

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 8: THE EULER PRODUCT INSIDE THE CRITICAL STRIP

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ABSTRACT. Inside the critical strip the partial Euler product $E_x(s) = \prod_{p \leq x} (1 - p^{-s})^{-1}$ diverges because of the pole. Removing the pole with the exponential integral, $E_x^* = E_x \exp(E_1((s-1) \log x))$, gives an exact product form of the explicit formula, $\log E_x^* = \log \zeta + \sum_{\rho} E_1((s-\rho) \log x) + \dots$, in which each zero is a mode of size $x^{\beta-\sigma} / \log x$. The corrected product therefore converges exactly to the right of the zeros: RH holds if and only if $E_x^* \rightarrow \zeta$ in $\frac{1}{2} < \sigma < 1$, and at a simple zero $E_x^*(\rho) \log x = \zeta'(\rho) e^{-\gamma_0}$ times the modes of the other zeros. The central identity is that a partial Euler product is the cross-energy C_2 of the partial sums of ζ restricted to smooth integers; the cross-energy of the primes alone obeys the trichotomy of the integers with the normalization $\log x$. As functions of $u = \log x$ the modes form the Euler helix, a circle for each zero on the line and a spiral for each zero off it. With the primes up to 10^8 the helix detects displacements of about $4 \cdot 10^{-5}$ of the zeros near a given height; sieving the partial sums turns the cross-energy wave into the helix; and the Laguerre expressions of ξ are moments of the helix, where we correct the second one by a term from conjugate pairs. We also treat an apparent abscissa read from the smoothed product, an off-line zero of the Davenport–Heilbronn function as an interference of two Euler products, a damped energy with abscissa of convergence $2 \sup \beta - 1$, with its discrete form, an explicitly diagonalized energy matrix of $\psi(k) - k$, the heat flow and the collisions of its zeros, the pairing of the 1-points of ζ with the zeros of ζ' , and the Euler product as a quantum state whose coherence does not depend on t . Each statement carries its status. We do not prove the Riemann hypothesis.

1. INTRODUCTION

1.1. Context. This paper belongs to a series on the Riemann hypothesis (RH) built on the cross-energy of consecutive Dirichlet partial sums. With $s = \sigma + it$, $X_N(s) = \sum_{k \leq N} k^{-s}$ and

$$C_2(n, s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma} = 2 \operatorname{Re} (X_{n-1}(s) \overline{n^{-s}}),$$

Part 1 [CE1] proves the trichotomy $\lim_n C_2(n, s) = 1/(\frac{1}{4} + t^2)$, 0 , $+\infty$ for $\sigma = \frac{1}{2}$, $> \frac{1}{2}$, $< \frac{1}{2}$ (Theorem A there), the additive representation $\zeta = \lim_N [X_N - \frac{1}{2} N^{-s} - N^{1-s}/(1-s)]$ and the C-transform [CE1, Sect. 2]; the prime energy law is [CE5, Thm. B], and the structural limits of the method are in [CE16, Sect. 2]. Part 11 [CE11] studies the Euler–Caceres bound, built on an exact smoothed form $\zeta = E_X^* Z_X$ of the hybrid Euler–Hadamard product of Gonek, Hughes and Keating [GHK]. Both papers use the primes additively, through $\Lambda(n)$, or through a factorization. This paper asks what the partial Euler product itself does inside the strip, where it diverges.

The divergence comes from the pole: the sum $\sum_{p \leq x} p^{-s}$ contains the term $\int^x u^{-s} du / \log u$, of size $x^{1-\sigma} / ((1-s) \log x)$; on the critical line at $t = 20$ the ratio E_x / ζ takes the values 0.92, 1.60, 0.35, 0.58, 84.5 at $x = 10^3, 10^4, 10^5, 10^6, 2 \cdot 10^6$. Removing this term with the exponential integral gives a product E_x^* whose error is, zero by zero, a superposition of modes $x^{\rho-s} / ((s-\rho) \log x)$ (Proposition 2). This is a reorganization of the classical explicit formula, and its smoothed, truncated form is the

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hybrid product of [GHK]; the exact smoothed factorization, its zero charges ([CE11, Prop. 25]) and the bound built on it are in [CE11] and are not reproduced here. What the present paper adds is a collection of consequences and measurements of the sharp product form, many of them with the Davenport–Heilbronn function f as a control, together with several observations on the Euler product that do not fit elsewhere in the series.

1.2. Results. Table 1 lists the main statements and their status. The proved statements are elementary consequences of the explicit formula, of Hurwitz’s theorem, or of classical facts, and we say so where they are classical in substance. The remaining material is numerical or heuristic, and is labelled as such where it appears.

statement	where	status
explicit formula in product form	Prop. 2, Lemma 1	proved (classical reorganization; domain and branches explicit)
$\text{RH} \Leftrightarrow E_x^* \rightarrow \zeta$ in $\frac{1}{2} < \sigma < 1$	Cor. 3	proved
the corrected product at a zero	Prop. 4	proved; the decay is a DRH-type statement, not proved
Euler product as a C_2 energy	Thm. 6, Rem. 7	proved; the central identity
trichotomy for the prime zeta function	Prop. 8	proved (from the PNT)
apparent abscissa	Lemma 9,	identity proved; detector numerical
	Sect. 4.1	
an off-line zero as interference	Sect. 4.2	numerical
the Euler helix and its energy	Sect. 5	identity proved; energy heuristic
rigidity, sieved wave, prime operators	Rems. 11–13	identities proved; numerical
Laguerre expressions as helix moments	Prop. 14	proved for the model sum (corrected)
collisions under the heat flow	Sect. 6.2	pair law elementary; numerical
damped energy	Prop. 15	proved; detection cost numerical
energy matrix of the Chebyshev errors	Rem. 16	identities exact; equivalence classical
1-points, zeros of ζ' and of P	Rem. 17	observations
series over the derivatives	Sect. 9	elementary; rejected
the Euler state	Prop. 18	proved

TABLE 1. Results and status.

Three topics that belong to the Euler product are proved elsewhere in the series and only summarized here, with the citation: the partial Euler products over $\mathbb{F}_q[T]$ (Section 2, [CE17, Prop. 10]), amplification and the parity barrier (Section 8, [CE17, Sect. 7.5]), and the prime translations as isometries of Weil’s form without a closed finite set of lags (Section 11, [CE12, Sect. 9]).

1.3. What is new and what is classical. The equivalence of RH with the analyticity of $\log \zeta$ in $\sigma > \frac{1}{2}$ is classical, and so are the facts used in the proofs: the explicit formula [In, T], Landau’s theorem on Dirichlet integrals with nonnegative coefficients [In], Hurwitz’s theorem, the Hadamard product, and the theory of the de Bruijn–Newman constant [Ne, RT, Pm, PT]. Conrad formulated the Deep Riemann Hypothesis for L -functions without a pole [Co] (see also [Sh]); the decay $|E_x^*(\rho)| \log x \rightarrow |\zeta'(\rho)|e^{-\gamma_0}$ at a zero of ζ is the analogue of his statement for ζ with the pole correction, and its constant $e^{-\gamma_0}$ is the one that appears in his work. What is new here is the systematic use of the sharp product form as a geometric object (the Euler helix), its relation to the cross-energy of [CE1] (Theorem 6 and Proposition 8, Remark 12), the corrected moment identity for the Laguerre expressions (Proposition 14), and the measurements, with controls, collected in Sections 2–10. None of them constrains the position of the zeros.

The statements of the paper are of three kinds, and we keep them apart. (1) *Classical identities, reorganized*: the explicit formula in product form (Proposition 2, for which we give a complete proof

with its domain, branches and order of summation), the equivalence of RH with the convergence of the corrected products (Corollary 3), the damped energy (Proposition 15) and the integral representations. (2) *New structural connections*: the partial Euler product as the cross-energy of the partial sums over the smooth integers (Theorem 6, the central identity of the paper), the trichotomy for the primes alone (Proposition 8), the sieved wave (Remark 12) and the moment identity for the Laguerre expressions (Proposition 14). (3) *Heuristic and numerical statements*: the energy of the Euler helix (Remark 10), the reduction of the Laguerre hierarchy to that energy, the double-sum picture of the damped energy, the deformed cross-energy under the heat flow, and all measurements. (4) *Constraints on the zeros*: there are none. No statement of the paper excludes a zero with $\beta > \frac{1}{2}$, and the computations detect displacements of the zeros near a given height without bounding β in general.

1.4. Notation. $\rho = \beta + i\gamma$ runs over the nontrivial zeros of ζ , counted with multiplicity and ordered by $|\gamma|$ when a sum over them is only conditionally convergent; $\Theta = \sup \beta$; $\gamma_0 = 0.5772\dots$ is Euler's constant; $C_B = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots$; $\lambda = s(1-s)$. $E_1(w) = \int_1^\infty e^{-wv} v^{-1} dv$ is the exponential integral, continued to $\mathbb{C} \setminus (-\infty, 0]$; for $\sigma > 1$, $E_1((s-1)\log x) = \int_x^\infty u^{-s} du / \log u$, and $E_1(w) = -\gamma_0 - \log w + \text{Ein}(w)$ with Ein entire. The cross-energy wave of [CE1] is $W_n(s) = C_2(n, s) + iS_2(n, s) = 2X_{n-1}(s)n^{-s}$. The Davenport–Heilbronn function is $f = c_1 L(s, \chi) + c_2 L(s, \bar{\chi})$, χ a complex character modulo 5; it has the functional equation of an L -function of conductor 5 and zeros off the line, among them $\rho_f = 0.808517 + 85.699348i$ [DH], [CE16, Sect. 2].

2. THE CORRECTED PARTIAL PRODUCT

Let

$$E_x(s) = \prod_{p \leq x} (1-p^{-s})^{-1}, \quad E_x^*(s) = E_x(s) \exp(E_1((s-1)\log x)), \quad R_x(s) = \sum_{k \geq 2} \sum_{x^{1/k} < p \leq x} \frac{p^{-ks}}{k}.$$

Domain and branches. Let $\mathcal{D}$ be the half-plane $\sigma > 0$ with the segment $(0, 1]$ and the horizontal half-lines $\{\rho - r : r \geq 0\}$ removed, one for each nontrivial zero ρ . Each half-line runs from ρ to the boundary $\sigma = 0$, so $\mathcal{D}$ is simply connected; it contains $\sigma > 1$ and no zero of ζ . On $\mathcal{D}$ we use the following branches: $\log \zeta$ is the continuation from $\sigma > 1$, where $\log \zeta(s) = \sum_n \Lambda(n)n^{-s} / \log n$; $\log E_x(s) = -\sum_{p \leq x} \text{Log}(1-p^{-s})$ with the principal logarithm, legitimate since $|p^{-s}| < 1$; and E_1 is the principal branch on $\mathbb{C} \setminus (-\infty, 0]$, so that $\log E_x^* = \log E_x + E_1((s-1)\log x)$. For $s \in \mathcal{D}$ none of the arguments $(s-1)\log x$, $(s-\rho)\log x$, $(s+2k)\log x$ lies on $(-\infty, 0]$, since $s-\rho \in (-\infty, 0]$ means that s lies on the half-line of ρ ; so all the terms below are analytic on $\mathcal{D}$. We use the bound, uniform in $|\arg w| \leq \frac{3}{4}\pi$, $|w| \geq 1$ [DLMF, §6.12],

$$(1) \quad E_1(w) = \frac{e^{-w}}{w} + r(w), \quad |r(w)| \leq C \frac{e^{-\text{Re } w}}{|w|^2},$$

and, for $\text{Re } w > 0$, $|E_1(w)| \leq e^{-\text{Re } w} / \text{Re } w$.

Lemma 1 (the explicit formula with a parameter). *Let $x > 1$ not be a prime power. For $s \neq 1$ not a zero of ζ ,*

$$\sum_{n \leq x} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) - \frac{x^{1-s}}{s-1} + \sum_{\rho} \frac{x^{\rho-s}}{s-\rho} + \sum_{k \geq 1} \frac{x^{-2k-s}}{s+2k},$$

where $\sum_{\rho} = \lim_{T \rightarrow \infty} \sum_{|\gamma| < T}$. *For every compact set $K \subset \mathbb{C}$, the partial sums over $|\gamma| < T$, with the finitely many terms of the zeros within distance 1 of K removed, converge uniformly on K as $T \rightarrow \infty$.*

Proof. This is the explicit formula for $\psi(x)$, the case $s = 0$, with a parameter. The standard proof [Da, Ch. 17], [In, Ch. IV] applies Perron's formula to $-\frac{\zeta'}{\zeta}(s+w)x^w/w$ on $\text{Re } w = c > 1 - \sigma$, $|\text{Im } w| \leq T$, and moves the contour to the left. The residues at $w = 0$, $w = 1 - s$, $w = \rho - s$ and $w = -2k - s$ give the four terms, and, with $c = 1 - \sigma + 1/\log x$ and, for each s , a height $T_s \in [T, T+1]$ such that $t \pm T_s$ are at distance $\gg 1/\log T$ from the ordinates, the remaining integrals are $O_K(x^{1-\sigma} \log^2(xT)/T + x^{-\sigma} \log x \min(1, x/(T\langle x \rangle)))$ uniformly for $s \in K$, where $\langle x \rangle$ is the distance from x to the nearest prime power. The zeros enter with $|\gamma - t| < T_s$. Replacing this by the fixed cutoff $|\gamma| < T$ changes $O_K(\log T)$ terms, each $O_K(x^{1-\sigma}/T)$ when the zeros within distance 1 of K are excluded, so the Cauchy criterion holds uniformly on K , whether or not K contains zeros. $\square$

Proposition 2 (explicit formula in product form). *Let $x > 1$ not be a prime power. On $\mathcal{D}$,*

$$(2) \quad \log E_x^*(s) = \log \zeta(s) + \sum_{\rho} E_1((s - \rho) \log x) + \sum_{k \geq 1} E_1((s + 2k) \log x) + R_x(s),$$

where $\sum_{\rho}$ is the limit as $T \rightarrow \infty$ of the sums over $|\gamma| < T$, locally uniformly on $\mathcal{D}$, and the sum over k is $O_{\sigma}(x^{-2-\sigma}/\log x)$ for $x \geq 2$.

Proof. Put $L = \log x$ and $w_{\rho} = (s - \rho)L$. *Convergence.* Let $K \subset \mathcal{D}$ be compact and $\sigma_K = \max_K \sigma$. For all but finitely many zeros, $|\gamma - t| \geq \max(\sigma_K + 1, 1/L)$ for every $s \in K$; for these, $|\text{Im } w_{\rho}| = |t - \gamma|L \geq |\sigma - \beta|L = |\text{Re } w_{\rho}|$ and $|w_{\rho}| \geq 1$, so $|\arg w_{\rho}| \leq \frac{3}{4}\pi$ and (1) gives

$$E_1(w_{\rho}) = \frac{x^{\rho-s}}{(s - \rho)L} + O\left(\frac{x^{1-\sigma}}{|s - \rho|^2 L^2}\right).$$

The error terms are summable uniformly on K , since $\sum_{\rho} |\gamma|^{-2} < \infty$; the main terms converge uniformly on K by Lemma 1; and the finitely many remaining terms are analytic on $\mathcal{D}$. Hence $Z(s) = \sum_{\rho} E_1(w_{\rho})$ is analytic on $\mathcal{D}$, and since $\frac{d}{ds} E_1((s - \rho)L) = L E_1'(w_{\rho}) = -x^{\rho-s}/(s - \rho)$, its derivative is $-\sum_{\rho} x^{\rho-s}/(s - \rho)$, in the same order of summation. The sum over k converges absolutely, with the stated bound, by $|E_1(w)| \leq e^{-\text{Re } w}/\text{Re } w$.

The derivative. Put $\Phi = \log E_x^* - \log \zeta - Z - \sum_k E_1((s + 2k)L) - R_x$, analytic on $\mathcal{D}$. Since $\log E_x(s) = \sum_{n \leq x} \Lambda(n)n^{-s}/\log n + R_x(s)$, we have $(\log E_x - R_x)' = -\sum_{n \leq x} \Lambda(n)n^{-s}$, and $\frac{d}{ds} E_1((s - 1)L) = -x^{1-s}/(s - 1)$. Hence

$$\Phi'(s) = -\sum_{n \leq x} \frac{\Lambda(n)}{n^s} - \frac{x^{1-s}}{s - 1} - \frac{\zeta'}{\zeta}(s) + \sum_{\rho} \frac{x^{\rho-s}}{s - \rho} + \sum_{k \geq 1} \frac{x^{-2k-s}}{s + 2k} = 0$$

by Lemma 1, so Φ is constant on the connected set $\mathcal{D}$.

The constant. Let $s = \sigma \rightarrow +\infty$ on the real axis, which lies in $\mathcal{D}$ for $\sigma > 1$. Then $\log E_x(\sigma)$, $E_1((\sigma - 1)L)$, $\log \zeta(\sigma)$, $R_x(\sigma)$ and the sum over k tend to 0. For $Z(\sigma)$, every w_{ρ} has $\text{Re } w_{\rho} > 0$ and (1) applies to all zeros: the error terms add up to $O(x^{1-\sigma} \sum_{\rho} |\sigma - \rho|^{-2}) \rightarrow 0$, and the main terms are $L^{-1}x^{-\sigma} \sum_{\rho} x^{\rho}/(\sigma - \rho)$. With $\frac{1}{\sigma - \rho} = -\frac{1}{\rho} + \frac{\sigma}{\rho(\sigma - \rho)}$, this sum is $-\sum_{\rho} x^{\rho}/\rho$, a constant (the explicit formula for ψ), plus $O(\sigma x \sum_{\rho} |\rho|^{-1} |\sigma - \rho|^{-1}) = O(x \log^2 \sigma)$, and $x^{-\sigma}$ times it tends to 0. So $\Phi = 0$. $\square$

By (1), for $|\gamma - t|$ large the term of a zero is

$$E_1((s - \rho) \log x) = \frac{x^{\rho-s}}{(s - \rho) \log x} \left(1 + O\left(\frac{1}{|s - \rho| \log x}\right)\right) :$$

it decays like $x^{\beta-\sigma}/\log x$ if $\beta < \sigma$ and grows if $\beta > \sigma$. This describes each term separately. The sum over the zeros converges only conditionally, in the order of Proposition 2, and the size of the sum cannot be read off from the sizes of its terms; every global statement below is proved for the

sum, not term by term. With this caveat, on a vertical line the corrected product sees exactly the zeros to its right.

