

# CROSS-ENERGY OF DIRICHLET PARTIAL SUMS.

## PART 4: ZEROS OF TRUNCATIONS: SPIRALS, SECTIONS AND THE CIRCLES OF THE PRIMES

PEDRO CACERES

**ABSTRACT.** We study what the zeros of three truncations of the Riemann zeta function, and the zeros of  $\zeta$  seen on the Euler factors of single primes, say about the real parts of the zeros. For a simple zero  $\rho = \beta + i\gamma$ , the nearby zero  $\rho_n$  of  $F_n(s) = X_{n-1}(s) - n^{1-s}/(1-s)$ ,  $X_n$  the partial sums, satisfies  $\rho_n = \rho + n^{-\rho}/(2\zeta'(\rho)) + O_\rho(n^{-2\beta} \log n + n^{-\beta-1})$ : a logarithmic spiral of radius  $n^{-\beta}/(2|\zeta'(\rho)|)$ . On the critical line the trajectory crosses the line  $\gamma \log 2/\pi + \vartheta$  times,  $|\vartheta| \leq 1$ , in every range  $(N, 2N]$  with  $N \geq N_0(\rho)$  (no uniform  $N_0$  known), and sampling at the integers sees every crossing; off the line it stops crossing. With the endpoint term  $\frac{1}{2}N^{-s}$  restored the zeros form helices of radius  $|\rho|N^{-\beta-1}/(12|\zeta'(\rho)|)$  around the lines through the zeros, with explicit constants and hypotheses certified in interval arithmetic for 100 zeros. An off-line pair converges at two rates whose product does not depend on  $\beta$ : the spirals measure  $\beta$  but do not force it. For the sections  $\sum_{k \leq N} k^{-s}$  the real parts of the zeros are governed by a polygon on a torus with one free direction per prime: a new prime never lowers the right edge, the powers 4, 8, 16 lower it (in interval arithmetic), and numerically every prime power  $p^k$ ,  $k \geq 2$ , up to 128 except  $11^2$  does. The “ellipse” of the author’s 2024 preprint is an exact circle, the Euler factor of a “prime”  $e$ ; we correct it and show that a fraction  $1 - \text{gd}(\frac{1}{2})/\pi = 84.7\%$  of the zeros lie left of its centre (85.0% observed). Under RH, on the circle of a true prime, Landau’s formula makes the zeros lean toward the angle  $\pi$  (under GRH toward  $\pi + \arg \chi(p)$  for  $L(s, \chi)$ ); the bias lives in the exact positions and is blind to  $\beta$ . We do not prove the Riemann hypothesis.

### 1. INTRODUCTION

**1.1. Truncations and circles.** Write  $s = \sigma + it$ ,  $X_N(s) = \sum_{k \leq N} k^{-s}$  for the partial sums of the Riemann zeta function, and, as in Part 1 [CE1],  $X_N^C(s) = X_N(s) - \frac{1}{2}N^{-s}$  and  $Y_N(s) = N^{1-s}/(1-s)$ . In the critical strip  $X_N$  and  $Y_N$  both diverge and  $\zeta(s) = \lim_N [X_N^C(s) - Y_N(s)]$ ; this additive representation, the cross-energy  $C_2(n, s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma}$  built on it, and the helix that  $C_2$  describes at a fixed point  $s$  (Theorem A, [CE1, Thm. 3.2]) are the subject of [CE1]. This paper looks at the other side of the same identities: at the *zeros* of three truncations of  $\zeta$ ,

$$\begin{aligned} F_n(s) &= X_{n-1}(s) - Y_n(s), & S_N(s) &= X_N(s), \\ D_N(s) &= X_N^C(s) - Y_N(s) = X_N(s) - \frac{1}{2}N^{-s} - \frac{N^{1-s}}{1-s}, \end{aligned}$$

and at what the zeros of  $\zeta$  look like on the circles  $|p^{-s}| = p^{-\sigma}$  of the Euler factors  $(1 - p^{-s})^{-1}$ .

The three truncations differ from  $\zeta$  by terms of increasing order. By Euler–Maclaurin summation,  $F_n = \zeta - \frac{1}{2}n^{-s} + O_s(n^{-\sigma-1})$  and  $D_N = \zeta - \frac{s}{12}N^{-s-1} + O_s(N^{-\sigma-3})$ , while  $S_N$  differs from  $\zeta$  by the divergent term  $Y_N$  and has zeros far from those of  $\zeta$ . In both convergent cases the leading error is a multiple of  $n^{-s}$ , which on the vertical line  $\text{Re } s = \sigma$  runs over the circle  $|w| = n^{-\sigma}$ , and

---

*Date:* October 2026. Part 4 of the series *Cross-energy of Dirichlet partial sums* (seventeen parts; the plan of the series is in Part 1 [CE1]).

*2020 Mathematics Subject Classification.* Primary 11M26; Secondary 11M06, 30C15, 11Y35.

*Key words and phrases.* Riemann zeta function, partial sums, zeros of sections, Euler–Maclaurin summation, Rouché’s theorem, Turing’s method, Kronecker’s theorem, Landau’s formula, Euler factors, Davenport–Heilbronn function.

a simple zero  $\rho$  of  $\zeta$  is followed by a zero of the truncation that turns around  $\rho$  on a scaled copy of the curve  $n \mapsto n^{-\rho}$ : a logarithmic spiral of radius proportional to  $n^{-\beta}$  (Sections 2 and 3). The rate of convergence is the real part of the zero. For the sections  $S_N$ , the substitution  $x_p = p^{-s}$  turns  $S_N$  into a polynomial in one variable per prime  $p \leq N$ , and Kronecker’s theorem turns the question of where its zeros lie into a question about closed polygons on the torus of the prime circles (Section 4). Finally, the zeros of  $\zeta$  themselves can be placed on the circle of one Euler factor at the angle  $\gamma \log p$ ; Landau’s formula says what they see there (Section 6). The case  $p = e$ , which is not a prime, is the “ellipse” of the author’s preprint [E24], and Section 5 is its erratum.

**1.2. What is classical and what is new.** The approximation of zeros of  $\zeta$  by zeros of truncations is classical in several forms. Zeros of the sections  $S_N$  were studied by Turán [Tu], who showed that their absence from  $\sigma > 1$  for all large  $N$  would imply the Riemann hypothesis (RH), by Montgomery [Mo83] and Montgomery and Vaughan [MV], who located the right edge of their zeros asymptotically, and numerically by Borwein, Fee, Ferguson and van der Waall [BFFW]; Gonek and Ledoan [GL] count them, and Platt and Trudgian [PT16] decided for which  $N$  they reach  $\sigma > 1$ . Landau’s formula [Lan] and its consequences for the distribution of  $\gamma\alpha$  modulo 1 (Hlawka [Hl], Ford and Zaharescu [FZ]) are classical, as is Turing’s method [Tg] for verifying RH up to a given height. What we add is the following: the spiral description of the convergence of the zeros of  $F_n$  and  $D_N$ , with an explicit and checkable version; the observation that an off-line pair converges at two rates whose product is fixed, so that these truncations measure  $\beta$  without constraining it; a crossing count that is a version of Turing’s method in which each zero carries the weight  $\gamma \log 2/\pi$ ; the reduction of the right edge of the zeros of  $S_N$  to a polygon condition with one free side per large prime, with the prime step and the behaviour of prime powers; and the correction of [E24], together with the comparison between the circle of  $e$ , the circles of the primes and the twisted circles of a Dirichlet  $L$ -function. Table 1 separates the two kinds of statements; “new” means new as far as we know, and several of the new statements are elementary once formulated.

| statement                                    | status                                         | origin                                           |
|----------------------------------------------|------------------------------------------------|--------------------------------------------------|
| Proposition 1 (spiral of a zero)             | proved                                         | new in this form; standard perturbation argument |
| Theorems 5 and 6 (crossings)                 | proved, no uniform $N_0(\rho)$                 | new                                              |
| Lemma 9 (Backlund)                           | proved                                         | classical [Ed]                                   |
| Proposition 10 (explicit spiral)             | proved; hypotheses certified in Arb, 100 zeros | new                                              |
| Proposition 12 (torus)                       | proved                                         | classical (Bohr, Kronecker)                      |
| Propositions 13, 14 (prime step, polygons)   | proved                                         | new, elementary                                  |
| Proposition 15 (powers of two)               | computer-assisted, Arb                         | new                                              |
| Propositions 16, 18 (circle of $e$ )         | proved                                         | erratum to [E24]; elementary                     |
| Theorem 19, Proposition 21 (Landau)          | classical; twisted case in outline             | classical [Lan]                                  |
| Corollary 20 (bias on the circle of a prime) | conditional on RH                              | reading of [Lan]                                 |
| heat-flow Landau law                         | proved in [CE15]                               | summary of Part 15                               |

TABLE 1. What is proved, and what is classical.

