

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 2: THE TRANSITION AT THE CRITICAL LINE AND HARDY'S FUNCTION

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ABSTRACT. This is Part 2 of a series on the cross-energy $C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma}$ of the partial sums X_N of ζ . At a zero $\frac{1}{2} + it$, $C_2(\cdot, \rho)$ is monotone beyond $t/(2\pi - \eta)$, and its last decrease follows a Fresnel law $N_{\text{last}} = \frac{t}{2\pi} + \kappa\sqrt{t} + O(1)$ with $\kappa = -0.2077978646\dots$, exact in floor form for 119 of 120 zeros tested; beyond it the crossing is unique. The windows of monotonicity near a zero involve $\zeta^{(m)}(\rho)$, and their total over the zeros is Gonek's sum $J_{-1}(T)$. On the critical line the helix of $X_{N-1}(s)N^{-\bar{s}}$ has a natural orientation whose signed radius is Hardy's $Z(t)$, phase-locking at $\log N$ extracts $2Z(t)$, and the Riemann–Siegel formula and Hardy's theorem read naturally in these coordinates; a scan of Z up to 10^5 shows no extremum without a crossing, while the Davenport–Heilbronn function has eight. The cross-energy of a mollified series isolates the mechanism of Levinson and Conrey and shows why it stops at a positive proportion. We do not prove the Riemann hypothesis.

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1. INTRODUCTION

Let $X_N(s) = \sum_{k \leq N} k^{-s}$ be the partial sums of the Dirichlet series of ζ , with $s = \sigma + it$. The *cross-energy* of consecutive partial sums is

$$C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} (X_{N-1}(s) N^{-\bar{s}}) = 2N^{-\sigma} \sum_{n < N} n^{-\sigma} \cos \left(t \log \frac{N}{n} \right).$$

It records how the newest term N^{-s} interacts with all the previous ones. This part studies C_2 near the critical line: the transition of $C_2(\cdot, \rho)$ at a zero, its monotonicity windows, and the reading of Hardy's function, the Riemann–Siegel formula and the mollifier methods in cross-energy coordinates.

Throughout, $\rho = \beta + i\gamma$ is a nontrivial zero of ζ , $\Theta = \sup \beta$, $\lambda = s(1-s)$, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, and

$$C_B = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots,$$

the sum taken in symmetric order [Da]; under RH, $\rho(1-\rho) = |\rho|^2 = \frac{1}{4} + \gamma^2$. We write $Y_N(s) = N^{1-s}/(1-s)$ for the continuous envelope of X_N and $X_N^C = X_N - \frac{1}{2}N^{-s}$.

This paper is Part 2 of the series *Cross-energy of Dirichlet partial sums* (seventeen parts; the plan of the series is in Part 1 [CE1]). It uses the notation and the basic results of Part 1, in particular [CE1, Theorem A], which are cited with the prefix [CE1].

1.1. Main results. The transition at the critical line (Section 2). At a zero $\frac{1}{2} + it$, $C_2(\cdot, \rho)$ is monotone for $N \geq \max(N_\eta, t/(2\pi - \eta))$ (Theorem 2.5), and the last decrease follows a Fresnel law, $N_{\text{last}} = \frac{t}{2\pi} + \kappa\sqrt{t} + O(1)$ with $\kappa = -0.2077978646\dots$ (Theorem 2.10), exact in floor form for 119 of 120 zeros tested; the crossing is unique beyond N_{last} (Corollary 2.13). The window law for the monotonicity of C_2 near a zero involves $\zeta^{(m)}(\rho)$, and its total over the zeros is Gonek's sum $J_{-1}(T)$ (Theorem 2.18, Proposition 2.20).

Hardy's function (Section 3). On the critical line the helix of $X_{N-1}(s)N^{-\bar{s}}$ has a natural orientation whose signed radius is $Z(t)$, and phase-locking at $\log N$ extracts $2Z(t)$ (Proposition 3.1, Theorem 3.2); the Riemann–Siegel formula and Hardy's theorem read naturally in these coordinates, and the scan of Z up to 10^5 shows no extremum without a crossing. The cross-energy of the mollified series isolates the Levinson–Conrey mechanism (Theorem 3.9).

1.2. What is new and what is classical. The Fresnel transition law, the monotonicity windows and their relation with Gonek's sum, the orientation of the helix, phase-locking and the mollified cross-energy are, as far as we know, new in this form, with complete proofs except for Lemma 2.8, which is verified numerically. Hardy's function, the Riemann–Siegel formula, Turing's method, Hardy's theorem and the mollifier methods of Levinson and Conrey are classical and cited as such. We do not prove the Riemann hypothesis.

2. THE TRANSITION AT THE CRITICAL LINE

At a zero $\rho = \frac{1}{2} + it$ of ζ on the critical line, Part 1 [CE1] shows that $C_2(n, \rho)$ is eventually strictly increasing, that it crosses the axis once at $n^* = \lambda + \frac{1}{12} + \frac{1}{180\lambda} + \dots$ with $\lambda = \rho(1-\rho) = \frac{1}{4} + t^2$, and that it tends to $1/\lambda$. The effective range given there for the monotone expansion is $n \gtrsim 0.057 t^{5/3}$ ([CE1, Corollary 3.5(ii)]), and the crossing law is proved for $n \geq t^{3/2}$ ([CE1, Proposition 3.6]). Numerically, however, $C_2(\cdot, \rho)$ is increasing from just below $n = t/2\pi$, the length of the approximate functional equation. This section explains that threshold and proves what happens there. At a zero the head of the Dirichlet series is minus its tail, and Poisson summation of the tail has a stationary point exactly when $n < t/2\pi$. Above the threshold every Fourier mode is non-stationary and C_2 is monotone

(Theorem 2.5). In a window of width $\sqrt{t}$ around it the stationary mode produces a Fresnel profile, which locates the last decrease at $t/2\pi + \kappa\sqrt{t} + O(1)$ with a universal constant $\kappa = -0.2077978646 \dots$ (Theorem 2.10). The second half of the section collects three further pieces of finite- n geometry: the maximum that a hypothetical zero off the line would produce, the monotonicity windows just off a zero, which measure $|\zeta^{(m)}(\rho)|$ and the multiplicity, and the pole-free difference energy of the integers and the primes, with three filters that any rigidity statement has to pass.

Throughout, $\psi(u) = \{u\} - \frac{1}{2}$ is the sawtooth, $\zeta(s, x) = \sum_{n \geq 0} (n+x)^{-s}$ is the Hurwitz zeta function, and $C_2(x, s)$ for real $x \geq 1$ is the Hurwitz extension of [CE1, Proposition 3.6], defined through $X_{x-1}(s) = \zeta(s) - \zeta(s, x)$. Until the subsection on off-line zeros, $\rho = \frac{1}{2} + it$ with $t > 0$ and $\zeta(\rho) = 0$, and

$$u_0 = \frac{t}{2\pi}, \quad w(x) = \frac{\pi\sqrt{2}(x - u_0)}{\sqrt{t}}, \quad a(u) = u^{-3/2}, \quad \varphi_k(u) = 2\pi ku - t \log u. \quad (2.1)$$

2.1. The tail at a zero. Proposition 2.1 (tail representation). Let $\zeta(\rho) = 0$. For every real $x \geq 1$,

$$C_2(x, \rho) = -2 \operatorname{Re} [x^{-\bar{\rho}} \zeta(\rho, x)]; \quad (2.2)$$

for integers N this reads $C_2(N, \rho) = -2 \operatorname{Re} [N^{-\bar{\rho}} \sum_{n \geq N} n^{-\rho}]$, the tail being understood by analytic continuation. At a zero the accumulated head $X_{N-1}(\rho)$ of the Dirichlet series is exactly minus its tail.

Proof. $C_2(x, s) = 2 \operatorname{Re} [X_{x-1}(s) x^{-\bar{s}}]$ and $X_{x-1}(\rho) = \zeta(\rho) - \zeta(\rho, x) = -\zeta(\rho, x)$. $\square$

Status: proved.

The first-order Euler–Maclaurin formula for the Hurwitz function is exact:

$$\zeta(s, x) = \frac{x^{1-s}}{s-1} + \frac{1}{2}x^{-s} - s \int_0^\infty \psi(v) (x+v)^{-s-1} dv \quad (\sigma > 0, x > 0).$$

On $\sigma = \frac{1}{2}$ the first two terms give $-2 \operatorname{Re} [x^{-\bar{s}} x^{1-s} / (s-1)] = 2 \operatorname{Re} \frac{1}{1-s} = \frac{1}{\lambda}$ and $-2 \operatorname{Re} [\frac{1}{2} x^{-\bar{s}} x^{-s}] = -\frac{1}{x}$, with no oscillating factor, because $x^{-\bar{s}} x^{-s} = x^{-1}$. Everything else comes from the sawtooth, whose Fourier series $\psi(v) = -\sum_{k \neq 0} e^{2\pi i k v} / (2\pi i k)$ converges boundedly. Since $e^{2\pi i k(u-x)} u^{-\rho-1} = e^{-2\pi i k x} a(u) e^{i\varphi_k(u)}$ and $x^{-\bar{\rho}} e^{-2\pi i k x} e^{i\varphi_k(x)} = x^{-1/2}$, the following holds.

Lemma 2.2 (Fourier–tail representation). Let $\zeta(\rho) = 0$, $\rho = \frac{1}{2} + it$. For every real $x \geq 1$,

$$C_2(x, \rho) = \frac{1}{\lambda} - \frac{1}{x} + \sum_{k \neq 0} E_k(x), \quad E_k(x) = -2 \operatorname{Re} \left[\frac{\rho}{2\pi i k} x^{-1/2} \mathcal{I}_k[a](x) \right], \quad (2.3)$$

where $\mathcal{I}_k[f](x) = \int_x^\infty f(u) e^{i(\varphi_k(u) - \varphi_k(x))} du$. Each E_k is a smooth function of x with no oscillating factor, and the series converges absolutely for every $x \geq 1$, uniformly on $x \geq \frac{3}{4}u_0$.

Proof. Insert the Fourier series of ψ in the exact formula above; term-by-term integration is justified because $u^{-3/2} \in L^1[x, \infty)$ and the partial sums of the Fourier series are uniformly bounded. Then apply (2.2) and the two evaluations above. For $|k| > t/(\pi x)$ the phase satisfies $|\varphi'_k| \geq \pi|k|$ on $[x, \infty)$, and the bounds in the proof of Lemma 2.6 below give $E_k(x) = O(x^{-2}k^{-2} + t x^{-3}|k|^{-3})$, which is summable; on $x \geq \frac{3}{4}u_0$ this holds for every $k \neq 0, 1$. $\square$

Status: proved.

The threshold. The phase of $\mathcal{I}_k$ has derivative $\varphi'_k(u) = 2\pi k - t/u$. For $u \geq x$ it vanishes for some $k \neq 0$ if and only if $x < u_0 = t/2\pi$, and then only for $k = 1$, at $u = u_0$. So $N = t/2\pi$ is the exact threshold: above it every mode is non-stationary and can be expanded by integration by parts; below it the mode $k = 1$ has a stationary point, which produces the dual sum of the approximate functional equation. All the estimates below rest on three facts about the integrals $\mathcal{I}_k$.

Lemma 2.3 (oscillatory integrals). Let $\varphi = \varphi_k$ with $\varphi' \neq 0$ on $[x, \infty)$, put $Df = (f/\varphi)'$, $X = 1/(u\varphi')$, $Y = \varphi''/\varphi'^2 = t/(u^2\varphi'^2)$ and $\varepsilon(u) = |X(u)| + |Y(u)|$, and write $\mathcal{I} = \mathcal{I}_k$. For smooth f with $f, f' = O(u^{-3/2})$:

$$\mathcal{I}[f](x) = \frac{if(x)}{\varphi'(x)} + i\mathcal{I}[Df](x), \quad \frac{d}{dx}\mathcal{I}[f](x) = \varphi'(x)\mathcal{I}[Df](x). \quad (2.4)$$

Moreover $D^m a = a P_m(X, Y)$ with P_m a homogeneous polynomial of degree m whose coefficients depend only on m , so that $|D^m a| \leq C_m a \varepsilon^m$, and

$$\mathcal{I}[a] = \frac{ia}{\varphi'} - \frac{Da}{\varphi'} - \frac{iD^2a}{\varphi'} - i\mathcal{I}[D^3a], \quad |\mathcal{I}[D^m a](x)| \leq C_m \int_x^\infty a \varepsilon^m du. \quad (2.5)$$

Proof. The first identity in (2.4) is one integration by parts, with $e^{i\varphi} = (e^{i\varphi})'/(i\varphi')$. Differentiating under the integral gives $\frac{d}{dx}\mathcal{I}[f] = -f(x) - i\varphi'(x)\mathcal{I}[f](x)$; inserting the first identity, the terms $\pm f(x)$ cancel. For the symbol calculus, $a'/\varphi' = -\frac{3}{2}aX$, $X'/\varphi' = -X^2 - XY$ and $Y'/\varphi' = -2XY - 2Y^2$ (using $\varphi'' = t/u^2$), so $D(aP) = a[-\frac{3}{2}XP - YP - (X^2 + XY)\partial_X P - (2XY + 2Y^2)\partial_Y P]$, again a times a homogeneous polynomial of one degree more; induct on m . (2.5) is the first identity of (2.4) applied three times. $\square$

Status: proved.

The second identity in (2.4) is the key point: an x -derivative of one of these integrals is one order *smaller* than the integral, so no oscillating factor appears after differentiation.

2.2. Monotonicity above the threshold. Lemma 2.4 (uniform expansion above the threshold).

Let $0 < \eta < 2\pi$. Uniformly for zeros $\rho = \frac{1}{2} + it$ and real $x \geq t/(2\pi - \eta)$,

$$C_2(x, \rho) = \frac{1}{\lambda} - \frac{1}{x} + \frac{\Upsilon(t/x)}{x^2} + O_\eta(x^{-3}), \quad (2.6)$$

where, for $0 < \tau < 2\pi$,

$$\Upsilon(\tau) = -\left[\frac{1}{4\sin^2(\tau/2)} + \frac{1}{\tau^2} - \frac{\tau \cos(\tau/2)}{4\sin^3(\tau/2)}\right]. \quad (2.7)$$

Υ extends analytically to $\tau = 0$ with $\Upsilon(\tau) = -\frac{1}{12} - \frac{\tau^2}{80} + O(\tau^4)$, and $\Upsilon(\tau) \sim -\frac{\pi}{2} \sin^{-3}(\tau/2) \rightarrow -\infty$ as $\tau \uparrow 2\pi$.