Corollary 3. *Let $\Omega = \{s : \frac{1}{2} < \sigma < 1\}$. The following are equivalent: (i) RH; (ii) $E_x^* \rightarrow \zeta$ locally uniformly on Ω as $x \rightarrow \infty$; (iii) for every compact $K \subset \Omega$ there are C_K and x_K with $|E_x^*(s)/\zeta(s) - 1| \leq C_K x^{1/2-\sigma} \log x$ for $s \in K$ and $x \geq x_K$.*

Proof. Since $\exp(E_1(w)) = e^{-\gamma_0} e^{\text{Ein}(w)}/w$ is single-valued, $E_x^*(s) = E_x(s) e^{-\gamma_0} e^{\text{Ein}((s-1)\log x)}/((s-1)\log x)$ is meromorphic on $\sigma > 0$, with a single simple pole at $s = 1$, and it has no zeros there. For $\sigma > 1$, since $E_1((s-1)\log x) = \int_x^\infty u^{-s} du / \log u$ and $\log \zeta(s) - \log E_x(s) + R_x(s) = \sum_{n>x} \Lambda(n)n^{-s} / \log n$,

$$(3) \quad \frac{E_x^*(s)}{\zeta(s)} = \exp(R_x(s) - I_x(s)), \quad I_x(s) = \int_x^\infty \frac{u^{-s}}{\log u} d\Delta(u), \quad \Delta(u) = \psi(u) - u.$$

Integrating by parts,

$$I_x(s) = -\frac{\Delta(x)x^{-s}}{\log x} + \int_x^\infty \Delta(u) u^{-s-1} \left(\frac{s}{\log u} + \frac{1}{\log^2 u} \right) du.$$

(i) $\Rightarrow$ (iii). Assume RH. Then $|\Delta(u)| \leq \frac{1}{8\pi} u^{1/2} \log^2 u$ for $u \geq 74$ [Sc], and Δ is bounded on $[2, 74]$. So the last integral converges absolutely and locally uniformly in $\sigma > \frac{1}{2}$ and is analytic there. Both sides of (3) are meromorphic on $\sigma > \frac{1}{2}$, where ζ has no zeros, so (3) holds on $\{\sigma > \frac{1}{2}\} \setminus \{1\}$, and in particular on Ω . For $s \in K$ and $\alpha = \sigma - \frac{1}{2}$, with $\int_x^\infty u^{-\alpha-1} \log u du = x^{-\alpha}(\frac{\log x}{\alpha} + \frac{1}{\alpha^2})$,

$$|I_x(s)| \leq C x^{1/2-\sigma} \log x + C(|s|+1) x^{1/2-\sigma} \left(\frac{\log x}{\sigma - \frac{1}{2}} + \frac{1}{(\sigma - \frac{1}{2})^2} \right) \leq C_K x^{1/2-\sigma} \log x.$$

Further, $R_x(s)$ is a sum of p^{-ks}/k over prime powers $p^k > x$ with $k \geq 2$. There are at most $C\sqrt{y}$ such prime powers up to y , so partial summation gives $|R_x(s)| \leq C\sigma x^{1/2-\sigma}/(\sigma - \frac{1}{2})$. Since $|e^z - 1| \leq 2|z|$ for $|z| \leq 1$, (iii) follows. (iii) $\Rightarrow$ (ii) is immediate. (ii) $\Rightarrow$ (i). Each E_x^* is analytic and zero-free on Ω , which does not contain $s = 1$. By Hurwitz's theorem a locally uniform limit is zero-free or identically zero, so ζ has no zeros in Ω , and by the functional equation none with $0 < \beta < \frac{1}{2}$ either. $\square$

This is the product form of the classical equivalence of RH with the analyticity of $\log \zeta$ in $\sigma > \frac{1}{2}$. The same argument, with $\Delta(u) = O(u^{\Theta+\varepsilon})$, shows that $E_x^*/\zeta \rightarrow 1$ locally uniformly on $\sigma > \Theta$, and on no larger half-plane.

s	x	measured $\log E_x^* - \log \zeta$	sum over 682 zeros plus R_x
$0.8 + 20i$	$2 \cdot 10^6$	$-0.00068 - 0.00019i$	$-0.00070 - 0.00015i$
$0.7 + 20i$	$2 \cdot 10^6$	$-0.00285 - 0.00113i$	$-0.00292 - 0.00095i$
$0.6 + 20i$	10^4	$-0.01512 - 0.04052i$	$-0.01427 - 0.03802i$
$0.6 + 20i$	$2 \cdot 10^6$	$-0.01161 - 0.00629i$	$-0.01194 - 0.00552i$
$0.7 + 50i$	$2 \cdot 10^6$	$-0.00422 + 0.01193i$	$-0.00403 + 0.01198i$

TABLE 2. Proposition 2 with all primes $p \leq 2 \cdot 10^6$ and the 682 zeros with $|\gamma| < 600$.

Numerical check. Table 2 compares the two sides of (2). The remaining discrepancy is the contribution of the zeros above height 600. The error of the Euler product is, point by point, the superposition of the modes of the zeros (Figure 1, left); conversely, removing the computed modes from E_x^* reproduces ζ to about 1% at $0.6 + 20i$, $0.75 + 50i$ and at $\frac{1}{2} + 20i$ on the critical line. Status: numerical (euler/run.tables.py, euler/fig6.py).

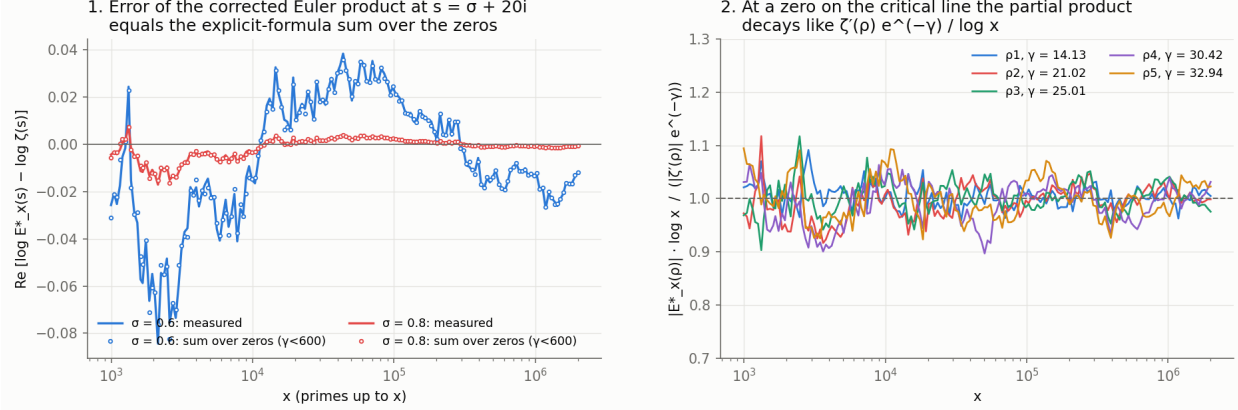


FIGURE 1. Left: the error $\text{Re}[\log E_x^*(s) - \log \zeta(s)]$ at $s = \sigma + 20i$ for $\sigma = 0.6$ and 0.8 (lines), against the sum over the zeros with $|\gamma| < 600$ plus R_x (circles). Right: $|E_x^*(\rho)| \log x / (|\zeta'(\rho)| e^{-\gamma_0})$ at the first five zeros.

2.1. Relation to the hybrid product. Gonek, Hughes and Keating [GHK, Thm. 1] proved $\zeta = P_X Z_X (1 + o(1))$ with P_X a smoothed Euler product over the primes up to X and Z_X a product over the zeros near s , and Bui, Gonek and Milinovich [BGM] used it for the moments of $\zeta'(\rho)$. Weighting (2) with a smooth density g in $\log x$, compactly supported in $(0, \infty)$, gives the exact factorization $\zeta = E_g^* Z_g$, $Z_g(s) = \prod_{\rho} e^{-K_g(s-\rho)}$, $K_g(w) = \int g(u) E_1(wu) du$, valid on all of $\mathbb{C}$ with an absolutely convergent product over the zeros [CE11, Lemma 8]. The factorization is due to [GHK]; its exact form, the potential theory of $\log |Z_g|$ ([CE11, Sect. 11.1, Prop. 25]) and the Euler–Caceres bound built on it are in [CE11]. In this paper we use the sharp form (2), and the smoothed form only in Lemma 9, Remark 5 and the next paragraph.

A finite-height look at the splitting. Gonek, Hughes and Keating conjectured that the moments of ζ on the critical line split as the product of the moments of the two factors, $\frac{1}{T} \int_T^{2T} |\zeta|^{2k} \sim \frac{1}{T} \int_T^{2T} |P_X|^{2k} \cdot \frac{1}{T} \int_T^{2T} |Z_X|^{2k}$, when X and T tend to infinity with $X = O((\log T)^{2-\varepsilon})$ [GHK]. We measured the corresponding statistics for the exact factors E_g^*, Z_g of [CE11, Lemma 8], with g the normal density of mean $\log X$, $X = 1000.5$, and standard deviation 0.4 (strictly its truncation to $(0, \infty)$; the mass removed is e^{-150}), on the grid $t \in [100, 1900]$ of step 0.2 ; Z_g is built from the zeros with $|\gamma - t| \leq 20$, and $E_g^* Z_g$ reproduces $\zeta(\frac{1}{2} + it)$ to $2 \cdot 10^{-7}$. On the ranges $[100, 500)$, $[500, 1000)$ and $[1000, 1900)$ the correlation of $\log |E_g^*|$ and $\log |Z_g|$ is $+0.015$, $+0.028$ and $+0.035$; the variance of $\log |E_g^*|$ is 1.10 , 1.15 , 1.15 , and that of $\log |Z_g|$ is 0.25 , 0.30 , 0.40 ; and the ratios $\langle |\zeta|^2 \rangle / (\langle |E_g^*|^2 \rangle \langle |Z_g|^2 \rangle)$ and $\langle |\zeta|^4 \rangle / (\langle |E_g^*|^4 \rangle \langle |Z_g|^4 \rangle)$ are 0.73 , 0.67 , 0.63 and 0.38 , 0.31 , 0.25 . So the two factors are almost uncorrelated on the logarithmic scale, the prime factor carries most of the variance, with a variance that stays constant while that of the zero factor grows with t , and the second and fourth moments do not split; the ratios move away from 1 as t grows. This is not evidence against the conjecture: X is fixed and far above $(\log T)^2$, outside its range, which at $X = 1000$ begins only at heights beyond e^{31} , and E_g^* is a smoothed and completed form of P_X . Status: numerical (`helix_split/helix_split.py`).

2.2. At a zero.

Proposition 4 (the corrected product at a zero). *Let $\rho_* = \beta_* + i\gamma_*$, $\gamma_* \neq 0$, be a simple zero such that no other zero has ordinate γ_* and real part larger than β_* , and let $x > 1$ not be a prime power.*

Then, with the logarithms of (2),

$$(4) \quad E_x^*(\rho_*) \log x = \zeta'(\rho_*) e^{-\gamma_0} \exp \left(\sum_{\rho \neq \rho_*} E_1((\rho_* - \rho) \log x) + \sum_{k \geq 1} E_1((\rho_* + 2k) \log x) + R_x(\rho_*) \right).$$

Proof. By the hypothesis, a small disc B around ρ_* meets no half-line removed from $\mathcal{D}$ except that of ρ_* . On B the terms $E_1((s - \rho) \log x)$, $\rho \neq \rho_*$, are analytic, and by the proof of Proposition 2 (Lemma 1 on the compact set $\bar{B}$) their sum converges uniformly on $\bar{B}$; so it is continuous at ρ_* , and so are the sum over k and R_x . The two remaining terms, $\log \zeta(s)$ and $E_1((s - \rho_*) \log x) = -\gamma_0 - \log((s - \rho_*) \log x) + \text{Ein}((s - \rho_*) \log x)$, jump by $2\pi i$ and $-2\pi i$ across the half-line of ρ_* , since ρ_* is simple, so their sum extends analytically to $B \setminus \{\rho_*\}$. Let $s \rightarrow \rho_*$ in (2): $\log \zeta(s) = \log \zeta'(\rho_*) + \log(s - \rho_*) + o(1)$ and $E_1((s - \rho_*) \log x) = -\gamma_0 - \log(s - \rho_*) - \log \log x + o(1)$, so the two singular terms combine to $\log \zeta'(\rho_*) - \gamma_0 - \log \log x$, up to a multiple of $2\pi i$ which disappears on exponentiating. $\square$

Hence $|E_x^*(\rho_*)| \log x \rightarrow |\zeta'(\rho_*)| e^{-\gamma_0}$ if and only if the real part of the sum over the other zeros in (4) tends to 0. For $\beta_* = \frac{1}{2}$ this is a statement of the type of the Deep Riemann Hypothesis at ρ_* , the analogue for ζ , with the pole correction, of the conjecture that Conrad formulated for L -functions without a pole [Co] (the term $R_x(\rho_*)$ tends to 0 by the prime number theorem); the constant $e^{-\gamma_0}$ is the one of [Co], see also [Sh]. We do not prove the decay, and we do not claim that it is equivalent to Conrad's conjecture. For the first zero, $\zeta'(\rho_1) = 0.7833 + 0.1247i$ and $|\zeta'(\rho_1)| e^{-\gamma_0} = 0.4453$. At the first five zeros the ratio $|E_x^*(\rho)| \log x / (|\zeta'(\rho)| e^{-\gamma_0})$ stays within 1 ± 0.1 for $10^3 \leq x \leq 2 \cdot 10^6$, and its fluctuations are reproduced by the other 681 zeros of the list to about 0.005 (Figure 1, right). For ρ_1 , at $x = 10^3, 10^4, 10^5, 10^6, 2 \cdot 10^6$, the ratio is 1.021, 1.001, 1.017, 0.975, 1.005, against 1.035, 1.002, 1.013, 0.979, 1.007 predicted by (4). The uncorrected product behaves differently: $|E_x(\rho_1)|$ is 0.023, 0.013, 0.417, $1.4 \cdot 10^{-4}$ at $x = 10^4, 10^5, 10^6, 2 \cdot 10^6$, because the pole term oscillates with amplitude $x^{1/2} / (|1 - \rho_1| \log x)$. Status: (4) is proved; the limit is conditional on the Deep Riemann Hypothesis at ρ_* ; the numbers are computed (euler/).

Logarithmic form. Let $D_x(s) = \sum_{p \leq x} p^{-s} - \int_2^x u^{-s} du / \log u$ and $Q(s) = \sum_p \sum_{k \geq 2} p^{-ks} / k$, which converges for $\sigma > \frac{1}{2}$, and on the line for $t \neq 0$ by the prime number theorem. Since $\int_2^x u^{-s} du / \log u = E_1((s - 1) \log 2) - E_1((s - 1) \log x)$, (4) gives, under the same proviso on the other zeros,

$$\text{Re } D_x(\rho_*) = -\log \log x + C_{\rho_*} + o(1), \quad C_{\rho} = \log |\zeta'(\rho)| - \gamma_0 - \text{Re } Q(\rho) - \text{Re } E_1((\rho - 1) \log 2).$$

The term $-\log \log x$ is the resonance of the zero with itself, $x^{\rho_* - s} = 1$ at $s = \rho_*$. For the first three zeros $C_{\rho} = -1.142, -0.406, -0.076$, against $\text{Re } D_x(\rho) + \log \log x = -1.131, -0.436, -0.095$ at $x = 10^7$; the differences oscillate with the other zeros and decrease like $1 / \log x$. At the point $\frac{1}{2} + i\gamma$ below a hypothetical zero $\frac{1}{2} + \delta + i\gamma$ the same term becomes $-\int_2^x u^{\delta-1} du / \log u \sim -x^{\delta} / (\delta \log x)$. Status: derivation conditional as stated; numbers computed.

Remark 5 (the smoothed form at a zero). The same limit in the smoothed factorization [CE11, Lemma 8], with $\int g = 1$, $\psi_g(v) = \int_v^\infty g$ and $\log E_g^*(s) = \sum_{n \geq 2} \Lambda(n) \psi_g(\log n) n^{-s} / \log n + K_g(s - 1) - \sum_{k \geq 1} K_g(s + 2k)$, uses $K_g(w) = -\gamma_0 - \log w - \int g(u) \log u du + O(w)$ and gives, under the hypothesis of Proposition 4,

$$\zeta'(\rho_*) = E_g^*(\rho_*) e^{\gamma_0 + \int g(u) \log u du} \prod_{\rho \neq \rho_*} e^{-K_g(\rho_* - \rho)}.$$

The factor $e^{\int g \log u}$, about $\log X$ for g concentrated near $\log X$, replaces $\log x$, and the product over the other zeros converges absolutely. This is an exact and unconditional form of the relation $\zeta'(\rho) = P_X(\rho) Z'_X(\rho) (1 + o(1))$ of [BGM], proved there under RH. With a Gaussian g (whose mass on $u \leq 0$ is negligible, as above) and the zeros within distance 25 of γ_* , the right side gives $\zeta'(\rho_1) = 0.7832965119 + 0.1246998297i$, and the relative error at the first five zeros is between

$3 \cdot 10^{-11}$ and $9 \cdot 10^{-11}$. Status: the identity is proved (it is immediate from [CE11, Lemma 8]); the numbers are computed (`euler_local/v2/numerics.py`).

2.3. The Davenport–Heilbronn control. The analogue of Proposition 2 holds for f , with $\log f(s) = \sum_{n \geq 2} b(n)n^{-s}/\log n$ for large σ , no pole term and a mode for each zero of f . At the height 85.6993 of ρ_f we computed the relative error $|\exp(\sum_{n \leq x} b(n)n^{-s}/\log n)/f(s) - 1|$:

σ	$x = 10^3$	10^4	10^5	$2 \cdot 10^5$
0.95 (right of ρ_f)	0.258	0.146	0.087	0.075
0.90 (right of ρ_f)	0.535	0.336	0.224	0.200
0.75 (left of ρ_f)	1.891	1.564	1.375	1.334
0.70 (left of ρ_f)	1.301	1.152	1.077	1.062

The series converges to the right of the off-line zero and fails to its left, as the modes predict. The equivalence of convergence with the absence of zeros to the right is a property of logarithmic Dirichlet series in general, not of Euler products. What distinguishes ζ is that its coefficients $\Lambda(n)/\log n$ are nonnegative and supported on prime powers; for f , 72% of $\sum_{n \leq 2 \cdot 10^5} |b(n)|/n$ lies on integers that are not prime powers [C5]. Status: numerical (`euler/dhlog.py`).

2.4. Function fields. Over $\mathbb{F}_q[T]$ the computation is finite and exact. For a Dirichlet character χ over $\mathbb{F}_q[T]$, nontrivial on $\mathbb{F}_q^*$, $L(u, \chi) = \prod_j (1 - \alpha_j u)$ is a polynomial [Ro], and with $u = q^{-s}$ the line $\sigma = \frac{1}{2}$ is the circle $|u| = q^{-1/2}$. The partial Euler product E_m over the monic irreducibles of degree at most m satisfies $\log E_m(u) = \log L(u, \chi) + \sum_{n > m} (\sum_j \alpha_j^n) u^n/n + R_m(u)$, where R_m collects the powers of degree greater than m of the primes of degree at most m , and $E_m \rightarrow L(\cdot, \chi)$ locally uniformly on $|u| < q^{-1/2}$ if and only if $|\alpha_j| \leq \sqrt{q}$ for every j ; both statements are proved in [CE17, Prop. 10]. The modes are $(\alpha_j u)^n/n$, the analogues of $x^{\rho-s}/((s-\rho)\log x)$ with $x = q^m$, and Weil’s theorem $|\alpha_j| = \sqrt{q}$ [W] is exactly the statement that the partial products converge inside the disc; positivity enters there through the point counts of the curve behind L , and the tower of base changes supplies the uniformity [CE17, Sects. 7.4–7.5]. A smoothed version, with the analogue of the Euler–Caceres bound, is in [CE11, Sect. 13.1]. Status: proved in [CE17].

3. THE EULER PRODUCT AS A CROSS-ENERGY

For $P \geq 2$ let $1 = n_1 < n_2 < \dots$ be the P -smooth integers, those whose prime factors are all at most P , and put $X_k^{(P)}(s) = \sum_{j \leq k} n_j^{-s}$ and $C_2^{(P)}(n_k, s) = 2 \operatorname{Re} (X_{k-1}^{(P)}(s) \overline{n_k^{-s}})$.