**1.3. Results and their status.** The status of each result is stated where it appears; here is the list.

- *The spiral of a zero* (Proposition 1, proved): the zero  $\rho_n$  of  $F_n$  near a simple zero  $\rho$  satisfies  $\rho_n = \rho + n^{-\rho}/(2\zeta'(\rho)) + O_\rho(n^{-2\beta} \log n + n^{-\beta-1})$ . Numerically the exponents are  $-0.494$  and  $-0.485$  at  $\rho_1, \rho_2$ , and  $-0.809, -0.191$  at the off-line pair of the Davenport–Heilbronn function, whose rates differ by  $2(\beta - \frac{1}{2})$  (Figure 1).

- *Crossings of the spiral* (Theorem 5, continuous crossings, and Theorem 6, sampling at the integers, both proved; Proposition 7, no crossings off the line, proved): for a simple zero on the critical line and  $N \geq N_0(\rho)$ , the trajectory  $\operatorname{Re} \rho_x - \frac{1}{2}$  has exactly  $\gamma \log 2/\pi + \vartheta$  zeros in  $(N, 2N]$ ,  $|\vartheta| \leq 1$ , all simple and more than one unit apart, so that the sequence  $\operatorname{Re} \rho_n - \frac{1}{2}$  changes sign the same number of times; off the line it stops changing sign. The threshold  $N_0(\rho)$  depends on  $\rho$ , and the proof gives no uniform bound for it; consecutive zeros are more than one unit apart once  $N \geq \gamma$  as well. Numerically for the first 29 zeros and for the Davenport–Heilbronn function (Figure 2); for finitely many zeros this is a Turing-type count (Remark 8).
- *The explicit spiral* (Lemma 9, classical; Proposition 10, proved): under three checkable hypotheses  $D_N$  has exactly one zero  $\rho_N$  near  $\rho$  and  $\rho_N = \rho + \rho N^{-\rho-1}(1 + \theta_N)/(12\zeta'(\rho))$  with an explicit bound for  $\theta_N$ . The proposition is a theorem conditional on (H1)–(H3); separately, the hypotheses are certified in interval arithmetic for the first 100 zeros and all  $N \geq N_0^{\text{cert}}(\rho)$ ,  $0.39|\rho| \leq N_0^{\text{cert}} \leq 0.64|\rho|$  (computer-assisted). For the Davenport–Heilbronn function the corrected Hurwitz pieces converge at the rate  $M^{-(\beta+2)}$  (Remark 11, proved in outline, and numerical; Figure 3).
- *Zeros of the sections* (Proposition 12, proved, after Bohr and Kronecker; Proposition 13, the prime step, proved; Proposition 14, polygon reduction, proved; Proposition 15,  $e(2^k) < e(2^k - 1)$  for  $k = 2, 3, 4$ , computer-assisted in interval arithmetic): the right edge  $e(N)$  of the real parts of the zeros of  $S_N$  rises at every prime and falls at every prime power  $p^k$ ,  $k \geq 2$ , up to 128 except  $121 = 11^2$  (numerical, Figure 4); the mechanism for the powers of 2 is heuristic, and we conjecture  $e(2^k) < e(2^k - 1)$  for every  $k \geq 2$ .
- *The circle of  $e$*  (Propositions 16 and 18, proved; Remark 17, proved): the curve of [E24] is the circle with centre  $r^2/(1 - r^2)$  and radius  $r/(1 - r^2)$ ,  $r = e^{-\alpha}$ ; equidistributed angles put a fraction  $1 - \operatorname{gd}(\alpha)/\pi$  of the zeros left of its centre, 84.71% at  $\alpha = \frac{1}{2}$  against 85.04% observed for the 341 zeros with  $0 < \gamma < 600$  (Figure 5). The critical circle is the fixed circle of the involution  $s \mapsto 1 - \bar{s}$ , and this is all that the symmetry gives.
- *The circles of the primes* (Theorem 19, classical [Lan]; Corollary 20, conditional on RH; Proposition 21, proved in outline; Remark 22): under RH the zeros lean toward  $\pi$  on the circle of every prime power and are unbiased on the circle of any other base; under GRH, for  $L(s, \chi)$  they lean toward  $\pi + \arg \chi(p)$ . Without RH, Landau’s formula weights each angle by  $x^{\beta-1/2}$ , and no statement about the angles alone follows. Numerically (Table 2, Figure 6) the bias survives a random displacement of each zero by a tenth of the mean spacing and is absent for points with the local statistics of the Gaussian unitary ensemble.
- *Landau’s formula under the heat flow* (Section 6, a summary of Part 15 [CE15]: proved there for fixed  $n$  and  $0 < t \leq \frac{1}{2}$  [CE15, Thm. 11.1], and tested on exactly computed zeros of  $H_t$ ): along the de Bruijn–Newman deformation the bias on the circle of a prime decays like  $p^{-(t/4)\log(\gamma/2\pi p)}$  and rotates by  $-\frac{\pi}{8}t \log p$ , and the zeros acquire a bias on the circles of composite numbers  $n$ , proved wherever the deformed von Mangoldt function satisfies  $\Lambda_t(n) > 0$ , that is for prime powers and all squarefree  $n$ , and numerical for the other composites ( $n \leq 2 \cdot 10^4$ ).

None of these results bears on RH beyond what is said: each truncation reads off  $\beta$  for a zero that has been located, and none excludes  $\beta \neq \frac{1}{2}$ . Section 7 compares the circles that occur here with those of Li’s criterion, of graphs and of rounded primes, and Section 8 lists open problems.

**1.4. Relation to earlier work of the author.** The material of Sections 2–4 and 6 appeared in Section 2 of version 3 of [CP]; the material of Section 5 and 7 in Part VIII of an earlier series [C8]. Both are collected and revised here. Two corrections with respect to version 3 of [CP] are made: the angle of the spiral is  $-\gamma \log n - \arg \zeta'(\rho)$ , not  $\pi - \gamma \log n - \arg \zeta'(\rho)$ , so that the sign in the crossing formula is  $+$  (checked numerically, see Section 2), and the error term of Proposition 1

carries a factor  $\log n$ . Section 5 is the erratum [Er] to [E24]. Li's coefficients, which Part VIII computed on a third circle, are in [CE5]; the Ihara zeta function of regular graphs and the primes rounded to a common scale are in Part 17 [CE17].

**1.5. Notation.** Throughout  $\rho = \beta + i\gamma$  is a nontrivial zero of  $\zeta$ ,  $\rho_1 = \frac{1}{2} + i\gamma_1$ ,  $\gamma_1 = 14.1347\dots$ ,  $\Lambda$  is von Mangoldt's function and  $N(T)$  the number of zeros with  $0 < \gamma \leq T$ . The Davenport–Heilbronn function is  $f(s) = \sum_n a(n)n^{-s}$  with  $a$  periodic modulo 5,  $a(1) = 1$ ,  $a(2) = \kappa$ ,  $a(3) = -\kappa$ ,  $a(4) = -1$ ,  $a(5) = 0$ ,  $\kappa = (\sqrt{10 - 2\sqrt{5}} - 2)/(\sqrt{5} - 1)$  [DH]; it satisfies a functional equation of the same type as  $\zeta$ , has no Euler product and has zeros off the critical line, the first at  $0.808517 + 85.699348i$  and its partner  $0.191483 + 85.699348i$  [CE17]. In Section 6,  $\chi$  is a Dirichlet character (in [CE3] the same letter denotes the factor of the functional equation). Part 11 [CE11] writes  $D_N$  for the partial sum  $X_N$ ; here  $D_N$  is the corrected sum  $X_N^C - Y_N$ .