Proof. Put $\tau = t/x$, so $\varphi'_k(x) = 2\pi k - \tau$. For $u \geq x$ we have $|\varphi'_1(u)| \geq \eta$ and $|\varphi'_k(u)| \geq \pi|k|$ for $k \neq 0, 1$, hence $\varepsilon(u) \leq C_\eta/(|k|u)$ for every $k \neq 0$. Insert (2.5) into (2.3) and take the four terms in turn. The term ia/φ' contributes $-2\operatorname{Re}[\rho/(2\pi k)]x^{-2}/(2\pi k - \tau) = -x^{-2}/(2\pi k(2\pi k - \tau))$, since $\operatorname{Re} \rho = \frac{1}{2}$. The term $-Da/\varphi'$ is real, so only $\operatorname{Re}[\rho/(2\pi ik)] = t/(2\pi k)$ survives; with $Da(x) = -x^{-5/2}[\frac{1}{2}(2\pi k - \tau)^{-1} + 2\pi k(2\pi k - \tau)^{-2}]$ it contributes $-x^{-2}[\tau(2\pi k)^{-1}(2\pi k - \tau)^{-2} + 2\tau(2\pi k - \tau)^{-3}]$. The term $-iD^2a/\varphi'$ carries no factor t and contributes $O_\eta(x^{-4}|k|^{-3})$, and the remainder contributes $O(|\rho|x^{-1/2}|k|^{-1} \int_x^\infty a \varepsilon^3) = O_\eta(t x^{-4}|k|^{-4}) = O_\eta(x^{-3}|k|^{-4})$. All four are summable over k . The k -sums of the first two terms are evaluated with $\sum_{k \in \mathbb{Z}} (\tau - 2\pi k)^{-1} = \frac{1}{2} \cot \frac{\tau}{2}$, $\sum_{k \in \mathbb{Z}} (\tau - 2\pi k)^{-2} = \frac{1}{4} \csc^2 \frac{\tau}{2}$ and its derivative $\sum_{k \in \mathbb{Z}} (\tau - 2\pi k)^{-3} = \frac{1}{8} \csc^2 \frac{\tau}{2} \cot \frac{\tau}{2}$, after the partial fractions $\frac{1}{a(a-\tau)} = \frac{1}{\tau}(\frac{1}{a-\tau} - \frac{1}{a})$ and $\frac{1}{a(a-\tau)^2} = \frac{1}{\tau^2}(\frac{1}{a} - \frac{1}{a-\tau}) + \frac{1}{\tau(a-\tau)^2}$ with $a = 2\pi k$. The first sum is $-x^{-2}[\tau^{-2} - \frac{1}{2\tau} \cot \frac{\tau}{2}]$, the second is $-x^{-2}[\frac{1}{2\tau} \cot \frac{\tau}{2} + \frac{1}{4} \csc^2 \frac{\tau}{2} - \frac{\tau}{4} \cos \frac{\tau}{2} \csc^3 \frac{\tau}{2}]$, and their sum is $x^{-2}\Upsilon(\tau)$. The expansion at $\tau = 0$ follows from $\frac{x^2}{\sin^2 x} = 1 + \frac{x^2}{3} + \frac{x^4}{15} + \dots$ and $\frac{x \cos x}{\sin x} = 1 - \frac{x^2}{3} - \frac{x^4}{45} - \dots$. $\square$

Status: proved.

The limit $\tau \rightarrow 0$ recovers the fixed- t Euler–Maclaurin expansion [CE1, (3.12)] of [CE1, Corollary 3.5] (and [CE1, (3.16)] of [CE1, Proposition 3.8]): $\Upsilon(0)/x^2 = -\frac{1}{12x^2}$ is its Bernoulli term, and $-\frac{\tau^2}{80x^2} = -\frac{t^2}{80x^4}$ is the leading part of $\operatorname{Re}(\rho)_3/(360x^4)$. Lemma 2.4 is that expansion resummed in the variable $\tau = t/x$, which stays uniform up to $\tau = 2\pi - \eta$ (Table 1).

zero t	$x = 1.3 u_0$	$x = 2 u_0$	$x = 3 u_0$
1001.349483	-3.7367 / -3.7465	-0.3505 / -0.3505	-0.1583 / -0.1583
5000.834381	-3.7008 / -3.7012	-0.3512 / -0.3512	-0.1582 / -0.1582

TABLE 1. $N^2(C_2(N, \rho) - \frac{1}{\lambda} + \frac{1}{N})$ against $\Upsilon(t/N)$ (Lemma 2.4), at the integer N nearest to $1.3u_0, 2u_0, 3u_0$. The deviation is of the size $O(1/N)$ of the remainder.
Script: `anc/fresnel/fresnel_transition.py` `upilon`.

Theorem 2.5 (monotonicity above the threshold). For every $\eta \in (0, 2\pi)$ there is N_η such that, for every zero $\rho = \frac{1}{2} + it$ with $t > 0$,

$$C_2(N+1, \rho) > C_2(N, \rho) \quad \text{for all integers } N \geq \max\left(N_\eta, \frac{t}{2\pi - \eta}\right). \quad (2.8)$$

In particular, for $t \geq (2\pi - \eta)N_\eta$ the trajectory is strictly increasing on the whole range $N \geq t/(2\pi - \eta)$.

Proof. By (2.6) at N and $N+1$, $\Delta C_2(N) = \frac{1}{N(N+1)} + \Delta[\Upsilon(t/N)N^{-2}] + O_\eta(N^{-3})$. By the mean value theorem $\Delta[\Upsilon(t/N)N^{-2}] = -\xi^{-3}(2\Upsilon(\tau_\xi) + \tau_\xi \Upsilon'(\tau_\xi))$ for some $\xi \in (N, N+1)$, $\tau_\xi = t/\xi$. Since Υ and Υ' are bounded on $[0, 2\pi - \eta]$, this is $O_\eta(N^{-3})$, and $\Delta C_2(N) = N^{-2}(1 + O_\eta(N^{-1})) > 0$ for $N \geq N_\eta$. $\square$

Status: proved. The constant N_η is not made explicit.

Theorem 2.5 replaces “eventually” in [CE1, Corollary 3.5] by a range proportional to t . The effective threshold of [CE1, Corollary 3.5(ii)], of order $0.057 t^{5/3}$, is weaker for large t but explicit; together they give an explicit range for small t and the sharp order $t/2\pi$ for large t . Near the threshold both Υ and Υ' are negative, so the correction $-\xi^{-3}(2\Upsilon + \tau \Upsilon')$ is *positive* and reinforces monotonicity; the expansion (2.6) loses its uniformity only when $x - u_0$ is of order $\sqrt{t}$, where the stationary point of φ_1 enters. That window is treated next.

2.3. The Fresnel window. Lemma 2.6 (non-stationary modes). Uniformly for zeros $\rho = \frac{1}{2} + it$ with $t \geq 1$ and real $x \geq \frac{3}{4}u_0$,

$$\sum_{k \neq 0,1} E_k(x) = O(t^{-2}), \quad \frac{d}{dx} \sum_{k \neq 0,1} E_k(x) = O(t^{-3}). \quad (2.9)$$

Proof. For $u \geq x \geq \frac{3}{4}u_0$ we have $t/u \leq \frac{8\pi}{3}$, hence $|\varphi'_k(u)| \geq \frac{2\pi}{3}|k|$ for $k \neq 0, 1$, $|X| \leq \frac{3}{2\pi|k|u}$, $|Y| \leq \frac{6}{\pi k^2 u}$ and $\varepsilon(u) \leq c_0/(|k|u)$ with $c_0 < 2.5$. Insert (2.5) into (2.3). The term ia/φ' contributes $-x^{-2}/(2\pi k \varphi'_k(x)) = O(x^{-2}k^{-2})$; the real term $-Da/\varphi'$ is multiplied by $t/(2\pi k)$ and contributes $O(t x^{-3}|k|^{-3})$; the term $-iD^2a/\varphi'$ contributes $O(x^{-4}|k|^{-4})$; the remainder contributes $O(t x^{-4}|k|^{-4})$. Summing over k gives $O(t^{-2})$. For the derivative, the three explicit terms are rational functions of x and $\varphi'_k(x)$, and each differentiation gains a factor $O(1/x)$; for the remainder, (2.4) gives $\frac{d}{dx}(x^{-1/2}\mathcal{I}[D^3a]) = O(|k|x^{-1/2} \int_x^\infty a\varepsilon^4) + O(x^{-3/2} \int_x^\infty a\varepsilon^3) = O(x^{-5}|k|^{-3})$. Summing over k gives $O(t x^{-4} + x^{-3}) = O(t^{-3})$. $\square$

Status: proved.

Put

$$F(w) = \int_w^\infty e^{iw^2} dv, \quad e^{-iw^2} F(w) = \Phi(w) + i\tilde{\Phi}(w),$$

$$\Phi_1(w) = 2w \Phi(w) + \frac{8}{3}w^3 \tilde{\Phi}(w) - \frac{4}{3}w^2. \quad (2.10)$$

Since $F' = -e^{iw^2}$, the pair $(\Phi, \tilde{\Phi})$ solves $\Phi' = 2w\tilde{\Phi} - 1$, $\tilde{\Phi}' = -2w\Phi$, with $\Phi(0) = \tilde{\Phi}(0) = \frac{\sqrt{\pi}}{2\sqrt{2}}$.

Proposition 2.7 (two-term Fresnel expansion). For every $W > 0$, uniformly for zeros $\rho = \frac{1}{2} + it$ with $t \geq t_0(W)$ and real x with $|w(x)| \leq W$,

$$C_2(x, \rho) = \frac{1}{\lambda} - \frac{1}{x} - \frac{2\sqrt{2}}{\sqrt{t}} \Phi(w) + \frac{\Phi_1(w)}{t} + O_W(t^{-3/2}), \quad w = w(x), \quad (2.11)$$

and the same expansion holds after differentiation in x , with $\frac{d}{dx} = \frac{\pi\sqrt{2}}{\sqrt{t}} \frac{d}{dw}$ and error $O_W(t^{-2})$.

Proof. By Lemma 2.6 only E_1 matters at this order. Put $u = u_0(1 + y)$; then $\varphi_1(u) - \varphi_1(u_0) = t q(y)$ with $q(y) = y - \log(1 + y)$. The map $\mu = \text{sign}(y)\sqrt{q(y)}$ is an analytic increasing bijection of $(-1, \infty)$ onto $\mathbb{R}$, with $y = \sqrt{2}\mu + \frac{2}{3}\mu^2 + O(\mu^3)$. With $v = \sqrt{t}\mu$, exactly,

$$\int_x^\infty a e^{i\varphi_1} du = \frac{e^{i\varphi_1(u_0)}}{\sqrt{u_0 t}} \int_{v_x}^\infty g(v/\sqrt{t}) e^{iv^2} dv, \quad g(\mu) = (1 + y(\mu))^{-3/2} y'(\mu), \quad (2.12)$$

where g does not depend on t , is analytic on the relevant range $\mu \geq \mu(-\frac{1}{4})$, satisfies $g^{(j)}(\mu) \ll (1 + |\mu|)^{-2-j}$, and has $g(\mu) = \sqrt{2} - \frac{5}{3}\mu + g_2\mu^2 + \mu^3\tilde{g}(\mu)$ with $\tilde{g}^{(j)} \ll (1 + |\mu|)^{-5-j}$. The polynomial part integrates in closed form in the Abel sense, $\int_\ell^\infty e^{iv^2} dv = F(\ell)$, $\int_\ell^\infty v e^{iv^2} dv = \frac{i}{2} e^{i\ell^2}$, $\int_\ell^\infty v^2 e^{iv^2} dv = \frac{i}{2}(\ell e^{i\ell^2} + F(\ell))$. For the rest, integrate $t^{-3/2} \int_{v_x}^\infty v^3 \tilde{g}(v/\sqrt{t}) e^{iv^2} dv$ by parts twice with $v e^{iv^2} = (e^{iv^2})'/2i$: the boundary terms are $O((1 + v_x^2)t^{-3/2})$, and the remaining integrals are $t^{-3/2} \int |\tilde{g}'| d\mu$ and $t^{-3/2} \int |\tilde{g}' + \mu\tilde{g}''| d\mu$, both $O(t^{-3/2})$. Hence, uniformly for $|v_x| \leq 2W$,

$$\int_{v_x}^\infty g\left(\frac{v}{\sqrt{t}}\right) e^{iv^2} dv = \sqrt{2} F(v_x) - \frac{5i}{6\sqrt{t}} e^{iv_x^2} + O_W(t^{-1}).$$

Now expand the remaining factors to relative order $t^{-1/2}$, uniformly for $|w| \leq W$, with $y_x = (x - u_0)/u_0 = \sqrt{2}w/\sqrt{t}$: $v_x = \sqrt{t}\mu(y_x) = w - \frac{\sqrt{2}}{3} \frac{w^2}{\sqrt{t}} + O(t^{-1})$, so $F(v_x) = F(w) + \frac{\sqrt{2}}{3} \frac{w^2}{\sqrt{t}} e^{iw^2} + O(t^{-1})$; $x^{-1/2} = u_0^{-1/2}(1 - \frac{w}{\sqrt{2t}} + O(t^{-1}))$; $e^{-i(\varphi_1(x) - \varphi_1(u_0))} = e^{-itq(y_x)} = e^{-iw^2}(1 + \frac{2\sqrt{2}}{3} \frac{iw^3}{\sqrt{t}} + O(t^{-1}))$; and $\rho/(2\pi i) = u_0(1 + O(t^{-1}))$. Inserting all of this in $E_1 = -2 \text{Re} \left[\frac{\rho}{2\pi i} x^{-1/2} e^{-i\varphi_1(x)} \int_x^\infty a e^{i\varphi_1} du \right]$ gives

$$E_1(x) = -\frac{2}{\sqrt{t}} \text{Re} \left[\left(1 - \frac{w}{\sqrt{2t}}\right) \left(1 + \frac{2\sqrt{2} iw^3}{3\sqrt{t}}\right) \left(\sqrt{2}(\Phi + i\tilde{\Phi}) + \frac{1}{\sqrt{t}}\left(\frac{2}{3}w^2 - \frac{5i}{6}\right)\right) \right] + O_W(t^{-3/2}),$$

whose terms of order $t^{-1/2}$ and t^{-1} are $-\frac{2\sqrt{2}}{\sqrt{t}}\Phi$ and $\frac{1}{t}(2w\Phi + \frac{8}{3}w^3\tilde{\Phi} - \frac{4}{3}w^2) = \frac{\Phi_1}{t}$. Together with Lemma 2.2 this is (2.11). For the derivative, $V(x) = x^{-1/2}\mathcal{I}_1[a](x)$ satisfies exactly, by (2.4), $V' = -(\frac{1}{2x} + i\varphi_1'(x))V - x^{-2}$, where $|\varphi_1'(x)| \ll W/\sqrt{t}$ in the window; substituting the expansion of V just obtained gives that of E_1' with error $O_W(t^{-2})$, and Lemma 2.6 handles the other modes. $\square$
Status: proved.

Lemma 2.8 (the Fresnel profile).

- (a) The largest zero of Φ' is $w^* = -0.9232215874\dots$; it is simple, with $\Phi''(w^*) = -6.66323\dots$ and $\Phi(w^*) = 1.63669\dots$
- (b) $\Phi' > 0$ on (w_2, w^*) , where $w_2 = -1.980395\dots$ is the next zero.
- (c) $\Phi' < 0$ on (w^*, ∞) . More precisely, for $w \geq 1.6$,

$$\Phi'(w) = -\frac{3}{4w^4} + r(w), \quad |r(w)| \leq \frac{15}{8w^6}. \quad (2.13)$$

Proof. For $w \geq 1.6$, four integrations by parts give $F(w) = e^{iw^2}[\frac{i}{2w} + \frac{1}{4w^3} - \frac{3i}{8w^5}] + \frac{15i}{8} \int_w^\infty e^{iv^2} v^{-6} dv$, and one more integration by parts of the last integral bounds the remainder in (2.13); in particular $\Phi' < 0$ for $w > \sqrt{5}/2$. On the compact interval $[w^* + 0.05, 1.6]$, a grid of $2 \cdot 10^5$ points gives $\max \Phi' \leq -0.0659$ with $\max |\Phi''| \leq 5.76$; since the grid step times $\max |\Phi''|$ is $7 \cdot 10^{-5}$, this gives $\Phi' < 0$ there. On $[w^*, w^* + 0.05]$, $\Phi'' \leq -5.75$, so Φ' is negative to the right of its simple zero. Parts (a) and (b) are checked in the same way: on $[w_2 + 0.05, w^* - 0.05]$ the

minimum of Φ' is 0.356 with $|\Phi''| \leq 25.2$ (step $4.8 \cdot 10^{-6}$), $\Phi'' \leq -6.66$ on $[w^* - 0.05, w^*]$ and $\Phi'' \geq 25.1$ on $[w_2, w_2 + 0.05]$. $\square$

Status: proved for $w \geq 1.6$; on the compact interval $[w_2, 1.6]$ the statements are verified numerically in double precision on a grid whose spacing is controlled by a bound on $|\Phi''|$, with a margin of four orders of magnitude, not in interval arithmetic (`anc/fresnel/fresnel_transition.py constants`).