Theorem 6 (the Euler product as a cross-energy; elementary). *For $\sigma > 0$ and $P \geq 2$,*

$$|E_P(s)|^2 = \prod_{p \leq P} (1 - p^{-2\sigma})^{-1} + \sum_{k \geq 2} C_2^{(P)}(n_k, s),$$

the series converging absolutely. For $\sigma > 1$, letting $P \rightarrow \infty$ gives $|\zeta(s)|^2 = \zeta(2\sigma) + \sum_{n \geq 2} C_2(n, s)$.

Proof. For $\sigma > 0$ the product $E_P(s) = \prod_{p \leq P} \sum_{m \geq 0} p^{-ms}$ of finitely many absolutely convergent geometric series equals $\sum_k n_k^{-s}$, with absolute convergence. Telescoping $|X_k^{(P)}|^2 - |X_{k-1}^{(P)}|^2 = n_k^{-2\sigma} + C_2^{(P)}(n_k, s)$ and letting $k \rightarrow \infty$ gives the identity, since $\sum_k n_k^{-2\sigma} = \prod_{p \leq P} (1 - p^{-2\sigma})^{-1}$; absolute convergence follows from $|C_2^{(P)}(n_k, s)| \leq 2n_k^{-\sigma} E_P(\sigma)$. For $\sigma > 1$ and fixed n , $C_2^{(P)}(n, s) = C_2(n, s)$ as soon as $P \geq n$, and $|C_2(n, s)| \leq 2n^{-\sigma} \zeta(\sigma)$ gives dominated convergence. $\square$

With $P = 7$ and the 14672 seven-smooth integers up to 10^{12} , the right side equals 0.886590 at $s = 1.5 + 20i$, as does $|E_7|^2$, and 0.761394 at $s = 0.6 + 20i$, against $|E_7|^2 = 0.761404$; the difference is the contribution of the smooth integers above 10^{12} (`smooth_energy/smooth_energy.py`). Each partial Euler product is therefore the cross-energy of the partial sums of ζ restricted to the P -smooth

integers. Inside the strip the limit $P \rightarrow \infty$ does not exist without the correction of Proposition 2: the energy over the smooth integers inherits the drift of size $x^{1-\sigma}/\log x$ that comes from the pole, which in the ordinary cross-energy is the envelope term $\operatorname{Re}(2/(1-s)) n^{1-2\sigma}$ of Theorem A of [CE1].

Remark 7 (the prime step, and what positivity gives). Theorem 6 is the structural link of this paper between the primes, the Euler product and the cross-energy of [CE1]: the partial Euler product is not only a product over the primes but the cross-energy of the partial sums of ζ over an arithmetic subsequence of the integers. Passing from the prime q to the next prime p adds to the q -smooth integers all their multiples by powers of p , and since $|E_p|^2 = |E_q|^2 |1 - p^{-s}|^{-2}$ and $D_P = \prod_{\ell \leq P} (1 - \ell^{-2\sigma})^{-1}$ satisfies $D_p = D_q (1 - p^{-2\sigma})^{-1}$, the total cross-energy changes by

$$\sum_{k \geq 2} C_2^{(p)}(n_k, s) - \sum_{k \geq 2} C_2^{(q)}(n_k, s) = |E_q(s)|^2 (|1 - p^{-s}|^{-2} - 1) - D_q ((1 - p^{-2\sigma})^{-1} - 1),$$

which is $2|E_q(s)|^2 \operatorname{Re} p^{-s} + O_\sigma(p^{-2\sigma} |E_q(s)|^2 + p^{-2\sigma} D_q)$: the new prime enters the energy linearly, through $\operatorname{Re} p^{-s}$, with the weight $|E_q|^2$. The identity of Theorem 6 is exact, but the positivity of its left side is automatic and gives no information on the zeros of ζ ; a constraint would need an inequality for the cross-energy over the smooth integers, uniform in P , that does not follow from $|E_P|^2 \geq 0$. We do not have one. *Status*: the identity is proved (elementary).

The increments. The increment of $\log |E_x|^2$ at a prime is $-2 \log |1 - p^{-s}| = 2 \operatorname{Re} p^{-s} + O(p^{-2\sigma})$, a prime analogue of C_2 . At a simple zero on the line, by Proposition 4, the increments up to x add up to $-2 \log \log x + 2 \log |\zeta'(\rho)| - 2\gamma_0 - 2 \operatorname{Re} E_1((\rho - 1) \log x)$ plus the modes of the other zeros; the pole term $-2 \operatorname{Re} E_1((\rho - 1) \log x)$, of size $x^{1/2}/\log x$, has to be kept.

3.1. The cross-energy of the prime zeta function. The construction of Theorem A can be carried out on the primes alone. Put $P_x(s) = \sum_{p \leq x} p^{-s}$ and, for a prime p ,

$$C_2^P(p, s) = |P_p(s)|^2 - |P_{p-1}(s)|^2 - p^{-2\sigma} = 2p^{-\sigma} \operatorname{Re}(p^{it} P_{p-1}(s)).$$

Proposition 8. *Let $0 < \sigma < 1$ and t real. As $p \rightarrow \infty$ through the primes,*

$$\log p \cdot C_2^P(p, s) = \frac{2(1-\sigma)}{(1-\sigma)^2 + t^2} p^{1-2\sigma} (1 + o(1)) + O(p^{-\sigma} \log p) + o(p^{1-2\sigma}).$$

Hence $\log p \cdot C_2^P(p, s)$ tends to $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$.

Proof. Let $\pi(u) = \operatorname{li}(u) + \mathcal{E}(u)$ with $\mathcal{E}(u) = O(ue^{-c\sqrt{\log u}})$ (prime number theorem with de la Vallée Poussin's error term). Partial summation gives

$$P_x(s) = \int_2^x \frac{u^{-s}}{\log u} du + \mathcal{E}(x)x^{-s} + s \int_2^x \mathcal{E}(u)u^{-s-1} du + c_0,$$

with a constant c_0 , and for $\sigma < 1$ the last three terms are $O(x^{1-\sigma} e^{-c'\sqrt{\log x}}) = o(x^{1-\sigma}/\log x)$. The first is $x^{1-s}/((1-s)\log x) (1 + O(1/\log x))$ plus a constant, by integration by parts. Insert this with $x = p^-$ in $C_2^P(p, s) = 2p^{-\sigma} \operatorname{Re}(p^{it} P_{p-1}(s))$: the main term gives $2p^{1-2\sigma} \operatorname{Re}(1/(1-s))/\log p$, the constants give $O(p^{-\sigma})$ and the rest $o(p^{1-2\sigma}/\log p)$. Multiply by $\log p$. At $\sigma = \frac{1}{2}$ the main coefficient is $1/(\frac{1}{4} + t^2)$ and the other terms are $o(1)$; for $\sigma > \frac{1}{2}$ everything tends to 0; for $\sigma < \frac{1}{2}$ the positive main term dominates. $\square$

This is the trichotomy of Theorem A on the primes, with the normalization $\log x$, the density of the primes. The finer structure is the explicit formula for $\pi(x)$: for s not a zero, $P_x(s) = P(s) + \operatorname{Li}(x^{1-s}) - \sum_\rho \operatorname{Li}(x^{\rho-s}) + O(x^{1/2-\sigma}/\log x)$, where $P(s) = \sum_k \mu(k)k^{-1} \log \zeta(ks)$ continues the prime zeta function and the error collects the prime squares; hence

$$C_2^P(x, s) = \frac{2(1-\sigma)x^{1-2\sigma}}{|1-s|^2 \log x} + 2x^{-\sigma} \operatorname{Re}(x^{it} P(s)) - \frac{2}{\log x} \operatorname{Re} \sum_\rho \frac{x^{\rho-2\sigma}}{\rho-s} + \cdots,$$

a pole term, a wave term carrying $P(s)$ in place of $\zeta(s)$, and a term from the zeros which is, to leading order, $(2/\log x) \operatorname{Re}(x^{-\sigma+it} H_s(\log x))$, with H_s the Euler helix of Section 5. The roles of the zeros are reversed with respect to C_2 : at a zero of ζ the wave term of C_2 vanishes, whereas P has a logarithmic singularity at ρ , so near a zero the wave term of C_2^P is largest. With the primes up to $2 \cdot 10^7$, the mean of $(\frac{1}{4} + t^2) \log p C_2^P(p, \frac{1}{2} + it)$ over $\log p \in [13.8, 16.8]$ is 0.856, 0.825 and 1.104 for $t = 5, 10$ and 20 , and 1.101, with much larger oscillations, at the ordinate $t = 14.1347$ (Figure 2a). The wave term dominates the pole term until x is of the order of $(t^2 |P(s)| \log x)^2$, about 10^8 for $t = 20$, which explains the slow convergence (Figure 2b). Off-line zeros contribute $x^{\beta-1}/\log x$ at $\sigma = \frac{1}{2}$, which tends to 0, so the pointwise limit does not see them. Status: Proposition 8 is proved; the decomposition is the explicit formula (classical); the numbers are from `c2_primezeta/c2_primezeta.py`.

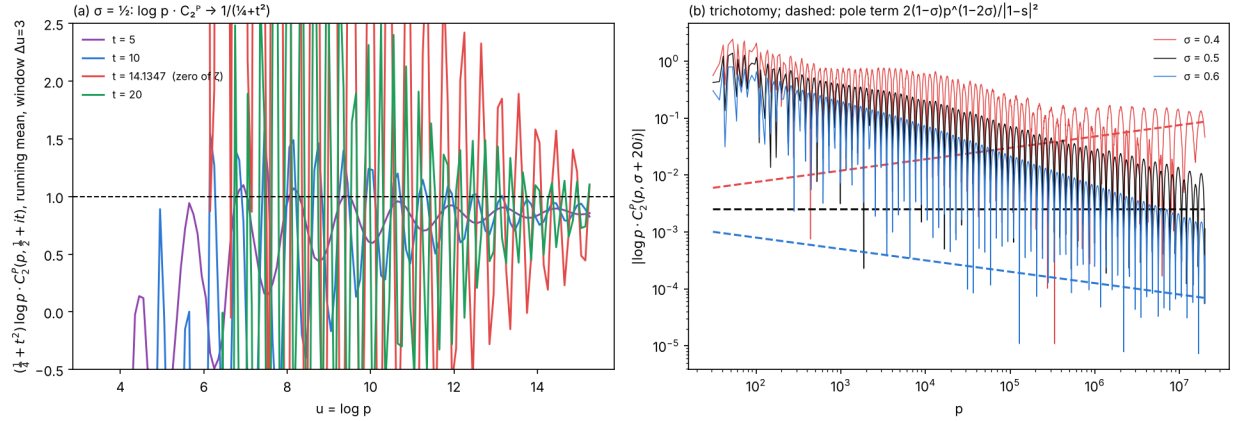


FIGURE 2. The cross-energy of the prime zeta function, primes up to $2 \cdot 10^7$. (a) Running mean, over windows of length 3 in $u = \log p$, of $(\frac{1}{4} + t^2) \log p C_2^P(p, \frac{1}{2} + it)$ for $t = 5, 10, 20$ and at the ordinate $t = 14.1347$; the limit is 1. (b) $|\log p C_2^P(p, \sigma + 20i)|$ for $\sigma = 0.4, 0.5, 0.6$ (solid) and the pole term $2(1 - \sigma)p^{1-2\sigma}/|1 - s|^2$ (dashed), still below the wave term in this range.

4. ABSCISSAE READ FROM THE PRIMES

The C-transform $C_N(s; a) = \sum_{n \leq N} a(n)n^{-s} - \epsilon_a \int_1^N x^{-s} dx$ of [CE1, Sect. 2] converges for $a = \Lambda$ exactly in $\sigma > \Theta$ [CE1, Prop. 2.6]; the difference $R_N(s) = \sum_{n \leq N} (\Lambda(n) - 1)n^{-s}$ of the transforms of the primes and of the integers satisfies, at a single real γ_0 , $\text{RH} \Leftrightarrow R_N(\frac{1}{2} + i\gamma_0) = O_\epsilon(N^\epsilon)$ [CE1, Props. 2.8 and 2.9], a classical criterion in the line of von Koch and Landau [In]; and the resonance exponent $\max_{10 \leq \gamma \leq 100} \log |R_N(\frac{1}{2} + i\gamma)| / \log N$ measured in [CE1] follows $\log \log N / \log N$ for ζ and $L(s, \chi)$ modulo 5, and $\beta - \frac{1}{2}$ for f . Here we add two local readings of the real parts.

4.1. The apparent abscissa. Let g be a smooth probability density compactly supported in $(0, \infty)$, as [CE11, Lemma 8] requires, $\psi_g(v) = \int_v^\infty g$, and

$$G(w) = \frac{1}{w} \int g(u) e^{-wu} du.$$

Lemma 9. For s not a zero, not 1 and not a trivial zero,

$$R(s; \Lambda_0) := -\frac{\zeta'}{\zeta}(s) - \sum_{n \geq 2} \Lambda(n) \psi_g(\log n) n^{-s} - G(s-1) = -\sum_{\rho} G(s-\rho) - \sum_{k \geq 1} G(s+2k).$$

Proof. Differentiate $\log \zeta = \log E_g^* - \sum_{\rho} K_g(\cdot - \rho)$ [CE11, Lemma 8] term by term (the sums converge locally uniformly) and use $K'_g(w) = \int g(u) u E'_1(wu) du = -G(w)$, since $E'_1(z) = -e^{-z}/z$. $\square$

In the computations g is the normal density with mean $\Lambda_0 = \log X$ and variance σ_g^2 , for which $G(w) = e^{-w\Lambda_0 + w^2\sigma_g^2/2}/w$. This density is not admissible: it puts mass on $u \leq 0$, and for it the sum over the trivial zeros in Lemma 9 diverges, since $|G(s + 2k)|$ grows like $e^{2k^2\sigma_g^2}$. The lemma applies to its smooth truncation to an interval $[a, b] \subset (0, \infty)$ around Λ_0 , renormalized. We use the closed form for the untruncated density and omit the trivial zeros: for $t \geq 10$ the terms with $2k < t$ are below $e^{-t^2\sigma_g^2/2}$ in relative size, the remaining ones are of the order of the truncated mass, and the truncation at $\Lambda_0 \pm 4.5\sigma_g$ changes the quantities computed by about 10^{-5} in relative terms, below the resolution of the fits. For a Gaussian each zero contributes $|G(s - \rho)| = e^{-(\sigma - \beta)\Lambda_0} e^{-(t - \gamma)^2\sigma_g^2/2} e^{(\sigma - \beta)^2\sigma_g^2/2} / |s - \rho|$, so the growth rate of $\log |R(\sigma + it; \Lambda_0)|$ in Λ_0 is $-(\sigma - \beta)$ for the largest β among the zeros within about $1/\sigma_g$ of the height t , and

$$\Theta(t) = \sigma + \frac{d}{d\Lambda_0} \log |R(\sigma + it; \Lambda_0)|$$

is an apparent abscissa at height t . For f we use the analogous expression with the coefficients $b(n)$ of $-f'/f = \sum b(n)n^{-s}$ and no pole term; we checked it numerically but do not prove it here, since the smoothed factorization is not established for f in [CE11]. The pole term $G(s - 1)$ has modulus $e^{-t^2\sigma_g^2/2}/|s - 1|$ at $\sigma = 1$, below $1.3 \cdot 10^{-15}$ for $t \geq 10$, $\sigma_g = 0.8$; it matters only near $t = 0$. A check of Lemma 9 at $s = 1 + 20i$, $1 + 40i$, $0.8 + 30i$, $\Lambda_0 = 5, 7$, with the prime powers up to $e^{\Lambda_0 + 6\sigma_g}$ and the 682 zeros with $|\gamma| < 600$, gives agreement to $6 \cdot 10^{-11} - 5 \cdot 10^{-10}$ (`theta/check_identity.py`).

With $\sigma = 1$, $\sigma_g = 0.8$, $4 \leq \Lambda_0 \leq 11$, the coefficients up to $7 \cdot 10^6$, and the slope fitted to the root mean square of R over windows of length 2 in Λ_0 , the apparent abscissa of ζ on $10 \leq t \leq 120$ has median 0.500, with 90% of the values in $[0.47, 0.54]$. For the Davenport–Heilbronn function it has a plateau at 0.806 around $t = 85.7$, against $\beta = 0.8085$, and one at 0.631 around $t = 114.2$, against $\beta = 0.6508$ (Figure 3). Two zeros on the line closer than about 1 produce a beat whose period in Λ_0 exceeds the window, and the fit reads it as growth: such pairs give narrow false peaks, at $t \approx 40.6$ for f , whose zeros 40.163 and 40.683 are 0.52 apart, and at $t \approx 88$ and 95 for ζ . They are separated from off-line zeros by their width, and would be removed by a longer range of Λ_0 or by demodulating at the frequency γ . The apparent abscissa is a local detector computed from the primes, not a bound valid at all heights. Status: Lemma 9 is proved; the computations are numerical (`theta/`, Hurwitz zeta functions by Euler–Maclaurin in double precision).

4.2. An off-line zero as an interference of two Euler products. The zero ρ_f is seen in the resonance exponent of [CE1] and in the apparent abscissa; it can also be read in the approximate functional equation. With $a(n)$ the periodic coefficients of f and χ_f the factor of its functional equation, at $x = y = \sqrt{5\gamma_f/2\pi} = 8.26$ the two halves $\sum_{n \leq x} a(n)n^{-\rho_f}$ and $\chi_f(\rho_f) \sum_{n \leq y} a(n)n^{\rho_f-1}$ have moduli 0.343 and 0.276, whereas on the critical line the two halves always have equal moduli and only a phase has to match. The truncated sum alone already vanishes at $0.724 + 85.704i$. Projected on the direction of the first half, the largest cancelling terms after $n = 1$ are $n = 4$ and $n = 6$, where $a(4) = -1 \neq a(2)^2$ and $a(6) = 1 \neq a(2)a(3)$ (Figure 4). If a is replaced by the completely multiplicative function with the same values at the primes, the truncated sum has no zeros off the line for $80 \leq t \leq 91$, and its value at ρ_f rises from 0.09 to 0.82. At ρ_f the two Euler products are nonzero, $|L(\rho_f, \chi)| = |L(\rho_f, \bar{\chi})| = 0.702$, and in opposite phase; their truncations to the primes up to 10^7 reproduce them to $2 \cdot 10^{-3}$. The zero of f is an interference of two Euler products [C5], and in the truncated sum it is carried by the coefficients at which multiplicativity fails. What this leaves is the question whether a Dirichlet polynomial with multiplicative coefficients can resonate at one frequency with amplitude $N^{\beta-1/2}$, $\beta > \frac{1}{2}$; by [CE1, Prop. 2.9] it is equivalent

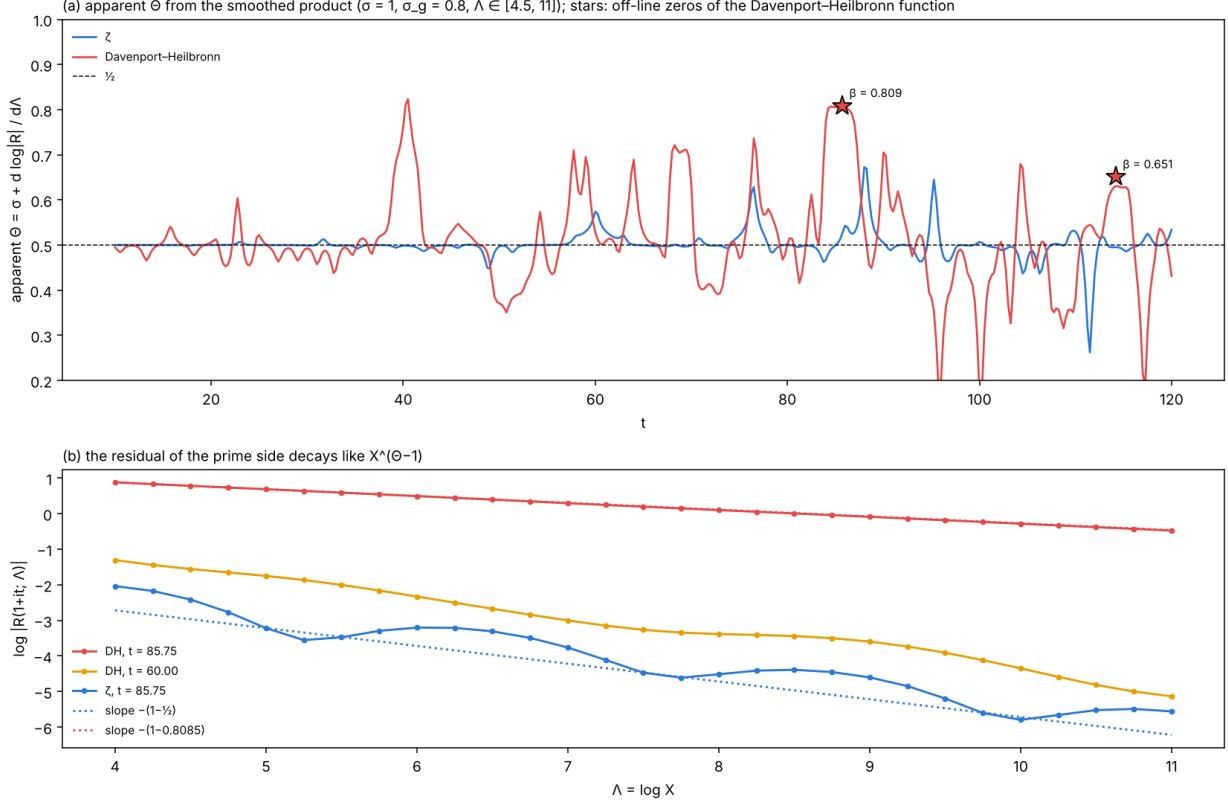


FIGURE 3. The apparent abscissa from the smoothed product. (a) $\Theta(t)$ for ζ and for the Davenport–Heilbronn function, $\sigma = 1$, $\sigma_g = 0.8$; stars: the off-line zeros of f below height 120. (b) $\log |R|$ against Λ_0 at $t = 85.75$ for both functions and at $t = 60$ for f , with the slopes $-(1 - \frac{1}{2})$ and $-(1 - 0.8085)$.

to RH, and it is not answered here. Status: numerical (`resonance/paso1.py`–`paso3.py`, `mpmath` for the values of L and f ; the figure from `resonance/fig_afe.py`).

