ up to a unimodular constant. The cross function is $g_{12}(x) = \int f_1(y) \overline{f_2(y-x)} dy = K(x-L')e^{-i\gamma_0 x}$, supported in $[L'-w, L'+w] \subset (0, \infty)$, so $g_{12}(-\log n) = 0$ and the prime part of $B(f_1, f_2)$ is $-\sum_n \Lambda(n) n^{-1/2} K(\log n - L') n^{-i\gamma_0}$. With $Y = -B(f_1, f_2)$ the first statement follows, and the 2×2 matrix $\begin{pmatrix} D & -Y \\ -\bar{Y} & D \end{pmatrix}$ has eigenvalues $D \pm |Y|$. For the sizes: the archimedean term of D is $\frac{1}{2\pi} \int |\hat{\varphi}_0(r-\gamma_0)|^2 (\text{Re } \psi(\frac{1}{4} + \frac{ir}{2}) - \log \pi) dr$ and $\text{Re } \psi(\frac{1}{4} + \frac{ir}{2}) = \log \frac{|r|}{2} + O(r^{-2})$; the pole terms involve

$\hat{\varphi}_0(\pm \frac{i}{2} - \gamma_0)$, which is $O(e^{w/4}\gamma_0^{-3})$ since $\hat{\varphi}_0(z)$ decays like $|z|^{-3}$ in horizontal strips. In position space the archimedean term of a function G supported in $(0, \infty)$ is $-\int_0^\infty e^{-x/2} G(x) \frac{2dx}{1-e^{-2x}}$ (up to the corresponding term for $G(-x)$) [CE11, Sect. 13.6], which for $G = g_{12}$ is $O(e^{-(L'-w)/2})$; the pole part of the cross term is $O(e^{L/2}\gamma_0^{-6})$ in the same way. $\square$

Status: proved (the size estimates in outline). At $\gamma_0 = 300, 800$ and 1200 the zero side and the prime side, without R , agree to four digits (`weil_height/weil_check.py`).

The form is thus a sum over the prime powers in a window around $e^{L'}$, twisted by $n^{-i\gamma_0}$, against an archimedean term of size $\log \gamma_0$. A zero at distance δ from the line makes these prime powers resonate with amplitude $e^{\delta L'}$, and this gives the following detection statement, proved on the zero side.

Proposition 6 (detection of an isolated zero). *Let $w = 5$, $k = 2\pi/w$, and let $\varphi_{1,-1}$, L' and $L = L' + w$ be as above. Put*

$$c(T) = \frac{\log T}{2\pi} + 2(0.111 \log T + 0.275 \log \log T + 2.45) + 1, \quad S_w = \sum_{j \in \mathbb{Z}} \sup_{[j, j+1]} |\hat{\varphi}_0|^2 \leq 25.23.$$

Let $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$ be a zero of ζ with $0 < \delta < \frac{1}{2}$, let $H \geq 3$ with $\gamma_0 - H - 1 \geq e$, and assume that no zero off the critical line other than ρ_0 and $1 - \bar{\rho}_0$ has $|\gamma - \gamma_0| \leq H$. Then, for every $L' > w$,

$$(5) \quad W(\varphi_{1,-1}) \leq -2w^2 \sinh^2 \frac{\delta L'}{2} + 4S_w c(\gamma_0 + H + 1) + \frac{8k^4}{q^2} e^{L/2} \Theta(\gamma_0, H),$$

where $q = 1 - (k^2 + \frac{1}{4})/H^2$ and $\Theta(\gamma_0, H) = 2 \sum_{j \geq \lfloor H \rfloor} c(\gamma_0 + j + 2) j^{-6}$. Consequently:

- (a) if $0 < \delta < \frac{1}{2}$ and $A > 1/(10\delta)$ are fixed and $H = (\log \gamma_0)^A$, then for $\gamma_0 \geq \gamma_1(\delta, A)$ the right side of (5) is negative for some L' with $\delta L' = \log \log \gamma_0 + O_\delta(1)$, so that the prime powers up to $(C_\delta \log \gamma_0)^{1/\delta}$ suffice to detect ρ_0 ;
- (b) for $\delta \geq 0.2$ and $H = (\log \gamma_0)^2$, and for $\delta \geq 0.1$ and $H = (\log \gamma_0)^3$, the right side of (5) is negative, for a suitable $L' \leq 42.8$, respectively $L' \leq 85.6$, at every height $10^6 \leq \gamma_0 \leq 10^{1000}$.