Lemma 2.9 (monotonicity beyond the window). There are absolute constants W_0 and t_0 such that, for every zero $\rho = \frac{1}{2} + it$ with $t \geq t_0$, the Hurwitz extension satisfies $\frac{d}{dx}C_2(x, \rho) > 0$ for every real x with $w(x) \geq W_0$. In particular $\Delta C_2(N, \rho) > 0$ whenever $w(N) \geq W_0$.

Proof. Let $x = u_0 + \delta$ with $\delta \geq W_0 \sqrt{t}/(\pi \sqrt{2})$. For $u \geq x$, $\varphi'_1(u) = 2\pi(u - u_0)/u > 0$, $X = \frac{1}{2\pi(u - u_0)}$, $Y = \frac{t}{4\pi^2(u - u_0)^2}$, and $\varepsilon(u) \leq \varepsilon(x) \leq \frac{1}{2w^2} + \frac{1}{\sqrt{2}tw}$. Split E_1 by (2.5) into $E^{(0)} + E^{(1)} + E^{(2)} + E^{(R)}$. The first two are explicit:

$$E^{(0)} = -\frac{1}{4\pi^2 x \delta}, \quad E^{(1)} = -\frac{t}{\pi x} \left[\frac{3}{8\pi^2 \delta^2} + \frac{t}{8\pi^3 \delta^3} \right].$$

Both are negative and increase towards 0; in particular $(E^{(0)})' > 0$ and

$$(E^{(1)})' \geq \frac{3t}{4\pi^3 \delta^4}. \quad (2.14)$$

The piece $E^{(2)}$ carries no factor t and its derivative is $O(\varepsilon t^{-1})(E^{(1)})'$. By (2.4)–(2.5), $|(E^{(R)})'| \ll t(x^{-3/2} \int_x^\infty a\varepsilon^3 + x^{-1/2} \varphi'_1(x) \int_x^\infty a\varepsilon^4)$, and with $\int_x^\infty a(u)(u - u_0)^{-n} du \leq x^{-3/2} \delta^{1-n}/(n-1)$ a direct comparison with (2.14) gives $|(E^{(R)})'| \leq C(W_0^{-2} + t^{-1/2})(E^{(1)})'$. By Lemma 2.6 the other modes contribute $O(tx^{-4} + x^{-3}) = o(x^{-2})$ to the derivative. Therefore

$$\frac{d}{dx}C_2(x, \rho) \geq \frac{1}{x^2}(1 - o(1)) + (E^{(0)})' + \left(1 - \frac{C}{W_0^2} - \frac{C}{\sqrt{t}}\right)(E^{(1)})' > 0$$

for W_0 and t large, and $\Delta C_2(N) = \int_N^{N+1} \frac{d}{dx}C_2(x, \rho) dx > 0$. $\square$

Status: proved. At the zero $t = 20000.128165\dots$ the ratio of $\Delta C_2(N)$ to the explicit value $x^{-2} + (E^{(0)} + E^{(1)})'$ at $x = N + \frac{1}{2}$ is 0.913, 0.987, 1.000, 1.004, 0.996, 0.987 for $w = 3, 5, 10, 20, 50, 150$ [C1]: the deviation at small w has the predicted size $O(w^{-2})$, and at large w it comes from the modes $k \neq 0, 1$.

The range of Lemma 2.9 contains $x \geq t/(2\pi - \eta)$, so it gives a second proof of Theorem 2.5 for $t \geq t_0$. Together with Proposition 2.7 it determines the last decrease.

Theorem 2.10 (transition law). Let $\zeta(\frac{1}{2} + it) = 0$ and $N_{\text{last}}(t) = \max\{N \geq 1 : C_2(N+1, \frac{1}{2} + it) < C_2(N, \frac{1}{2} + it)\}$.

(a) For every W , uniformly for $|w(N + \frac{1}{2})| \leq W$,

$$\Delta C_2(N) = -\frac{4\pi}{t} \left[\Phi'(w_{N+\frac{1}{2}}) - \frac{\sqrt{2}}{4\sqrt{t}} \Phi'_1(w_{N+\frac{1}{2}}) + O_W(t^{-1}) \right], \quad w_{N+\frac{1}{2}} = w(N + \frac{1}{2}). \quad (2.15)$$

(b) As $t \rightarrow \infty$,

$$N_{\text{last}}(t) = \frac{t}{2\pi} + \kappa \sqrt{t} + O(1), \quad \kappa = \frac{w^*}{\pi \sqrt{2}} = -0.2077978646\dots, \quad (2.16)$$

and $\Delta C_2(N) > 0$ for every N with $w(N + \frac{1}{2}) \geq w^* + Ct^{-1/2}$, for an absolute constant C .

(c) With

$$c_1 = \frac{\Phi'_1(w^*)}{4\pi \Phi''(w^*)} = 0.0513\dots, \quad (2.17)$$

one has $N_{\text{last}} + \frac{1}{2} = \frac{t}{2\pi} + \kappa\sqrt{t} + c_1 + O(t^{-1/2})$, in the sense that $\Delta C_2(N) < 0$ at the integers just to the left of this point and $\Delta C_2(N) > 0$ for all larger N . Consequently

$$N_{\text{last}}(t) = \left\lfloor \frac{t}{2\pi} + \kappa\sqrt{t} - \frac{1}{2} + c_1 \right\rfloor \quad (2.18)$$

whenever the fractional part of $\frac{t}{2\pi} + \kappa\sqrt{t} - \frac{1}{2} + c_1$ is at distance $\gg t^{-1/2}$ from 0 and 1.

- (d) $C_2(N_{\text{last}} + 1, \frac{1}{2} + it) = -2\sqrt{2}\Phi(w^*)t^{-1/2} + O(t^{-1}) = -4.6292\dots t^{-1/2} + O(t^{-1})$, the minimum of the trajectory on $N > N_{\text{last}}$.

Proof. (a) $\Delta C_2(N) = \int_N^{N+1} \frac{d}{dx} C_2(x, \rho) dx$. By the differentiated form of (2.11), $\frac{d}{dx} C_2 = \frac{1}{x^2} - \frac{4\pi}{t} \Phi'(w) + \frac{\pi\sqrt{2}}{t^{3/2}} \Phi'_1(w) + O_W(t^{-2})$; the term x^{-2} is $O(t^{-2})$, and averaging a smooth function of w over $[N, N+1]$, where $dw/dx = \pi\sqrt{2}/\sqrt{t}$, changes it by $O(t^{-1})$ relative to its value at $N + \frac{1}{2}$. (b) By Lemma 2.8(a)–(b) and (2.15), for $w_{N+\frac{1}{2}} \in [w^* - 0.6, w^* - Ct^{-1/2}]$ the bracket is positive, so $\Delta C_2 < 0$; such N exist, since consecutive values of w differ by $\pi\sqrt{2}/\sqrt{t}$. For $w_{N+\frac{1}{2}} \in [w^* + Ct^{-1/2}, W_0 + 1]$ the bracket is negative: near w^* because $\Phi'(w) \leq -5(w - w^*)$, further out because Φ' is bounded away from 0 on compact parts of (w^*, ∞) (Lemma 2.8(c)). For $w(N) \geq W_0$, Lemma 2.9 gives $\Delta C_2 > 0$. Hence $N_{\text{last}} + \frac{1}{2}$ lies within $O(1)$ of the point where $w = w^*$, which is $u_0 + \kappa\sqrt{t}$. (c) Near w^* the bracket equals $\Phi''(w^*)(w - w^*) - \frac{\sqrt{2}}{4\sqrt{t}} \Phi'_1(w^*) + O(t^{-1}) + O((w - w^*)^2)$, which vanishes at $w = w^* + \frac{\sqrt{2}\Phi'_1(w^*)}{4\Phi''(w^*)\sqrt{t}} + O(t^{-1})$; in terms of N , through $N - u_0 = w\sqrt{t}/(\pi\sqrt{2})$, this is the shift c_1 . Since $N_{\text{last}} + \frac{1}{2}$ is the last half-integer before the sign change of ΔC_2 , (2.18) follows. (d) By (b), $w(N_{\text{last}} + 1) = w^* + O(t^{-1/2})$, and $\Phi'(w^*) = 0$; insert this in (2.11), where $1/\lambda$ and $1/x$ are $O(t^{-1})$. Monotonicity on $N > N_{\text{last}}$ is the definition of N_{last} . $\square$

Status: proved, with one lemma verified numerically: the proof uses Lemma 2.8 on the compact interval $[w_2, 1.6]$, which is checked on a double-precision grid and not in interval arithmetic. Parts (b)–(d) are asymptotic; t_0 and the constants in $O(1)$ are not made explicit.

Remark 2.11 (numerical confirmation). The constants are $w^* = -0.9232215874$, $\kappa = -0.2077978646$, $\Phi''(w^*) = -6.66323$, $\Phi'_1(w^*) = -4.29904$ and $c_1 = 0.05134$ (`anc/fresnel/fresnel.transition.py constants`; κ also from `anc/fresnel/c2_numerics.py transition`, the script of [C1]).

- (i) *The second-order term.* At a zero, $q(N) = \sqrt{t} [\sqrt{t}(C_2(N) - \frac{1}{\lambda} + \frac{1}{N}) + 2\sqrt{2}\Phi(w(N))]$ should equal $\Phi_1(w(N)) + O(t^{-1/2})$ by (2.11). Table 2 shows the deviation shrinking by a factor of about $4 = \sqrt{16}$ between the two heights, as predicted.

	$t = 20000.128165$		$t = 320000.040965$	
w	$q(N)$	$\Phi_1(w(N))$	$q(N)$	$\Phi_1(w(N))$
≈ -0.92	-3.0667	-3.0559	-3.0231	-3.0207
≈ 0	-0.0055	-0.0047	+0.0039	+0.0040
≈ 1	-0.0224	-0.0221	-0.0219	-0.0219
≈ 2	-0.0750	-0.0750	-0.0752	-0.0752

TABLE 2. The second-order Fresnel term of Proposition 2.7 at the integer N nearest to the given w (the column of Φ_1 uses the exact $w(N)$). Script: `anc/fresnel/fresnel.transition.py profile`.

- (ii) *The floor law.* On 120 zeros, the first 20 above each of $t = 100, 300, 10^3, 3 \cdot 10^3, 10^4, 3 \cdot 10^4$, formula (2.18) gives the observed N_{last} for 112 zeros without the term c_1 and for 119 with it (Table 3). The only miss is the zero $t = 1005.543420$, where the fractional part in (2.18) is 0.9992, within 10^{-3} of an integer, the boundary case excluded in (c). On a second sample of

240 zeros spread over $150 < t < 30000$ (Figure 1b), (2.18) is exact for 238; the two misses, at $t = 9517.248$ and 16386.840 , are again boundary cases, with fractional parts 0.996 and 0.997.

formula	exact predictions of N_{last}
$\lfloor t/2\pi + \kappa\sqrt{t} - \frac{1}{2} \rfloor$	112/120
$\lfloor t/2\pi + \kappa\sqrt{t} - \frac{1}{2} + c_1 \rfloor$, $c_1 = 0.0513$	119/120

TABLE 3. The floor law (2.18) on 120 consecutive zeros in six groups of 20 ($100 < t < 30030$). Script: `anc/fresnel/fresnel_transition.py floor`.

- (iii) *The profile and the limit.* Figure 1a shows $-t\Delta C_2(N)/4\pi$ against $w(N + \frac{1}{2})$ at two zeros, on the curve $\Phi'(w)$ of (2.15); Figure 1b shows $(N_{\text{last}} - t/2\pi)/\sqrt{t}$ approaching κ inside the band allowed by (2.18). By (d), $\sqrt{t}C_2(N_{\text{last}} + 1)$ equals -4.929 , -4.695 , -4.646 at the zeros $t = 1001.35$, 20000.13 , 320000.04 , against the limit -4.6292 .
- (iv) *Consistency with the observations of Part 1* [CE1]. For the 341 zeros with $t < 600$, the last decrease was observed at $n \leq 0.952t/2\pi$, with median $0.932t/2\pi$. Formula (2.18) gives $N_{\text{last}} = 89 = 0.932u_0$ at $t = 600$, and $1 + 2\pi\kappa/\sqrt{t} = 0.947$ there.

Status: numerical (`anc/fresnel/fresnel_transition.py`).

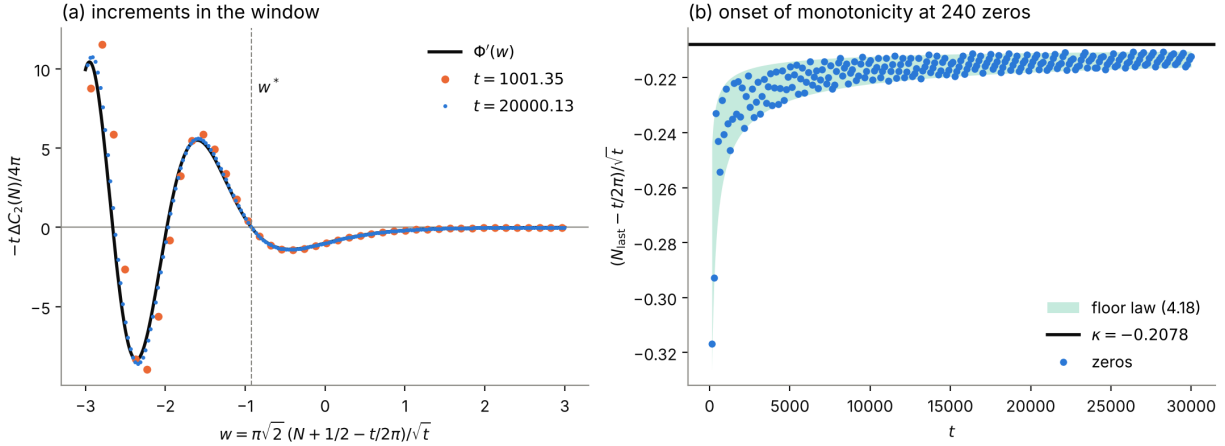


FIGURE 1. The Fresnel transition (Theorem 2.10). (a) The increments $-t\Delta C_2(N)/4\pi$ at the zeros $t = 1001.35$ and $t = 20000.13$, against $w = \pi\sqrt{2}(N + \frac{1}{2} - t/2\pi)/\sqrt{t}$, and the limit profile $\Phi'(w)$; the last decrease occurs where the profile last crosses 0, at w^* . (b) $(N_{\text{last}} - t/2\pi)/\sqrt{t}$ at 240 zeros with $150 < t < 30000$, the constant κ , and the band $\kappa + (c_1 - \frac{3}{2})/\sqrt{t} \leq \cdot \leq \kappa + (c_1 - \frac{1}{2})/\sqrt{t}$ allowed by the floor law (2.18). Linear axes. Scripts: `anc/fresnel/fresnel_transition.py figdata` and `anc/fresnel/fig_fresnel_transition.py`.