5. THE EULER HELIX

Proposition 2 has a geometric form. For $s \in \mathcal{D}$ with $t \neq 0$ and $u = \log x$ put

$$H_s(u) = u \left(\sum_{n \leq e^u} \frac{\Lambda(n)}{n^s \log n} + E_1((s-1)u) - \log \zeta(s) \right),$$

the corrected partial product without the term R_x , minus its limit, multiplied by u . By (2), exactly,

$$(5) \quad H_s(u) = \sum_{\rho} u E_1((s-\rho)u) + \sum_{k \geq 1} u E_1((s+2k)u) = \sum_{\rho} \frac{e^{(\rho-s)u}}{s-\rho} \left(1 + O\left(\frac{1}{|s-\rho|u}\right) \right) + O(e^{-(2+\sigma)u}).$$

The first equality in (5) is exact, with the order of summation of Proposition 2; the second describes the individual terms by (1), for $|\gamma - t|$ large, and is not a bound for the sum, which converges only conditionally. Each zero contributes a vector of length $e^{(\beta-\sigma)u}/|s-\rho|$ that turns with angular velocity $\gamma - t$. With u as a third coordinate this is a helix of constant radius if $\beta = \sigma$, an expanding spiral if $\beta > \sigma$ and a contracting one if $\beta < \sigma$. On the critical line, in the frame that turns with

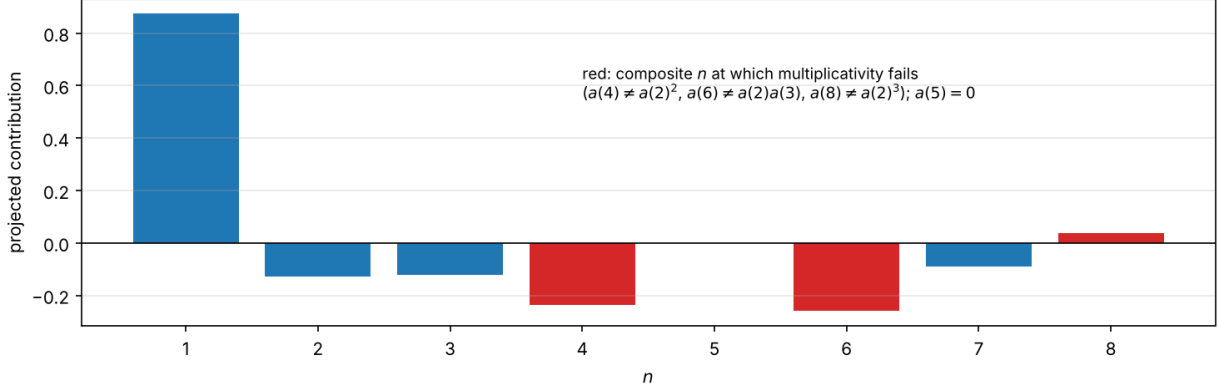


FIGURE 4. The terms $n \leq 8$ of the approximate functional equation of the Davenport–Heilbronn function at its zero $0.8085 + 85.699i$, $x = y = 8.26$: each term $a(n)n^{-\rho} + \chi_f(\rho)a(n)n^{\rho-1}$ projected on the direction of the first half. Red: composite n at which multiplicativity fails.

e^{itu} , the pair $\rho, \bar{\rho}$ with $\rho = \frac{1}{2} + i\gamma$ contributes

$$-i \left(\frac{e^{i\gamma u}}{t - \gamma} + \frac{e^{-i\gamma u}}{t + \gamma} \right),$$

two circles turning in opposite directions, that is, an ellipse with semi-axes $|t - \gamma|^{-1} \pm |t + \gamma|^{-1}$. The pairs of zeros on the line are ellipses, as for the prime signal of [CE5, Sect. 2], now centred at the point s . The *truncation helix* of [CE4], formed by the zeros of $X_N - \frac{1}{2}N^{-s} - N^{1-s}/(1-s)$ in $(\sigma, t, \log N)$, is a different object: it lives on the additive side and spirals onto each zero, while the Euler helix lives on the multiplicative side and is centred at an arbitrary point.

Remark 10 (energy of the Euler helix). Assume RH and let $s = \frac{1}{2} + it$ with t not an ordinate. If the mean square of H_s is computed term by term in (5), one obtains

$$\lim_{U \rightarrow \infty} \frac{1}{U} \int_0^U |H_s(u)|^2 du = \sum_{\gamma} \frac{m_{\gamma}^2}{(t - \gamma)^2} \geq \sum_{\gamma} \frac{m_{\gamma}}{(t - \gamma)^2} = -\frac{d^2}{dt^2} \log |\xi(\frac{1}{2} + it)|,$$

where the sums run over the distinct ordinates, of both signs, and m_{γ} is the multiplicity. The first equality is Parseval’s identity for the almost periodic sum, whose frequencies $\gamma - t$ are distinct; the second follows from the Hadamard product, in which each zero contributes $m_{\gamma} \log |t - \gamma|$ to $\log |\xi(\frac{1}{2} + it)|$. Equality holds if and only if the zeros are simple, as in Theorem B of [CE5]. If a zero has $\beta > \frac{1}{2}$, its spiral grows like $e^{(\beta - \frac{1}{2})u}$, and term by term the mean square is infinite; this is heuristic, like the rest of the remark. We do not justify the interchange of the sum over the zeros with the mean.

At $t = 20$, H_s computed from the prime powers up to 10^8 agrees with the sum over the 682 zeros with $|\gamma| < 600$ to a root mean square of 0.056 on $8 < u < 18.4$ when both are sampled on a grid of step 0.005, against a root mean square of 1.02 for H_s itself; the discrepancy is a sampling effect (Remark 11). The running mean of $|H_s|^2$ is 1.045 at $u = 18.4$. The sum over the zeros is 1.074, or 1.077 with the tail estimate for $|\gamma| > 600$, and $-(\log |\xi|)''(20) = 1.077$ computed directly from ξ . At $\sigma = 0.3$ and 0.7 , $|H_s|$ grows and decays like $e^{\pm 0.2u}$ (Figure 5). Seen from $\sigma = 0.3$ every zero lies to the right and produces an expanding spiral, as an off-line zero would on the critical line. The construction shows the Euler product inside the strip as a superposition of circles, one per zero, but it does not exclude spirals. Status: (5) is proved; the energy identity is heuristic (interchange not justified); the computations are numerical (euler/euler_helix.py, euler/fig_euler_helix.py).

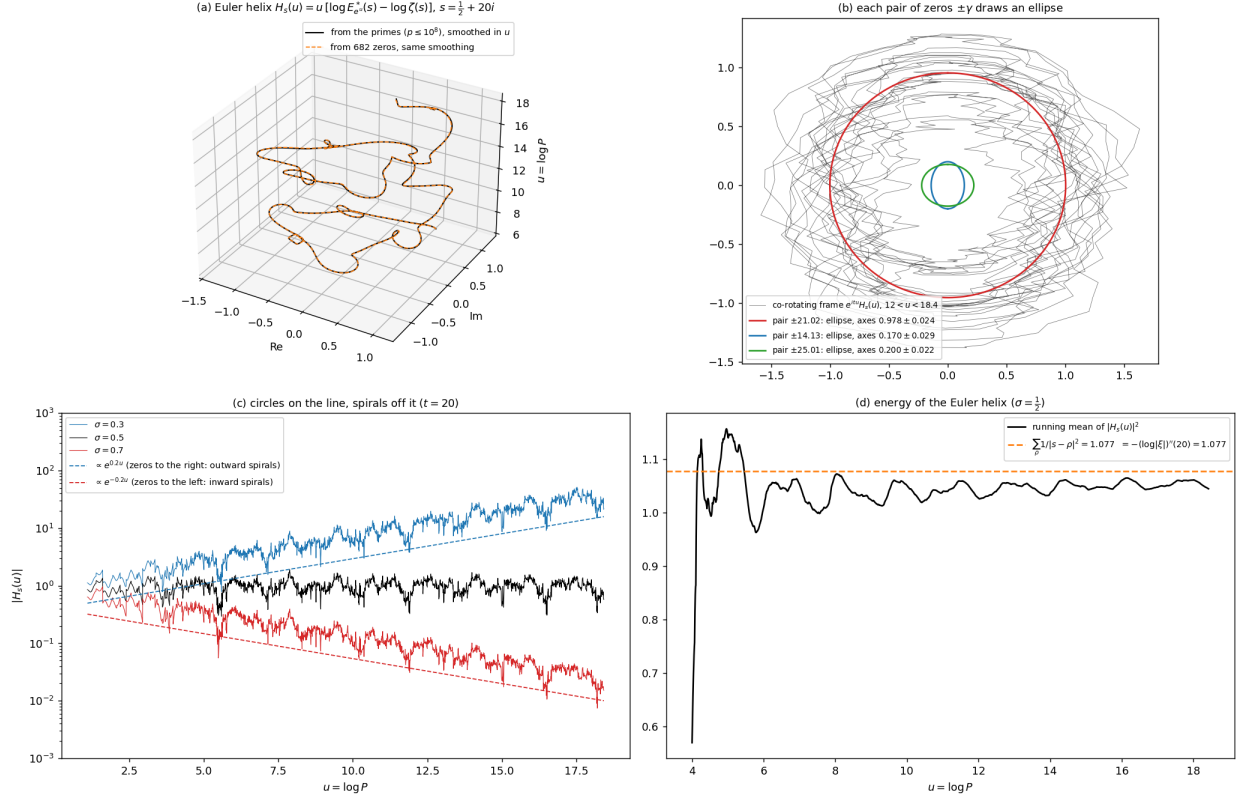


FIGURE 5. The Euler helix. (a) $H_s(u)$ at $s = \frac{1}{2} + 20i$ from the prime powers up to 10^8 (solid) and from the 682 zeros with $|\gamma| < 600$ (dashed), both smoothed in u . (b) In the frame turning with e^{itu} , the curve $e^{itu}H_s(u)$ for $12 < u < 18.4$ and the ellipses of the pairs ± 21.02 , ± 14.13 , ± 25.01 . (c) $|H_s(u)|$ at $t = 20$ for $\sigma = 0.3, 0.5, 0.7$. (d) Running mean of $|H_s|^2$ at $s = \frac{1}{2} + 20i$ against $\sum_{\rho} |s - \rho|^{-2} = -(\log |\xi|)''(20) = 1.077$.

Remark 11 (rigidity of the Euler helix). Since (5) is an identity, the agreement between primes and zeros is limited by the computation, not by the primes. H_s is a step function in u , and the zeros with $|\gamma| > 600$, which are not in the sum, contribute oscillations of every frequency; sampled at isolated points they alias onto the grid. We therefore average H_s exactly over bins of width $h = 5 \cdot 10^{-4}$, using for the pole term and for each zero the primitive

$$\int u E_1(cu) du = \frac{u^2}{2} E_1(cu) - \frac{e^{-cu}}{2} \left(\frac{u}{c} + \frac{1}{c^2} \right),$$

and convolve with a Gaussian of width 0.02 in u , which removes the contribution of the zeros with $|\gamma \mp t| > 300$. For 45 heights $16.5 \leq t \leq 117.7$ and the prime powers up to 10^8 , the smoothed H_s then agrees with the smoothed sum over the zeros to a relative root mean square between $2 \cdot 10^{-7}$ and $8 \cdot 10^{-6}$ (median $2.3 \cdot 10^{-6}$), with no growth on $10 \leq u \leq 18.3$ (Figure 6a).

Replace a zero $\frac{1}{2} + i\gamma$ and its conjugate by the pairs $\frac{1}{2} \pm \delta + i\gamma$ and $\frac{1}{2} \pm \delta - i\gamma$, each of multiplicity $\frac{1}{2}$, which respects the functional equation. The first-order terms in δ cancel, and the sum over the zeros changes by

$$\frac{\delta^2}{2} \partial_{\beta}^2 [u E_1((s - \rho)u) + u E_1((s - \bar{\rho})u)] + O(\delta^4), \quad \partial_{\beta}^2 u E_1((s - \rho)u) = u^3 E_1''((s - \rho)u).$$

A symmetric pair is therefore seen by the helix only at second order. We fit the residual $H_s - \sum_{\rho}$ by least squares on the second derivatives in β of the three zeros nearest to t , together with their

first derivatives in γ , after the same smoothing; the coefficient of the second derivative estimates $\delta^2/2$. On the actual data these coefficients are noise, and twice their standard deviation over the 45 heights defines a threshold $\delta_{\min}$. With the primes up to x , $\delta_{\min} = 1.3 \cdot 10^{-4}, 0.9 \cdot 10^{-4}, 0.5 \cdot 10^{-4}, 0.6 \cdot 10^{-4}, 0.4 \cdot 10^{-4}$ for $\log x = 10, 12, 14, 16, 18.3$, a decrease close to $(\log x)^{-1.9}$ (Figure 6b). When the zero at 21.022 is split by $\delta = 0.003, 0.01, 0.03$ in synthetic data that keep the true residual, the fit returns 0.0030, 0.0100 and 0.0291 for every x between e^{10} and 10^8 ; the last value is low by the term of order δ^4 (Figure 6c). Without smoothing, a direct comparison of root mean squares could not separate the split zeros from the true ones below $\delta \approx 0.03$; that threshold came from the sampling, not from the second-order nature of the effect.

With the primes up to 10^8 the Euler helix thus detects a symmetric displacement $\delta \gtrsim 4 \cdot 10^{-5}$ of one of the zeros nearest to a given height. This is a statistical detection threshold (twice the standard deviation of the fitted coefficients over 45 heights, in double precision), not a bound for β . This does not approach RH, for three reasons. (i) A measurement bounds $|\beta - \frac{1}{2}|$ but never gives $\beta = \frac{1}{2}$. Symmetry closes this gap locally: if a disc centred on the critical line contains exactly one zero counted with multiplicity, then ρ and $1 - \bar{\rho}$ both lie in it, so $\beta = \frac{1}{2}$. This is the logic of Turing's method [Tu]; with interval arithmetic and rigorous bounds for the zeros outside the model the computation would verify, from the primes alone, that the zeros near height t lie on the line. (ii) The verification is local: the number of primes needed grows with t . (iii) The fit compares the helix with a list of known zeros; a zero outside the list appears only as a residual, and excluding it requires control of all zeros at once. The smoothed H_s is Weil's explicit formula [We] evaluated on Gaussian test functions, and RH is equivalent to the positivity of Weil's functional on all test functions, which no finite family reaches. Status: numerical (**rigidity**/).

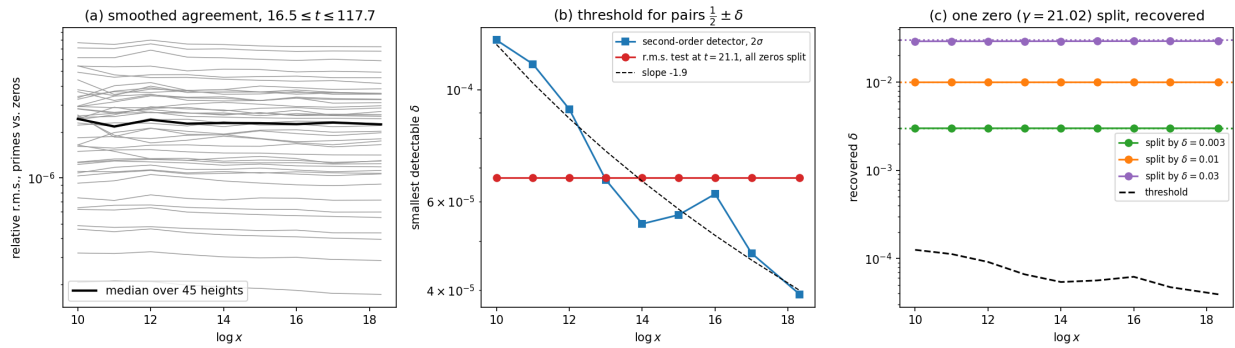


FIGURE 6. Rigidity of the Euler helix, with the prime powers up to $x \leq 10^8$. (a) Relative root mean square of the difference between the smoothed H_s computed from the primes and from the zeros, at 45 heights $16.5 \leq t \leq 117.7$ (grey) and their median (black). (b) Threshold $\delta_{\min}$ of the second-order detector for symmetric pairs $\frac{1}{2} \pm \delta$ (squares), with a fitted power law, and the threshold of a direct root-mean-square comparison when every zero is split, at $t = 21.1$ (circles). (c) The zero at $\gamma = 21.022$ split by $\delta = 0.003, 0.01, 0.03$ in synthetic data, and the values of δ recovered by the detector.