## 2. THE SPIRAL OF A ZERO

We use Euler–Maclaurin summation in the form of [CE1, Sect. 2] and [T, Sect. 2.1]: for an integer  $n \geq 1$ ,  $X_{n-1}(s) = \zeta(s) - \zeta(s, n)$ , where  $\zeta(s, x)$  is the Hurwitz zeta function, and

$$(1) \quad \zeta(s, x) = \frac{x^{1-s}}{s-1} + \frac{1}{2}x^{-s} + \frac{s}{12}x^{-s-1} + O_K(x^{-\sigma-3})$$

uniformly for  $s$  in a compact set  $K \subset \{\sigma > -3\} \setminus \{1\}$  and  $x \geq 1$ ; the remainder is  $-\frac{s(s+1)(s+2)}{6} \int_0^\infty \tilde{B}_3(v)(x+v)^{-s-3} dv$ , with  $\tilde{B}_3$  the periodic Bernoulli function, and since  $\tilde{B}_3$  has mean 0 one more integration by parts gives the stated order. The function  $E(s, x) = \zeta(s, x) - x^{1-s}/(s-1)$  is analytic at  $s = 1$  as well, and the expansion of  $E$  holds on compact subsets of  $\sigma > -3$ . Hence

$$(2) \quad F_n(s) = X_{n-1}(s) - \frac{n^{1-s}}{1-s} = \zeta(s) - \frac{1}{2}n^{-s} - \frac{s}{12}n^{-s-1} + O_K(n^{-\sigma-3}).$$

**Proposition 1** (the spiral of a zero). *Let  $\rho = \beta + i\gamma$  be a simple zero of  $\zeta$  with  $0 < \beta < 1$ . There are  $\delta > 0$  and  $n_1$ , depending on  $\rho$ , such that for  $n \geq n_1$  the function  $F_n$  has exactly one zero  $\rho_n$  in the disc  $|s - \rho| < \delta$ , and*

$$\rho_n = \rho + \frac{n^{-\rho}}{2\zeta'(\rho)} + O_\rho(n^{-2\beta} \log n + n^{-\beta-1}).$$

*Thus  $\rho_n$  moves on a logarithmic spiral around  $\rho$ , at the angle  $-\gamma \log n - \arg \zeta'(\rho)$  and at the distance  $n^{-\beta}/(2|\zeta'(\rho)|)$  to leading order; the rate of convergence is  $n^{-\beta}$ .*

*Proof.* Choose  $\delta < \min(\beta, 1 - \beta)/2$  so small that  $\rho$  is the only zero of  $\zeta$  in  $|s - \rho| \leq \delta$ , and let  $c = \min_{|s-\rho|=\delta} |\zeta(s)| > 0$ . By (2),  $|F_n - \zeta| \leq n^{-\beta+\delta}$  on the closed disc for  $n$  large, which is  $< c$ ; by Rouché's theorem  $F_n$  has exactly one zero  $\rho_n$  in the disc. Write  $\varepsilon = \rho_n - \rho$  and  $d = \zeta'(\rho)$ . Since  $|\zeta(s)| \geq |d||s - \rho|/2$  for  $|s - \rho|$  small,  $0 = F_n(\rho_n)$  and (2) give first  $|\varepsilon| = O_\rho(n^{-\beta})$ . Then  $\zeta(\rho_n) = d\varepsilon + O_\rho(|\varepsilon|^2)$  and

$$\frac{1}{2}n^{-\rho_n} = \frac{1}{2}n^{-\rho}e^{-\varepsilon \log n} = \frac{1}{2}n^{-\rho}(1 + O(|\varepsilon| \log n)),$$

so that (2) at  $\rho_n$  reads  $d\varepsilon = \frac{1}{2}n^{-\rho} + O_\rho(n^{-2\beta} \log n + n^{-\beta-1})$ . □

*Status:* proved. Version 3 of [CP] gave the angle as  $\pi - \gamma \log n - \arg \zeta'(\rho)$  and the error without the factor  $\log n$ ; the sign is fixed by the computation above, and numerically, at  $\rho_1$  with  $\zeta'(\rho_1) = 0.783297 + 0.124700i$ , the displacement  $\rho_n - \rho_1$  for  $n = 10^3$ ,  $3 \cdot 10^3$  and  $10^4$  differs in argument from  $n^{-\rho_1}/(2\zeta'(\rho_1))$  by  $-0.021\pi$ ,  $0.006\pi$  and  $-0.018\pi$ , not by  $\pi$  (`zerospiral/sign_check.py`).

*Remark 2* (an off-line pair, and the helix of the cross-energy). A zero on the critical line is its own reflection under  $s \mapsto 1 - \bar{s}$ , and its spiral closes at the rate  $n^{-1/2}$ , circling the line. An off-line pair  $\rho, 1 - \bar{\rho}$  gives two spirals with the rates  $n^{-\beta}$  and  $n^{-(1-\beta)}$ ; their product is  $n^{-1}$  whatever  $\beta$  is, so the functional equation imposes no constraint here. Proposition 1 measures  $\beta$  for a zero that has been located, but does not force  $\beta = \frac{1}{2}$ . It complements the helix of [CE1, Thm. 3.2]: at a point  $s$ ,  $X_{n-1}(s)\overline{n^{-s}}$  turns around the centre  $n^{1-2\sigma}/(1-s)$  at the radius  $|\zeta(s)|n^{-\sigma}$  with the angle  $t \log n + \arg \zeta(s)$ ; at a zero this radius vanishes, and what remains turning with the angle  $\gamma \log n$  is the spiral of the approximate zero. *Status*: proved (consequence of Proposition 1).

*Remark 3* (the normalized displacement and the cross-energy at  $\rho_n$ ). Put  $\mathcal{J}_n(\rho) = 2\zeta'(\rho)(\rho_n - \rho)$ . Proposition 1 gives  $\mathcal{J}_n(\rho) = n^{-\rho} + O_\rho(n^{-2\beta} \log n + n^{-\beta-1})$ , that is,

$$n^\rho \mathcal{J}_n(\rho) = 1 + O_\rho(n^{-\beta} \log n + n^{-1}),$$

for every simple zero in the strip. This normalization removes the dependence on  $\beta$  of the leading term, and with it the information that the spiral carries about  $\beta$ . On the side of the cross-energy there is an exact identity at the zeros of the truncation. Since  $F_n(\rho_n) = 0$  means  $X_{n-1}(\rho_n) = Y_n(\rho_n) = n^{1-\rho_n}/(1-\rho_n)$ , and  $C_2(n, s) = 2 \operatorname{Re}(X_{n-1}(s)\overline{n^{-s}})$ ,

$$C_2(n, \rho_n) = 2n^{1-2\sigma_n} \operatorname{Re} \frac{1}{1-\rho_n} = \frac{2(1-\sigma_n)n^{1-2\sigma_n}}{|1-\rho_n|^2}, \quad \sigma_n = \operatorname{Re} \rho_n.$$

At  $\rho_n$  the point  $X_{n-1}\overline{n^{-s}}$  of the helix of [CE1, Thm. 3.2] sits exactly at its centre  $n^{1-2\sigma}/(1-s)$ , and the phase  $\gamma \log n$  enters  $C_2(n, \rho_n)$  only through  $\rho_n$ , to leading order through  $\sigma_n = \beta + \operatorname{Re}(n^{-\rho}/(2\zeta'(\rho))) + \dots$ , so that

$$\log C_2(n, \rho_n) = (1-2\beta) \log n + \log \frac{2(1-\beta)}{|1-\rho|^2} - \operatorname{Re}(n^{-\rho}/\zeta'(\rho)) \log n + o(n^{-\beta} \log n).$$

The identity holds to 22 digits in the computations of `spiral_crossings/continuous_check.py`. *Status*: proved (the identity is exact); like the spiral itself, it reads  $\beta$  and does not constrain it.

We followed  $\rho_n$  by Newton's method for  $100 \leq n \leq 2 \cdot 10^5$  (`zerospiral/spz.py`, Figure 1). For  $\rho_1$  the distance  $|\rho_n - \rho_1|$  decays like  $n^{-0.494}$  (least squares for  $n \geq 10^3$ ) and equals  $1.43 \cdot 10^{-3}$  at  $n = 2 \cdot 10^5$ , as predicted by  $\frac{1}{2}n^{-1/2}/|\zeta'(\rho_1)| = 1.41 \cdot 10^{-3}$ ; for  $\rho_2$  the exponent is  $-0.485$ . For the Davenport–Heilbronn function the same perturbation argument applies to its partial sums  $\sum_{k \leq n} a(k)k^{-s}$ , with  $n$  a multiple of 5: there is no pole term, and since  $\sum_{b=1}^4 a(b) = 0$  the tail is  $cn^{-s} + O(|s|n^{-\sigma-1})$  with  $c = -\frac{1}{5} \sum_b b a(b) \neq 0$ , so the rate is again  $n^{-\beta}$ . Its on-line zero at  $\gamma = 43.0813$  converges like  $n^{-0.499}$ , the off-line zero  $0.8085 + 85.699i$  like  $n^{-0.809}$  and its partner  $0.1915 + 85.699i$  like  $n^{-0.191}$ , against the predicted exponents  $-0.5$ ,  $-0.8085$  and  $-0.1915$ . The two rates of the pair differ by  $0.618$ , against  $2(\beta - \frac{1}{2}) = 0.617$  (further details in Appendix A). *Status*: numerical.

**2.1. Crossings of the spiral.** The spiral can be read without knowing its centre. To speak of crossings between consecutive integers we extend  $F_n$  to real  $x \geq 1$  by

$$F_x(s) = \zeta(s) - E(s, x), \quad E(s, x) = \zeta(s, x) - \frac{x^{1-s}}{s-1},$$

which agrees with  $F_n$  at the integers. We separate the statement into a continuous one, about the trajectory  $x \mapsto \rho_x$  (Theorem 5), and a discrete one, about its samples at the integers (Theorem 6). Both need, besides (1), the size of the  $s$ - and  $x$ -derivatives of  $E$ .