Proof. We use the zero side $W(\varphi) = \sum_\rho \hat{g}(\gamma_\rho)$, with multiplicity, where $\hat{g}(z) = \hat{\varphi}(z) \overline{\hat{\varphi}(\bar{z})}$. Since $\cos^2(\pi x/w) = \frac{1}{2}(1 + \cos(kx))$ on $|x| \leq \frac{w}{2}$,

$$\hat{\varphi}_0(z) = -\frac{k^2 \sin(zw/2)}{z(z^2 - k^2)},$$

which is even, entire and real on the real line, and $|\hat{\varphi}_0(z)| \leq e^{|\operatorname{Im} z|w/2} k^2 / (|z||z^2 - k^2|)$. Also $\hat{\varphi}_{1,-1}(\gamma_0 + z) = 2i\hat{\varphi}_0(z) \sin(zc)$ with $c = L'/2$. Write $\Delta = \gamma - \gamma_0$.

The zero ρ_0 . An off-line pair $\frac{1}{2} \pm \delta' + i(\gamma_0 + \Delta)$ has $\gamma_\rho = \gamma_0 + \Delta \mp i\delta'$, and since $\hat{g}(\bar{z}) = \overline{\hat{g}(z)}$ it contributes $2 \operatorname{Re} \hat{g}(\gamma_0 + \Delta - i\delta') = -8 \operatorname{Re} \{ \hat{\varphi}_0(\Delta - i\delta')^2 \sinh^2((\delta' + i\Delta)L'/2) \}$. For ρ_0 this is $-8\hat{\varphi}_0(-i\delta)^2 \sinh^2(\delta L'/2) \leq -2w^2 \sinh^2(\delta L'/2)$, because $\hat{\varphi}_0(-i\delta) = \int \varphi_0(x) \cosh(\delta x) dx \geq \int \varphi_0 = \frac{w}{2}$.

Counting zeros. By the Riemann–von Mangoldt formula $N(T) = \theta(T)/\pi + 1 + S(T)$, with $\theta'(t) \leq \frac{1}{2} \log \frac{t}{2\pi} + t^{-2}$, and Trudgian's bound $|S(T)| \leq 0.111 \log T + 0.275 \log \log T + 2.450$ for $T \geq e$ [Tr], every interval $[T-1, T)$ with $T-1 \geq e$ contains at most $c(T)$ zeros. By the symmetry $\gamma \mapsto -\gamma$, and since there are no zeros with $|\gamma| < 14$, an interval of length 1 at distance j from γ_0 , on either side, contains at most $c(\gamma_0 + j + 2)$ zeros.

Nearby zeros. By hypothesis every zero other than $\rho_0, 1 - \bar{\rho}_0$ with $|\Delta| \leq H$ lies on the line, and contributes $4\hat{\varphi}_0(\Delta)^2 \sin^2(\Delta L'/2) \leq 4\hat{\varphi}_0(\Delta)^2$. Grouping these zeros by the unit intervals $[j, j+1)$ that contain Δ , each of which contains at most $c(\gamma_0 + H + 1)$ of them, their total is at most $4S_w c(\gamma_0 + H + 1)$. The bound $S_w \leq 25.23$ follows from $|\hat{\varphi}_0(r)| \leq \int \varphi_0 = \frac{w}{2}$ on the four unit intervals in $[-2, 2]$ and $|\hat{\varphi}_0(r)| \leq k^2/(|r|(r^2 - k^2))$ for $|r| \geq 2$: $S_w \leq 4(\frac{w}{2})^2 + 2 \sum_{j \geq 2} (k^2/(j(j^2 - k^2)))^2$. A grid

of step $2.5 \cdot 10^{-5}$ on $|r| \leq 20$, corrected by the Lipschitz constant $w \int |x| \varphi_0$ of $\hat{\varphi}_0^2$, gives the sharper value $S_w \leq 18.04$ (a floating-point computation, `weil_height/detect_explicit.py`), which we use only in the numerical part (b).

Distant zeros. For $|\Delta| \geq H \geq 3$ and $|\operatorname{Im} z| = \delta' < \frac{1}{2}$, with $z = \Delta - i\delta'$, we have $|z| \geq |\Delta|$ and $|z^2 - k^2| \geq \Delta^2 - \frac{1}{4} - k^2 \geq q\Delta^2$. So $|\hat{\varphi}_0(z)|^2 \leq e^{\delta'w} k^4 / (q^2 \Delta^6)$, and with $|\sinh(a + ib)|^2 \leq \cosh^2 a \leq e^{2|a|}$ every zero with $|\Delta| \geq H$, on or off the line and of either sign of γ , contributes at most $8k^4 e^{\delta'(w+L')}/(q^2 \Delta^6) \leq 8k^4 e^{L'/2}/(q^2 \Delta^6)$, counting both members of an off-line pair. Summing with the counting bound gives the last term of (5). This proves (5).