Remark 12 (the sieved wave is the Euler helix). The wave $W_n(s) = 2X_{n-1}(s)\overline{n^{-s}}$ of [CE1] and the Euler helix are one object. Remove from the partial sums the multiples of the primes $p \leq y$. By multiplicativity,

$$X_{n-1}^{(y)}(s) = \sum_{\substack{k < n \\ (k, P(y))=1}} k^{-s} = \sum_{d|P(y)} \mu(d) d^{-s} X_{\lceil n/d \rceil - 1}(s), \quad P(y) = \prod_{p \leq y} p,$$

an identity that has no analogue for the Davenport–Heilbronn function. By the expansion of X_N of [CE1, Sect. 3] applied to each term, the sieved wave $W_n^{(y)} = 2X_{n-1}^{(y)}\overline{n^{-s}}$ has the centre $2c_y n^{1-2\sigma}/(1-s)$ with $c_y = \prod_{p \leq y} (1 - \frac{1}{p})$, a rotating term of amplitude $2|\zeta_y(s)|n^{-\sigma}$ with $\zeta_y(s) = \zeta(s) \prod_{p \leq y} (1 - p^{-s})$, and a correction of order $n^{-2\sigma}$, periodic modulo $P(y)$. At $s = \frac{1}{2} + 20i$ the radius of $W^{(y)}$ near $n = 10^5$ is 1.155, 1.924, 1.352 for $y = 1, 3, 7$, against $|\zeta_y(s)| = 1.148, 1.914, 1.345$ (Figure 7a). A finite Euler product does not remove zeros: at ρ_1 the sieved waves collapse like n^{-1} for every y (Figure 7b). Removing the pole correction from the free amplitude leaves the helix: by Proposition 2,

$$A_y^*(s) := \zeta_y(s) e^{-E_1((s-1)\log y)} = \exp\left(-\frac{H_s(\log y)}{\log y} - R_y(s)\right).$$

With the primes up to 10^6 the two sides agree to about 1%, the accuracy of the grid on which H_s was stored. At $t = 20$, $|A_y^* - 1|$ is 0.035 and 0.005 for $y = 10^3$ and 10^6 at $\sigma = 0.7$, decreases like $1/\log y$ at $\sigma = \frac{1}{2}$, and grows at $\sigma = 0.3$ (Figure 7c). By Corollary 3, RH holds if and only if $A_y^*(s) \rightarrow 1$ locally uniformly on Ω . Sieving the wave by the Euler product turns its free amplitude into the Euler helix and adds no information; the obstruction is again the pole correction, which has to be removed by hand. Status: the identities are proved; the computations are numerical (`sieved_wave/`).

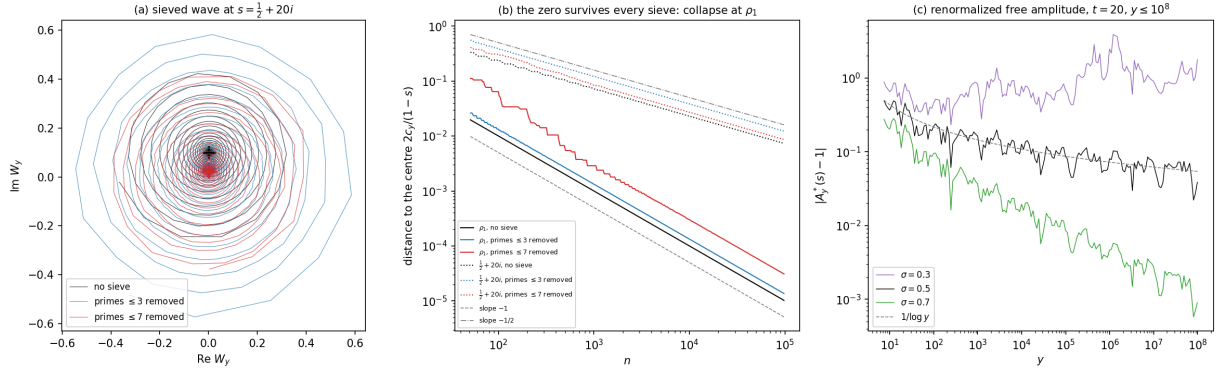


FIGURE 7. The sieved wave. (a) $W^{(y)}$ at $s = \frac{1}{2} + 20i$ with no sieve and with the primes ≤ 3 and ≤ 7 removed; crosses: the centres $2c_y/(1-s)$. (b) Distance of $W^{(y)}$ to its centre (maximum over blocks of 210 consecutive n) at ρ_1 (solid) and at $\frac{1}{2} + 20i$ (dotted). (c) $|A_y^*(s) - 1| = |e^{-H_s(\log y)/\log y - R_y(s)} - 1|$ at $t = 20$ for $\sigma = 0.3, 0.5, 0.7$, from the prime powers up to 10^8 .

Remark 13 (prime operators and the functional equation). Extend the wave to real arguments by $W(x, s) = 2X_{[x]-1}(s)\overline{x^{-s}}$. Removing the multiples of a prime p acts on the sieved wave by

$$W^{(y \cup \{p\})}(n, s) = W^{(y)}(n, s) - p^{-2\sigma} W^{(y)}(n/p, s),$$

because the multiples of p below n contribute $p^{-s} X_{[n/p]-1}^{(y)}(s)$ and $\overline{n^{-s}} = \overline{p^{-s}} \overline{(n/p)^{-s}}$; the identity holds to 10^{-15} at $s = \frac{1}{2} + 20i$, $0.7 + 33i$ and ρ_1 for $p = 2, 3, 5, 7$. The phase t cancels: on the critical line each prime acts by the real operator $1 - p^{-1}T_p$, $T_p W(n) = W(n/p)$, the same at every height. On a mode n^λ the operator multiplies by $1 - p^{-2\sigma-\lambda}$, and the formal product over all primes is $1/\zeta(2\sigma+\lambda)$: it is $1/\zeta(1) = 0$ on the centre ($\lambda = 1-2\sigma$) and $1/\zeta(s)$ on the rotating term ($\lambda = -\sigma+it$), and each single factor vanishes on a mode $n^{-2\sigma}$. Equivalently $W(n, s) = \sum_{m < n} m^{-2\sigma} (T_m[2x^{-s}])(n)$

is a positive combination of dilates of the elementary wave $2x^{-\bar{s}}$. For the Davenport–Heilbronn function no such operator exists: the best fit of the sieved wave by $W_f - cT_p W_f$ leaves a relative residual 1.00 for $p = 2, 3, 7$. For $L(s, \chi)$ the operator is $1 - \chi(p)p^{-2\sigma}T_p$; the structure is that of every Euler product, so an argument for RH resting on it alone would apply to the whole Selberg class.

Combined with the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, the sieve gives

$$\chi_y(s) = \frac{\zeta_y(s)}{\zeta_y(1-s)} = \chi(s) \prod_{p \leq y} \frac{1-p^{-s}}{1-p^{s-1}}.$$

On the critical line $|\chi_y(s)| = 1$ for every y , since $1-s = \bar{s}$. Off the line the factor is erratic: at $s = 0.7 + 20i$, $|\chi_y| = 0.79, 0.70, 1.53, 0.72, 19.5, < 10^{-3}, 3 \cdot 10^{-4}$ for $y = 1, 10, \dots, 10^6$. The reflected point $1-\sigma = 0.3$ lies to the left of the critical line, where the Euler product diverges whether or not RH holds, because the zeros on the line lie to its right. The helix measures this: $|H_s(u)/u|$ decreases from 0.147 to 0.0025 as $y = e^u$ goes from 10^2 to 10^8 at $s = 0.7 + 20i$, and grows from 0.79 to 2.94, like $y^{0.2}/\log y$, at $s = 0.3 + 20i$. The functional equation therefore pairs the half of the strip where the Euler product converges, provided there are no zeros to the right, with the half where it always diverges; at an off-line pair the relation reads $0 = \chi_y \cdot 0$. Combining the two structures would require control of the Euler product on the divergent side. Status: the identities are proved; the computations are numerical (`sieved_wave/operators.py`).

6. HEAT FLOW AND THE LAGUERRE HIERARCHY

The properties that could force the zeros onto the line can be tested on the heat-flow deformation

$$H_t(z) = \int_{-\infty}^{\infty} e^{tu^2} \Phi(u) \cos(zu) du, \quad H_0(z) = \xi(\tfrac{1}{2} + iz),$$

with Φ the kernel of de Bruijn [dB]. There is a constant Λ , the de Bruijn–Newman constant, such that, in the normalization of [dB, Ne], the deformed function has only real zeros if and only if $t \geq \Lambda$; with the normalization $H_0(z) = \xi(\frac{1}{2} + iz)$ of this section ($\tau = 4t$, see below), H_t has only real zeros if and only if $t \geq \Lambda/4$. Rodgers and Tao proved $\Lambda \geq 0$ [RT]; the upper bound $\Lambda \leq 0.22$ of the Polymath project [Pm] was improved to $\Lambda \leq 0.2$ by Platt and Trudgian [PT], and $\Lambda \leq 0.174$ is claimed in [CE13]; and RH is the statement $\Lambda \leq 0$. Hence H_t has non-real zeros for every $t < 0$, and a property that implies RH must hold at $t = 0$ and fail for every $t < 0$.

Computation. We computed H_t and its derivatives by the trapezoidal rule (step $\frac{1}{150}$, 60 digits) on $2 < x < 58$. All 12 zeros there are real for $t \geq -0.45$. The first collision is that of the closest pair, 48.005 and 49.774, at $t_c = -0.469477$ and $x = 49.67$; for smaller t the number of real zeros falls to 10, 7 and 5 at $t = -0.5, -1, -1.5$ (Figure 8a). Three properties were compared. (i) The first Laguerre expression $L_1 = H_t'^2 - H_t H_t''$ is positive for $t > t_c$ and changes sign at the collision point linearly in $t - t_c$ (Figure 8b). (ii) The energy slope of [CE10, Sect. 3], in the form $\partial_y |H_t(x + iy)|^2 \geq 0$ for $y > 0$, fails exactly where non-real zeros exist. (iii) The Jensen polynomials $J^{d,0}$ of H_t [GORZ] with $d \leq 120$ remain hyperbolic at $t = -0.6, -1$ and -1.5 , although H_t has non-real zeros at height about 50. The local properties detect the collision exactly; the low-degree global ones do not see it. Inside any finite window ξ has a margin, here $|t_c| = 0.47$; since $\Lambda \geq 0$, for every $t < 0$ some pair collides at a greater height, so collisions occur arbitrarily close to $t = 0$ and no finite computation decides the question. Status: numerical (`heat/, spectral/`).

Higher expressions. The generating identity $|F(x + iy)|^2 = \sum_{n \geq 0} L_n(F)(x) y^{2n}$, $L_n(F) = \sum_{k=0}^{2n} (-1)^{k+n} F^{(k)} F^{(2n-k)} / (k! (2n-k)!)$, shows that a real entire function F of order less than 2 (more generally, of genus at most 1 after removing a Gaussian factor) belongs to the Laguerre–Pólya class, and so has only real zeros, if and only if $L_n(F)(x) \geq 0$ for all n and all real x [CV]. A pair $x_0 \pm iy_0$ contributes $-2/y_0^2$ to $-(\log |F|)''$ at x_0 , while real zeros with mean spacing Δ

contribute about π^2/Δ^2 , so pairs with $y_0 > (\sqrt{2}/\pi)\Delta$ pass the test $L_1 \geq 0$; for ζ this would apply to off-line zeros above height about 2000, where $\Delta < 1$. On the deformation the non-real zeros $47.07 + 4.10i$ and $55.66 + 5.01i$ at $t = -4$ are first detected by L_2 , and $49.89 + 6.53i$ at $t = -6$ by L_3 ; all others by L_1 (Figure 8c). Status: the identity is classical; the rest numerical.

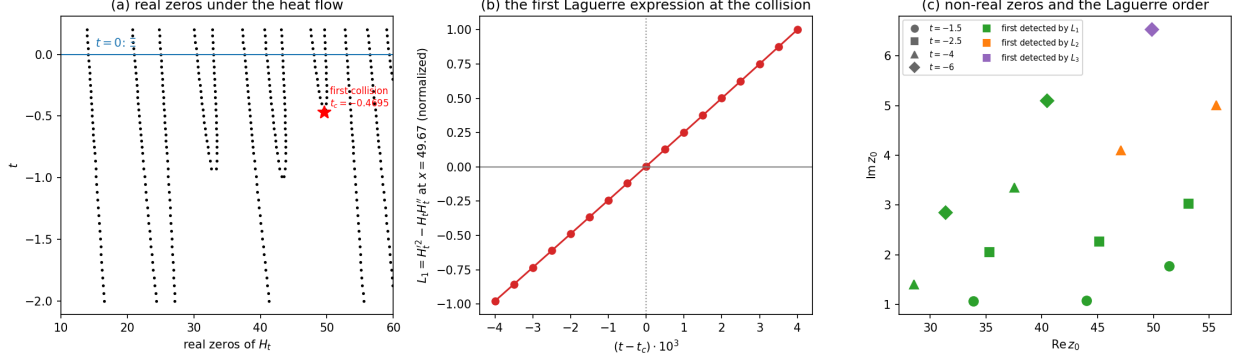


FIGURE 8. Heat flow and the Laguerre hierarchy. (a) Real zeros of H_t in $10 < x < 60$ for $-2 \leq t \leq 0.2$; the star marks the first collision, at $t_c = -0.4695$. (b) $L_1 = H_t'^2 - H_t H_t''$ at $x = 49.67$ for $|t - t_c| \leq 0.004$, normalized. (c) Non-real zeros z_0 of H_t in the upper half plane for $t = -1.5, -2.5, -4, -6$, coloured by the lowest n for which $L_n < 0$ near $\text{Re } z_0$.

6.1. The Laguerre expressions as moments of the helix. On the side of the primes the hierarchy concerns moments of the Euler helix. Assume RH and simple zeros, fix $s = \frac{1}{2} + it$ with t not an ordinate, and let

$$\mathcal{H}(u) = \sum_{\rho} a_{\rho} e^{i(\gamma-t)u}, \quad a_{\rho} = \frac{1}{s - \rho} = \frac{1}{i(t - \gamma)},$$

the leading part of (5), as an almost periodic function (Besicovitch class, since $\sum |a_{\rho}|^2 < \infty$), and $M_{2m} = \lim_U \frac{1}{U} \int_0^U |\mathcal{H}|^{2m}$. Since $\sum |a_{\rho}|^2 < \infty$ but $\sum |a_{\rho}| = \infty$, $\mathcal{H}$ is in the Besicovitch class B^2 but not known to be in B^4 ; we define M_4 as the limit of the means for the truncations $\mathcal{H}_N$ to the N zeros nearest to t . Under the linear-independence hypothesis below, the quadruple formula of the proof is exact for each N , and it shows that $\mathcal{H}_N$ is a Cauchy sequence in B^4 , the tail contributing at most a constant times $(\sum_{\text{tail}} w_{\gamma})^2 + \sum_{\text{tail}} |c_{\gamma}| \sum |c_{\gamma}|$. Put $w_{\gamma} = (t - \gamma)^{-2}$ for the ordinates of both signs, e_n for the elementary symmetric functions of the w_{γ} , and $c_{\gamma} = 1/(\gamma^2 - t^2)$ for $\gamma > 0$.

Proposition 14. *Under these assumptions $L_n(\Xi)/\Xi^2 = e_n(w)$ at $x = t$, where $\Xi(z) = \xi(\frac{1}{2} + iz)$, and $L_1/\Xi^2 = M_2$. If moreover the positive ordinates are linearly independent over $\mathbb{Q}$, then*

$$\frac{L_2}{\Xi^2} = \frac{1}{2}(M_4 - M_2^2) - 2 \left[\left(\sum_{\gamma > 0} c_{\gamma} \right)^2 - \sum_{\gamma > 0} c_{\gamma}^2 \right].$$

Proof. By the Hadamard product $\Xi(z) = \Xi(0) \prod_{\gamma > 0} (1 - z^2/\gamma^2)$, so $|\Xi(t + iy)|^2/\Xi(t)^2 = \prod_{\gamma} (1 + y^2 w_{\gamma}) = \sum_n e_n(w) y^{2n}$, the product over the ordinates of both signs; compare with the generating identity. The frequencies $\gamma - t$ are distinct, so $M_2 = \sum |a_{\rho}|^2 = e_1(w)$. For M_4 , the mean of $\mathcal{H}^2 \bar{\mathcal{H}}^2$ is the sum of $a_1 a_2 \bar{a}_3 \bar{a}_4$ over the quadruples with $(\gamma_1 - t) + (\gamma_2 - t) = (\gamma_3 - t) + (\gamma_4 - t)$, that is $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$. Under linear independence of the positive ordinates the solutions are $\{\gamma_1, \gamma_2\} = \{\gamma_3, \gamma_4\}$, which give $2e_1^2 - p_2$ with $p_2 = \sum w_{\gamma}^2$, and $\{\gamma_1, \gamma_2\} = \{\gamma, -\gamma\}$, $\{\gamma_3, \gamma_4\} = \{\gamma', -\gamma'\}$ with $\gamma \neq \gamma'$ positive, which give $4 \sum_{\gamma \neq \gamma'} c_{\gamma} c_{\gamma'}$, since $a_{\gamma} a_{-\gamma} = -1/(t^2 - \gamma^2) = c_{\gamma}$ is real. Hence $\frac{1}{2}(M_4 - M_2^2) = \frac{1}{2}(e_1^2 - p_2) + 2 \sum_{\gamma \neq \gamma'} c_{\gamma} c_{\gamma'} = e_2 + 2[(\sum c_{\gamma})^2 - \sum c_{\gamma}^2]$. $\square$

The second term is missing from the statement $L_2/\xi^2 = \frac{1}{2}(M_4 - M_2^2)$ of version 3 of [CP], which overlooked the coincidences $(\gamma - t) + (-\gamma - t) = (\gamma' - t) + (-\gamma' - t)$ between conjugate pairs. At $s = \frac{1}{2} + 20i$ the exact value is $L_2/\xi^2 = e_2(w) = 0.12026$; the correction is 0.0017 with the 341 positive ordinates below 600, so that $\frac{1}{2}(M_4 - M_2^2) \approx 0.1220$; and $\frac{1}{2}(M_4 - M_2^2)$ computed from the prime powers up to 10^8 on windows $[u_0, 18.4]$, $8 \leq u_0 \leq 12$, lies between 0.115 and 0.120, a scatter that does not resolve the difference. A direct time average of the model sum with the 29 positive ordinates below 100 gives $\frac{1}{2}(M_4 - M_2^2) = 0.10805$, against $e_2 = 0.10698$ and e_2 plus the correction $= 0.10805$ (`spectral/moment_correction.py`). Status: Proposition 14 is proved for the model sum $\mathcal{H}$; its identification with the mean of the helix computed from the primes is the unjustified interchange of Remark 10.

The moments are automatically finite and nonnegative when all zeros are on the line, because the helix is then a sum of circles. An off-line zero does not make L_2 negative on the helix side; it turns a circle into a growing spiral and makes every moment infinite. In the language of the primes the Laguerre hierarchy therefore reduces, heuristically, to the finiteness of the energy of the Euler helix (Remark 10) and of the prime signal (Theorem B of [CE5]); the higher orders add expressions in the moments and the pair products c_γ , but no new constraint. Status: heuristic.

6.2. Collisions, the deformed cross-energy and survival. In the normalization of this section $H_0(z) = \xi(\frac{1}{2} + iz)$, so the zeros of H_0 are the ordinates $\pm\gamma$, and $\partial_t H_t = \partial_s^2 H_t$ in the variable $s = \frac{1}{2} + iz$. In the normalization $H_0(z) = \frac{1}{8}\xi(\frac{1}{2} + \frac{iz}{2})$ of [Pm] and [CE15] the time is $\tau = 4t$.