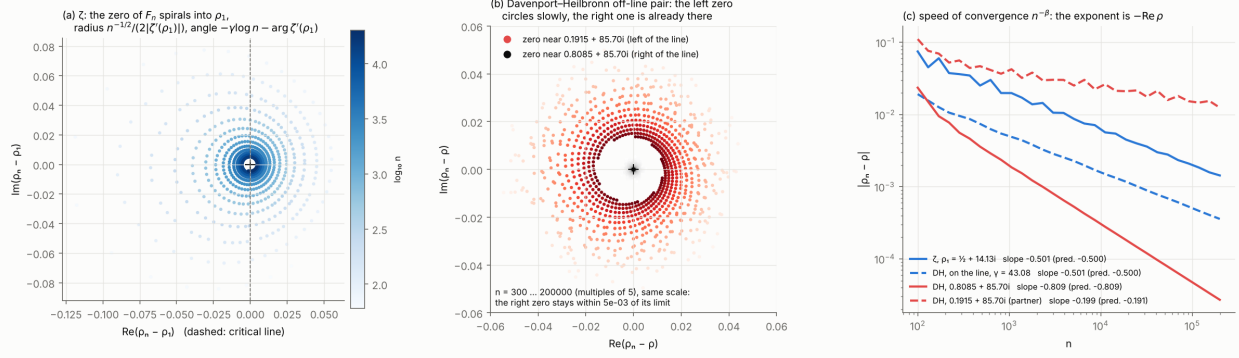


FIGURE 1. The spiral of a zero (Proposition 1). (a) The zero  $\rho_n$  of  $F_n$  near  $\rho_1$ , as  $\rho_n - \rho_1$ , for  $60 \leq n \leq 20000$ , coloured by  $\log_{10} n$ . (b) The two zeros of the Davenport–Heilbronn partial sums near its off-line pair at height 85.699, on the same scale ( $300 \leq n \leq 2 \cdot 10^5$ , multiples of 5): the zero left of the critical line circles slowly, the zero right of it stays close to its limit. (c)  $|\rho_n - \rho|$  against  $n$  ( $n$  a multiple of 5), with the slopes fitted on this grid for  $n \geq 10^3$  (they differ from those quoted in the text, fitted on another grid, by at most 0.008) and the predicted slopes  $-\text{Re } \rho$ .

**Lemma 4** (the error term and its derivatives). *Let  $K$  be a compact subset of the strip  $0 < \sigma < 1$ . Uniformly for  $s \in K$  and  $x \geq 3$ ,*

- (a)  $E(s, x) = \frac{1}{2}x^{-s} + O_K(x^{-\sigma-1})$ ,
- (b)  $\partial_s E(s, x) = -\frac{1}{2}x^{-s} \log x + O_K(x^{-\sigma-1} \log x)$ ,
- (c)  $\partial_x E(s, x) = x^{-s} - s\zeta(s+1, x) = -\frac{s}{2}x^{-s-1} + O_K(x^{-\sigma-2})$ .

*Proof.* (a) is (1). (b) Let  $K'$  be the closed  $\frac{1}{\log 3}$ -neighbourhood of  $K$ , a compact subset of  $\sigma > -1$ . The function  $E(s, x)$  is analytic in  $s$  (the poles at  $s = 1$  of  $\zeta(s, x)$  and of  $x^{1-s}/(s-1)$  cancel), and by (1), which holds uniformly on  $K'$ ,  $r(s, x) = E(s, x) - \frac{1}{2}x^{-s}$  satisfies  $|r(s', x)| \leq Cx^{-\text{Re } s'-1}$  for  $s' \in K'$  and  $x \geq 1$ . Cauchy's estimate on the circle  $|s' - s| = 1/\log x$ , on which  $x^{-\text{Re } s'} \leq e x^{-\sigma}$ , gives  $|\partial_s r(s, x)| \leq C e x^{-\sigma-1} \log x$ , and  $\partial_s(\frac{1}{2}x^{-s}) = -\frac{1}{2}x^{-s} \log x$ . (c) For  $\sigma > 1$ ,  $\partial_x \zeta(s, x) = -s \sum_{k \geq 0} (k+x)^{-s-1} = -s\zeta(s+1, x)$ ; both sides are analytic in  $s \neq 1$  (the pole of  $\zeta(s+1, x)$  at  $s = 0$  is cancelled by the factor  $s$ ), so the identity holds for all  $s \neq 1$ . With  $\partial_x[x^{1-s}/(s-1)] = -x^{-s}$  this gives the first equality. By (1) for  $\zeta(s+1, x)$ ,  $s\zeta(s+1, x) = x^{-s} + \frac{s}{2}x^{-s-1} + O_K(x^{-\sigma-2})$ .  $\square$

By Lemma 4(a) the proof of Proposition 1 applies to  $F_x$  verbatim: for  $x \geq x_1(\rho)$  there is exactly one zero  $\rho_x$  of  $F_x$  in  $|s - \rho| < \delta$ , and  $\rho_x = \rho + x^{-\rho}/(2\zeta'(\rho)) + O_\rho(x^{-2\beta} \log x + x^{-\beta-1})$ .

**Theorem 5** (continuous crossings). *Let  $\rho = \frac{1}{2} + i\gamma$  be a simple zero of  $\zeta$ ,  $d = \zeta'(\rho)$ ,  $\phi(x) = \gamma \log x + \arg d$  and  $u(x) = \text{Re } \rho_x - \frac{1}{2}$ . There are  $X(\rho)$  and  $C(\rho)$  such that, for  $x \geq X(\rho)$ :*

- (i)  $\rho_x$  is a real-analytic function of  $x$ , and

$$u(x) = \frac{\cos \phi(x)}{2|d|\sqrt{x}} + e(x), \quad |e(x)| \leq C(\rho) \frac{\log x}{x}, \quad |e'(x)| \leq C(\rho) \frac{\log x}{x^2};$$

- (ii) the zeros of  $u$  are simple, and they are in increasing bijection  $y_k \leftrightarrow x_k$  with the zeros  $x_k$  of  $\cos \phi$ , with  $|\phi(y_k) - \phi(x_k)| \leq \tau(y_k)$ , where  $\tau(x) = C(\rho)x^{-1/2} \log x$ ;
- (iii) there is  $N_0(\rho) \geq X(\rho)$  such that for  $N \geq N_0(\rho)$  the number of zeros of  $u$  in  $(N, 2N]$  is

$$\frac{\gamma \log 2}{\pi} + \vartheta, \quad |\vartheta| \leq 1.$$

*Proof. Step 1: the trajectory and its velocity.* Write  $\varepsilon(x) = \rho_x - \rho$  and  $\mu(x) = x^{-\rho}/(2d)$ , so that  $|\mu(x)| = x^{-1/2}/(2|d|)$  and  $\varepsilon = \mu + O_\rho(x^{-1} \log x)$ . By Lemma 4(b),  $\partial_s F_x(\rho_x) = \zeta'(\rho_x) - \partial_s E(\rho_x, x) = d + O_\rho(|\varepsilon| + x^{-1/2} \log x) = d(1 + O_\rho(x^{-1/2} \log x))$ , which does not vanish for  $x$  large. Since  $F_x(s)$  is real-analytic in  $x$  and analytic in  $s$ , the implicit function theorem makes  $\rho_x$  real-analytic, and differentiating  $F_x(\rho_x) = 0$  gives

$$\varepsilon'(x) = \frac{\partial_x E(\rho_x, x)}{\partial_s F_x(\rho_x)}.$$

By Lemma 4(c) and  $x^{-\rho_x} = x^{-\rho} e^{-\varepsilon \log x} = x^{-\rho}(1 + O_\rho(x^{-1/2} \log x))$ ,

$$\partial_x E(\rho_x, x) = -\frac{\rho + \varepsilon}{2} x^{-\rho_x-1} + O_\rho(x^{-5/2}) = -\frac{\rho}{2} x^{-\rho-1} + O_\rho(x^{-2} \log x),$$

and dividing by  $\partial_s F_x(\rho_x)$ , whose relative error is  $O_\rho(x^{-1/2} \log x)$ , gives  $\varepsilon'(x) = -\rho x^{-\rho-1}/(2d) + O_\rho(x^{-2} \log x) = \mu'(x) + O_\rho(x^{-2} \log x)$ . The real part of  $\mu$  is  $\cos \phi(x)/(2|d|\sqrt{x})$ , and (i) follows with  $e = \operatorname{Re}(\varepsilon - \mu)$ .