Remark 2.12 (precision of the ordinates). For $N \gg t$ the increments $\Delta C_2 \asymp N^{-2}$ are tiny. If the ordinate is off by Δt , then $|\zeta(\frac{1}{2} + it)| \approx |\zeta'(\rho)|\Delta t$, and a zeta wave with increments of size $|\zeta|tN^{-3/2}$ reappears, which eventually beats N^{-2} . At the zero $t = 20000.128165335744$, the trajectory has no decrease on $[u_0 + 5\sqrt{t}, 3u_0]$, and none with the ordinate rounded to 10^{-8} ; with the ordinate rounded to 10^{-6} it has 2532 decreases there, the first at $N = 1.22u_0$. The computations of Theorem 2.10 and Remarks 2.11–2.12 therefore use ordinates obtained from an Euler–Maclaurin evaluation of $Z(t)$ with eight Bernoulli corrections and $2t + 60$ terms, refined by Brent’s method

to the limit of double precision; the result is stable to the last digit under changes of the number of terms. (The ordinate 20000.128165334787 used in [C1] is off by 10^{-9} ; this does not affect any conclusion.) Status: numerical (`anc/fresnel/fresnel.transition.py` precision).

Corollary 2.13 (unique crossing beyond the transition). There is t_0 such that for every zero $\rho = \frac{1}{2} + it$ with $t \geq t_0$ the sequence $C_2(N, \rho)$, $N > N_{\text{last}}(t)$, has exactly one sign change, at the crossing $n^*(\rho) = \lambda + \frac{1}{12} + \frac{1}{180\lambda} + \dots$ of [CE1, Propositions 3.6 and 3.8]. The Hurwitz extension $x \mapsto C_2(x, \rho)$ is strictly increasing on $x \geq u_0 + \kappa\sqrt{t} + C$, with C absolute, and has exactly one zero there.

Proof. By Theorem 2.10(b), $C_2(\cdot, \rho)$ is strictly increasing on $N \geq N_{\text{last}} + 1$; by the differentiated form of (2.11), Lemma 2.8(c) and Lemma 2.9, $\frac{d}{dx}C_2(x, \rho) > 0$ for $w(x) \geq w^* + Ct^{-1/2}$, that is for $x \geq u_0 + \kappa\sqrt{t} + C$. A strictly increasing sequence or function changes sign at most once. By [CE1, Proposition 3.6], for $t \geq t_0$ there is a sign change at $n^*(\rho) \geq t^{3/2}$, which lies in both ranges; and by Theorem 2.10(d) and $\lim C_2 = 1/\lambda > 0$ it is a change from negative to positive. $\square$

Status: proved, with one lemma verified numerically (the dependence on Lemma 2.8 is inherited from Theorem 2.10); t_0 is not effective.

Thus the uniqueness of the crossing, proved in [CE1, Proposition 3.6] for $n \geq t^{3/2}$ and $t \geq t_0$, holds for large t on the whole range $n > t/2\pi + \kappa\sqrt{t} + O(1)$, essentially down to the threshold of the approximate functional equation. Below the transition the head $X_{N-1}(\rho)$ is a short Dirichlet polynomial and $C_2(\cdot, \rho)$ oscillates; the counts of its sign changes there are given in Part 1 [CE1].

Remark 2.14 (the three regimes of a zero-conditioned trajectory). Theorems 2.5 and 2.10, together with the Riemann–Siegel formula in the form of Section 3, organize the trajectory $N \mapsto C_2(N, \frac{1}{2} + it)$ at a critical zero into three regimes (Table 4). The first row is not a regime of the trajectory itself: the Riemann–Siegel main sum appears as a truncated cross-energy $C_2^{[m]}(x^*, s)$ evaluated at the non-integer point $x^* = e^{\theta(t)/t} \approx \sqrt{t/2\pi}e$ with the sum running to $m = \lfloor \sqrt{t/2\pi} \rfloor > x^*$, so it is not $C_2(N, s)$ for any N (Proposition 3.3). What the trajectory itself does for $N \ll t/2\pi$ is oscillate, because $X_{N-1}(\rho)$ is then a short sum, dual through the approximate functional equation to one of length $t/(2\pi N)$. At a point of the line that is not a zero, the zeta wave $2Z(t)N^{-1/2} \cos(t \log N - \theta(t))$ of Section 3 is added in every regime; its increments are of order $N^{-3/2}$ and destroy monotonicity. The ordered structure is specific to zeros. Status: the table assembles Theorems 2.5 and 2.10 and Proposition 3.3; the description of the short-sum regime is heuristic.

regime	range of N	behaviour	governing term
short sums	$N \ll t/2\pi$	oscillation; Riemann–Siegel main sum = $\sqrt{x^*} C_2^{[m]}(x^*)$ off the trajectory	dual sum of the approximate functional equation
transition	$ N - t/2\pi \lesssim \sqrt{t}$	Fresnel profile; last decrease at $t/2\pi + \kappa\sqrt{t}$; minimum $-4.63 t^{-1/2}$	stationary point of the mode $k = 1$
far field	$N \geq (1 + \varepsilon) t/2\pi$	strictly increasing; crossing at $\lambda + \frac{1}{12} + \dots$; limit $1/\lambda$	$\Upsilon(t/N)$, then Euler–Maclaurin

TABLE 4. The three regimes of $C_2(\cdot, \rho)$ at a zero $\rho = \frac{1}{2} + it$.

2.4. The maximum at a hypothetical zero off the line. If $\zeta(\rho) = 0$ with $\rho = \beta + i\gamma$, $\beta \neq \frac{1}{2}$, the wave term of [CE1, Theorem 3.2 (Theorem A)] is absent, and by [CE1, Proposition 3.6], for real $x \geq 1$, $C_2(x, \rho) = 2x^{-2\beta}g_\rho(x)$ with $g_\rho(x) = -\text{Re}[x^\rho \zeta(\rho, x)]$ and

$$g_\rho(x) = \frac{x}{2\nu} - \frac{1}{2} - \frac{\beta}{12x} + \frac{\text{Re}(\rho)_3}{720x^3} + O(|\rho|^5 x^{-4}), \quad \nu = \nu(\rho) = \frac{1}{A_\rho} = \frac{|1 - \rho|^2}{2(1 - \beta)},$$

where $\nu(\rho)$ is the crossing scale of [CE1, Proposition 3.6], $n^*(\rho) = \nu(\rho) + \frac{\beta}{6} + O(\gamma^{-2})$. Since $\partial_x \zeta(s, x) = -s \zeta(s+1, x)$, the same Euler–Maclaurin argument expands g'_ρ , and the expansion may be differentiated term by term. [CE1, Remark 3.7] locates, to leading order, the maximum that the trajectory has when $\beta > \frac{1}{2}$; the following proposition proves it, with its uniqueness and an error term.

Proposition 2.15 (supercritical maximum). Let $\zeta(\rho) = 0$ with $\rho = \beta + i\gamma$, $\beta = \frac{1}{2} + a$, $0 < a < \frac{1}{2}$. There is an absolute γ_0 such that for $\gamma \geq \gamma_0$ the function $x \mapsto C_2(x, \rho)$ has exactly one critical point x_m in $x \geq \gamma^{3/2}$, a maximum, and

$$x_m = \left(1 + \frac{1}{2a}\right) \nu(\rho) (1 + O(\gamma^{-2})), \quad C_2(x_m, \rho) = \frac{x_m^{-2\beta}}{2a} (1 + O(\gamma^{-2})). \quad (2.19)$$

In particular $x_m/n^*(\rho) = (1 + \frac{1}{2a})(1 + O(\gamma^{-2}))$.

Proof. $\frac{d}{dx}[x^{-2\beta} g_\rho] = x^{-2\beta-1} h(x)$ with $h = x g'_\rho - 2\beta g_\rho = -\frac{a}{\nu} x + \beta + e(x)$, where, by the expansion and its derivative, $|e(x)| \ll x^{-1} + \gamma^3 x^{-3} + \gamma^5 x^{-4} \ll \gamma^{-1}$ and $|e'(x)| \ll x^{-2} + \gamma^3 x^{-4} + \gamma^5 x^{-5}$ on $x \geq \gamma^{3/2}$. For $\gamma \geq \gamma_0$, $|e| \leq 0.01 \leq \beta/50$, so $h > 0$ where $\frac{a}{\nu} x \leq \beta - 0.01$ and $h < 0$ where $\frac{a}{\nu} x \geq \beta + 0.01$. In between, $x \asymp \nu/a$ with $\nu \geq \gamma^2/2$, so $|e'(x)| \ll (a/\nu)\gamma^{-2}$; hence h is strictly decreasing there and has exactly one zero, $x_m = \frac{\beta\nu}{a}(1 + O(\gamma^{-2})) = \frac{2\beta}{2\beta-1}\nu(1 + O(\gamma^{-2}))$. Then $g_\rho(x_m) = \frac{\beta}{2a} - \frac{1}{2} + O(\gamma^{-2}) = \frac{1}{2(2\beta-1)} + O(\gamma^{-2})$, which gives the value. Finally $n^*(\rho) = \nu(1 + O(\gamma^{-2}))$. $\square$

Status: proved.

Thus a hypothetical trajectory with $\beta > \frac{1}{2}$ crosses upward at $n^* \approx \nu(\rho)$, rises to a finite maximum at $(1 + \frac{1}{2a})n^*$ and decays to 0^+ . As $a \downarrow 0$ the maximum escapes to infinity, and at $a = 0$ it is replaced by the asymptote $1/\lambda$; for $\beta < \frac{1}{2}$ the trajectory crosses upward and grows without bound. Like the crossing law, the proposition is a property of the Hurwitz tail at the point ρ : it describes what an off-line zero would look like and does not exclude one.

2.5. Monotonicity windows just off a zero. Let $\rho = \frac{1}{2} + i\gamma$, $\gamma > 0$, be a zero of multiplicity $m \geq 1$, and put

$$a_m(\rho) = \frac{|\zeta^{(m)}(\rho)|}{m!}, \quad s_\varepsilon = \frac{1}{2} + i(\gamma + \varepsilon), \quad N_0(\varepsilon) = \frac{1}{4 a_m(\rho)^2 |s_\varepsilon|^2 |\varepsilon|^{2m}}.$$

At s_ε the wave of [CE1, Theorem 3.2 (Theorem A)] is present, with amplitude $|\zeta(s_\varepsilon)| = a_m |\varepsilon|^m (1 + O(\varepsilon))$, and by Part 1 [CE1] monotonicity fails eventually. The following lemma measures the competition between the wave and the monotone part.

Lemma 2.16 (first difference on the critical line). For $s = \frac{1}{2} + it$ in a compact set and integers $N \geq t$,

$$\Delta C_2(N, s) = \frac{1}{N(N+1)} - 2|\zeta(s)| |s| N^{-3/2} \sin(t \log N + \arg \zeta(s) + \vartheta) + R_N, \quad (2.20)$$

where $\cos \vartheta = t/|s|$, $\sin \vartheta = 1/(2|s|)$, with $|R_N| \leq C(|\zeta(s)| N^{-5/2} + N^{-3})$, C uniform on the compact set.

Proof. $C_2(x, s) = 2 \operatorname{Re}[\zeta(s) x^{-\bar{s}}] - 2 \operatorname{Re}[x^{-\bar{s}} \zeta(s, x)]$. On the line the second part is $\frac{1}{\lambda} - \frac{1}{x} - \frac{1}{12x^2} + O(x^{-4})$ by the Euler–Maclaurin expansion [CE1, (3.12)], which holds at every point of the line for the part without $\zeta(s)$, together with its derivative; its increment is $\frac{1}{N(N+1)} + O(N^{-3})$. The first part is $f(x) = 2|\zeta(s)| x^{-1/2} \cos(t \log x + \arg \zeta(s))$, with $f'(x) = -2|\zeta(s)| |s| x^{-3/2} \sin(t \log x + \arg \zeta(s) + \vartheta)$ and $f'' = O(|\zeta(s)| x^{-5/2})$, so $\Delta f(N) = f'(N) + O(|\zeta(s)| N^{-5/2})$. $\square$

Status: proved.

Definition 2.17. Let $N_1 = N_1(\gamma)$ be such that $C_2(\cdot, \rho)$ is strictly increasing on $[N_1, \infty)$; for large γ one may take $N_1 = N_{\text{last}}(\gamma) + 1 \approx \gamma/2\pi + \kappa\sqrt{\gamma}$ (Theorem 2.10). The *monotonicity window*

at $\gamma + \varepsilon$ is

$$N^*(\varepsilon) = \min\{N \geq N_1 : C_2(N+1, s_\varepsilon) < C_2(N, s_\varepsilon)\}.$$

It is finite for small $\varepsilon \neq 0$: since $\zeta(s_\varepsilon) \neq 0$, the wave term of (2.20), of order $N^{-3/2}$, eventually dominates N^{-2} , and $\Delta C_2(\cdot, s_\varepsilon)$ changes sign infinitely often.

Theorem 2.18 (window law). For every $\delta \in (0, 1)$ there is $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| < \varepsilon_0$

$$(1 - \delta) N_0(\varepsilon) \leq N^*(\varepsilon) \leq (1 + \delta) e^{2\pi/\gamma} N_0(\varepsilon), \quad N_0(\varepsilon) = \frac{(m!)^2}{4 |\zeta^{(m)}(\rho)|^2 |s_\varepsilon|^2 |\varepsilon|^{2m}}. \quad (2.21)$$

Proof. The key identity is $2|\zeta(s_\varepsilon)| |s_\varepsilon| N^{-3/2} = N^{-2} \sqrt{N/N_0} (1 + O(\varepsilon))$. *Lower bound.* For $N_1 \leq N \leq N_2$, with $N_2 \geq \gamma$ fixed, $C_2(\cdot, s_\varepsilon) \rightarrow C_2(\cdot, \rho)$ uniformly on this finite range, and $\Delta C_2(N, \rho) > 0$ there, so no decrease occurs for small ε . For $N_2 \leq N \leq (1 - \delta)N_0$, (2.20) gives $\Delta C_2 \geq N^{-2} [1 - \sqrt{1 - \delta} (1 + O(\varepsilon)) - O(N^{-1}) - O(|\varepsilon|^m N^{-1/2})] > 0$ once $N_2 = N_2(\delta)$ is large and ε small. *Upper bound.* For $N \geq (1 + \delta)N_0$ the wave term exceeds $N^{-2}(1 + O(N^{-1}))$, and dominates R_N , as soon as $\sin(\gamma_\varepsilon \log N + \arg \zeta(s_\varepsilon) + \vartheta) > (1 + \delta)^{-1/2}(1 + o(1))$, with $\gamma_\varepsilon = \gamma + \varepsilon$; this is an arc of phases of length $\ell(\delta) > 0$. The phase increases by $\gamma_\varepsilon \log(1 + 1/N) \rightarrow 0$ per step and by 2π over every interval $[M, Me^{2\pi/\gamma_\varepsilon}]$, so some $N \leq (1 + \delta)N_0 e^{2\pi/\gamma} (1 + o(1))$ lands in the arc, and there $\Delta C_2(N) < 0$. Replacing δ by a smaller value absorbs the factors $1 + o(1)$. $\square$

Status: proved.