The deformed cross-energy. On a Dirichlet term the flow acts by $e^{t\partial_s^2} n^{-s} = b_n n^{-s}$ with $b_n = e^{t \log^2 n}$, the coefficients of [Pm] in this normalization. For the partial sums $\sum_{k \leq n} b_k k^{-s}$ the centre term of the cross-energy C_2 of [CE1] becomes, by the Euler–Maclaurin formula with the tail evaluated at its end point, $2b_n^2 n^{1-2\sigma} \operatorname{Re}(1/(1-s+2t \log n))$ in place of $2n^{1-2\sigma} \operatorname{Re}(1/(1-s))$: the critical line of the trichotomy of Theorem A of [CE1] is moved to $\sigma = \frac{1}{2} + t \log n$, and for $t < 0$ the series converges in the whole plane and the pole is smoothed. With the gamma factor, the n -th term of the Riemann–Siegel sum at height γ is shifted by $t(\log \frac{\gamma}{2\pi} - \log n)$, which is the mechanism of [Pm] by which the zeros become real for $t > 0$. These statements, with a numerical check of the deformed expansion at $\tau = -0.2$, are in [CE15, Sect. 7 and Fig. 2(d)], where the drift $\frac{1}{4} \log \frac{\gamma}{2\pi}$ of the effective line per unit of τ is identified with the speed at which an isolated off-line zero is absorbed. Status: heuristic, as in [CE15].

Collisions. Under the flow a simple real zero moves by $\dot{z}_k = 2 \sum_{j \neq k} 1/(z_k - z_j)$ [CSV], [Pm]. For an isolated pair at distance D this gives $\frac{d}{dt} D^2 = 8$, so the pair collides at $t_c \approx -D^2/8$ ($\tau_c \approx -D^2/2$); in a lattice of equally spaced zeros the separations do not change, so collisions need irregular spacing. We computed the 10142 zeros below height 10^4 (Riemann–Siegel with three correction terms, refined below height 1000 by Euler–Maclaurin; the Lehmer pair 7005.062866, 7005.100565 is resolved) and integrated the flow backwards, with the 40 nearest zeros on each side of a pair mobile and the others frozen. The Lehmer pair, $D = 0.03770$, collides at $t_c = -1.778 \cdot 10^{-4}$ ($\tau_c = -7.11 \cdot 10^{-4}$), within 0.1% of $-D^2/8$, and the next closest pair, at 5229.20, at $-2.341 \cdot 10^{-4}$; for the five closest pairs below 10^4 the law $-D^2/8$ holds to within 0.6%. At low height it is cruder: the first collision of Figure 8, $t_c = -0.4695$ for $D = 1.769$, is 1.2 times $-D^2/8$. This is the classical use of Lehmer pairs for lower bounds of Λ [CSV], and, since $\Lambda \geq 0$, it is the picture of collisions accumulating at 0^- .

The number of consecutive gaps below height 10^4 with $D^2/8 < |t|$, that is, of the pairs that would collide by time t if they were isolated, is 16, 114 and 747 for $|t| = 0.0022, 0.0075$ and 0.025 ($\tau = 0.0087, 0.03, 0.1$), against 29.0, 178.8 and 963.2 predicted by the Wigner surmise for the GUE with the local density $\frac{1}{2\pi} \log \frac{\gamma}{2\pi}$. The ratios 0.55, 0.64, 0.78 are the known deficit of small gaps at finite height [BBLM].

The first collision of a window. The working notes behind this paper read the collision as collective. We integrated backwards the 26 consecutive zeros of each of 489 windows between heights 100 and 10^4 , with all other zeros frozen (their field exact within distance 60 and linearized beyond), up to the first collision. The central pair of the window collides first in 3.7% of the windows, and the median collision time is 0.136 times $-D^2/8$ for the central gap D , in agreement with the notes. But the pair that collides first is the closest pair of the window in 98% of the windows, at a time between 1.02 and 1.16 times $-D_{\min}^2/8$ (10th to 90th percentile; median 1.07), and 3.7% is close to the share $\frac{1}{25}$ to be expected if the closest pair always collided first. So the backward collision time is not a property of a given pair, since it is fixed by the closest pair in the neighbourhood, but the isolated-pair law describes that pair to within a few per cent; we do not find a collective effect beyond this. As a consequence the gap predictor of [CE11, Sect. 11.1], computed from the primes alone, follows the gaps but not the collision times of the windows. Normalizing the collision time by the square of the local density gives a quantity that, according to the same notes, is positive for ζ , for $L(s, \chi)$ modulo 5 and at the zeros of the Davenport–Heilbronn function on the line, and negative at its off-line zeros: a perfect separation, but a tautological one, since a negative collision time is a non-real zero at $t = 0$. Status: the pair law is elementary; the rest is numerical (`collisions/zeros_rs.py`, `collisions/collisions.py`), except the normalized times for $L(s, \chi)$ and f , which are from the author’s working notes.

Survival. The opposite question, how long an off-line zero of ζ would survive the forward flow, is answered in [CE15]: unconditionally $\Lambda \geq \delta^2/(70 \log \gamma \log \log \gamma)$ if $\zeta(\frac{1}{2} + \delta + i\gamma) = 0$, with a sharper bound linear in δ under local regularity; for a single pair the interaction term of its exact equation at $t = 0$ is the energy slope $\partial_\sigma \log |\xi|$ of [CE10, Sect. 3] of the other zeros, which predicts the survival times of the off-line zeros of the Davenport–Heilbronn function to within 7%. That paper supersedes the earlier note on the absorption bound. The constant in the law $\Lambda \leq C/\log X$ for upper bounds on Λ obtained from the verification of RH up to height X is studied in [CE14].

7. HOLOMORPHIC AND NON-HOLOMORPHIC SUMS; THE DAMPED ENERGY

Sums over the zeros that are holomorphic in ρ , such as $\sum_\rho 1/(\rho(1-\rho)) = 2\sum_\rho 1/\rho = C_B$, are given by the explicit formula as linear expressions in the primes; they are finite and computable, but they do not see β . Non-holomorphic sums see β : the pair identity of [CE5, Prop. 2.5] gives $\sum_\rho |\rho|^{-2} \geq C_B$ with equality if and only if RH holds, and the crossing point of the cross-energy is $n^*(\rho) = 1/(2 \operatorname{Re} \frac{1}{1-\rho}) + \frac{\beta}{6} + O(\gamma^{-2})$ [CE1]. An off-line pair at height T changes $\sum |\rho|^{-2}$ by about $2(2\beta - 1)^2/T^4$, while it moves the crossing points by $(2\beta - 1)^2 T^2/(2\beta(1 - \beta))$: $2 \cdot 10^{-6}$ against 17 for $\beta = 0.6$ at $T = 14$, and $8 \cdot 10^{-54}$ against $8 \cdot 10^{18}$ for $\beta = 0.501$ at $T = 10^{12}$. From the primes, non-holomorphic sums are reached only through quadratic quantities, the energies of Theorem B and Remark 10. In the coordinate $w = 1 - 1/\rho$ of Li’s criterion [Li], RH says $|w_\rho| = 1$; the functional equation acts by $w \mapsto 1/\bar{w}$, which is holomorphic, while the reflection in the critical line acts by $w \mapsto 1/\bar{w}$, and any quantity that measures the distance of a zero from the line must contain $w\bar{w}$.

The damped energy interpolates between the two kinds. Let $S(u) = e^{-u/2}(\psi(e^u) - e^u)$ and

$$\mathcal{E}(\varepsilon) = \int_2^\infty e^{-\varepsilon u} S(u)^2 du.$$

Proposition 15. *The abscissa of convergence of the Laplace integral $\mathcal{E}(\varepsilon)$ is $2\Theta - 1$. In particular RH holds if and only if $\mathcal{E}(\varepsilon) < \infty$ for every $\varepsilon > 0$.*

Proof. If $\varepsilon > 2\Theta - 1$, then $\psi(x) - x = O(x^{\Theta+\delta})$ for every $\delta > 0$ [In] gives $S(u)^2 = O(e^{(2\Theta-1+2\delta)u})$ and $\mathcal{E}(\varepsilon) < \infty$. Conversely let $\mathcal{E}(\varepsilon) < \infty$ and $a > \varepsilon/2$. By Cauchy–Schwarz, $\int_2^\infty e^{-au} |S(u)| du \leq \mathcal{E}(\varepsilon)^{1/2} (\int_2^\infty e^{-(2a-\varepsilon)u} du)^{1/2} < \infty$, and $\int_2^\infty e^{-au} S(u) du = \int_{e^2}^\infty (\psi(x) - x) x^{-a-3/2} dx$. With $z = a + \frac{1}{2}$ this says that the Mellin integral $\int_{e^2}^\infty (\psi(x) - x) x^{-z-1} dx$ converges absolutely in $\operatorname{Re} z > \frac{1+\varepsilon}{2}$, where it is analytic; for $\operatorname{Re} z > 1$ it equals $-\zeta'(z)/(z\zeta(z)) - 1/(z-1)$ minus an entire function, which

has a pole, with residue $-m_\rho/\rho \neq 0$, at every zero ρ of multiplicity m_ρ . Hence $\Theta \leq \frac{1+\varepsilon}{2}$, that is, $\varepsilon \geq 2\Theta - 1$. Since the integrand is nonnegative, the set of ε with $\mathcal{E}(\varepsilon) < \infty$ is a half-line, and the two directions show that its endpoint is $2\Theta - 1$. $\square$

By the explicit formula, $S(u) = -\sum_\rho e^{(\rho-\frac{1}{2})u}/\rho - e^{-u/2} \log 2\pi + O(e^{-5u/2})$, so formally $\mathcal{E}(\varepsilon)$ is a double sum over the zeros with denominators $\varepsilon + 1 - \rho - \bar{\rho}'$, and an off-line zero gives a pole at $\varepsilon = 2\beta - 1$, at a position that does not depend on its height. With the prime powers up to 10^8 , $\mathcal{E}(\varepsilon)$ on $2 \leq u \leq \log 10^8$ agrees with the double sum over the 682 zeros with $|\gamma| < 600$, the constant term and the diagonal contribution of the higher zeros to within 0.3 to 0.6% for $0 \leq \varepsilon \leq 1$ (Figure 9a).

The price is the size of the primes needed. We added to the zeros a pair $\frac{1}{2} \pm \delta + iT$ (with conjugates) and compared the mean energy on $[2, U]$ with that of the same pair placed on the line. The difference reaches 10% at $U = 13, 39$ and 115 for $T = 27.5$ and $\delta = 0.1, 0.03, 0.01$; at $U = 25.5, 81$ and 240 for $T = 100.3$; and at $U = 43.5$ and 145 for $T = 500.7$ and $\delta = 0.1, 0.03$ (Figure 9b,c). This follows $x^* = e^{U^*} \approx (cT^2)^{1/(2\delta)}$, roughly $T^{1/\delta}$: an energy detects an off-line zero at any height, but only with the primes up to about $T^{1/\delta}$. Since RH is verified up to $T = 3 \cdot 10^{12}$ [PT], detecting there a zero with $\delta = 0.01$ would need x of order 10^{1250} . That an off-line zero influences $\psi(x)$ only for x beyond a power $1/\delta$ of its height is the classical heuristic; here it is measured on the energy. Status: Proposition 15 is proved (classical in substance: Landau's theorem in Cauchy-Schwarz form); the double-sum picture is formal; the computations are numerical (**damped/**).

Remark 16 (a discrete form: the energy matrix of the Chebyshev errors). Put $a_n = \Lambda(n) - 1$ and $A(k) = \sum_{n \leq k} a_n = \psi(k) - k$. Exchanging the sums gives

$$\mathcal{Q}_N = \sum_{k \leq N} A(k)^2 = a^\top W_N a, \quad W_N(m, n) = N + 1 - \max(m, n).$$

In reversed coordinates W_N is the matrix $\min(i, j)$, whose inverse is the second-difference matrix, so the diagonalization is explicit:

$$\lambda_j = \frac{1}{4 \sin^2 \frac{(2j-1)\pi}{4N+2}} \sim \frac{4N^2}{(2j-1)^2 \pi^2}, \quad v_j(n) = \frac{2}{\sqrt{2N+1}} \sin((N+1-n)\theta_j), \quad \theta_j = \frac{(2j-1)\pi}{2N+1},$$

and $\mathcal{Q}_N = \sum_j \lambda_j c_j^2$, $c_j = \langle a, v_j \rangle$, with $\sum_j c_j^2 = \|a\|^2 = \sum_{n \leq N} (\Lambda(n) - 1)^2 \sim N \log N$. Then RH holds if and only if $\mathcal{Q}_N = O_\varepsilon(N^{2+\varepsilon})$ for every $\varepsilon > 0$. One direction follows from $\psi(x) - x = O(x^{1/2} \log^2 x)$. For the other, $(\psi(x) - x)^2 \leq 2A(k)^2 + 2$ on $[k, k+1)$, so $\int_1^{N+1} (\psi(x) - x)^2 dx \leq 2\mathcal{Q}_N + 2N$. Cauchy-Schwarz on dyadic ranges then makes the Mellin integral of $\psi(x) - x$ absolutely convergent in $\operatorname{Re} z > \frac{1}{2} + \frac{\varepsilon}{2}$, and one concludes as in Proposition 15; this is the discrete form of that proposition and is classical in substance. By Parseval the modes above J contribute at most $\lambda_{J+1} \|a\|^2 \ll N^3 \log N / J^2$, so RH is equivalent to the same bound for the modes $j \leq N^{1/2+\delta}$ alone, for any fixed $\delta > 0$. Under RH each fixed mode satisfies $|c_j| \leq (\mathcal{Q}_N / \lambda_j)^{1/2} = O_\varepsilon(j N^{\varepsilon/2})$, whereas Cauchy-Schwarz alone gives $|c_j| \leq \|a\| \asymp (N \log N)^{1/2}$. For $N = 10^6$, $\mathcal{Q}_N / N^2 = 0.0219$; the first 10, 100 and 1000 modes carry 25%, 82% and 97% of $\mathcal{Q}_N$; $c_1, \dots, c_5 = 0.016, 0.059, 0.100, 0.137, 0.066$; and $\max_{j \leq 1000} |c_j| / (2j - 1) = 0.076$. The Parseval bound for the modes above 1000 is still 54 times $\mathcal{Q}_N$, so at this size the high modes are controlled by it only from J of order $(N \log N)^{1/2}$ on. The spectral form localizes the problem but does not remove it: by summation by parts each c_j is a smoothed sum of $\psi(n) - n$ at the frequency $\theta_j \asymp j/N$, that is, on the principal major arc, where the explicit formula brings back the terms N^ρ . Status: the identities are exact and elementary; the equivalence is classical in substance; the numbers are numerical (**damped/spectral_WN.py**).

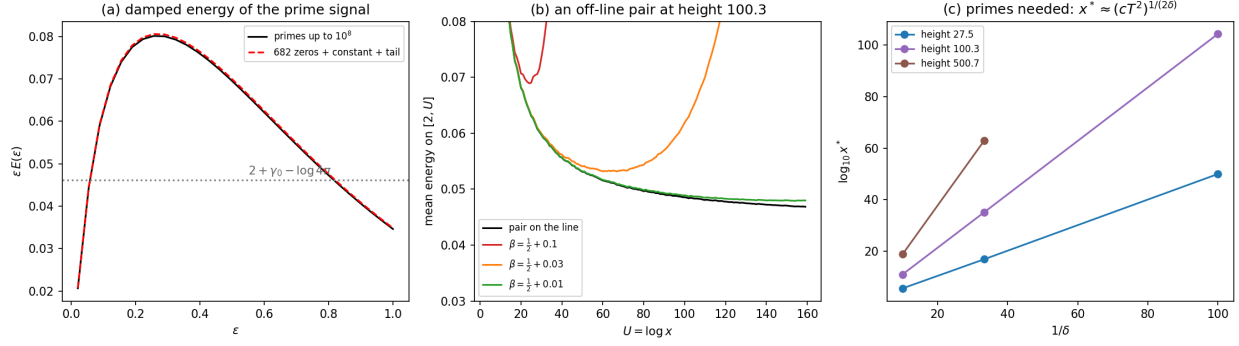


FIGURE 9. The damped energy. (a) $\varepsilon\mathcal{E}(\varepsilon)$ on $2 \leq u \leq \log 10^8$ from the prime powers (solid) and from the zeros with the constant term and the tail (dashed); the dotted line is $C_B = 2 + \gamma_0 - \log 4\pi$, the limit as $\varepsilon \rightarrow 0$ when the range of u is infinite. (b) Mean energy on $[2, U]$ with a pair at height 100.3 on the line (black) and at $\beta = \frac{1}{2} + \delta$. (c) The size $x^* = e^{U^*}$ of the primes at which the mean energy exceeds its value under RH by 10%, against $1/\delta$.

8. AMPLIFICATION AND PARITY

For curves over finite fields the analogue of RH is proved by amplification: a bound $|N_k - q^k - 1| \leq Cq^{k/2}$ for the point counts over $\mathbb{F}_{q^k}$, valid for all k , forces every inverse root of the numerator of the zeta function to have modulus $\sqrt{q}$ [St, Bo73, De]. Over $\mathbb{Q}$ the powers α^k become x^ρ , $N_k - q^k - 1$ becomes $x - \psi(x)$, and one side suffices: by Landau's theorem on Mellin transforms of nonnegative functions, RH holds if and only if $\psi(x) \leq x + O(x^{1/2} \log^2 x)$, an upper bound for the primes. Sieves give upper bounds by positivity, but Selberg's Λ^2 sieve stops at the parity barrier, and removing it needs bilinear information of type II, which the cross-energy $C_2(n, s)$, bilinear only in the additive sense, does not supply. The one-sided bound with its proof, the sieve computations on $(10^8, 10^8 + y]$ and the corresponding figure are in [CE17, Prop. 11 and Sect. 7.5]. Status: classical; the computations are in [CE17].

9. ZEROS OF $\log \zeta$, OF ζ' AND OF THE PRIME ZETA FUNCTION

The Euler product has no zeros, and the zeros of ζ are singularities of its logarithm. The zeros of $\log \zeta$ are the 1-points of ζ , the solutions of $\zeta(s) = 1$; those of $(\log \zeta)' = \zeta'/\zeta$ are the zeros of ζ' , which by Speiser's theorem [Sp] lie in $\sigma < \frac{1}{2}$ only if RH fails, and whose distance to the line is $\log \log T / \log T$ on average [LM]. Their numbers satisfy $N_1(T) = N(T) - \frac{T}{2\pi} \log 2 + O(\log T)$ and $N'(T) = \frac{T}{2\pi} \log \frac{T}{4\pi} - \frac{T}{2\pi} + O(\log T)$ [GS, Bt], so there are as many 1-points as zeros of ζ' , and fewer than zeros of ζ by one every $2\pi / \log 2 = 9.06$ units of height.

Remark 17 (the pairing of 1-points and zeros of ζ'). Below $t = 350$ there are 169 zeros of ζ , 132 1-points and 132 zeros of ζ' . Of the 168 gaps between consecutive zeros of ζ , 38 contain no 1-point, 129 one, and one gap, $(317.73, 318.85)$, contains two, at $0.614 + 318.149i$ and $1.283 + 318.833i$; the counting formula predicts 38.6 empty gaps. The 1-points and the zeros of ζ' are paired one to one at almost the same height: $|t' - t_w| \leq 0.19$ (median 0.083), the zero s' of ζ' lies to the right of its 1-point w in all 132 cases, and $|s' - w|$ lies between 0.32 and 1.06, with mean 0.77, 0.63 and 0.56 on $t < 100$, $100 < t < 200$ and $200 < t < 350$ (Figure 10). At the zeros of ζ' with $t < 220$, ζ is close to a positive real number, with $|\arg \zeta| \leq 0.62$ and median $|\zeta| = 0.84$.