(a) Choose L' with $2w^2 \sinh^2(\delta L'/2) = 2 \cdot 4S_w c(\gamma_0 + H + 1)$. Since $c(\gamma_0 + H + 1) \sim c(\gamma_0) \asymp \log \gamma_0$, this gives $e^{\delta L'} \asymp \log \gamma_0$, so $\delta L' = \log \log \gamma_0 + O_\delta(1)$ and $e^{L'/2} \ll_\delta (\log \gamma_0)^{1/(2\delta)}$. Also $c(\gamma_0 + j + 2) \ll \log \gamma_0 + \log(1 + j)$, so $\Theta(\gamma_0, H) \ll \log \gamma_0 \cdot H^{-5} = (\log \gamma_0)^{1-5A}$. The last term of (5) is therefore $\ll (\log \gamma_0)^{1/(2\delta)+1-5A}$, which is $o(\log \gamma_0)$ because $5A > 1/(2\delta)$, while the first two terms add up to $-4S_w c(\gamma_0 + H + 1) \asymp -\log \gamma_0$. (In fact $e^{\delta L'} \approx 16S_w c/w^2 \approx 4.4 \log \gamma_0$, so the constant C can be taken absolute.) The prime powers that enter are those up to $e^L = e^w e^{L'}$.

(b) is a computation, with $S_w \leq 18.04$. The negative term of (5) does not depend on γ_0 ; the near term increases with γ_0 and with H ; the last term increases with γ_0 and decreases with H , as does $1/q^2$. For each half-decade $[g_1, g_2] \subset [10^6, 10^{1000}]$ we therefore evaluate the near term at g_2 with $H = (\log g_2)^A$, and the last term with c at g_2 and $H = (\log g_1)^A$, which bounds the right side on the whole interval, and take the first L' on a grid of step 0.01 at which the bound is below -1% of the near term. At these L' the ratio of the last term to the near term is at most $3 \cdot 10^{-7}$. The computation is in double precision; the required margin of 1% is more than 10^{10} times the rounding error, and the conclusion extends to every $\delta_1 \geq \delta$, for which the negative term is larger. $\square$

Status: (5) and (a) are proved; $\gamma_1(\delta, A)$ is effective but we have not computed it; (b) is a double-precision computation with a margin of 1% (script `detect_explicit.py` in `weil_height/`, with its output file).

The isolation hypothesis prevents the phases of nearby off-line zeros from cancelling the negative term; we do not know how to remove it, and Turán's power sum method [Tu] is the natural tool. An isolation radius $(\log \gamma_0)^A$ with $A > 1/(10\delta)$ suffices for every $\delta \in (0, \frac{1}{2})$, at the price of a larger γ_1 ; for $\delta = 0.05$ the bound (5) with $A = 3$ is negative at $\gamma_0 = 10^{20}$, 10^{40} and 10^{100} but not yet at 10^6 or 10^{12} , and with $A = 2 = 1/(10\delta)$ at none of these heights. The bound is a worst case: for $\delta = 0.2$ it gives $L = 26.6, 28.4, 30.1, 36.7$ at $\gamma_0 = 10^6, 10^{12}, 10^{20}, 10^{100}$, and for $\delta = 0.1$ ($A = 3$), $L = 48.3, 51.9, 55.2, 68.4$ at the same heights. The proposition uses the fixed choice $\varphi_{1,-1}$, for which $W = 2(D + \operatorname{Re} Y)$, rather than the minimising (a, b) of Lemma 5, and it does not depend on the size estimates of that lemma.

Measured thresholds. Measured thresholds are smaller. Inserting a pair $\frac{1}{2} \pm \delta + i\gamma$ into the zeros of ζ , rescaled to the density $\frac{1}{2\pi} \log \frac{\gamma}{2\pi}$ for $\gamma > 2000$ (the results do not change when the window of zeros or the band of test functions is widened), the general form becomes negative at $L^* \approx \log(\gamma/2\pi) + 0.8$ at small heights, where L^* depends little on δ , as in Section 6, and at $L^* \approx \log \log(\gamma/2\pi)/\delta$ at large heights for $\delta = 0.3$ and 0.4 , with $L^*\delta$ about 0.5 below this for $\delta = 0.2$. At $\gamma = 10^{12}$ and $\delta = 0.2$, $L^* = 13.65$ for the general form and 16.05 for the two windows (Figure 7a). An isolated zero with $\beta \geq 0.7$ at height γ is thus seen by the prime powers up to a power of $\log \gamma$, against $\gamma^{1/\delta}$ for the window energy of the prime signal [CE5, Sect. 2]. Status: numerical (`weil_height/weil_umbra.py`, `weil_ley.py`, `weil_robust.py`, `weil_det2b.py`).