*Step 2: the phase variable.* Put  $v(x) = 2|d|\sqrt{x}u(x) = \cos \phi(x) + w(x)$  with  $w = 2|d|\sqrt{x}e$ . By (i),  $|w| \leq C_1 x^{-1/2} \log x$  and  $|w'| \leq |d|x^{-1/2}|e| + 2|d|\sqrt{x}|e'| \leq C_2 x^{-3/2} \log x$ . Since  $\phi'(x) = \gamma/x$ , the function  $W(\phi) = w(x(\phi))$  satisfies  $|dW/d\phi| = (x/\gamma)|w'| \leq (C_2/\gamma)x^{-1/2} \log x$ . Let  $\kappa(x) = \max(C_1, C_2/\gamma)x^{-1/2} \log x$ , which decreases for  $x > e^2$ , and choose  $X(\rho) \geq \max(x_1, e^2)$  with  $\kappa(X) \leq \frac{1}{4}$  and, enlarging it, with  $\phi(X) \in \pi\mathbb{Z}$ , so that  $|\cos \phi(X)| = 1$ . Then for  $x \geq X$ , as a function of  $\phi$ ,  $V(\phi) = \cos \phi + W(\phi)$  with  $|W|, |W'| \leq \kappa \leq \frac{1}{4}$ . Let  $c_k = \frac{\pi}{2} + k\pi$  and  $J_k = [c_k - \frac{\pi}{6}, c_k + \frac{\pi}{6}]$ . Outside  $\bigcup_k J_k$ ,  $|\cos \phi| > \frac{1}{2} > |W|$ , so  $V \neq 0$ . On  $J_k$ ,  $|V'| \geq |\sin \phi| - |W'| \geq \frac{\sqrt{3}}{2} - \frac{1}{4} > 0$ , so  $V$  is strictly monotone; at the two ends of  $J_k$ ,  $|\cos \phi| = \frac{1}{2} > |W|$  and  $\cos \phi$  takes opposite signs. Hence  $V$  has exactly one zero  $\psi_k$  in each  $J_k$  with  $c_k > \phi(X)$  (these  $J_k$  lie in  $[\phi(X), \infty)$  by the choice of  $X$ ), it is simple, and  $V$  has no other zeros on  $[\phi(X), \infty)$ . At  $\psi_k$ ,  $|\cos \psi_k| = |W(\psi_k)| \leq \kappa$ , and since  $|\cos(c_k + t)| = |\sin t| \geq 2|t|/\pi$  for  $|t| \leq \frac{\pi}{2}$ ,  $|\psi_k - c_k| \leq \frac{\pi}{2}\kappa$ . With  $y_k = x(\psi_k)$  and  $x_k = x(c_k)$  this is (ii), with  $C(\rho)$  enlarged.

*Step 3: counting.* The zeros of  $u$  in  $(N, 2N]$  are the  $y_k$  with  $\psi_k \in (\phi(N), \phi(N) + L]$ ,  $L = \gamma \log 2$ . Assume  $N_- = Ne^{-\pi/(8\gamma)} \geq X(\rho)$  and put  $\tau_N = \frac{\pi}{2}\kappa(N_-) \leq \frac{\pi}{8}$ . By Step 2,  $|\psi_k - c_k| \leq \frac{\pi}{8}$  for every  $k$ . If  $c_k > \phi(N)$ , or if  $\psi_k > \phi(N)$ , then  $\psi_k > \phi(N) - \frac{\pi}{8}$ , that is  $y_k > N_-$ , so  $\kappa(y_k) \leq \kappa(N_-)$  and  $|\psi_k - c_k| \leq \tau_N$ . Hence every  $k$  with  $c_k \in (\phi(N) + \tau_N, \phi(N) + L - \tau_N]$  has  $\psi_k \in (\phi(N), \phi(N) + L]$ , and every  $k$  with  $\psi_k \in (\phi(N), \phi(N) + L]$  has  $c_k \in (\phi(N) - \tau_N, \phi(N) + L + \tau_N]$ . So the number  $Z$  of zeros of  $u$  in  $(N, 2N]$  lies between the numbers of points of  $\frac{\pi}{2} + \pi\mathbb{Z}$  in the half-open intervals  $(\phi(N) + \tau_N, \phi(N) + L - \tau_N]$  and  $(\phi(N) - \tau_N, \phi(N) + L + \tau_N]$ . A half-open interval of length  $\ell$  contains  $\ell/\pi + \vartheta$  such points with  $|\vartheta| < 1$ , so  $|Z - L/\pi| < 1 + 2\tau_N/\pi$ . If  $L/\pi \in \mathbb{Z}$ , every integer at distance  $< 1 + 2\tau_N/\pi \leq \frac{5}{4}$  from  $L/\pi$  is at distance  $\leq 1$ . Otherwise let  $f \in (0, 1)$  be the fractional part of  $L/\pi$ ; the integers at distance  $< 1 + 2\tau_N/\pi$  from  $L/\pi$  are  $\lfloor L/\pi \rfloor$  and  $\lceil L/\pi \rceil$ , at distance  $< 1$ , as soon as  $2\tau_N/\pi < \min(f, 1 - f)$ . Choosing  $N_0(\rho)$  with  $N_0 e^{-\pi/(8\gamma)} \geq X(\rho)$  and large enough for this condition (which only becomes easier as  $N$  grows, since  $\kappa(N_-)$  decreases) gives (iii).  $\square$

**Theorem 6** (sampling at the integers). *Let  $\rho$  be as in Theorem 5 and let  $N \geq \max(N_0(\rho), \gamma)$  be an integer. Then consecutive zeros of  $u$  in  $[N, \infty)$  are more than one unit apart, and the number of integers  $n \in [N, 2N)$  for which  $u(n)u(n+1) < 0$  or  $u(n+1) = 0$  equals the number of zeros of  $u$  in  $(N, 2N]$ . In particular the sequence  $\operatorname{Re} \rho_n - \frac{1}{2}$ ,  $N \leq n \leq 2N$ , changes sign  $\gamma \log 2/\pi + \vartheta$  times,  $|\vartheta| \leq 1$  (a value 0 counted as a change).*

*Proof.* By Step 2,  $|\psi_k - c_k| \leq \frac{\pi}{2}\kappa(y_k) \leq \frac{\pi}{8}$ , so  $\psi_{k+1} - \psi_k \geq \pi - \frac{\pi}{4} \geq \frac{\pi}{2}$ . As  $\psi_{k+1} - \psi_k = \gamma \log(y_{k+1}/y_k) \leq \gamma(y_{k+1} - y_k)/y_k$ , we get  $y_{k+1} - y_k \geq \pi y_k/(2\gamma) > 1$  for  $y_k \geq N \geq \gamma$ . So each interval  $(n, n+1]$ ,  $n \geq N$ , contains at most one zero of  $u$ , and it is simple. If it contains one in its interior,  $u(n)$  and  $u(n+1)$  are nonzero (a zero at  $n$  or  $n+1$  would be a second zero within distance

1) and of opposite signs; if it contains  $n + 1$ , then  $u(n + 1) = 0$ ; if it contains none,  $u(n + 1) \neq 0$  and  $u$  does not change sign on  $[n, n + 1]$ , so  $u(n)u(n + 1) \geq 0$ . Theorem 5(iii) gives the count.  $\square$

*Status:* Theorems 5 and 6 are proved. The constants  $X(\rho)$ ,  $C(\rho)$  and  $N_0(\rho)$  are effective in principle but not computed. They depend on  $\rho$  through  $\delta$  and  $|\zeta'(\rho)|$ , through the constants of Lemma 4 on a disc around  $\rho$ , and, in Step 3, through the fractional part of  $\gamma \log 2/\pi$ . The bound  $|\vartheta| \leq 1$  is therefore an asymptotic statement for each fixed zero; the proof gives no bound for  $N_0(\rho)$  that is uniform in  $\rho$  (Open problem 1 asks for a uniform version). Numerically, for  $\rho_1$  on  $[10^3, 2 \cdot 10^3]$  and  $[4 \cdot 10^3, 8 \cdot 10^3]$ , and for  $\rho_5, \rho_{10}$  on one dyadic range each, the normalized errors  $x|e|/\log x$  and  $x^2|e'|/\log x$  stay below 0.46 and 14, consecutive zeros of  $u$  are at least 113 apart, and the continuous and the sampled counts agree (3, 7, 11 against  $\gamma \log 2/\pi = 3.12, 7.27, 10.98$ ; `spiral_crossings/continuous_check.py`).

**Proposition 7** (no crossings off the line). *If  $\beta \neq \frac{1}{2}$ , then  $\operatorname{Re} \rho_x - \frac{1}{2}$  has the sign of  $\beta - \frac{1}{2}$  for all large  $x$ . To leading order this holds as soon as  $x^{-\beta}/(2|\zeta'(\rho)|) < |\beta - \frac{1}{2}|$ , that is, for  $x > x_0 = (2|\zeta'(\rho)| |\beta - \frac{1}{2}|)^{-1/\beta}$ .*

*Proof.*  $\operatorname{Re} \rho_x - \frac{1}{2} = (\beta - \frac{1}{2}) + O_\rho(x^{-\beta})$ , and the leading term of the oscillation has amplitude  $x^{-\beta}/(2|\zeta'(\rho)|)$ .  $\square$

*Status:* proved, with the threshold  $x_0$  given to leading order.