The mechanism is the first term of the Dirichlet series. Write $\zeta(s) - 1 = 2^{-s}H(s)$ with $H(s) = 1 + (2/3)^s + (2/4)^s + \dots$; then $\zeta' = 2^{-s}(H' - H \log 2)$, so $\zeta'(s) = 0$ if and only if $H'/H = \log 2$, and

near a zero w of H , with R the regular part of H'/H at w , this gives $s' - w \approx 1/(\log 2 - R(s'))$, a displacement that is nearly real and positive when R is small.

The zeros of the prime zeta function $P(s) = \log \zeta(s) - \frac{1}{2} \log \zeta(2s) - \dots$, studied in their half-plane of absolute convergence in [CE9], follow the 1-points: below $t = 100$ we located 18, each within 0.08 to 0.59 of a 1-point (one more, next to the 1-point $0.19 + 95.3i$, is expected at $\sigma < 0.13$, near the natural boundary $\sigma = 0$ of P , where we did not locate it). The singularities of P are the points ρ/k and $1/k$ for squarefree k ; RH is equivalent to all of them lying on the lines $\sigma = 1/(2k)$, a restatement by dilation.

None of these families sees the zeros of ζ : they are determined by the first terms of the Dirichlet series, and the Davenport–Heilbronn function shows the same structure. Below $t = 200$ it has 108 1-points and 109 zeros of f' (the extra one at $2.881 + 199.99i$, whose 1-point lies above 200), paired one to one at almost the same height, $|t' - t_w| \leq 0.28$ (median 0.052), with the zero of f' to the right of its 1-point in every pair and $|s' - w|$ between 0.24 and 1.12 (mean 0.60 for $t < 100$, 0.47 for $100 < t < 200$). Its first coefficient after $a(1) = 1$ is $a(2) = \kappa = 0.284\dots$, so $f - 1 = \kappa 2^{-s} H_f(s)$ with $H_f(s) \rightarrow 1$, and $N_f(T) - N_1(T) = \frac{T}{2\pi} \log 2 + O(\log T)$ as for ζ : 22.1 predicted and 22 observed at $T = 200$. Because its zeros are denser, 20 of its 126 gaps contain two 1-points. The pairing is therefore a property of the Dirichlet series, not of the Euler product or of RH.

Both families are visible in one sum over the primes. With the pole term removed, $P_x(s) - \text{Li}(x^{1-s})$ is $P(s)$ plus the term from the zeros of Section 3; computed from the primes up to 10^6 on the critical line, its logarithm has a peak at each of the 21 ordinates of zeros of ζ below 80, the logarithmic singularities of P , and its deepest minima at the ordinates of the zeros of P closest to the line, such as $0.51 + 68.5i$ and $0.59 + 65.6i$ (Figure 11). At finite x every peak is finite, and a zero at distance δ from the line would also produce a peak, lower by an amount that depends on δ ; the picture is the explicit formula and does not separate ζ from the Davenport–Heilbronn function. Status: the counting formulas and Speiser’s theorem are known; the pairing, its mechanism and the numbers are observations (`onepoints/`; for f , `onepoints/dh_np.py` evaluates the Hurwitz zeta function by Euler–Maclaurin in double precision, checked against `mpmath`).

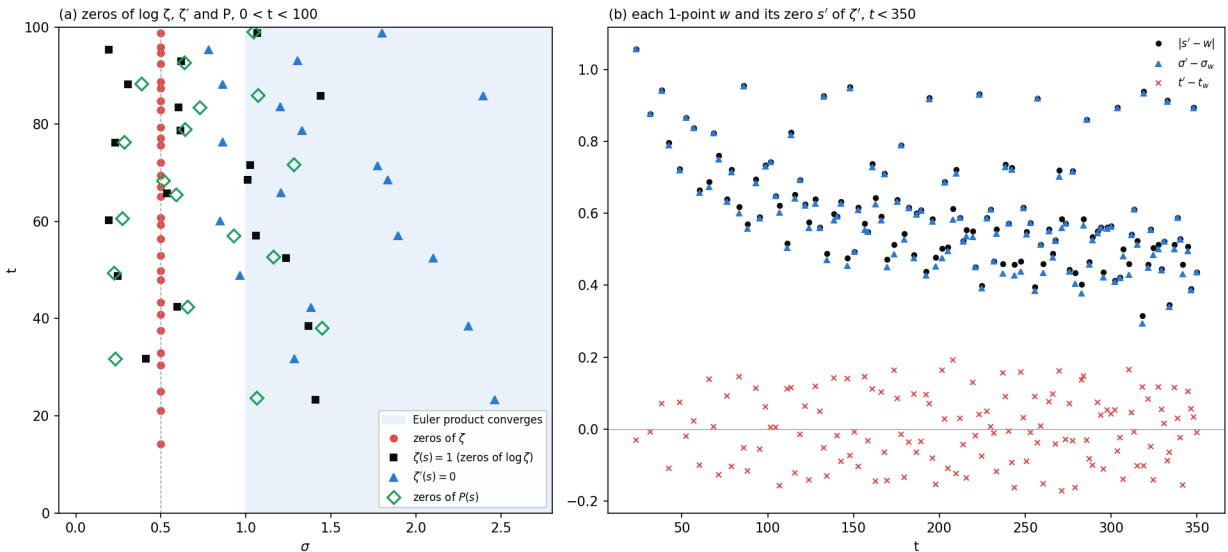


FIGURE 10. Zeros of $\log \zeta$, ζ' and P . (a) $0 < t < 100$: zeros of ζ (circles), 1-points (squares), zeros of ζ' (triangles), zeros of the prime zeta function (diamonds); shaded: $\sigma > 1$. (b) For each 1-point w with $t < 350$ and its zero s' of ζ' : distance $|s' - w|$, horizontal and vertical displacements.

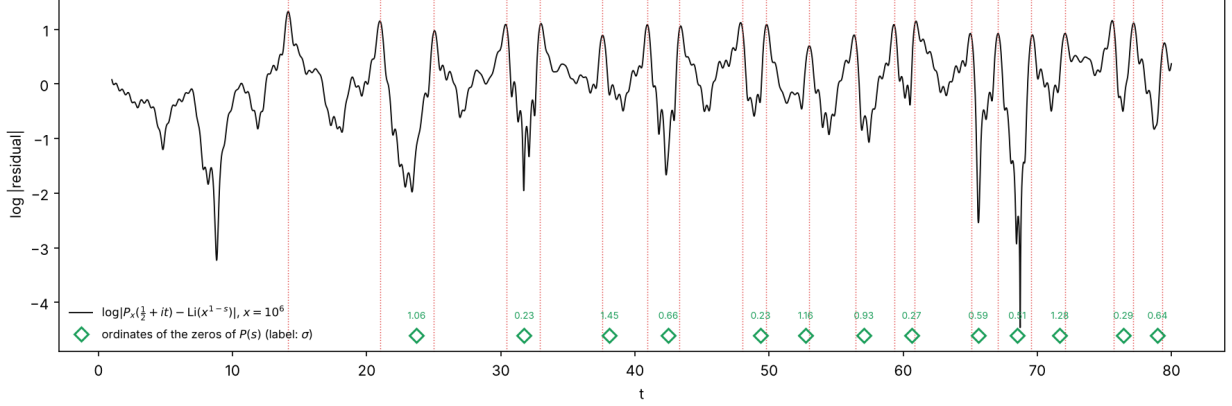


FIGURE 11. Both families in one prime sum. $\log |P_x(\frac{1}{2} + it) - \text{Li}(x^{1-s})|$ from the primes up to $x = 10^6$: peaks at the ordinates of the zeros of ζ (dotted), minima at the ordinates of the zeros of P (diamonds, labelled by their abscissa).

Higher derivatives. Passing to higher derivatives does not help. Under RH each $\zeta^{(k)}$ has at most finitely many non-real zeros in $\sigma < \frac{1}{2}$ [LM], the proportion of the zeros of $\xi^{(k)}$ on the critical line tends to 1 as $k \rightarrow \infty$ [Cy83], and by the Gauss–Lucas theorem information passes from ξ to ξ' , not back. Two families built from derivatives were considered in the working notes behind this paper and rejected. (i) $F_m = 1 - \zeta - \sum_{k=0}^m \zeta^{(k+1)}/\zeta^{(k)}$, which for $m = 0$ is the function $1 - \zeta - \zeta'/\zeta$ of [CE1, Prop. 2.8], in which the pole cancels. For $m \geq 1$ the terms with $k \geq 1$ have no arithmetic meaning, $\zeta^{(k)}$ has no functional equation, and the pole returns: $\zeta^{(k+1)}/\zeta^{(k)} = -(k+1)/(s-1) + O(1)$ at $s = 1$, so F_m has residue $\frac{(m+1)(m+2)}{2} - 1$ there. (ii) $S(s) = \sum_{k \geq 1} (-1)^{k+1}/\zeta^{(k)}(s)$. Since $\zeta(s) - 1/(s-1)$ is entire, $\zeta^{(k)}(s)/k! = (-1)^k (s-1)^{-k-1} (1 + o(1))$ as $k \rightarrow \infty$ for fixed $s \neq 1$, so the terms are eventually of size $|s-1|^{k+1}/k!$ and the series converges absolutely at every point that is not a zero of some $\zeta^{(k)}$, in particular at the zeros of ζ ; at ρ_1 the asymptotic size is reached only for k near 60, and $S(\rho_1) = 0.5486 - 8.9681i$. S has poles at the zeros of all the derivatives, among them the zero $2.4632 + 23.2983i$ of ζ' and the zero $-0.3551 + 3.5908i$ of ζ'' . By Speiser’s theorem [Sp] only a zero of ζ' in $0 < \sigma < \frac{1}{2}$ would contradict RH, while zeros of $\zeta^{(k)}$, $k \geq 2$, in $\sigma < \frac{1}{2}$ are compatible with it; so a pole of S to the left of the line says nothing about RH, and S carries no arithmetic. Status: the residue and the convergence are elementary; the numbers are computed from Taylor coefficients obtained by the fast Fourier transform of ζ on circles, stable under changes of the radius (onpoints/derivative_series.py). The working notes give $S(\rho_1) = 0.5558 - 8.9676i$, which differs from our value in the third digit.

10. THE EULER PRODUCT AS A QUANTUM STATE

Let L be the operator of multiplication by $\log n$ on $\ell^2(\mathbb{N})$ with basis $|n\rangle$, and P_N the projection onto the span of $|n\rangle$, $n \leq N$. For $\sigma > \frac{1}{2}$ (respectively $N < \infty$ and any σ) let $\psi_s = \zeta(2\sigma)^{-1/2} \sum_n n^{-s} |n\rangle$ (respectively $\psi_{s,N} = P_N \sum_n n^{-s} |n\rangle$, normalized).

Proposition 18 (the Euler state). (i) $\sum_n n^{-s} |n\rangle \in \ell^2(\mathbb{N})$ if and only if $\sigma > \frac{1}{2}$, and then $\psi_s = e^{-itL} \psi_\sigma$; likewise $\psi_{s,N} = e^{-itL} \psi_{\sigma,N}$.

(ii) Write $\ell^2(\mathbb{N}) = \bigotimes_p \ell^2(\mathbb{N}_0)$ by $|n\rangle = \bigotimes_p |v_p(n)\rangle$ (restricted product with respect to $|0\rangle$). Then $e^{-itL} = \bigotimes_p e^{-itL_p}$ with $L_p |k\rangle = k \log p |k\rangle$. Consequently every function of a state that is invariant under diagonal unitaries in the basis $|n\rangle$ (the populations, every coherence measure in this basis, the entropy of the populations) and every function invariant under local unitaries for a bipartition

of the primes (entanglement entropies, Schmidt coefficients) takes the same value on ψ_s and ψ_σ , and on $\psi_{s,N}$ and $\psi_{\sigma,N}$, for every t .

(iii) The relative entropy of coherence of ψ_s is the entropy of the populations $n^{-2\sigma}/\zeta(2\sigma)$:

$$C(\sigma) = \log \zeta(2\sigma) - 2\sigma \frac{\zeta'}{\zeta}(2\sigma) = \frac{1}{2\sigma - 1} - \log(2\sigma - 1) + 1 - \gamma_0 + O(2\sigma - 1) \quad (\sigma \rightarrow \frac{1}{2}^+),$$

and for the truncation on the line $C_N(\frac{1}{2}) = \frac{1}{2} \log N + \log \log N - \frac{\gamma_0}{2} + O(1/\log N)$.

(iv) For $\sigma > 0$ and the alternating vector $a_N = N^{-1/2} \sum_{n \leq N} (-1)^{n+1} |n\rangle$,

$$\left(N \sum_{n \leq N} n^{-2\sigma} \right)^{1/2} \langle a_N | e^{-itL} | \psi_{\sigma,N} \rangle = \sum_{n \leq N} (-1)^{n+1} n^{-s} = (1 - 2^{1-s})\zeta(s) + O_s(N^{-\sigma}),$$

uniformly for s in compact subsets of $\sigma > 0$. In particular the zeros of ζ in $0 < \sigma < 1$ are the points at which the rescaled overlap tends to 0.

Proof. (i) $\|\sum n^{-s}|n\rangle\|^2 = \sum n^{-2\sigma}$, and $n^{-s} = e^{-it \log n} n^{-\sigma}$; P_N is diagonal and commutes with L . (ii) By unique factorization $\log n = \sum_p v_p(n) \log p$, so $e^{-it \log n} = \prod_p e^{-it v_p(n) \log p}$, which is the stated product; for a bipartition $S \cup S^c$ of the primes it is $U_S \otimes U_{S^c}$ with diagonal factors, and since P_N commutes with e^{-itL} , also $\psi_{s,N} = (U_S \otimes U_{S^c})\psi_{\sigma,N}$, although $\psi_{\sigma,N}$ is not a product state. (iii) With $p_n = n^{-2\sigma}/\zeta(2\sigma)$, $-\sum p_n \log p_n = \log \zeta(2\sigma) + 2\sigma \sum p_n \log n$ and $\sum n^{-b} \log n = -\zeta'(b)$; the expansion follows from $\zeta(b) = \frac{1}{b-1} + \gamma_0 + O(b-1)$ and $\frac{\zeta'}{\zeta}(b) = -\frac{1}{b-1} + \gamma_0 + O(b-1)$ with $b = 2\sigma$. For the truncation, $\sum_{n \leq N} n^{-1} = \log N + \gamma_0 + O(N^{-1})$ and $\sum_{n \leq N} n^{-1} \log n = \frac{1}{2} \log^2 N + O(1)$. (iv) The left side equals $\sum_{n \leq N} (-1)^{n+1} n^{-s}$ by definition. Grouping consecutive terms, $|(2k-1)^{-s} - (2k)^{-s}| \leq |s|(2k-1)^{-\sigma-1}$, so the tail beyond N is $O(|s|N^{-\sigma}/\sigma) + O(N^{-\sigma})$, and the full series is $(1 - 2^{1-s})\zeta(s)$ for $\sigma > 0$. $\square$

Numerically, $C(\sigma)$ differs from the expansion by 0.039, 0.019, 0.0038, 0.0004 at $\sigma = 0.6, 0.55, 0.51, 0.501$, and $C_{4096}(\frac{1}{2}) = 6.066$ against 5.988 from the first three terms. $C(\sigma) \sim 1/(2\sigma - 1)$ is the thermal entropy of the gas of primes [Ju, BC] at inverse temperature 2σ ; ψ_s is normalizable exactly for $\sigma > \frac{1}{2}$, and the critical line is the boundary of the Hilbert space. This boundary is a consequence of the convergence of $\sum n^{-2\sigma}$ and holds for every t ; it does not say that the zeros lie on it, which is RH. Part (ii) says that the arithmetic of the zeros cannot be read from any intrinsic property of the Euler state: for $N = 4096$ on $\sigma = \frac{1}{2}$ the relative entropy of coherence 6.066379389410, the ℓ_1 -coherence 1799.345 [BCP] and the entanglement entropy between the prime 2 and the odd primes, 0.233336, coincide to twelve digits at $t = 0, 14.13, 20$ and 37.59. Only overlaps with vectors that are not diagonal in the multiplicative structure, such as a_N , depend on t . Status: Proposition 18 is proved; the numbers are computed (`quantum/coh.py`, `quantum/dyn.py`).

Remark 19 (coherence between characters). The alternating state is the frequency $\frac{1}{2}$ of the additive Fourier basis, and the zeros on the line are the times at which the evolved state becomes orthogonal to it: for $N = 4096$ the fidelity $|\langle a | e^{-itL} | \psi_{1/2,N} \rangle|^2$ drops to 10^{-9} – 10^{-10} at the ten zeros with $t < 50$, while at $\sigma = 0.6$ it stays above 10^{-6} (Figure 12b). For the other frequencies a/q the overlap is $F(s, a/q) = \sum_n e^{2\pi i n a/q} n^{-s}$, and every zero found with $t < 40$ lies off the line, with $\max |\beta - \frac{1}{2}| = 0.66, 0.73, 0.79$ for $q = 3, 4, 6$ and $0.77, 0.78, 0.81$ for $\frac{1}{5}, \frac{2}{5}, \frac{1}{8}$ (Figure 12c), in accordance with [DH]. The additive character $e^{2\pi i n a/q}$ is a combination of $\sum_{d|q} \#\{\text{primitive characters mod } d\}$ primitive L -functions, which is 1 only for $q \leq 2$, 2 for $q = 3, 4, 6$ and 4 for $q = 5, 8$ (Figure 12d). Measured against a single multiplicative character the orthogonality times lie on the line; measured against a coherent superposition of characters they do not. The dependence on the amount of superposition, for $\xi + \varepsilon \Lambda(\cdot, \chi_5)$, is treated in [CE16, Props. 2.4 and 2.5]: the distant zeros sit at $\beta \approx 2 \log(1/\varepsilon)/\log 5$ and zeros land on the line through square-root collisions, the same law as in the heat flow of

Section 6. Status: numerical (quantum/fig-coh.py, quantum/lerch.py); the counting of primitive characters is elementary.