A fixed family. The thresholds above are for test functions adapted to the height. For a fixed one-parameter family they are much larger. Let $\varphi = \mathbf{1}_{[-a/2, a/2]}$, so that $g = (a - |u|)_+$ and $\hat{g}(r) = 4 \sin^2(ra/2)/r^2$. For ζ the zero side $W(a)$, computed from the 10142 zeros below height 10^4 with the mean value of the tail, has mean $2C_B = 0.0924$ over $0 < a < 120$ and minimum 0.0405 at $a = 67.55$. An off-line zero $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$, added with $1 - \bar{\rho}_0$ and the conjugates, adds

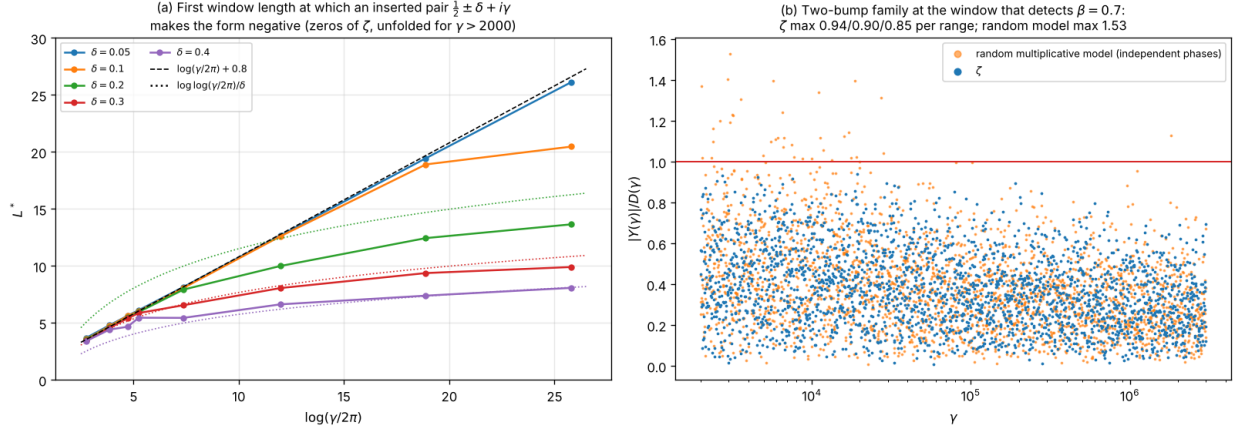


FIGURE 7. Weil's form at large height. (a) First window length L^* at which a pair $\frac{1}{2} \pm \delta + i\gamma$ inserted into the zeros of ζ makes the general form negative, against $\log(\gamma/2\pi)$; dashed: $\log(\gamma/2\pi) + 0.8$; dotted: $\log \log(\gamma/2\pi)/\delta$. (b) $|Y|/D$ for the two-window family at the window that detects $\beta = 0.7$: ζ and the random multiplicative model.

$4 \operatorname{Re} \hat{\varphi}(\gamma_0 - i\delta)^2$ with $\hat{\varphi}(z) = 2 \sin(za/2)/z$. For $\rho_0 = 0.6 + 1000i$ this is below $3.4 \cdot 10^{-5}$ in absolute value for $a \leq 18.4$, that is, for all windows reached by the prime powers up to 10^8 , and below $1.6 \cdot 10^{-3}$ for $a \leq 60$, and the form first becomes negative at $a = 97.8$. Since $|\hat{\varphi}(\gamma_0 - i\delta)|^2 \approx e^{\delta a}/\gamma_0^2$, the threshold is about $a^* = (2 \log \gamma_0 + \log \frac{C_B}{2})/\delta$, which gives 100 at $\gamma_0 = 1000$ and 537 for a zero with $\delta = 0.1$ just above the height $3 \cdot 10^{12}$ to which RH is verified: prime powers up to about 10^{233} , against a power of $\log \gamma_0$ for the adapted families. As with Skewes' bound for the first sign change of $\pi(x) - \operatorname{li}(x)$ [Sk] and the disproof of the Mertens conjecture [OtR], positivity on every window that can be computed is consistent with an off-line zero. Status: the estimate of a^* is elementary; the numbers are computed (`weil_height/two_worlds.py`).