For the first 29 zeros of  $\zeta$  ( $\gamma < 100$ ) and  $10^4 \leq n \leq 2 \cdot 10^4$  the number of crossings differs from  $\gamma \log 2/\pi$  by less than 1 (3 against 3.12 at  $\rho_1$ , 16 against 15.90 at  $\gamma = 72.07$ , 22 against 21.81 at  $\gamma = 98.83$ ); the total is 397 against 398.1. For the Davenport–Heilbronn function, with  $n$  a multiple of 5, the on-line zero at  $\gamma = 80.33$  gives 18 crossings against 17.7 for  $N = 500, 2000$  and 8000, and the two zeros of the off-line pair at height 85.699 give none (`spiral_crossings/wind.py`, `spiral_crossings/dhw.py`, Figure 2). *Status:* numerical.

*Remark 8* (a Turing-type count). For finitely many zeros, and  $N$  larger than all their thresholds  $N_0(\rho)$ , the counts of Theorems 5 and 6 add up to a version of Turing’s method [Tg] in which each zero carries the weight  $\gamma \log 2/\pi$ : an off-line pair removes about  $2\gamma \log 2/\pi$  crossings. It is not yet a verification method, since  $N_0(\rho)$  is neither explicit nor uniform; with a uniform and explicit  $N_0$ , it would verify RH up to the height computed and no further, like Turing’s method. The threshold  $\log x_0 = \beta^{-1} \log(1/(2|\zeta'(\rho)| |\beta - \frac{1}{2}|))$  is much shorter than the detection length of the truncated Poisson functional [CE10, Sect. 3], about  $4.05/(\beta - \sigma) + 4.5$  in  $\log x$  at abscissa  $\sigma$ , because  $F_n$  evaluates  $\zeta$  itself and not the primes. For a close off-line pair  $|\zeta'(\rho)|$  is of order  $|\beta - \frac{1}{2}|$ , and  $x_0$  grows like  $|\beta - \frac{1}{2}|^{-4}$ . *Status:* heuristic comparison.

### 3. THE SPIRAL WITH EXPLICIT CONSTANTS

Proposition 1 uses the truncation  $F_n$ , whose error  $-\frac{1}{2}n^{-s}$  fixes the rate  $n^{-\beta}$ . Restoring the end-point term gives the corrected sum  $D_N = X_N^C - Y_N$  of the additive representation [CE1, Prop. 2.3], whose zeros converge faster and with a leading term that can be controlled explicitly. Put

$$T_k(s, N) = \frac{B_{2k}}{(2k)!} s(s+1) \cdots (s+2k-2) N^{-s-2k+1},$$

with the Bernoulli numbers  $B_{2k}$ .

**Lemma 9** (Backlund’s bound). *For  $N \geq 1$ ,  $K \geq 1$  and  $\sigma > -2K - 1$ ,*

$$\zeta(s) = D_N(s) + \sum_{k=1}^K T_k(s, N) + R_K, \quad |R_K| \leq \frac{|s+2K+1|}{\sigma+2K+1} |T_{K+1}(s, N)|.$$

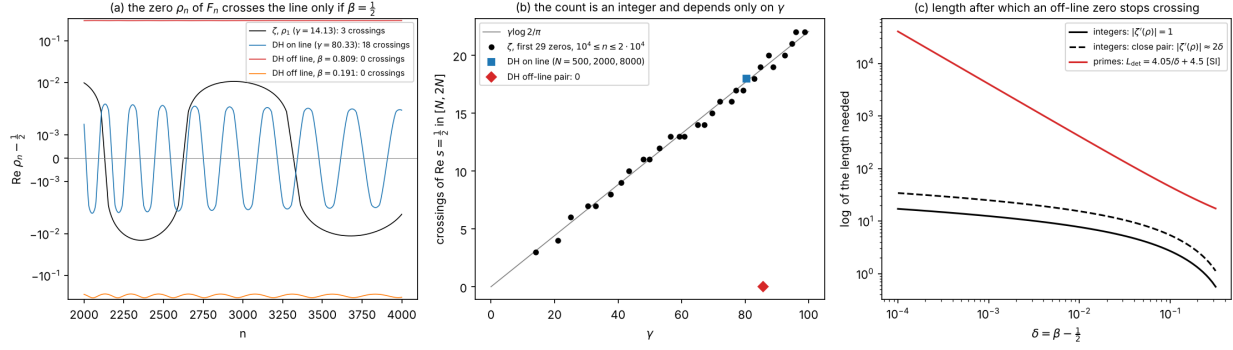


FIGURE 2. Crossings of the spiral (Theorems 5, 6 and Proposition 7). (a)  $\text{Re } \rho_n - \frac{1}{2}$  for  $2000 \leq n \leq 4000$ : the first zero of  $\zeta$ , an on-line zero of the Davenport–Heilbronn function, and the two zeros of its off-line pair at height 85.699 (symmetric logarithmic scale). (b) Crossings of the critical line for  $10^4 \leq n \leq 2 \cdot 10^4$  against  $\gamma \log 2/\pi$ . (c) The length  $\log x_0$  after which an off-line zero at distance  $\delta = \beta - \frac{1}{2}$  from the line stops crossing, for  $|\zeta'(\rho)| = 1$  and for a close pair, against the detection length of the truncated Poisson functional of [CE10, Sect. 3].

Consequently  $\zeta - D_N = T_1(1 + \eta)$  with  $T_1 = \frac{s}{12}N^{-s-1}$  and

$$|\eta| \leq \bar{\eta}_N(s) = \frac{|(s+1)(s+2)|}{60N^2} + \frac{|s+5|}{\sigma+5} \cdot \frac{|(s+1)(s+2)(s+3)(s+4)|}{2520N^4}.$$

*Status:* classical [Ed, Sect. 6.4]; the second statement is the case  $K = 2$ , with  $|T_2/T_1| = |(s+1)(s+2)|/(60N^2)$  and  $|T_3/T_1| = |(s+1) \cdots (s+4)|/(2520N^4)$ . Numerically the remainder is at most 0.26 and 0.39 times Backlund’s bound for  $K = 1, 2$  at the points tested (`helix_explicit/em_check.py`).

**Proposition 10** (the explicit spiral). *Let  $\rho = \beta + i\gamma$  be a simple zero of  $\zeta$ ,  $d = \zeta'(\rho)$ , and put*

$$q_N = \frac{|\rho|N^{-\beta-1}}{12}, \quad r_N = \frac{3q_N}{|d|}, \quad M_2 = \max_{|s-\rho| \leq r_N} |\zeta''(s)|, \quad \bar{\eta} = \max_{|s-\rho| \leq r_N} \bar{\eta}_N(s).$$

*Assume*

- (H1)  $M_2 r_N \leq |d|/2$ ,
- (H2)  $\bar{\eta} \leq \frac{1}{4}$ ,
- (H3)  $(1 + r_N/|\rho|)N^{r_N} \leq \frac{3}{2}$ .

*Then  $D_N$  has exactly one zero  $\rho_N$  in  $|s - \rho| < r_N$ , and*

$$\rho_N = \rho + \frac{\rho N^{-\rho-1}}{12 \zeta'(\rho)} (1 + \theta_N), \quad |\theta_N| \leq \left(1 + \frac{r_N}{|\rho|}\right) N^{r_N} (1 + \bar{\eta}) - 1 + \frac{9M_2 q_N}{2|d|^2}.$$

*For  $N \geq 3$  (for  $N \geq 2$  if  $\beta \geq 0.45$ ) the three hypotheses are monotone in  $N$ : once they hold, they hold for all larger  $N$ .*