Corollary 2.19 (window exponent). As $\varepsilon \rightarrow 0$, $\log N^*(\varepsilon)/\log(1/|\varepsilon|) \rightarrow 2m$. The zero is simple if and only if the window grows like $|\varepsilon|^{-2}$, and then $N^*(\varepsilon) \asymp 1/(4|\zeta'(\rho)|^2 |\rho|^2 \varepsilon^2)$.

Proof. Take logarithms in (2.21). $\square$

Status: proved.

The geometry of the detector of Part 1 [CE1] thus measures $|\zeta^{(m)}(\rho)|$ and the multiplicity of each zero. The upper factor $e^{2\pi/\gamma}$ is the worst case of the phase alignment: the decrease occurs at the first $N \geq N_0$ where the phase of the wave reaches the arc, so $N^*(\varepsilon) = (1 + o(1))N_0(\varepsilon)$ as $\gamma \rightarrow \infty$. For 25 zeros spread over $14 < \gamma < 600$ (every fourteenth of the 341 zeros with $\gamma \leq 600$) we chose ε with $N_0(\varepsilon) = 5 \cdot 10^4$ and measured N^* by direct summation, and the exponent by comparing ε with $\sqrt{10}\varepsilon$ (Table 5). Every ratio N^*/N_0 lies in $[1, e^{2\pi/\gamma}]$, and the mean exponent over the 25 zeros is 2.0004: every zero tested is simple, as measured purely through the geometry of C_2 . Status: numerical (`anc/windows/c2_part3_numerics.py windows`).

γ	$ \zeta'(\rho) $	ε	N^*	N^*/N_0	$e^{2\pi/\gamma}$	exponent
14.135	0.793	$1.99 \cdot 10^{-4}$	63888	1.278	1.560	1.935
65.113	2.288	$1.50 \cdot 10^{-5}$	54787	1.096	1.101	2.011
129.579	2.344	$7.36 \cdot 10^{-6}$	52260	1.045	1.050	2.021
209.577	4.118	$2.59 \cdot 10^{-6}$	50146	1.003	1.030	1.980
327.444	5.441	$1.26 \cdot 10^{-6}$	50259	1.005	1.019	2.000
433.889	6.721	$7.67 \cdot 10^{-7}$	50509	1.010	1.015	2.000
515.435	7.294	$5.95 \cdot 10^{-7}$	50206	1.004	1.012	2.001
593.975	4.097	$9.19 \cdot 10^{-7}$	50050	1.001	1.011	1.994

TABLE 5. Monotonicity windows (Theorem 2.18) at 8 of the 25 zeros tested, with $N_0(\varepsilon) = 5 \cdot 10^4$. Script: `anc/windows/c2_part3_numerics.py windows` (the script of [C3]).

Proposition 2.20 (total window and Gonek's conjecture). Assume RH and that all zeros are simple. For each zero put $\omega_+(\rho) = \limsup_{\varepsilon \rightarrow 0} \varepsilon^2 N^*(\varepsilon)$ and $\omega_-(\rho) = \liminf_{\varepsilon \rightarrow 0} \varepsilon^2 N^*(\varepsilon)$. Then $1 \leq 4|\zeta'(\rho)|^2 |\rho|^2 \omega_-(\rho) \leq 4|\zeta'(\rho)|^2 |\rho|^2 \omega_+(\rho) \leq e^{2\pi/\gamma}$, and the total normalized windows satisfy

$$\mathcal{W}_\pm(T) = \sum_{0 < \gamma \leq T} 4|\rho|^2 \omega_\pm(\rho) = (1 + o(1)) J_{-1}(T), \quad J_{-1}(T) = \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2}. \quad (2.22)$$

Consequently Gonek's conjecture $J_{-1}(T) \sim \frac{3}{\pi^3}T$ [Go89], which random matrix theory also predicts [HKO], is equivalent to $\mathcal{W}_{\pm}(T) \sim \frac{3}{\pi^3}T$, and the lower bound $J_{-1}(T) \geq (\frac{3}{2\pi^3} - \epsilon)T$ of Milinovich and Ng [MN12] gives $\mathcal{W}_{-}(T) \geq (\frac{3}{2\pi^3} - \epsilon)T$ for $T \geq T_{\epsilon}$.

Proof. The bounds on $\omega_{\pm}$ are Theorem 2.18 with $m = 1$ and $|s_{\epsilon}| \rightarrow |\rho|$. Hence $\mathcal{W}_{\pm}(T) = \sum_{0 < \gamma \leq T} c_{\rho} |\zeta'(\rho)|^{-2}$ with $1 \leq c_{\rho} \leq e^{2\pi/\gamma} \rightarrow 1$. Since $J_{-1}(T) \rightarrow \infty$ (for instance by [MN12]), the zeros below any fixed height contribute $o(J_{-1}(T))$, and above it $c_{\rho} - 1$ is uniformly small. $\square$

Status: conditional on RH (and on the simplicity of the zeros, which the windows test zero by zero).

The equivalence says that the average sharpness of the zero detector of Part 1 [CE1] is a random-matrix quantity, and gives a geometric way to measure $J_{-1}(T)$ zero by zero. At the heights computed here the agreement is already within 12% (Table 6). The old Part III [C3] compared J_{-1} with $6T/\pi^3$ and found ratios near $\frac{1}{2}$; the correct constant of Gonek's conjecture is $3/\pi^3$ [Go89, MN12].

T	zeros	$J_{-1}(T)$	$3T/\pi^3$	ratio
100	29	10.79	9.68	1.115
200	79	20.46	19.35	1.057
400	202	38.19	38.70	0.987
600	341	56.02	58.05	0.965

TABLE 6. The total window: $J_{-1}(T)$ against Gonek's $3T/\pi^3$. Script: `anc/windows/c2_part3_numerics.py gonek`.

2.6. The pole-free difference energy and three filters. The integers and the primes carry the same wave of the pole: with $X_N^{\Lambda}(s) = \sum_{n \leq N} \Lambda(n)n^{-s}$, both $X_N(s)$ and $X_N^{\Lambda}(s)$ grow like $Y_N(s) = N^{1-s}/(1-s)$. Their difference is therefore free of the pole ([CE1, Proposition 2.8], where $1 + \lim_N [X_N^{\Lambda} - X_N] = F(s) = 1 - \zeta - \zeta'/\zeta$), and its energy is a quadratic object of the same type as $|X_N|^2$, with coefficients $1 - \Lambda(n)$:

$$E_N^{-}(s) = |X_N(s) - X_N^{\Lambda}(s)|^2 = \left| \sum_{n \leq N} (1 - \Lambda(n))n^{-s} \right|^2, \quad (2.23)$$

$$E_N^{-} - E_{N-1}^{-} - (1 - \Lambda(N))^2 N^{-2\sigma} = 2(1 - \Lambda(N))N^{-\sigma} \sum_{j < N} (1 - \Lambda(j))j^{-\sigma} \cos\left(t \log \frac{N}{j}\right).$$

Let $\Theta = \sup \operatorname{Re} \rho$ over the nontrivial zeros, so that RH is $\Theta = \frac{1}{2}$.

Proposition 2.21 (difference energy). (a) The envelope $Y_N(s)$ cancels identically in $X_N - X_N^{\Lambda}$. (b) For $\sigma > \Theta$, $E_N^{-}(s) \rightarrow |\zeta(s) + \zeta'(s)/\zeta(s)|^2$. (c) For $\sigma < \Theta$, $\sup_N E_N^{-}(s) = \infty$. In particular, RH holds if and only if $E_N^{-}(s)$ is bounded in N for every s with $\sigma > \frac{1}{2}$.

Proof. (a) The summatory function of $1 - \Lambda(n)$ is $[x] - \sum_{n \leq x} \Lambda(n) = O(x^{\Theta+\epsilon})$ [In]: the main terms x cancel, and with them the envelopes Y_N of the two partial sums. (b) By partial summation, $\sum (1 - \Lambda(n))n^{-s}$ converges for $\sigma > \Theta$, to the continuation of $\zeta(s) + \zeta'(s)/\zeta(s)$. (c) If the partial sums were bounded at some s_0 with $\sigma_0 < \Theta$, the series would converge for $\sigma > \sigma_0$ and represent $\zeta + \zeta'/\zeta$ there, which is impossible because this function has poles at the zeros with real part in $(\sigma_0, \Theta]$. $\square$

Status: proved (classical in substance; (b) is [CE1, Proposition 2.8] read as an energy).

So E^{-} is the first pole-free cross-energy that separates ζ from its controls, but the natural rigidity statement about it, boundedness for $\sigma > \frac{1}{2}$, is RH. Under RH, at a critical zero of multiplicity m one expects $E_N^{-}(\rho) = m^2 \log^2 N (1 + o(1))$ outside a set of logarithmic density 0 (proved in outline in [C3]; a fit at ρ_1 gives slope 0.963 against $\log^2 N$), and for the Epstein function of $m^2 + 100n^2$

the analogous energy grows at the rate forced by its real zero $\beta_0 = 0.8134\dots$ [C3]. The example suggests three tests that any proposed route to RH through partial-sum trajectories has to pass. Call a property $\mathcal{S}(G)$ of the partial sums of a Dirichlet series G a *rigidity statement* if $\mathcal{S}(\zeta)$ implies RH.

Definition 2.22 (three filters). A rigidity statement $\mathcal{S}$ passes

(F1) *the control filter* if $\mathcal{S}$ fails for the controls that share the additive structure of ζ but have zeros off the line, such as the Davenport–Heilbronn function [DH] and the Epstein zeta function Z_Q of $Q = m^2 + 100n^2$, which has a pole at $s = 1$, nonnegative coefficients and a real zero $\beta_0 = 0.8134\dots$ in $(\frac{1}{2}, 1)$ ([CE16, Proposition 2.1]; real zeros of Epstein zeta functions go back to [BG]); equivalently, any proof of $\mathcal{S}(\zeta)$ must use a property these controls lack, the Euler product;

(F2) *the pole filter* if, when $\mathcal{S}$ is an inequality with nonnegative coefficients and kernel, it is not implied for $\sigma \leq 1$ by the contribution of the pole at $s = 1$ alone (the pole barrier, [CE5, Proposition 2.21] and Part 16 [CE16]);

(F3) *the non-tautology filter* if, when $\mathcal{S}$ is rewritten through the explicit formula as a condition on the zero modes $x^{\rho-s}$, it does not reduce to “no mode has $\text{Re } \rho > \frac{1}{2}$ ”, that is, it admits a verification that does not pass through the location of the zeros.

Proposition 2.23 (positivity–pole dichotomy). Let $G(s) = \sum a_n n^{-s}$ with $a_n \geq 0$ and abscissa of convergence σ_c . Then G is singular at the real point $s = \sigma_c$ [Lan05]. Hence a nonnegative-coefficient object either has a real singularity at σ_c , which dominates every positivity statement for $\sigma < \sigma_c$ (the pole barrier), or is analytic on the whole real half-line $(\frac{1}{2}, \infty)$, and then $\sigma_c \leq \frac{1}{2}$. In particular, if some $a_n \geq 0$ gave a Dirichlet series analytic on the real half-line $(\frac{1}{2}, \infty)$ and singular at every zero of ζ with $\text{Re } \rho > \frac{1}{2}$, RH would follow.

Proof. The first sentence is Landau’s theorem. If G is analytic on $(\frac{1}{2}, \infty)$, the singularity at σ_c forces $\sigma_c \leq \frac{1}{2}$. In the last case, G would converge for $\sigma > \frac{1}{2}$ and so could not be singular at a zero with $\text{Re } \rho > \frac{1}{2}$. $\square$

Status: classical [Lan05].

Remark 2.24 (verdicts). Arguments that use only the C_2 structure (the anatomy, the trichotomy, monotonicity, the crossing law) cannot prove the boundedness of C_2 at every zero, which is equivalent to RH by [CE1, Corollary 3.5(iii)]: the Epstein control has the same structure, including the pole and the trichotomy, and a real zero $\beta_0 > \frac{1}{2}$ at which its cross-energy tends to 0 ([CE16, Theorem 2.2]); so such a proof fails (F1). The 3-4-1 positivity of the prime cross-energy ([CE5, Theorem 2.17]) passes (F1), since it fails for the Epstein function of $m^2 + 5n^2$ ([CE5, Proposition 2.18]), but at $\sigma = \frac{1}{2}$ it fails (F2): there it is implied by the pole alone ([CE5, Proposition 2.21], and Proposition 2.23 in general). The arithmetic energy law $\sum_{n \leq N} C_2^\Lambda(n, \frac{1}{2}) = 4N + O(N^{1/2+\epsilon})$ ([CE5, Theorem 2.23], equivalent to RH; von Koch) and the boundedness of E_N^- for $\sigma > \frac{1}{2}$ (Proposition 2.21) pass (F1) and (F2) and fail (F3): both are equivalent to RH through the abscissa Θ . The transition law and the window law are theorems; they pass because they are blind off the line and imply nothing about RH. The only positivity route through (F2) is the one in Proposition 2.23, and no such construction is known: the obvious candidates, such as $\zeta(s)/\zeta(2s) = \sum |\mu(n)|n^{-s}$, have their singularities in the wrong place. A new candidate passing all three filters would be a genuinely new mechanism, and would presumably extend to L -functions with an Euler product. Status: heuristic.

The natural attempt at the construction of Proposition 2.23 starts from $F = 1 - \zeta - \zeta'/\zeta$ ([CE1, Proposition 2.8]), which is singular exactly at the zeros of ζ in $\sigma > 0$ but has coefficients $\Lambda(n) - 1$ of both signs, and tries to restore positivity by adding a harmless series with nonnegative coefficients. This is impossible.

Proposition 2.25 (no positive correction). Let $A(s) = \sum_{n \geq 1} a_n n^{-s}$ with $a_n \geq 0$, convergent in some half-plane, and suppose that its sum continues analytically to a neighbourhood of the

real half-line $[1, \infty)$ (for instance, that A is holomorphic in $\sigma > \frac{1}{2}$). Then $F + A$ does not have nonnegative coefficients.

Proof. The coefficient of n^{-s} in $F + A$ is $\Lambda(n) - 1 + a_n$ for $n \geq 2$; at the integers $n \geq 2$ that are not prime powers it is $a_n - 1$, so nonnegative coefficients require $a_n \geq 1$ there. By Landau's theorem [Lan05] the abscissa of convergence σ_c of A is a singularity of A on the real axis; since A is analytic near $[1, \infty)$, $\sigma_c < 1$, so $\sum_n a_n/n$ converges. But $\sum_{n \leq x} 1/n = \log x + O(1)$ while $\sum_{p^k \leq x} p^{-k} = \log \log x + O(1)$, so $\sum_n a_n/n \geq \sum_{n \text{ not a prime power}} 1/n = \infty$, a contradiction. $\square$

Status: proved.

So the route “coefficient positivity plus Landau's theorem” through F is closed: any correction that restores the sign of the coefficients has coefficients of both signs, or is itself singular on the real axis at or beyond $s = 1$, and in the first case it needs its own proof of analyticity in $\sigma > \frac{1}{2}$, which is the original problem. Together with the pole filter (F2) this shows that the positivity that Landau's theorem needs is destroyed by the same subtraction of the integers that removes the pole.