The converse direction. The converse asks for positivity of the two-window family at every height, with the window that detects $\delta = 0.2$. For ζ we computed $|Y|/D$ at 2400 heights in $[2 \cdot 10^3, 3 \cdot 10^6]$, with $L \approx 11$ to 13 and $w = L/4$ (prime powers up to $5.6 \cdot 10^5$). On $[2 \cdot 10^3, 2 \cdot 10^4]$, $[2 \cdot 10^4, 2 \cdot 10^5]$ and $[2 \cdot 10^5, 3 \cdot 10^6]$ the mean is 0.43, 0.38, 0.32 and the maximum 0.94, 0.90, 0.85, with 99th percentile 0.84 overall, so the margin $1 - |Y|/D$ is at least 6 to 15%. Replacing $p^{-i\gamma}$ by independent uniform phases, the random multiplicative model has the same mean, but maximum 1.53, 1.32, 1.13, 99th percentile 1.07, and $|Y| > D$ at 1.5% of the heights (Figure 7b). At this window positivity is not a consequence of the statistics of the primes; it is carried by the exact positions of the zeros. High moments of Y , which the mean value theorem for Dirichlet polynomials gives exactly since the polynomial is short, bound only the measure of the set of heights where $|Y| > D$, a statement of density type. Status: numerical (`weil_height/weil_medirP.py`, `fig_large_height.py`).

9. PRIME TRANSLATIONS AND THE DIFFERENCE SET

Where RH is proved, an operation transports the problem from one scale to the next and conserves a norm, so that one verification covers every scale: over function fields this is Frobenius, unitary on a finite-dimensional cohomology that it maps to itself [De]. Over $\mathbb{Q}$ the primes supply the operations. In the variable $u = \log x$ the dilation by p is the translation $T_{\log p}$; on the zero side it multiplies $\hat{\varphi}(\gamma)$ by the phase $p^{i\gamma}$.

Lemma 7 (prime translations are isometries). *For every $\varphi \in C_c^\infty(\mathbb{R})$ and every $x \in \mathbb{R}$, $W(T_x\varphi) = W(\varphi)$; in particular $W(T_{\log p}\varphi) = W(\varphi)$ for every prime p . If $W \geq 0$, the completion of $C_c^\infty(\mathbb{R})$ modulo the null space under W is a Hilbert space on which the translations act unitarily.*

Proof. The first statement is Proposition 1(iv): W depends on φ only through $g = \varphi * \tilde{\varphi}$, which translation does not change. The second follows because translations preserve W and the null space. $\square$

Status: proved (elementary). The second statement is conditional: the Hilbert space exists only if $W \geq 0$, which is RH, so it is a consequence of the positivity to be proved and not an independent object. Under RH the spectrum of the generator of these translations is the set of ordinates γ , which are real; this is the construction of Weil and Connes.

What fails is closure. By Proposition 1(iii) the form sees φ only through g , whose support is the difference set $S - S$ of the support S of φ , and its margin is governed by $S - S$, not by the dimension of the space or by the arithmetic of the translations. Table 6 gives the smallest eigenvalue, with 50 digits from the prime side in basis (d). More functions on the same window do not change the margin; a single translate of the window lowers it by eleven orders of magnitude, close to the value for the full window of length $1.25 + \log 2$, and it makes no difference whether the shift is $\log 2$ or 0.6, which is not the logarithm of a prime. A single translate adds the lags $\pm(x \pm 1.25)$ to $S - S$, new prime powers enter the form through them, and positivity has to be established again with no margin inherited from the smaller set (Section 5). Status: numerical (`transport/hp_transport.py`, results in `transport/tr_*.json`).

space	functions	$\lambda_{\min}$
window of length 1.25	24	$3.7 \cdot 10^{-10}$
the same window, twice the band	48	$3.0 \cdot 10^{-10}$
window + its translate by $\log 2$	48	$6.9 \cdot 10^{-21}$
window + its translate by 0.6	48	$7.2 \cdot 10^{-21}$
full window of length $1.25 + \log 2$	38	$2.5 \cdot 10^{-26}$

TABLE 6. Weil's form for ζ on spaces of trigonometric functions on one window and on a window with one translate (50 digits, prime side).