*Proof.* On  $|s - \rho| = r_N$ , Taylor’s formula and (H1) give  $|\zeta(s)| \geq |d|r_N - M_2 r_N^2/2 \geq \frac{3}{4}|d|r_N = \frac{9}{4}q_N$ . By Lemma 9,  $|\zeta(s) - D_N(s)| \leq |T_1(s)|(1 + \bar{\eta})$ , and  $|T_1(s)| = \frac{|s|}{12}N^{-\sigma-1} \leq q_N(1 + r_N/|\rho|)N^{r_N}$ , since  $|s| \leq |\rho| + r_N$  and  $\sigma \geq \beta - r_N$ . By (H2) and (H3),  $|\zeta - D_N| \leq \frac{3}{2} \cdot \frac{5}{4}q_N = \frac{15}{8}q_N < \frac{9}{4}q_N$  on the circle. By Rouché’s theorem  $D_N$  and  $\zeta$  have the same number of zeros in the disc, and  $\zeta$  has exactly one, since  $|\zeta(s) - d(s - \rho)| \leq M_2 r_N^2/2 < |d|r_N$  on the circle. Write  $\varepsilon = \rho_N - \rho$ , so  $|\varepsilon| < r_N$ . The equation  $\zeta(\rho_N) = T_1(\rho_N)(1 + \eta(\rho_N))$  becomes

$$d\varepsilon + \kappa = T_1(\rho) \left(1 + \frac{\varepsilon}{\rho}\right) N^{-\varepsilon} (1 + \eta(\rho_N)), \quad |\kappa| \leq \frac{M_2 |\varepsilon|^2}{2},$$

because  $T_1(\rho + \varepsilon) = T_1(\rho)(1 + \varepsilon/\rho)N^{-\varepsilon}$ . Dividing by  $T_1(\rho)$ , whose modulus is  $q_N$ , gives  $\varepsilon = \frac{T_1(\rho)}{d}(1 + \theta_N)$  with, exactly,

$$1 + \theta_N = \left(1 + \frac{\varepsilon}{\rho}\right)N^{-\varepsilon}(1 + \eta(\rho_N)) - \frac{\kappa}{T_1(\rho)}.$$

Here  $|\varepsilon/\rho| \leq r_N/|\rho|$ ,  $|N^{-\varepsilon} - 1| = |e^{-\varepsilon \log N} - 1| \leq e^{|\varepsilon| \log N} - 1 \leq N^{r_N} - 1$ , and  $|\eta(\rho_N)| \leq \bar{\eta}_N(\rho_N) \leq \bar{\eta}$  because  $\rho_N$  lies in the disc. Hence

$$|\theta_N| \leq \left| \left(1 + \frac{\varepsilon}{\rho}\right)N^{-\varepsilon}(1 + \eta) - 1 \right| + \frac{|\kappa|}{q_N} \leq \left(1 + \frac{r_N}{|\rho|}\right)N^{r_N}(1 + \bar{\eta}) - 1 + \frac{M_2 r_N^2}{2q_N},$$

using  $|(1+a)(1+b)(1+c) - 1| \leq (1+|a|)(1+|b|)(1+|c|) - 1$  (expand the product) with  $a = \varepsilon/\rho$ ,  $b = N^{-\varepsilon} - 1$ ,  $c = \eta(\rho_N)$ , and  $M_2 r_N^2/(2q_N) = 9M_2 q_N/(2|d|^2)$ . As  $N$  grows,  $q_N$  and  $r_N$  decrease, hence so do the disc,  $M_2$  and  $\bar{\eta}$  (each  $\bar{\eta}_N(s)$  decreases in  $N$ ), and (H1), (H2) are preserved. For (H3),  $r_N \log N$  is a constant times  $g(N) = N^{-\beta-1} \log N$ , and  $g'(N) = N^{-\beta-2}(1 - (\beta+1) \log N) < 0$  for  $N > e^{1/(\beta+1)}$ ; this threshold is at most  $e^{2/3} < 2$  for  $\beta \geq \frac{1}{2}$ , less than 2 for  $\beta > 1/\log 2 - 1 = 0.4427$ , and less than 3 for every  $\beta > 0$ . So both factors of (H3) decrease, and (H3) is preserved.  $\square$

*Status:* proved, as a theorem conditional on (H1)–(H3). The verification of the hypotheses for given  $\rho$  and  $N$  is a separate step, which we now make rigorous for the first 100 zeros.

**Certification of (H1)–(H3) in interval arithmetic.** We used the ball arithmetic of Arb [Jo] through `python-flint`, at 96 bits (`helix_explicit/arb_hypotheses.py`). The zeros  $\rho_k$ ,  $k \leq 100$ , are Arb's rigorous enclosures, and  $d = \zeta'(\rho_k)$  is the first-order coefficient of the Taylor series of  $\zeta$  at the ball  $\rho_k$ . By the maximum principle  $M_2$  is at most the maximum of  $|\zeta''|$  on the circle  $|s - \rho| = r$ , centred at the ball  $\rho_k$ , with  $r$  an upper bound for  $r_N$  plus the radius of the ball. We cut the circle into 256 arcs with midpoints  $c_j$ ; on the arc of  $c_j$ ,  $|s - c_j| \leq h = \pi r/256$  and

$$|\zeta''(c_j + u)| \leq \sum_{k < 10} \frac{|\zeta^{(k+2)}(c_j)|}{k!} h^k + \frac{12 \cdot 11 B h^{10}}{\rho_0^{12}}, \quad \rho_0 = \frac{1}{4},$$

with the Taylor coefficients at  $c_j$  computed as balls, and  $B$  an Arb upper bound for  $|\zeta|$  on the circle  $|w - \rho| = r + h + \rho_0$  (the integral form of Taylor's remainder with Cauchy's estimate for  $\zeta^{(12)}$ , and the maximum principle for  $B$ ), obtained by evaluating  $\zeta$  on 400 boxes that cover that circle. The bound  $\bar{\eta}$  uses  $|s + j| \leq |\rho + j| + r$  and  $\sigma + 5 \geq \beta - r + 5$ . For each zero, the first  $N$  at which (H1)–(H3) are certified is  $N_0^{\text{cert}}$ , and by monotonicity they hold for all  $N \geq N_0^{\text{cert}}$ . We find

$$0.391 |\rho| \leq N_0^{\text{cert}} \leq 0.636 |\rho| \quad (k \leq 100),$$

from  $N_0^{\text{cert}} = 9$  at  $\rho_1$  to 129 at  $\rho_{100}$ ;  $N_0^{\text{cert}}$  exceeds by at most 1 the floating-point estimate of `teorema.py`, in the same folder, which samples  $|\zeta''|$  at 48 points. At  $N = N_0^{\text{cert}}$  the certified bound for  $|\theta_N|$  is between 0.34 and 1.16, and it tends to 0 as  $N \rightarrow \infty$ . *Status:* computer-assisted (rigorous). The comparison of the bound with the actual  $|\theta_N|$  is numerical: for  $N_0 \leq N \leq \max(4N_0, 3000)$  the measured  $|\theta_N|$ , with  $\rho_N$  computed by Newton's method in double precision, is at most 0.32 times the bound (`helix_explicit/teorema.py`). The normalized displacement  $(\rho_N - \rho)N^{3/2}$  runs on a circle of radius  $|\rho|/(12|\zeta'(\rho)|)$ , measured 1.4856 against 1.4860 at  $\rho_1$  (`helix_explicit/helice.py`), and  $|\rho_N - \rho|$  decays with slopes between  $-1.502$  and  $-1.508$  for the seven zeros with  $\gamma < 42$  (`helix_explicit/eje3.py`). In the space  $(\sigma, t, \log N)$  the zeros of  $D_N$  thus form helices of radius  $|\rho|N^{-\beta-1}/(12|\zeta'(\rho)|)$  around the vertical lines  $\{(\beta, \gamma)\} \times \mathbb{R}$  through the zeros  $\rho = \beta + i\gamma$  of  $\zeta$  (Proposition 10 does not assume  $\beta = \frac{1}{2}$ ; for the zeros computed here  $\beta = \frac{1}{2}$ , and Remark 11 shows helices around off-line axes), while the zeros of  $X_N$  alone, without the pole and endpoint corrections, wander over the strip (Figure 3). *Status:* numerical.

*Remark 11* (off-line pairs and the Davenport–Heilbronn function). An off-line pair would give two helices with the rates  $N^{-(\beta+1)}$  and  $N^{-(2-\beta)}$ , whose product is  $N^{-3}$  whatever  $\beta$  is; as for Proposition 1, the functional equation imposes no constraint, and Proposition 10 measures  $\beta$  without forcing it. For the Davenport–Heilbronn function,  $f(s) = 5^{-s} \sum_{b=1}^4 a(b)\zeta(s, \frac{b}{5})$ , we apply the same correction to each Hurwitz piece:

$$F_M(s) = \sum_{b=1}^4 a(b)5^{-s} \left[ \sum_{k=0}^M \left(k + \frac{b}{5}\right)^{-s} - \frac{1}{2} \left(M + \frac{b}{5}\right)^{-s} - \frac{(M + b/5)^{1-s}}{1-s} \right].$$

By (1) for  $\zeta(s, \frac{b}{5})$ , each bracket is  $\zeta(s, \frac{b}{5}) - \frac{s}{12} \left(M + \frac{b}{5}\right)^{-s-1} + O(M^{-\sigma-3})$ . Since  $\sum_b a(b) = 0$  the terms of order  $M^{-s-1}$  cancel, and expanding  $\left(M + \frac{b}{5}\right)^{-s-1}$  gives

$$\rho_M - \rho = -\frac{5^{-\rho} \rho(\rho+1) \left(\sum_b a(b) \frac{b}{5}\right)}{12f'(\rho)} M^{-\rho-2} (1 + o(1)),$$

with rate  $M^{-(\beta+2)}$ , by the argument of Proposition 1. *Status:* proved in outline (the  $o(1)$  is not made explicit). The measured slopes are  $-2.8065$  and  $-2.1900$  for the pair  $0.8085, 0.1915 + 85.699i$  (predicted  $-2.8085, -2.1915$ ),  $-2.6490$  and  $-2.3477$  for the pair at height  $114.163$  (predicted  $-2.6508, -2.3492$ ), and  $-2.4982$  for the on-line zero at  $83.109$ ; at  $M = 4000$  the modulus of the displacement agrees with the prediction to  $3 \cdot 10^{-4}$  in relative terms. The two zeros of a pair converge at rates that differ by  $2(\beta - \frac{1}{2})$ , measured  $0.6165$  against  $0.6170$  (`helix_explicit/dh_helice.py`). *Status:* numerical.