3. THE CROSS-ENERGY AND HARDY'S FUNCTION

Section 2 studied C_2 at the zeros on the critical line. At any other point of the line the wave of [CE1, Theorem 3.2 (Theorem A)] is present, and its amplitude and phase are those of Hardy's function. This section makes that link explicit. The helix of [CE1, Theorem A] has a natural orientation whose signed radius is $Z(t)$; along a phase-locked sequence of N the real trajectory $C_2(N, \frac{1}{2} + it)$ reads $2Z(t)$ exactly; the Riemann–Siegel main sum is a truncated cross-energy; and Hardy's theorem, the extrema of Z , the de Bruijn–Newman picture and the mollifiers of Selberg and Levinson can all be stated in this language. Most of the section is classical mathematics rewritten; we say so at each step, and keep the new statements separate.

Let

$$\theta(t) = \operatorname{Im} \log \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2} \log \pi, \quad Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right). \quad (3.1)$$

By the functional equation $Z(t)$ is real for real t , $|Z(t)| = |\zeta(\frac{1}{2} + it)|$, and $\zeta(\frac{1}{2} + it) = e^{-i\theta(t)} Z(t)$, so every sign change of Z is a zero of ζ of odd order on the critical line [T, §4.17]. As $t \rightarrow \infty$,

$$\theta(t) = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(t^{-1}), \quad \theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O(t^{-2}), \quad \theta''(t) = \frac{1}{2t} + O(t^{-3}). \quad (3.2)$$

Throughout, $s = \frac{1}{2} + it$ and $\lambda = s(1-s) = \frac{1}{4} + t^2$.

3.1. The orientation of the helix. Proposition 3.1 (signed radius). For $s = \frac{1}{2} + it$ and $n \geq 2$,

$$e^{i(\theta(t) - t \log n)} \sqrt{n} \left(\frac{1}{2} W_n(s) - \frac{1}{\frac{1}{2} - it} \right) = Z(t) - \frac{1}{2} e^{i(\theta(t) - t \log n)} n^{-1/2} + O(|s| n^{-3/2}). \quad (3.3)$$

Rotating the helix $\sqrt{n}(\frac{1}{2} W_n - \frac{1}{1-s})$ back by the angle $t \log n - \theta(t)$ therefore gives, up to $O(n^{-1/2})$, a real number, its *signed radius*, which is $Z(t)$ for every n .

Proof. By [CE1, Theorem 3.2 (Theorem A)], $\frac{1}{2} W_n(s) = X_{n-1}(s) n^{-\bar{s}} = \frac{1}{1-s} + \zeta(s) n^{-1/2} e^{it \log n} - \frac{1}{2} n^{-1} + O(|s| n^{-2})$ on the critical line. Multiply by $\sqrt{n} e^{i(\theta(t) - t \log n)}$ and use $e^{i\theta(t)} \zeta(s) = Z(t)$. $\square$

Status: proved.

Each zero on the critical line is a crossing of the axis by the signed radius. A pair of zeros $\rho, 1 - \bar{\rho}$ off the line adds two zeros to the Riemann–von Mangoldt count $N(T) = \theta(T)/\pi + 1 + S(T)$ but no crossing, so RH holds if and only if, for every T , the zeros of Z in $(0, T]$, counted with multiplicity, number $N(T)$. This is Turing's method for verifying RH [Tur53], read on the C_2 helix: it locates zeros on the line, but cannot exclude an off-line pair above the computed height. The signed radius computed from the partial sums with $n = 4000$ agrees with $Z(t)$, with imaginary part at most 0.008, the size of the term $\frac{1}{2} n^{-1/2}$ (Figure 2a); on $0 < t < 400$ it has 202 sign changes, the number of zeros of ζ there. For the Davenport–Heilbronn function f , with $Z_f(t) = e^{i\theta_f(t)} f(\frac{1}{2} + it)$ and

$\theta_f(t) = \text{Im} \left[\frac{s}{2} \log \frac{5}{\pi} + \log \Gamma\left(\frac{s+1}{2}\right) \right]$, the same construction gives 287 sign changes on $0 < t < 400$ against 303 zeros: the deficit is 16, its eight off-line pairs, and near the pair at height 85.699 the signed radius approaches the axis and turns back without crossing (Figure 2b,c). Status: numerical (`anc/helix/count.py`, `anc/helix/fighel.py`).

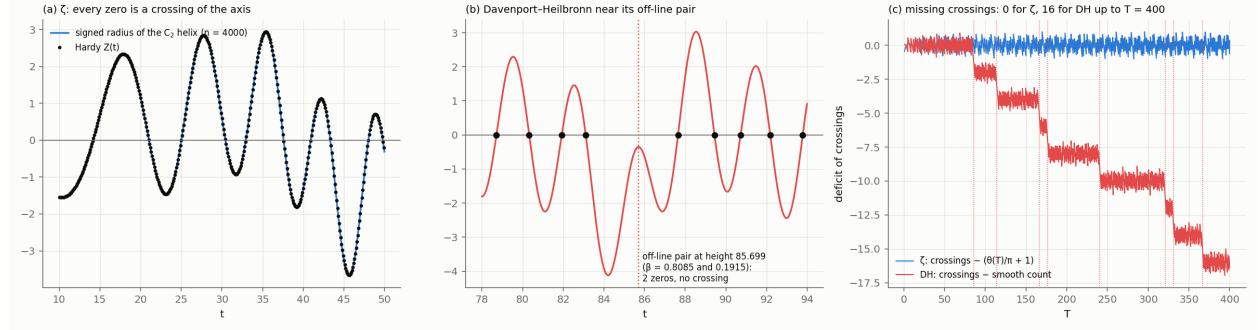


FIGURE 2. The orientation of the C_2 helix (Proposition 3.1). (a) For ζ , the signed radius of the helix computed from the partial sums ($n = 4000$, line) and Hardy's $Z(t)$ (dots). (b) The same for the Davenport–Heilbronn function near its off-line pair at height 85.699; dots mark the crossings. (c) Number of crossings up to T minus the smooth count, for ζ ($\theta(T)/\pi + 1$) and for the Davenport–Heilbronn function; dotted lines at the heights of its off-line zeros.

3.2. Phase-locking: the real trajectory reads $2Z(t)$. The helix uses the complex wave W_n . The real trajectory $C_2 = \text{Re } W_n$ alone already determines $Z(t)$, if N is chosen in phase. Put

$$\mathcal{R}_N(t) = \sqrt{N} \left[C_2\left(N, \frac{1}{2} + it\right) - \frac{1}{\lambda} + \frac{1}{N} + \frac{1}{12N^2} \right]. \quad (3.4)$$

Theorem 3.2 (phase-locking). For fixed $t > 0$,

$$\mathcal{R}_N(t) = 2Z(t) \cos(t \log N - \theta(t)) + O_t(N^{-7/2}). \quad (3.5)$$

Let $x_k(t) = \exp((\theta(t) + 2\pi k)/t)$ and $N_k(t) = \text{round}(x_k(t))$. Then

$$\lim_{k \rightarrow \infty} \mathcal{R}_{N_k(t)}(t) = 2Z(t). \quad (3.6)$$

In particular the sign of $Z(t)$, and with it the location of the zeros of odd order on the line, can be read from the real trajectory C_2 alone.

Proof. By [CE1, Theorem 3.2] and the Euler–Maclaurin expansion [CE1, (3.12)], whose terms other than the wave do not depend on $\zeta(s)$, at a point of the line $C_2(N, s) = \frac{1}{\lambda} - \frac{1}{N} + 2N^{-1/2} \text{Re}[\zeta(s)N^{it}] - \frac{1}{12N^2} + \frac{\text{Re}(s)_3}{360N^4} + O_t(N^{-5})$, and $2N^{-1/2} \text{Re}[\zeta(s)N^{it}] = 2N^{-1/2} Z(t) \text{Re } e^{i(t \log N - \theta(t))}$. This gives (3.5). At $N = N_k$, $t \log N_k = \theta(t) + 2\pi k + \epsilon_k$ with $|\epsilon_k| \leq t |\log N_k - \log x_k| \leq t/(x_k - 1) \rightarrow 0$, so $\cos(t \log N_k - \theta(t)) \rightarrow 1$. $\square$

Status: proved.

Theorem 3.2 is the far-field reading of Z , exact in the limit (Table 7). At a zero $Z(t) = 0$ and (3.5) is the monotone expansion [CE1, (3.12)]; in the transition window of Section 2 the term $2Z(t)N^{-1/2} \cos(t \log N - \theta(t))$, whose increments are of order $N^{-3/2}$, is what destroys the ordered three regimes at a non-zero point. On the critical line the reflected cross-energy C_2^{FR} of [CE3] reduces to C_2 , and [CE3] refines (3.5) to an error $O(N^{-11/2})$.

t	N_k	$\mathcal{R}_{N_k}(t)$	$2Z(t)$
17.5	11213	4.603691	4.603692
17.5	1193400	4.603692	4.603692
20.0	9604	2.295684	2.295685
20.0	1069126	2.295685	2.295685
25.0	10123	-0.029745	-0.029745
25.0	933322	-0.029745	-0.029745

TABLE 7. Phase-locking (Theorem 3.2): the residual (3.4) at the phase-locked N_k near 10^4 and 10^6 , against $2Z(t)$ computed independently. Script: `anc/hardy/hardy_checks.py` locking.

3.3. The Riemann–Siegel main sum as a truncated cross-energy. For $m \geq 1$ and real $x > 0$ define the truncated cross-energy

$$C_2^{[m]}(x, s) = 2x^{-s} \sum_{j \leq m} j^{-s} \cos\left(t \log \frac{x}{j}\right), \quad (3.7)$$

so that $C_2^{[N-1]}(N, s) = C_2(N, s)$.

Proposition 3.3 (Riemann–Siegel as a C_2 sum). Let $m(t) = \lfloor \sqrt{t/2\pi} \rfloor$ and

$$x^*(t) = e^{\theta(t)/t} = \sqrt{\frac{t}{2\pi e}} (1 + O(t^{-1})). \quad (3.8)$$

Then

$$Z(t) = \sqrt{x^*(t)} C_2^{[m(t)]}(x^*(t), \tfrac{1}{2} + it) + O(t^{-1/4}). \quad (3.9)$$

Proof. The Riemann–Siegel formula [Si32], [T, §4.16–4.17] states that $Z(t)$ equals $2 \sum_{n \leq m(t)} n^{-1/2} \cos(\theta(t) - t \log n) + O(t^{-1/4})$. Since $t \log x^* = \theta(t)$ exactly, $\sqrt{x^*} C_2^{[m]}(x^*, \tfrac{1}{2} + it) = 2 \sum_{j \leq m} j^{-1/2} \cos(\theta(t) - t \log j)$. The expansion (3.8) follows from (3.2). $\square$

Status: classical [Si32]; (3.9) is the Riemann–Siegel formula rewritten.

Two remarks keep this in proportion. First, nothing is gained analytically: the identity $t \log x^* = \theta(t)$ turns the phase $\theta(t) - t \log j$ of the Riemann–Siegel sum into the phase $t \log(x^*/j)$ of a cross-energy. Second, (3.9) is *not* a point on the trajectory $N \mapsto C_2(N, s)$: the sum runs to $m \approx \sqrt{t/2\pi}$, past the evaluation point $x^* \approx \sqrt{t/2\pi e}$ (since $\sqrt{e} > 1$), and x^* is not an integer. It is a truncated cross-energy evaluated off the trajectory, and the “near field” of Remark 2.14 is to be understood in that sense. Table 8 confirms the size $t^{-1/4}$ of the error.

t	m	x^*	$\sqrt{t/2\pi e}$	$\sqrt{x^*} C_2^{[m]}(x^*)$	$Z(t)$	difference	$t^{-1/4}$
100	3	2.410	2.420	2.23945	2.69270	-0.453	0.316
1000	12	7.649	7.652	1.11211	0.99779	0.114	0.178
5000	28	17.109	17.110	-0.70164	-0.80426	0.103	0.119

TABLE 8. The Riemann–Siegel main sum as a truncated cross-energy (Proposition 3.3); $Z(t)$ computed independently by Euler–Maclaurin. Script: `anc/hardy/hardy_checks.py` rs.

3.4. Hardy’s theorem in C_2 form. The classical proof of Hardy’s theorem compares $\int \zeta$ with $\int Z$ on $[T, 2T]$. The first mean value is the X – Y balance; the second uses the Riemann–Siegel sum of Proposition 3.3.

Lemma 3.4 (mean value from the X – Y balance). $\int_T^{2T} \zeta(\tfrac{1}{2} + it) dt = T + O(T^{1/2})$.

Proof. The balance $\zeta(s) = X_x(s) - Y_x(s) + O(x^{-\sigma})$ holds uniformly for $\sigma \geq \frac{1}{2}$ and $|t| \leq x$ [T, Thm 4.11]. Take $x = 2T$. Then $\int_T^{2T} X_x(\frac{1}{2} + it) dt = T + \sum_{2 \leq n \leq 2T} n^{-1/2} \int_T^{2T} n^{-it} dt = T + O(\sum_{2 \leq n \leq 2T} n^{-1/2} / \log n) = T + O(T^{1/2})$, while $|Y_x(\frac{1}{2} + it)| = x^{1/2} / |\frac{1}{2} - it| \ll T^{-1/2}$ and the remainder is $O(T^{-1/2})$ pointwise. $\square$

Status: classical [T, Thm 4.11].

Lemma 3.5 (mean value from the near field). $\int_T^{2T} Z(t) dt = O(T^{3/4})$.

Proof. By (3.9), $\int_T^{2T} Z dt = 2 \sum_{n \leq \sqrt{T/\pi}} n^{-1/2} \int_{I_n} \cos \varphi_n(t) dt + O(T^{3/4})$, with $I_n = [\max(T, 2\pi n^2), 2T]$ and $\varphi_n(t) = \theta(t) - t \log n$, so that $\varphi'_n(t) = \frac{1}{2} \log(t/2\pi n^2) + O(t^{-2})$ and $\varphi''_n(t) \asymp 1/t$. For $n \leq \sqrt{T/8\pi}$, $\varphi'_n \geq \frac{1}{2} \log 4 - o(1)$ on I_n , and the first-derivative test [T, Lemma 4.2] bounds each integral by $O(1)$; these n contribute $O(T^{1/4})$. For $\sqrt{T/8\pi} < n \leq \sqrt{T/\pi}$, the second-derivative test [T, Lemma 4.4] bounds each integral by $O(T^{1/2})$; there are $O(T^{1/2})$ such n , each with weight $n^{-1/2} \asymp T^{-1/4}$, so they contribute $O(T^{3/4})$. $\square$

Status: classical; much sharper bounds for $\int Z$ are known, and this weak form suffices.

Theorem 3.6 (Hardy's theorem, C_2 form). For every sufficiently large T there is $t \in (T, 2T)$ with $\zeta(\frac{1}{2} + it) = 0$. At every such t the trajectory $N \mapsto C_2(N, \frac{1}{2} + it)$ is eventually strictly increasing, from $N_{\text{last}}(t) + 1 = t/2\pi + \kappa\sqrt{t} + O(1)$ on when t is large (Theorem 2.10); it changes sign exactly once in that range, at $n^* = \lambda + \frac{1}{12} + \frac{1}{180\lambda} + \dots$ ([CE1, Proposition 3.6] and Corollary 2.13); and it converges to $1/\lambda$ at rate $1/N$.