For a curve over $\mathbb{F}_q$ all the translations are multiples of $\log q$, the difference set of a space closed under them stays in the lattice $(\log q)\mathbb{Z}$, the space is closed and finite-dimensional, and the margin on windows is the constant $2\log q$ of [CE10, Prop. 3.15]. Over $\mathbb{Q}$ the numbers $\log p$ are linearly independent, the translations generate a dense subgroup of $\mathbb{R}$, and every translate brings new lags. Nor can a common degree be imposed approximately: the best ℓ with $\log p \approx k_p \ell$ for all $p \leq P$ leaves an error $\varepsilon \approx \ell^{1+1/(n-1)}$ with $n = \pi(P)$; for $p \leq 13$, $\varepsilon = 0.10, 7 \cdot 10^{-3}, 9 \cdot 10^{-5}$ at $\ell \approx 0.33, 0.03, 0.001$. Keeping ε below a margin of 10^{-55} would require $\ell \approx 10^{-46}$, that is, $q = e^\ell \rightarrow 1$. Status: numerical (`transport/a_frobenius.py`). The difference between $\mathbb{Q}$ and function fields can therefore be stated in terms of the objects of this paper: the prime operators and Weil's norm exist and are compatible, and what is missing is a finite set of lags, closed under the prime translations, on which the norm can be checked once.

10. DISCUSSION AND OPEN PROBLEMS

In quadratic form the prime powers restore positivity. In linear inequalities the pole cancels the positivity of Λ [CE16, Sect. 2]; in Weil's form the pole is a rank-two term, and the primes produce the positivity and move the zeros of the minimiser onto the zeros of ζ . The margin, however, is empirically about $10^{-(5.42e^L - 12.6)}$ in converged discretizations of the windows computed ($L \leq 5$), an upper bound for $\lambda_{\min}(L)$, positivity on all windows is equivalent to RH, and the size of the

margin is fixed by the density of the zeros alone. In the windows tested, removing the most recent prime power makes the computed form negative. For the Davenport–Heilbronn function the same cancellation fails at $L^* \approx 3.425$, with an $O(1)$ negative eigenvalue soon after. Any proof must explain why, for ζ , the prime partial sums on the critical line never let the form become negative, and must use a property that the coefficients $b(n)$ of f lack: multiplicativity, $\Lambda \geq 0$, or support on the prime powers. Section 5 shows the difference in the minimising direction: for ζ every prime power term has the same sign and positivity is a coherent balance against the pole, while for f it rests on cancellation between coefficients of opposite sign, at composite n as well.

At large height the window is short, and positivity there is rigid. By Proposition 6 an isolated zero with $\beta = \frac{1}{2} + \delta$ at height γ is detected by the prime powers up to $(C \log \gamma)^{1/\delta}$, for γ large and under the isolation hypothesis. The related two-window positivity measured in Section 8 ($w = L/4$, $L \approx 11$ – 13 ; not the family of the proposition) holds for ζ with a margin of 6 to 15% at 2400 sampled heights up to $3 \cdot 10^6$, where RH is already verified [PT], but fails for a random multiplicative model with the same mean: it is not a property of the distribution of the primes, and an argument for it must use the arithmetic that fixes the positions of the zeros.

What the computations do not do. No computation here certifies positivity: apart from Proposition 4, which certifies a negative value for f , all are in floating point or in fixed high precision without interval arithmetic, positivity on a finite-dimensional space is not Weil’s criterion, and on the windows computed the form is governed by zeros that are known to lie on the line [PT]. The comparison with f shows that positivity on a fixed window, and the accuracy of the Connes minimiser, are compatible with off-line zeros.

Open problems.

- (1) Prove the density law (4), or at least an upper bound $\lambda_{\min}(L) \leq e^{-cL^2}$, for points of density $\frac{1}{2\pi} \log \frac{t}{2\pi}$ with a gap below the first ordinate (Remark 3); explain the law $10^{-(5.42e^L - 12.6)}$ of Section 3.5 and the cut-off $\Omega \approx 14e^L$ of the bases that reach it.
- (2) Remove the isolation hypothesis in Proposition 6, for example with Turán’s power sum method, and compute the threshold $\gamma_1(\delta, A)$ of part (a), which part (b) replaces for $\delta \geq 0.1$ only on $10^6 \leq \gamma_0 \leq 10^{1000}$.
- (3) Prove that $|Y| \leq D$ for the two-window family, with $w = L/4$ and the window that detects $\delta = 0.2$ (or with $w = 5$ and the window of Proposition 6), on a set of heights of density one, using the exact moments of the short Dirichlet polynomial Y , and identify the arithmetic input that excludes the exceptional heights of the random model.
- (4) Decide whether the Connes approximation of the zeros by the minimiser converges as $L \rightarrow \infty$, and whether the locking of a zero of the minimiser onto an off-line ordinate beyond L^* can be proved for f .
- (5) Explain the exponent δ^6 of the deficit after the removal of the most recent prime power.
- (6) Find a family of test functions over $\mathbb{Q}$ whose difference set is stable under the prime translations up to an error smaller than the margin, or prove that none exists.