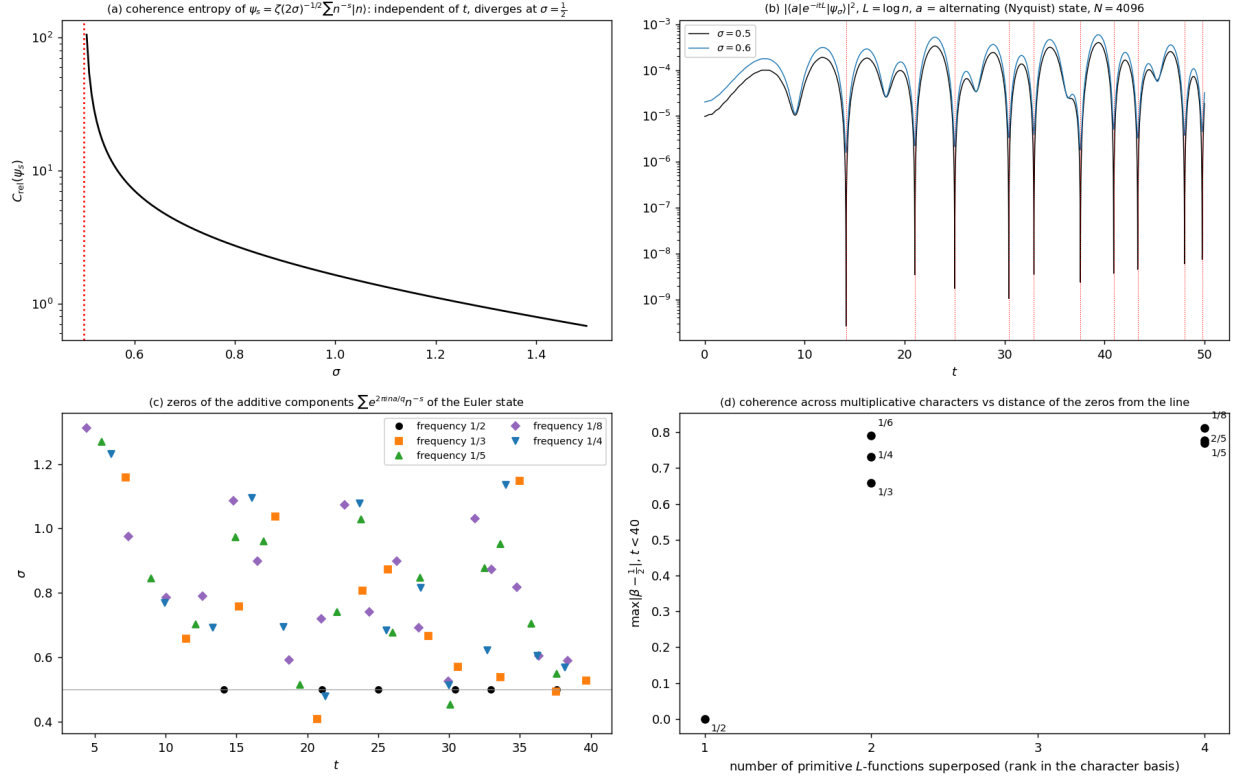


FIGURE 12. Coherence of the Euler state. (a) Relative entropy of coherence $C(\sigma)$ of ψ_σ . (b) $|\langle a|e^{-itL}|\psi_\sigma\rangle|^2$ for the alternating state, $N = 4096$; dotted: zeros. (c) Zeros of $\sum e^{2\pi i n a/q} n^{-s}$ with $t < 40$, $\beta \geq \frac{1}{2}$ shown, except the zeros of the local factors on $\sigma = 1$. (d) Number of primitive L -functions in the additive character against $\max|\beta - \frac{1}{2}|$.

Remark 20 (the wave as a quantum state). The cross-energy wave can be read in the same language. Sample $W_n(s)$ at 64 points with $\log n$ equally spaced in $[7, 9]$, form the 3×62 trajectory matrix $H_{jk} = W_{n_{j+k}}(s)$ and the density matrix $\varrho = HH^*/\text{tr} HH^*$; record the von Neumann entropy $S(\varrho)/\log 3 \in [0, 1]$, the ℓ_1 -coherence, and the drift κ of the dominant eigenvector v , defined by $v_{j+1}/v_j = e^{\kappa h}$ with h the step in $\log n$, so that a single mode $W \propto n^\kappa$ gives a pure state with drift κ . Since $X_{n-1}(s) = \zeta(s) - \zeta(s, n)$ with the Hurwitz function [CE10, Rem. 2.4], exactly

$$W_n(s) = 2\zeta(s)n^{-\bar{s}} - 2\zeta(s, n)n^{-\bar{s}},$$

a transient of amplitude $\zeta(s)$ and a steady part $-2n^{1-2\sigma}/(1-s) + O(n^{-2\sigma})$, the centre of Theorem A. At a zero the transient is absent and the state is pure with drift $\kappa = 1 - 2\beta$; elsewhere the second eigenvalue of ϱ decays like $n^{2\sigma-2}$ as the window moves, and at a zero like n^{-2} . The measured slopes of $\log \lambda_2$ are -0.962 at $\frac{1}{2} + 20i$ (predicted -1), -0.587 at $0.7 + i\gamma_1$ (-0.6) and -2.000 at ρ_1 (Figure 14a).

On $\sigma = \frac{1}{2}$ the entropy falls from 0.1 – 0.5 between zeros to 10^{-5} – 10^{-3} at every zero, with $|\kappa| \approx 10^{-4}$ (Figure 13a,b). For the Davenport–Heilbronn function the zeros are instead maximally mixed, $S = 0.96$ – 0.99 at the five zeros on the line with $80 < t < 90$ and $S = 0.990$ at ρ_f (Figure 13c):

without a pole there is no steady part, and at a zero only the periodic remainder is left. The purity at the zeros of ζ comes from the pole, not from the line. To separate the two, take $\zeta + 5f$, which has the pole, no functional equation, and zeros off the line. At its zeros the state is nearly pure ($S = 0.03\text{--}0.05$, the residue of the period 5), and the drift measures β : $\kappa = -0.2916, +0.1216, +0.1779, -0.0494$ against $1-2\beta = -0.2966, +0.1200, +0.1765, -0.0526$ at the zeros $0.648+85.569i$, $0.440+80.234i$, $0.412+82.019i$, $0.526+89.376i$ (Figure 13d). In this language a zero of a function with a pole is a pure state, stationary exactly when $\beta = \frac{1}{2}$, and RH says that every zero of ζ is a stationary pure state.

The family of pure states $s \mapsto W(s)/\|W(s)\|$ on $12 \leq n \leq 90$ has a Berry curvature [Be84], computed with gauge-invariant plaquettes on a grid of steps 0.0125 in σ and 0.025 in t (Figure 14b–d). Near a zero the state passes once around the Bloch sphere between the transient and the steady mode, so each zero carries one unit of flux: the flux/ 2π through $0.2 \leq \sigma \leq 0.95$, $10 \leq t \leq 50$ is 10.35, against 10 zeros, and rises in steps of 0.88 to 1.26 at the individual zeros. The zero of $\zeta + 5f$ at $\beta = 0.648$ carries its flux off the line. The flux counts zeros, as the argument principle does, but does not place them. Status: the splitting is an identity; the rest is numerical (quantum/qs.py, quantum/run.py, quantum/chan.py, quantum/berry.py).

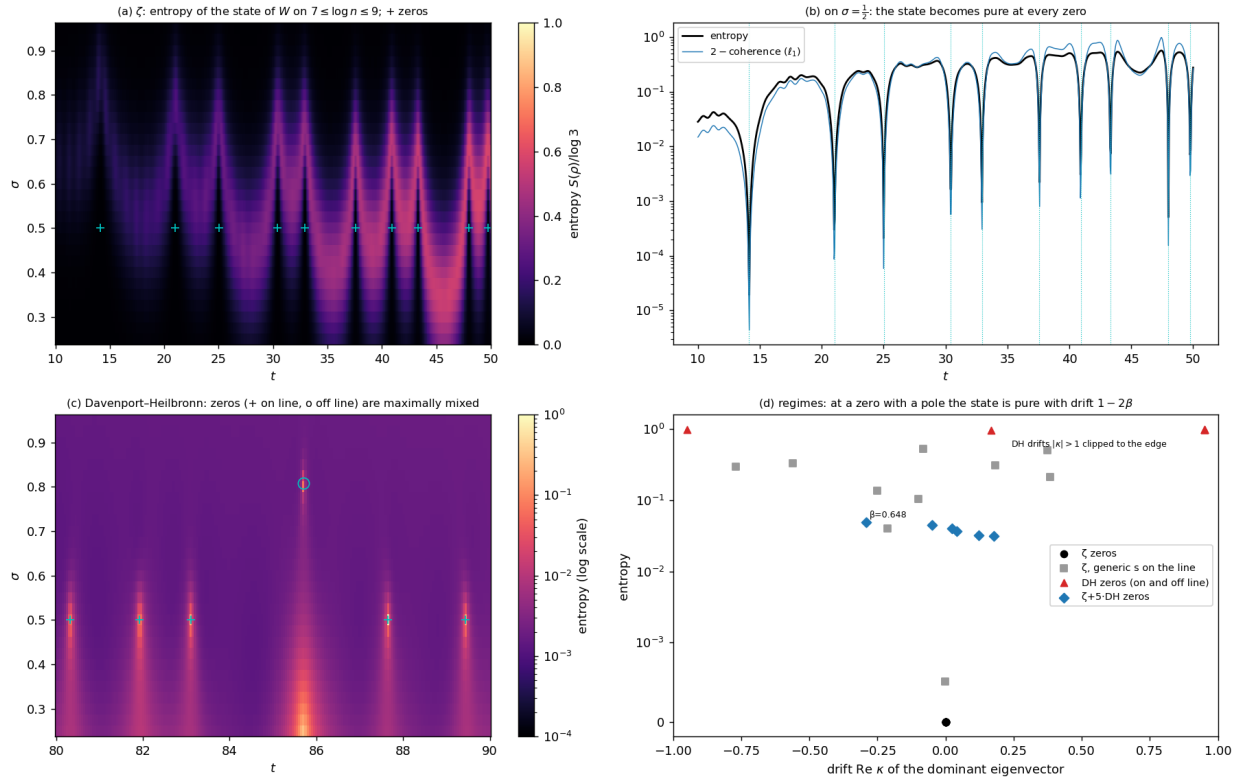


FIGURE 13. The wave as a quantum state, $7 \leq \log n \leq 9$. (a) Entropy $S(\rho)/\log 3$ for ζ in the (σ, t) plane; crosses: zeros. (b) Entropy and coherence on $\sigma = \frac{1}{2}$. (c) Entropy for the Davenport–Heilbronn function near its zero $0.8085 + 85.699i$ (circle) and its zeros on the line (crosses), logarithmic scale. (d) Entropy against the drift $\text{Re } \kappa$ at the zeros of ζ , at generic points of the line, at zeros of the Davenport–Heilbronn function (drifts beyond ± 1 drawn at the edge), and at the zeros of $\zeta + 5f$.

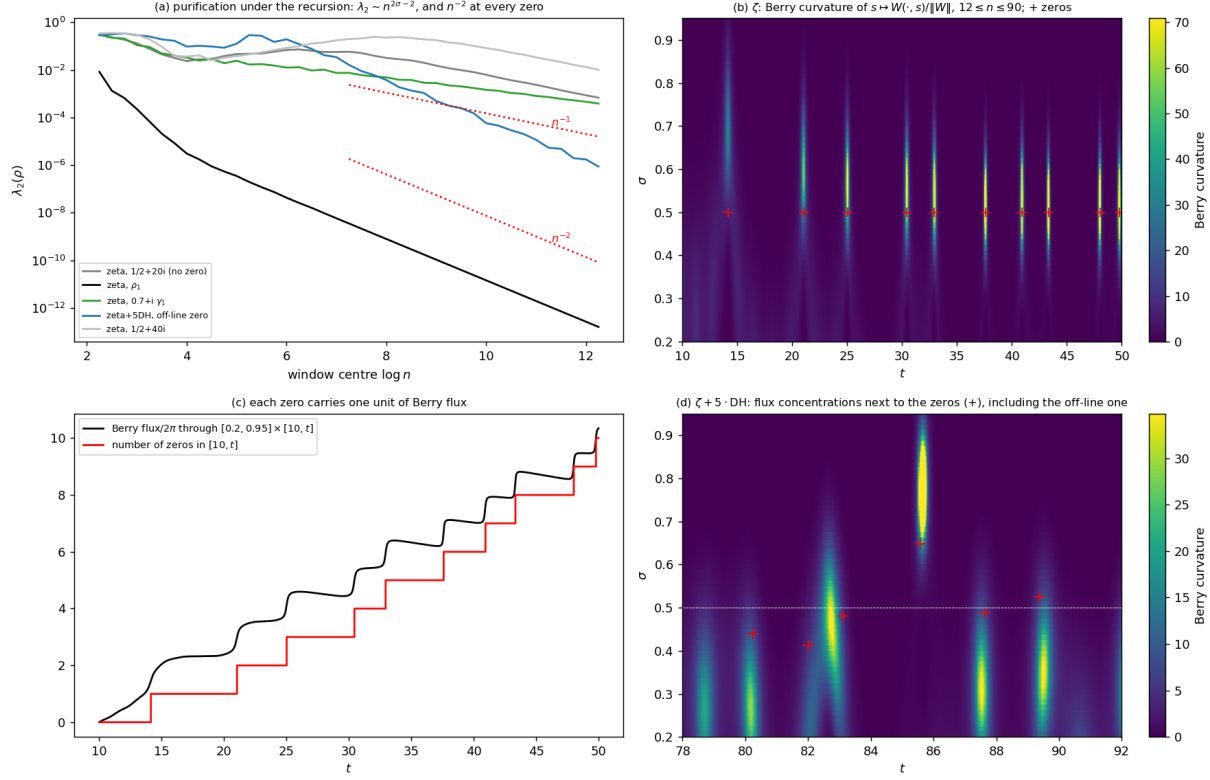


FIGURE 14. The recursion as a channel and the Berry curvature. (a) Second eigenvalue of ρ on windows of width 1.5 in $\log n$, at points with and without a zero. (b) Berry curvature of $s \mapsto W(s)/\|W(s)\|$, $12 \leq n \leq 90$, for ζ . (c) Flux/ 2π through $[0.2, 0.95] \times [10, t]$ and the number of zeros. (d) The same curvature for $\zeta + 5f$; crosses: its zeros.

11. TRANSPORT WITHOUT A CLOSED FINITE SPACE

A proof for all zeros needs either a limit at infinity or a compactification that brings infinity into a bounded region, and the computations of the series meet the same obstruction in both [CE16, CE11]. Where RH is proved, a third mechanism is used: an operation that transports the problem from one scale to the next and conserves a norm, as Frobenius does over function fields. Over $\mathbb{Q}$ the primes supply the operations: in the variable $\log x$ the dilation by p is a translation, which on the wave is the prime operator T_p of Remark 13, and every translation is an isometry of Weil's quadratic form [We], which depends on a test function only through its autocorrelation. What fails is closure: the margin of the form is governed by the difference set of the support of the test function, a single translate of a window lowers it by about eleven orders of magnitude, and no common degree can be imposed approximately on the numbers $\log p$. The isometry lemma, the 50-digit eigenvalues and the common-degree estimate are proved and computed in [CE12, Sect. 9] (Lemma 6 and Table 6 there). In the terms of this paper, the prime operators and Weil's norm exist and are compatible, and what is missing is a finite set of lags, closed under the prime translations, on which the norm can be checked once. Status: see [CE12].

12. DISCUSSION AND OPEN PROBLEMS

The Euler product does appear in the analytic continuation. Its partial products, corrected for the pole, converge in the strip exactly as far as the zeros allow, and their error is, zero by zero, the

sum of the modes $x^{\rho-s}/((s-\rho)\log x)$. The computations locate where the Euler product is used and where it is not.

Convergence is not the special ingredient. The logarithmic series of the Davenport–Heilbronn function converges where its own zeros allow (Section 2); what distinguishes ζ is the sign and support of $\Lambda(n)/\log n$. That sign is cancelled by the pole correction in every linear positivity argument inside the strip, the pole barrier of [CE16, Sect. 2].

From finite prime data the obstruction is uniformity, not resolution. With the primes up to 10^8 the Euler helix detects a symmetric displacement of about $4 \cdot 10^{-5}$ of a zero near a given height (a numerical detection threshold, not a bound), although symmetric off-line pairs enter only at second order (Remark 11); combined with the functional equation this can give exact local statements, as in Turing’s method, but it needs a list of the zeros and a number of primes that grows with the height.

Energies are expensive. Holomorphic sums over the zeros come linearly from the primes and do not see β ; non-holomorphic ones are reached only through energies, which see an off-line zero at height T only with the primes up to about $T^{1/\delta}$ (Section 7). The Laguerre hierarchy is a hierarchy of moments of the helix (Proposition 14).

Amplification needs an upper bound for the primes. Over $\mathbb{Q}$ the amplification that proves the analogue of RH for curves becomes the one-sided bound of [CE17, Prop. 11]. Sieves give upper bounds by positivity but stop at the parity barrier; removing it needs bilinear information, which the cross-energy does not provide.

The wave and the helix are one object, and the reflection pairs convergence with divergence. Sieving the partial sums by the primes up to y turns the free amplitude of the wave into the Euler helix (Remark 12); the primes act on all waves by the same real operator, and the functional equation, sieved, maps the half of the strip where the partial products converge onto the half where they always diverge (Remark 13).

Intrinsic quantities do not see t . The Euler state is a product state whose coherence and entanglement do not depend on t (Proposition 18); its zeros are orthogonality times with a vector outside the multiplicative structure, and they leave the line exactly when characters are superposed.

What is missing is closure. The prime translations are isometries of Weil’s form, but there is no finite set of lags closed under them (Section 11 and [CE12, Sect. 9]).

Two kinds of boundary at $\sigma = \frac{1}{2}$. The line $\sigma = \frac{1}{2}$ appears in three energies of this paper: the cross-energy C_2 of the integers [CE1], the cross-energy C_2^P of the primes (Proposition 8) and the damped energy $\mathcal{E}$ (Proposition 15, Remark 16). In the first two it comes from the growth $x^{1-2\sigma}$ of the pole term and holds at every point s , a zero or not. In the third it comes from the zeros. Theorem 6 links the Euler product to the first kind exactly; what is missing is a link from the first kind to the second, an inequality that turns information on the cross-energies into a bound for $\psi(x) - x$. Any such bound strong enough to give $\mathcal{E}(\varepsilon) < \infty$ for all $\varepsilon > 0$ is equivalent to RH.

We list questions that the computations suggest and that we cannot answer.

- Can the exact identities of Theorem 6 and Proposition 8 be combined into an unconditional bound for the damped energy $\mathcal{E}(\varepsilon)$, or for the low modes of Remark 16, that does not assume an estimate equivalent to RH?
- Is there a positive quantity built from the partial Euler products that does not pay the pole correction, for instance an average over twists in which the $t = 0$ term cancels?
- Can a Dirichlet polynomial with multiplicative coefficients resonate at one frequency with amplitude $N^{\beta-1/2}$, $\beta > \frac{1}{2}$? By [CE1, Prop. 2.9] this is equivalent to the negation of RH.
- Can the local verification of Remark 11 be made rigorous, with interval arithmetic and the symmetry argument of Turing, as a verification of RH near a given height from the primes alone?

- Is there any control of the Euler product on the divergent side $\sigma < \frac{1}{2}$ that is compatible with the prime operators and the functional equation?
- Is there, over $\mathbb{Q}$, a finite set of lags on which Weil’s form can be checked once and which is closed under the prime translations, or a substitute for it?

DATA AND CODE

The ancillary Python scripts (NumPy, SciPy, Matplotlib; mpmath where stated) reproduce the tables and figures; the file `anc/README.txt` lists, for each script, the section, table or figure it reproduces and its output. The computations are in double precision unless stated otherwise, without interval arithmetic; they illustrate the statements and are not used in the proofs. The ordinates of the zeros below 600 are in `anc/zeros600.npz`.

Provenance. This paper supersedes the preprint [C6] and contains the material of Section 6 of version 3 of [CP] on the Euler product that is not in the other parts of the series, together with the paragraph “Transport without a closed finite space” of Section 7 there, which is now proved and computed in [CE12, Sect. 9] and summarized in Section 11. The partial Euler products over $\mathbb{F}_q[T]$ and the material on amplification and parity are proved in [CE17] and summarized in Sections 2 and 8. The corrections with respect to those sources are the second term in Proposition 14, the sign of the pole term in Lemma 9 (which does not affect the computations), and the pole term in the sum of the prime increments at a zero (Section 3).

Use of AI tools. The proofs, computations and drafting of this paper, and of the preprints it absorbs, were carried out with the assistance of an AI system (Claude, Anthropic), under the author’s direction. The author checked the results and takes responsibility for them.

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