Proof. Suppose Z had constant sign on $[T, 2T]$. Then $\int_T^{2T} |Z| dt = |\int_T^{2T} Z dt| = O(T^{3/4})$ by Lemma 3.5, while $\int_T^{2T} |Z| dt = \int_T^{2T} |\zeta(\frac{1}{2} + it)| dt \geq |\int_T^{2T} \zeta(\frac{1}{2} + it) dt| = T + O(T^{1/2})$ by Lemma 3.4, a contradiction for large T . So Z changes sign in $(T, 2T)$ and, being continuous, vanishes there. The statements on C_2 are those of [CE1, Corollary 3.5], [CE1, Proposition 3.6], Theorem 2.10 and Corollary 2.13, which hold at every zero on the critical line. $\square$

Status: classical [Ha14]. The proof is the classical one, through the mean values of Lemmas 3.4–3.5; C_2 enters only in the statement of the conclusion, not in the proof.

On a grid of step 0.01, $\int_T^{2T} Z dt = 17.0$ and 28.6 for $T = 200$ and 500 , against $\int_T^{2T} |Z| dt = 329.7$ and 862.0 and $T^{3/4} = 53.2$ and 105.7 ; Z has 123 and 380 sign changes in $[T, 2T]$, which are all the zeros there ($N(400) - N(200) = 123$, $N(1000) - N(500) = 380$; the smooth count $\theta/\pi + 1$ gives 122.4 and 379.0). Each is a point where C_2 is eventually monotone and crosses once. Script: `anc/hardy/hardy_checks.py hardy`.

Remark 3.7 (scope). Theorem 3.6 reproduces Hardy's result of 1914 [Ha14] and does not advance it. An argument of this type cannot need the Euler product: Epstein zeta functions, which have none, also have infinitely many zeros on the critical line [PoT]. The quantitative improvements are classical: $\gg T$ critical zeros up to height T [HL21], a positive proportion [Se42], at least one third [Le74], at least two fifths [Co89], and more than five twelfths [PRZZ]. Reaching all of them is RH. Status: classical.

3.5. A zero detector on the critical line. Proposition 3.8 (detector). For $s = \frac{1}{2} + it$ put $D_n(t) = |X_n(s)|^2 - (n + \frac{1}{2})/\lambda$. Then

$$D_n(t) = 2\sqrt{n/\lambda} |\zeta(s)| \cos(t \log n + \arg \zeta(s) - \arg(\frac{1}{2} + it)) + |\zeta(s)|^2 + O_t(|\zeta(s)| n^{-1/2} + n^{-1}). \quad (3.10)$$

In particular $D_n(t) = O_t(1/n)$ if $\zeta(s) = 0$, and otherwise $D_n(t)$ oscillates in n with amplitude $2|\zeta(s)|\sqrt{n/\lambda}$.

Proof. By Euler–Maclaurin, $X_n(s) = Y_n(s) + \zeta(s) + \frac{1}{2}n^{-s} + O_t(n^{-3/2})$ with $Y_n(s) = n^{1-s}/(1-s)$. Expand $|X_n|^2$: $|Y_n|^2 = n/\lambda$, $2\operatorname{Re}[Y_n \cdot \frac{1}{2}\overline{n^{-s}}] = \operatorname{Re} \frac{1}{1-s} = \frac{1}{2\lambda}$, and $2\operatorname{Re}[Y_n \overline{\zeta(s)}]$ has modulus $2|\zeta(s)|\sqrt{n}/|1-s| = 2|\zeta(s)|\sqrt{n/\lambda}$ and phase as stated; the other terms are $|\zeta|^2$ and $O_t(|\zeta|n^{-1/2} + n^{-1})$. At a zero this is the straight-line law [CE1, (3.11)]. $\square$

Status: proved.

A single n is not enough to locate zeros: $D_n(t) = 0$ has many spurious solutions, because $D_n(t)$ oscillates in t with period about $2\pi/\log n$. Requiring smallness for several n removes them. With $\max_{n \in \{2000, 2500, 3000, 3500, 4000\}} |D_n(t)|$ on a grid of step 0.002 in $14 \leq t \leq 31$, the local minima below 10^{-2} occur exactly at 14.134, 21.022, 25.010 and 30.424, with values between $2 \cdot 10^{-4}$ and $5.4 \cdot 10^{-3}$ (limited by the grid), while the median elsewhere is 7.75 (`anc/hardy/hardy_checks.py detector`). The detector is valid but not efficient: each evaluation costs $O(n)$ operations, against $O(\sqrt{t})$ for the Riemann–Siegel formula of Proposition 3.3. It finds zeros on the critical line only, and each zero is detected by its own identity; a known zero gives no information about the next one.

The same mechanism is visible in C_2^2 . Along a zero on the line, $\lambda C_2(n, \rho) \approx 1 - \lambda/n$ is negative, vanishes once at the crossing $n^* \approx \gamma^2 + \frac{1}{3}$ of [CE1, Proposition 3.6] and increases to 1; the curves of different zeros coincide in the variable n/γ^2 . Along the critical line at fixed n , by [CE1, (3.7)], $C_2(n, \frac{1}{2} + it)^2$ oscillates under the envelope $(\lambda^{-1} + 2|\zeta(\frac{1}{2} + it)|n^{-1/2})^2$, which drops to λ^{-2} exactly at the zeros (Figure 3). The detector is confined to the critical line: off it, $\Delta C_2(N, s)$ has eventually the sign of $1 - 2\sigma$ at every point, zero or not ([CE1, Proposition 3.4(iii)]), and what distinguishes an off-line zero from its neighbours is only the absence of the oscillating term, that is $\zeta(s) = 0$. Status: follows from [CE1, Theorem 3.2] and [CE1, Proposition 3.4]; the figure is numerical (`anc/c2sq/c2_squared_linear.py`).

3.6. Extrema without crossing. An off-line pair appears on the critical line as an extremum of the signed radius that does not reach the axis: a positive local minimum or a negative local maximum of Z . Near a pair of zeros at height γ , $Z(t) \approx A((t - \gamma)^2 + \varepsilon)$ with

$$\varepsilon = \delta^2 \text{ for an off-line pair } \frac{1}{2} \pm \delta + i\gamma, \quad \varepsilon = -\frac{g^2}{4} \text{ for two zeros on the line at distance } g, \quad (3.11)$$

because $\xi(\frac{1}{2} + it)$, as a function of t , has the factor $(t - \gamma)^2 + \delta^2$ in the first case and $(t - \gamma)^2 - g^2/4$ in the second. RH is the statement that $\varepsilon \leq 0$ for every pair; a pair of close zeros on the line, a Lehmer pair [Leh56], is one where the margin $-\varepsilon$ is small. This is the local picture of the de Bruijn–Newman flow: under the backward heat flow of ξ such pairs collide and leave the line, the de Bruijn–Newman constant satisfies $\Lambda \geq 0$ [RT] and $\Lambda \leq 0.22$ [Pm], improved to $\Lambda \leq 0.2$ [PT] and, with optimised mollifiers and certified computation, to $\Lambda \leq 0.174$ [CE13], RH is $\Lambda = 0$, and Lehmer pairs give the classical lower bounds for Λ [CSV94]. Status: heuristic (the local model (3.11) is a second-order Taylor expansion; the statements on Λ are classical).

We scanned Z with the Riemann–Siegel formula (checked against Euler–Maclaurin to 10^{-6}) on $10 < t < 10^5$ with step 0.004. There are 138069 sign changes, against $\theta(10^5)/\pi + 1 = 138068.6$, and none of the 138069 extrema is without crossing. The lowest extremum is $Z = -5.5 \cdot 10^{-4}$ at $t = 71732.909$, between the zeros 71732.9015 and 71732.9159, a Lehmer pair; the next ones are at 78974.81, 66678.09 and 63137.22, and the classical pairs at 7005.06 and 17143.79 [Leh56] appear among the lowest twelve (Figure 4a). For the Davenport–Heilbronn function the scan of Z_f on $t < 400$ finds exactly eight extrema without crossing, each within 0.025 of the ordinate of one of its eight off-line pairs, and the height missing to reach the axis follows $|Z_f| \approx 2.81 \delta^2 \cdot \frac{1}{2} \log(\gamma/2\pi)$, within -5% and $+13\%$ for all eight (Figure 4b), in agreement with (3.11). Status: numerical (`anc/z_extrema/escaneo.py`, `anc/z_extrema/escaneo_dh.py`; figure by `anc/v3_additions_figs.py`).

3.7. The mollified cross-energy. The classical way arithmetic enters the study of critical zeros is mollification. Let $y \geq 2$ and let P be a polynomial with $P(0) = 0$ and $P(1) = 1$. The Möbius mollifier and the mollified series are

$$M_y(s) = \sum_{d \leq y} c_d d^{-s}, \quad c_d = \mu(d) P\left(\frac{\log(y/d)}{\log y}\right), \quad \zeta(s) M_y(s) = \sum_{n \geq 1} a_y(n) n^{-s}, \quad (3.12)$$

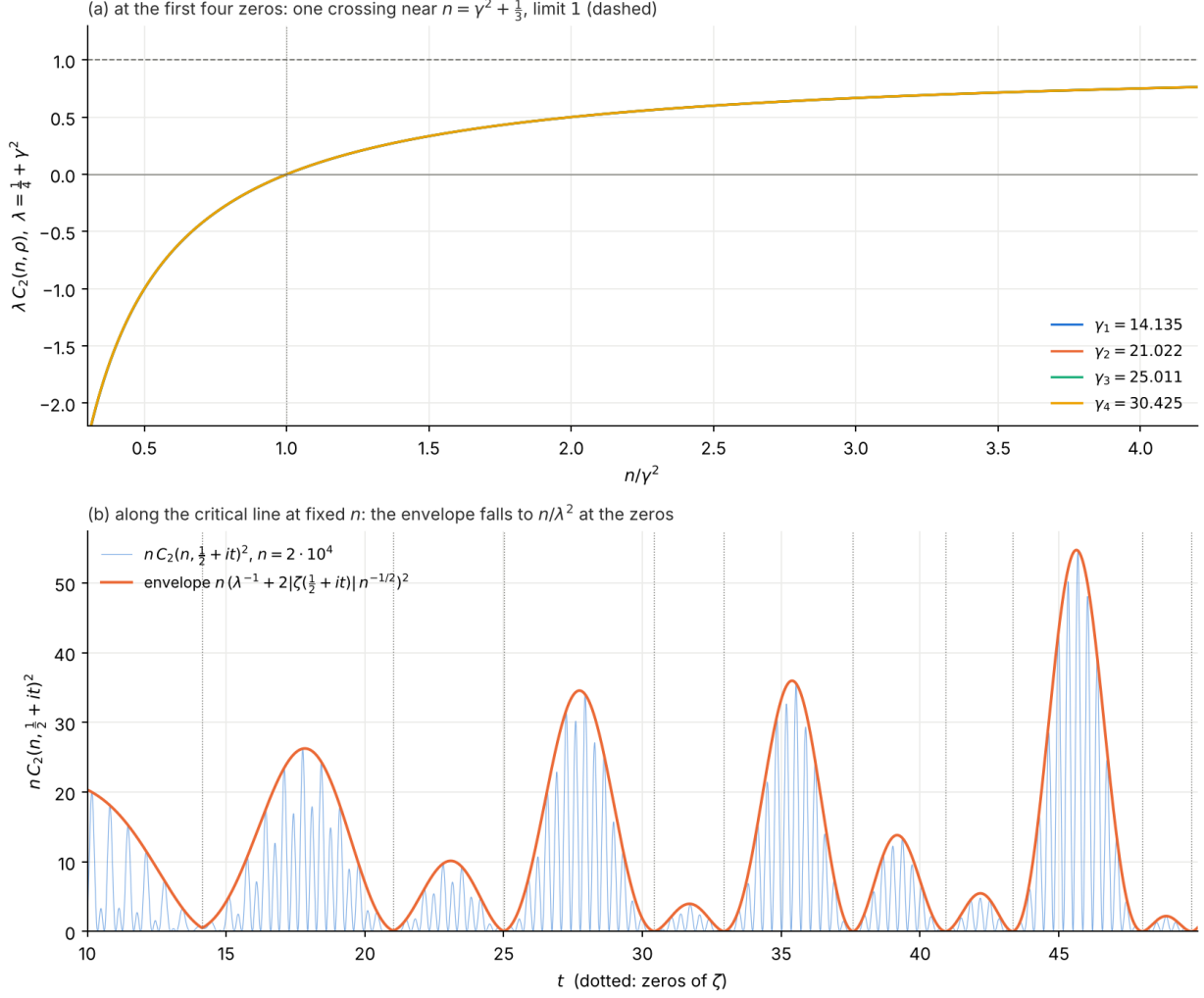


FIGURE 3. The zero detector in C_2^2 (linear axes). (a) $\lambda C_2(n, \rho)$ against n/γ^2 at the first four zeros, $\lambda = \frac{1}{4} + \gamma^2$; the four curves coincide to plotting accuracy, cross zero once near $n/\gamma^2 = 1$ and tend to 1 (dashed). (b) $n C_2(n, \frac{1}{2} + it)^2$ for $n = 2 \cdot 10^4$ and its envelope $n(\lambda^{-1} + 2|\zeta(\frac{1}{2} + it)|n^{-1/2})^2$, which falls to $n\lambda^{-2}$ at the zeros (dotted).

with $a_y(n) = \sum_{d|n, d \leq y} c_d$. With $P \equiv 1$ one would have $a_y(n) = \sum_{d|n} \mu(d) = 0$ for $2 \leq n \leq y$: the mollifier removes the small- n part of ζ , and a smooth P makes the removal tapered. Put

$$X_x^M(s) = \sum_{n \leq x} a_y(n) n^{-s} = \sum_{d \leq y} c_d d^{-s} X_{x/d}(s), \quad K_y(x, s) = 2 \operatorname{Re} [x^{-\bar{s}} X_x^M(s)]. \quad (3.13)$$

The sum in K_y includes the endpoint $n = x$; for $y < 2$ and integer $x = N$, $K_y(N, s) = C_2(N, s) + 2N^{-2\sigma}$. Let $\psi(u) = \{u\} - \frac{1}{2}$ and define the *Möbius-weighted sawtooth*

$$S_y(x) = \sum_{d \leq y} c_d \psi(x/d). \quad (3.14)$$

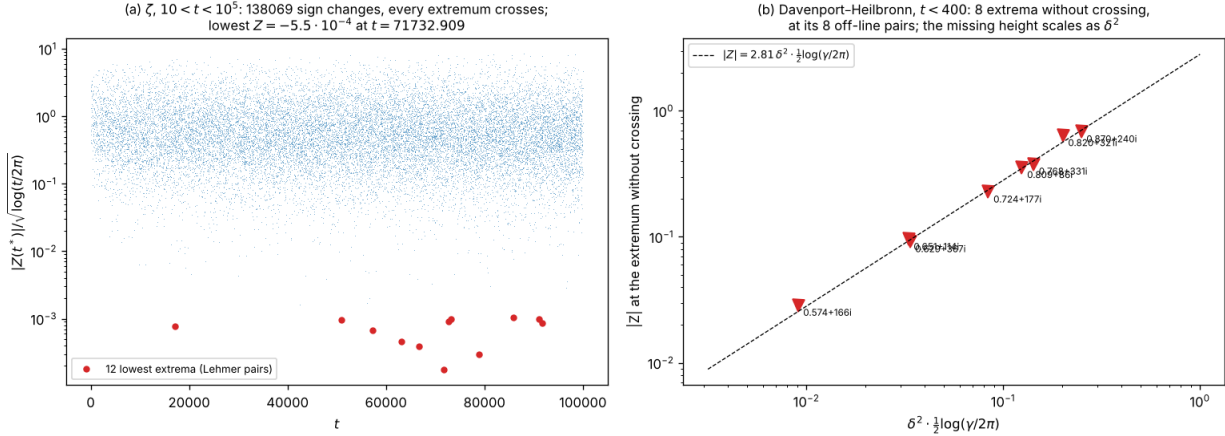


FIGURE 4. Extrema of Hardy's function. (a) $|Z(t^*)|/\sqrt{\log(t^*/2\pi)}$ at the extrema t^* for $10 < t < 10^5$ (a random subsample of 20000); dots: the twelve lowest, at Lehmer pairs. (b) The eight extrema without crossing of Z_f for $t < 400$, at the eight off-line pairs of the Davenport–Heilbronn function, against $\delta^2 \cdot \frac{1}{2} \log(\gamma/2\pi)$.