APPENDIX A. HIGH-PRECISION TABLES

Tables 7–9 give the data of Sections 3.5, 5 (Levinson recursion), 6 and 7, computed with the closed-form matrix of Section 3.5 and the scripts in `anc/weil_cf/`. All are floating-point computations at fixed precision, without interval arithmetic; agreement between two precisions and two basis sizes is the only check of their accuracy.

DATA AND CODE

The ancillary Python scripts (NumPy, SciPy, Matplotlib; mpmath for the high-precision computations) are in `anc/`, with one folder per topic; `anc/README.txt` lists, for each script, the section, table or figure it reproduces and its output file. The zeros of ζ with $0 < \gamma < 600$ (341 ordinates)

L	K	Ω/e^L	digits	$\lambda_{\min}$	$\log_{10} \lambda_{\min}$	fit
1.25	80	114.5	60	$2.92 \cdot 10^{-10}$	-9.53	
1.25	120	172.1	60	$2.87 \cdot 10^{-10}$	-9.54	
2	80	33.8	100	$6.79 \cdot 10^{-30}$	-29.17	-27.4
2	120	50.8	100	$6.26 \cdot 10^{-30}$	-29.20	-27.4
2.5	160	32.9	150	$5.34 \cdot 10^{-55}$	-54.27	-53.4
2.5	200	41.2	150	$5.06 \cdot 10^{-55}$	-54.30	-53.4
3	200	20.8	200	$5.14 \cdot 10^{-97}$	-96.29	-96.2
3	260	27.1	200	$4.69 \cdot 10^{-97}$	-96.33	-96.2
3.5	300	16.2	200	$4.90 \cdot 10^{-167}$	-166.31	-166.8
3.5	400	21.7	200	$3.96 \cdot 10^{-167}$	-166.40	-166.8
3.5	400	21.7	260	$3.96 \cdot 10^{-167}$	-166.40	-166.8
3.6	270	12.9	260	$1.08 \cdot 10^{-185}$	-184.97	-185.7
3.6	330	15.7	260	$7.68 \cdot 10^{-186}$	-185.11	-185.7
3.7	300	12.6	280	$1.95 \cdot 10^{-206}$	-205.71	-206.5
3.7	370	15.5	280	$1.30 \cdot 10^{-206}$	-205.89	-206.5
3.8	340	12.6	310	$1.88 \cdot 10^{-229}$	-228.72	-229.6
3.8	420	15.5	310	$1.12 \cdot 10^{-229}$	-228.95	-229.6
3.9	390	12.7	340	$6.24 \cdot 10^{-255}$	-254.21	-255.1
3.9	480	15.6	340	$3.86 \cdot 10^{-255}$	-254.41	-255.1
4	550	15.8	330	$3.05 \cdot 10^{-283}$	-282.52	-283.2
4	550	15.8	380	$3.05 \cdot 10^{-283}$	-282.52	-283.2
4	700	20.1	330	$2.49 \cdot 10^{-283}$	-282.60	-283.2
4.5	700	10.9	600	$7.29 \cdot 10^{-471}$	-470.14	-475.1
4.5	1000	15.5	600	$1.62 \cdot 10^{-475}$	-474.79	-475.1
4.5	1200	18.6	540	$1.33 \cdot 10^{-475}$	-474.88	-475.1
5	1700	14.4	950	$3.58 \cdot 10^{-793}$	-792.45	-791.5
5	2000	16.9	1050	$2.88 \cdot 10^{-793}$	-792.54	-791.5

TABLE 7. Smallest eigenvalue of Weil's form for ζ in the even cosine basis (b) with K functions and largest frequency $\Omega = (2K - 1)\pi/L$, matrix in closed form (Section 3.5). Rows with the same L : two or three basis sizes and, for $L = 3.5, 4, 4.5, 5$, two precisions. Last column: the fit $-(5.42 e^L - 12.6)$ for $L \geq 2$. All values are upper bounds for $\lambda_{\min}(L)$ (Proposition 1(v)).