Theorem 3.9 (anatomy of the mollified cross-energy). For fixed $s = \sigma + it$ with $0 < \sigma < 1$ and fixed y , as $x \rightarrow \infty$,

$$K_y(x, s) = \frac{2(1 - \sigma)M_y(1)}{|1 - s|^2} x^{1-2\sigma} + 2x^{-\sigma} \operatorname{Re} [\zeta(s)M_y(s) x^{it}] - 2x^{-2\sigma} S_y(x) + O_{s,y}(x^{-2\sigma-1}). \quad (3.15)$$

Proof. First-order Euler–Maclaurin with the sawtooth gives, for real $u \geq 1$, $X_u(s) = \frac{u^{1-s}}{1-s} + \zeta(s) - \psi(u)u^{-s} + O_s(u^{-\sigma-1})$. Insert this in (3.13) with $u = x/d$, and use $d^{-s}(x/d)^{1-s} = x^{1-s}/d$ and $d^{-s}(x/d)^{-s} = x^{-s}$: $X_x^M(s) = M_y(1) \frac{x^{1-s}}{1-s} + \zeta(s)M_y(s) - x^{-s}S_y(x) + O_{s,y}(x^{-\sigma-1})$. Multiply by $x^{-\bar{s}}$ and take $2 \operatorname{Re}$. $\square$

Status: proved.

Each of the three components of [CE1, Theorem 3.2] is modified by the arithmetic. The structural mode is scaled by $M_y(1)$; for $P(x) = x$, $M_y(1) = \frac{1}{\log y} \sum_{d \leq y} \frac{\mu(d)}{d} \log \frac{y}{d} = \frac{1+o(1)}{\log y}$ by the prime number theorem, so mollification *suppresses* the trace of the pole. The zeta wave has amplitude $2|\zeta(s)M_y(s)|x^{-\sigma}$ and is *flattened*. The endpoint term becomes the Möbius-weighted sawtooth S_y , so the arithmetic enters at the endpoint. Table 9 checks (3.15) with $y = 31$ and $P(x) = x$, for which $M_y(1) = 0.29180$ against $1/\log y = 0.29121$.

point	N	$K_y(N, s)$ (direct)	right side of (3.15)	sawtooth term $-2S_y(N)/N$
$t = 1000.5$ (not a zero)	20000	$+3.1148 \cdot 10^{-2}$	$+3.1148 \cdot 10^{-2}$	$+4.33 \cdot 10^{-6}$
	100000	$-1.8212 \cdot 10^{-3}$	$-1.8212 \cdot 10^{-3}$	$-4.38 \cdot 10^{-6}$
$t = 1001.349483$ (zero)	50000	$+9.5152 \cdot 10^{-6}$	$+9.5163 \cdot 10^{-6}$	$+9.24 \cdot 10^{-6}$
	100000	$-4.0800 \cdot 10^{-6}$	$-4.0778 \cdot 10^{-6}$	$-4.38 \cdot 10^{-6}$
	200000	$+2.6059 \cdot 10^{-6}$	$+2.6058 \cdot 10^{-6}$	$+2.32 \cdot 10^{-6}$

TABLE 9. The mollified cross-energy on the critical line ($y = 31$, $P(x) = x$) against Theorem 3.9. At the zero the sawtooth dominates. Script: `anc/hardy/hardy_checks.py` mollified; also `anc/fresnel/c2_numerics.py` mollified.

At a zero of ζ the wave vanishes and the mollified trajectory is governed by the oscillating sawtooth S_y (Table 9), so the monotonicity detector of [CE1, Corollary 3.5] and Section 2 does not transfer verbatim to K_y . It can be restored by smoothed (Riesz or Cesàro) partial sums, which

remove the sawtooth at the cost of explicit smoothing terms, or bypassed by using the wave itself, as follows.

Proposition 3.10 (phase-locked mollified residual). Write $M_y(\frac{1}{2} + it) = |M_y(\frac{1}{2} + it)|e^{i\alpha(t)}$. On the critical line the residual

$$\mathcal{R}_N^M(t) = \sqrt{N} \left[K_y(N, \tfrac{1}{2} + it) - \frac{M_y(1)}{\lambda} + \frac{2S_y(N)}{N} \right]$$

satisfies

$$\mathcal{R}_N^M(t) = 2Z(t) |M_y(\tfrac{1}{2} + it)| \cos(t \log N - \theta(t) + \alpha(t)) + O_{t,y}(N^{-3/2}), \quad (3.16)$$

and along N with $t \log N \equiv \theta(t) - \alpha(t) \pmod{2\pi}$, as in Theorem 3.2, $\mathcal{R}_N^M(t) \rightarrow 2Z(t) |M_y(\frac{1}{2} + it)|$.

Proof. On $\sigma = \frac{1}{2}$, $2(1 - \sigma)/|1 - s|^2 = \frac{1}{\lambda}$ and $\zeta(s)M_y(s) = e^{-i\theta(t)}Z(t)|M_y(s)|e^{i\alpha(t)}$; insert in (3.15). The locking argument is that of Theorem 3.2. $\square$

Status: proved.

The mollified cross-energy reads the sign of $Z(t)$ exactly as C_2 does, with amplitude $|Z(t)M_y(\frac{1}{2} + it)|$ in place of $|Z(t)|$. The effect of the mollifier on the amplitude is visible numerically: for $1000 \leq t \leq 1050$ (step 0.01, $y = 31$, $P(x) = x$) the mean square drops from 6.32 for $|\zeta(\frac{1}{2} + it)|$ to 2.21 for $|\zeta M_y(\frac{1}{2} + it)|$, and the spread of $\log |\cdot|$ between the 10th and 90th percentiles shrinks from 2.86 to 1.99 (percentiles $-1.54, 0.07, 1.32$ against $-1.20, 0.29, 0.79$). The large values of $|\zeta|$ are tamed, while the sign structure of Z is untouched (`anc/hardy/hardy_checks.py flattening`).

Remark 3.11 (the Selberg–Levinson mechanism in C_2 language). Hardy’s argument compares $\int Z$ with $\int |Z|$ on long intervals and detects only $\gg \log T$ sign changes up to T , because $|Z|$ fluctuates too much for the comparison to work on short intervals. Selberg’s idea [Se42], [T, Ch. X] is to compare on intervals of length $h \asymp 1/\log T$ after multiplying by a weight $|\phi(\frac{1}{2} + it)|^2$, where $\phi(s) = \sum_{\nu < X} \beta_\nu \nu^{-s}$ is a mollifier for $\zeta^{-1/2}$: $\beta_\nu = \alpha_\nu(1 - \log \nu / \log X)$ with $\sum \alpha_\nu \nu^{-s} = \zeta(s)^{-1/2}$. The criterion is

$$\int_t^{t+h} |Z(u)| |\phi(\tfrac{1}{2} + iu)|^2 du > \left| \int_t^{t+h} Z(u) |\phi(\tfrac{1}{2} + iu)|^2 du \right| \implies Z \text{ changes sign in } (t, t+h), \quad (3.17)$$

and the analytic input consists of mean values of $Z|\phi|^2$ and of $|\zeta\phi^2|^2$ over short intervals, that is, of the amplitude of a mollified wave and of its mean square. The dictionary with Proposition 3.10 is exact only after one adjustment. The phase-locked residual of ζM reads $2Z|M|$, so it reproduces the integrand of (3.17) for $M = \phi^2$, a Dirichlet polynomial of length X^2 with coefficients $\beta * \beta$, which approximates $1/\zeta$ but is not the Möbius mollifier (3.12). With $M = M_y$ the residual reads $Z|M_y|$, whereas (3.17) has the square of a mollifier of $\zeta^{-1/2}$; the Möbius mollifier $M_y \approx 1/\zeta$ is the one of Levinson [Le74] and Conrey [Co89], who use it in a different criterion, the argument principle for $(\zeta + \zeta'/\log T)M_y$ slightly to the left of the line. In every case the constant comes from a mean square of a mollified wave amplitude, the records being a positive proportion [Se42], $\frac{1}{3}$ [Le74], $\frac{2}{5}$ [Co89] and more than $\frac{5}{12}$ [PRZZ]. ([CE10, Remark 3.10] uses a mollifier of Möbius type, $P(x) = x$, to lower the mean energy in a density estimate; there it plays the role of M_y , not of Selberg’s ϕ .) Reproducing Selberg’s theorem in this language would need the short-interval second moment of $\mathcal{R}_N^M$ with $M = \phi^2$; at present the framework supplies a dictionary for the mechanism, not an improved estimate. Status: heuristic (dictionary); the cited results are classical.

Remark 3.12 (why proportions cannot reach RH). A proportion argument, however strong, cannot prove RH by itself. Bombieri and Hejhal [BH95] showed, under the generalized Riemann hypothesis and a hypothesis on the spacing of the zeros of the constituent L -functions, that linear combinations of Euler products with a common functional equation have 100% of their zeros on the critical line; yet such combinations, the Davenport–Heilbronn function and the Epstein zeta functions of class number larger than one, have infinitely many zeros off it [DH] (Part 16 [CE16]). Density one on the line is therefore compatible with infinitely many off-line zeros, and the last step

towards RH must exclude a sparse set of them, a zero-density problem at $\sigma = \frac{1}{2}$. In the cross-energy language the analysis leaves two components: the mollified wave detector of Proposition 3.10, where the methods of Selberg, Levinson and Conrey live and which finds zeros on the line, and the arithmetic energy of the primes, $\sum_{n \leq N} C_2^\Lambda(n, \frac{1}{2}) = 4N + O(N^{1/2+\epsilon})$ ([CE5, Theorem 2.23]), which excludes zeros off the line and is equivalent to RH (von Koch). Pushing the second to $\sigma = \frac{1}{2}$ is beyond current methods. Mollifiers built from the Euler product, and the bound that controls how far they can suppress the pole, are the subject of [CE11]. Status: heuristic; the inputs are classical.

4. DISCUSSION

The transition law and the reading of Hardy's function show what the cross-energy sees on the critical line.

1. *The transition is where a zero on the line becomes visible.* At a zero $\frac{1}{2} + it$ the cross-energy is eventually monotone, and the onset follows the Fresnel law $N_{\text{last}} = \frac{t}{2\pi} + \kappa\sqrt{t} + O(1)$ (Theorem 2.10); the windows of monotonicity near a zero carry the derivatives $\zeta^{(m)}(\rho)$, and their total over the zeros is Gonek's sum $J_{-1}(T)$ (Proposition 2.20). These are statements about zeros on the line; they do not distinguish the real part of a hypothetical zero off the line (Part 16 [CE16]).
2. *Hardy's function is the signed radius of the helix.* On the line the cross-energy is oriented, and its signed radius is $Z(t)$ (Proposition 3.1). The classical results that count zeros on the line (Hardy's theorem, Riemann–Siegel, Turing's method, and the proportions of Levinson and Conrey) read naturally in these coordinates. They are proved and do not reach beyond a positive proportion; the reason is that density one on the line is compatible with infinitely many off-line zeros (Remark 3.12).

4.1. Open problems.

- Is there an effective version of Corollary 2.13, with an explicit t_0 , so that the uniqueness of the crossing covers all zeros beyond the transition? Can Lemma 2.8, which is verified numerically, be proved?
- Under RH and simplicity the total monotonicity window equals Gonek's sum $J_{-1}(T)$ asymptotically (Proposition 2.20). Can the windows, which are defined zero by zero from the partial sums, give an upper bound for $J_{-1}(T)$ of the conjectured order $\frac{3}{\pi^3}T$ [Go89, HKO]?

We do not prove the Riemann hypothesis.

APPENDIX A. NUMERICAL METHODS AND DATA

All computations use double precision in Python (NumPy, SciPy), except where a number of digits is stated; those use mpmath or Python's `decimal` arithmetic. The zeros of ζ with $0 < \gamma < 600$ (341 ordinates) and the first 1000 ordinates were computed once and are reused throughout; where a statement depends on the precision of an ordinate (Remark 2.12), the ordinates are refined to 10^{-12} . ζ is evaluated by Euler–Maclaurin summation with Bernoulli corrections, and the Davenport–Heilbronn function f through Hurwitz zeta functions; its eight off-line zeros with $\gamma < 400$ are those listed in [CE17]. $\Lambda(n)$ is computed by sieve up to 10^8 (up to 10^9 for ψ). Every value in the tables was obtained by the script cited in its Status line, and every script prints the numbers quoted in the text.

- *Section 2.* The transition law, the profile Φ , the floor law on 120 and 240 zeros, and the precision test (`fresnel/fresnel_transition.py`, with the script of [C1] in `fresnel/c2_numerics.py`); the monotonicity windows and Gonek's sum (`windows/`).

- *Section 3.* Phase-locking, Riemann–Siegel, Hardy’s theorem, the detector, the mollified anatomy and its flattening (`hardy/hardy_checks.py`); the helix and Turing counts (`helix/`); the scan of $Z(t)$ up to 10^5 with step 0.004 (`z_extrema/`); the detector in C_2^2 (`c2sq/`).

How this work was produced. The cross-energy C_2 and the research program are the author’s, beginning with his preprints of 2017 and the series [C1]–[C8]. The author used Claude (Anthropic), an AI assistant, as a research and writing aid: to derive and check proofs, to write and run the Python computations, to test claims against the Davenport–Heilbronn control and against counterexamples, to search the literature, and to draft and edit the text and figures. In preparing this series, the material of the consolidated paper (version 3), of the earlier series [C1]–[C8] and of the author’s manuscript on the asymptotic balance of partial sums was inventoried result by result, regrouped into the seventeen parts of the series (Table 1 of [CE1]), and checked again; the corrections found in the process are stated where they apply and listed in Part 16 [CE16]. Several claims of the author’s earlier preprints were found to be incorrect and are not used. The author directed the work, checked the results and is responsible for the content.

Data availability. The scripts are included in the ancillary directory `anc/` of the source package, one subdirectory per topic as listed above. The earlier preprints [C1]–[C8] and the versions 1–4 of the consolidated paper are deposited on Zenodo under the DOIs listed in the references.

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