L	ζ					f (60 digits, $K = 200$)			
	K	digits	$\lambda_{\min}$	error γ_1	max. error	$\lambda_{\min}$ (neg.)	median error	zero near 85.7	pair ρ_1
3.6	330	260	$7.7 \cdot 10^{-186}$	$7.9 \cdot 10^{-182}$	$9.8 \cdot 10^{-134}$	$-2.7 \cdot 10^{-7}$ (1)	$4.9 \cdot 10^{-8}$	85.7305	$-5.8 \cdot 10^{-6}$
3.7	370	280	$1.3 \cdot 10^{-206}$	$1.4 \cdot 10^{-202}$	$4.7 \cdot 10^{-154}$	-0.026 (1)	$9.2 \cdot 10^{-3}$	85.7289	-0.25
3.8	420	310	$1.1 \cdot 10^{-229}$	$1.2 \cdot 10^{-225}$	$1.0 \cdot 10^{-176}$	-0.26 (2)	0.035	85.7203	-0.80
3.9	480	340	$3.9 \cdot 10^{-255}$	$4.2 \cdot 10^{-251}$	$8.1 \cdot 10^{-202}$	-0.45 (2)	0.045	85.7145	-0.84
4.0	550	380	$3.0 \cdot 10^{-283}$	$3.3 \cdot 10^{-279}$	$1.4 \cdot 10^{-229}$	-0.70 (2)	0.088	85.7083	-0.94

TABLE 8. The minimiser beyond the threshold of f (Section 7). For ζ : error of the first zero of $\hat{\varphi}$ against $\gamma_1 = 14.1347\dots$ and largest error over the 29 zeros below height 100. For f : smallest eigenvalue (number of negative eigenvalues), median error of the first ten zeros of f on the line, the real zero of $\hat{\varphi}$ next to the ordinate 85.6993 of $\rho_1 = 0.8085 + 85.6993i$, and the contribution $4 \operatorname{Re} \hat{\varphi}(\gamma_{\rho_1})^2 / \|\varphi\|^2$ of ρ_1 , $1 - \bar{\rho}_1$ and their conjugates to the minimal value.

	q	h	digits	break at L	κ at break	$1 - \kappa^2$ before	$\min(1 - \kappa^2)$
ζ	4	0.03	100	none ($L \leq 4.53$)			0.66
ζ	8	0.01	150, 220	none ($L \leq 4.51$)			0.68
f	4	0.03	60	3.75	4.62	0.08	0.08
f	4	0.01	80	3.57	14.27	0.13	0.13
f	4	0.005	80	3.54	-2.72	0.38	0.38
f	6	0.005	80	3.46	-2.36	0.47	0.47
f	8	0.01	80, 120	3.46	1.05	0.66	0.62
f	8	0.005	80	3.445	2.04	0.49	0.49
f	8	0.0025	80	3.4325	1.32	0.65	0.59
f	10	0.0025	80	3.4375	3.63	0.36	0.36
f	12	0.01	80	3.5	13.50	0.09	0.09
f	16	0.01	80	3.53	4.71	0.23	0.23
f	16	0.005	80	3.475	1.66	0.53	0.37

TABLE 9. Levinson–Durbin recursion for Weil’s form on the translates of a B-spline of order q and mesh h (Section 5). Break: first window length $(K + q)h$ at which $|\kappa_K| \geq 1$, i.e. the form on the first $K + 1$ translates is not positive. Last two columns: $1 - \kappa^2$ at the step before the break, and its minimum over all steps before it (over the whole range if there is no break).

are in `zeros600.npz`; the off-line zeros of f below height 400 in `offline.npy`. The high-precision computations of Sections 3.5, 5 (Levinson recursion), 6, 7 and Appendix A are in `anc/weil_cf/`, with every run in `anc/weil_cf/runs/`. The computations are in double precision unless a number of digits is stated, without interval arithmetic, except `weil_cf/certify_dh_upper.py` (Arb, Proposition 4); they illustrate the statements and are not used in the proofs.

Provenance. This paper supersedes the preprint [C7] and contains the material on Weil matrices, the Davenport–Heilbronn threshold, the Connes minimiser, the prime-by-prime computations, the two-window family and the transport computations of version 3 of [CP]; the energy slope as Weil’s functional at a Poisson kernel, its exact truncation and the finite-field margin are in [CE10, Sect. 3], and the fixed-interval computation with position-space archimedean term is in [CE11, Sect. 13.6]. Use of AI tools. The proofs, computations and drafting of this paper, and of the preprints that it absorbs, were carried out with the assistance of an AI system (Claude, Anthropic), under the author’s direction. The author checked the results and takes responsibility for them.

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