#### 4. ZEROS OF THE SECTIONS

For the sections  $S_N(s) = \sum_{k \leq N} k^{-s}$  there is no limit to approach, and the question is where their zeros lie. With  $x_p = p^{-s}$  for the primes  $p \leq N$ ,  $S_N$  is the value of the polynomial

$$P_N((x_p)_{p \leq N}) = \sum_{k \leq N} \prod_{p \leq N} x_p^{v_p(k)}$$

in  $\pi(N)$  variables. Write  $\mathbb{T}$  for the unit circle and, for real  $a$  and  $w = (w_p) \in \mathbb{T}^{\pi(N)}$ ,  $P_N(a, w) = P_N((p^{-a} w_p)_p)$ . Put

$$\begin{aligned} \mathcal{A}_N &= \{a \in \mathbb{R} : P_N(a, w) = 0 \text{ for some } w \in \mathbb{T}^{\pi(N)}\}, \\ e(N) &= \sup \mathcal{A}_N, \quad m_N(a) = \min_w |P_N(a, w)|. \end{aligned}$$

A zero of  $P_N(a, \cdot)$  is a closed polygon whose sides have the lengths  $k^{-a}$ ,  $k \leq N$ , and in which only the directions of the sides  $p^{-a}$  with  $p$  prime are free; the direction of a composite side is fixed by those of its prime factors.

**Proposition 12** (the torus). *The closure of the set of real parts of the zeros of  $S_N$  is  $\mathcal{A}_N$ .*

*Proof.* If  $S_N(a + it) = 0$ , then  $P_N(a, w) = 0$  with  $w_p = p^{-it}$ . Conversely let  $P_N(a, w) = 0$ . The function  $g(s) = P_N((p^{-s} w_p)_p)$  is a nonzero exponential polynomial (its term  $k = 1$  is 1) with a zero at  $a$ , which is isolated. Since the numbers  $\log p$  are linearly independent over  $\mathbb{Q}$ , Kronecker’s theorem gives real  $t_j$  with  $p^{-it_j} \rightarrow w_p$  for all  $p \leq N$ , so  $S_N(s + it_j) \rightarrow g(s)$  uniformly on compact sets, and by Hurwitz’s theorem  $S_N(s + it_j)$  has zeros  $s_j \rightarrow a$ . Hence  $a$  is a limit of real parts of zeros of  $S_N$ . Finally  $\mathcal{A}_N$  is closed, since  $\mathbb{T}^{\pi(N)}$  is compact.  $\square$

*Status:* proved; this is Bohr’s method for Dirichlet polynomials [T, Sect. 11.6]. For  $N = 2$  the zeros of  $1 + 2^{-s}$  are the points  $i(2m+1)\pi/\log 2$  and  $\mathcal{A}_2 = \{0\}$ . For  $N = 3$  the condition is that  $1, 2^{-a}$  and  $3^{-a}$  form a triangle, so  $\mathcal{A}_3 = [-1, 0.78788]$ , the right end being the root of  $2^{-a} + 3^{-a} = 1$ .

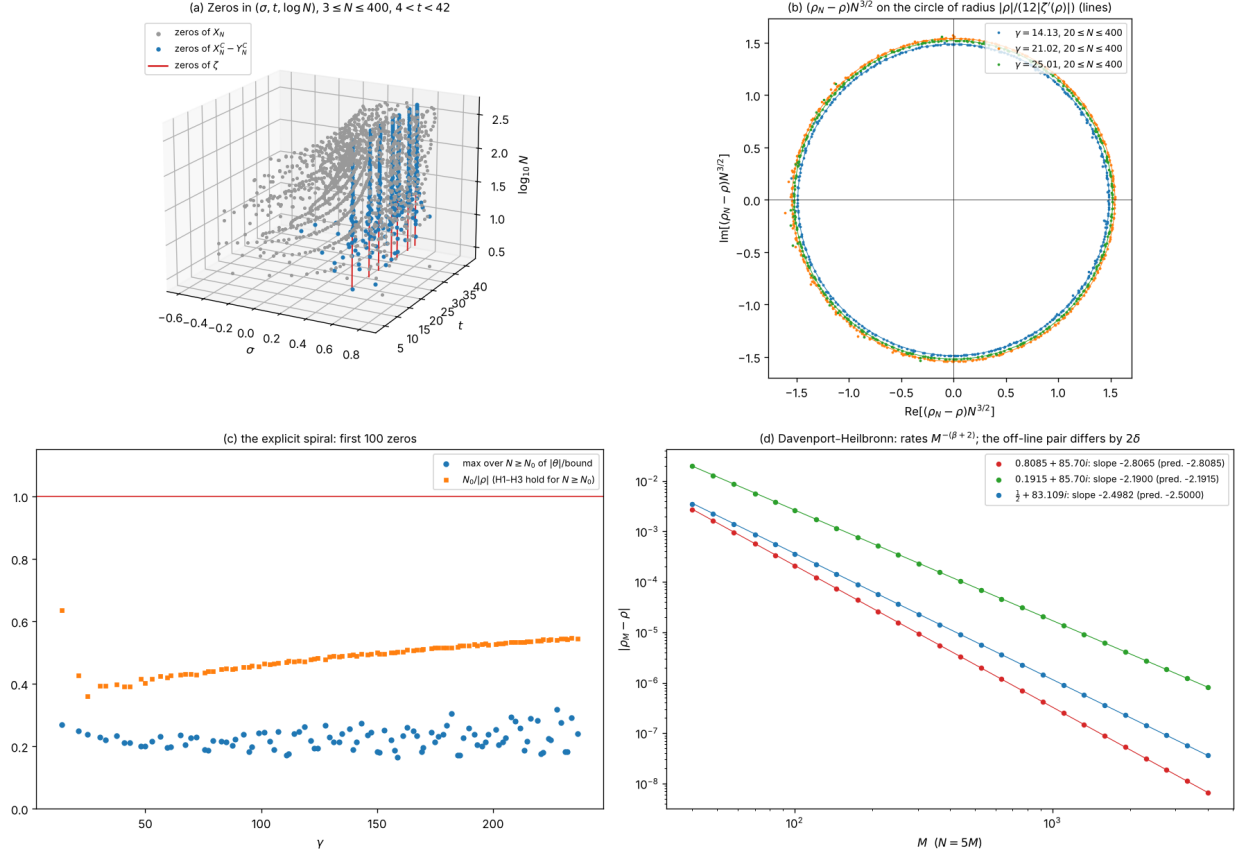


FIGURE 3. The explicit spiral (Proposition 10). (a) Zeros of  $X_N$  (grey) and of  $D_N = X_N^C - Y_N$  (blue) in  $(\sigma, t, \log_{10} N)$ ; vertical lines at the zeros of  $\zeta$ . (b)  $(\rho_N - \rho)N^{3/2}$  for the first three zeros and the circles of radius  $|\rho|/(12|\zeta'(\rho)|)$ . (c) For the first 100 zeros, the measured  $|\theta_N|$  divided by the bound of Proposition 10, and the ratio  $N_0/|\rho|$  from which the hypotheses hold in the floating-point check of `teorema.py` (the certified thresholds are at most one unit larger, with ratios 0.391–0.636). (d) Davenport–Heilbronn function:  $|\rho_M - \rho|$  for the two zeros of the off-line pair at height 85.699 and for an on-line zero, with the predicted displacements (lines), rates  $M^{-(\beta+2)}$ .

**Proposition 13** (the prime step). *If  $p$  is prime, then  $e(p) = \sup\{a \geq 0 : m_{p-1}(a) \leq p^{-a}\}$ ; in particular  $e(p) \geq e(p-1)$ .*

*Proof.* Let  $a \geq 0$ . The image of the torus under  $w \mapsto P_{p-1}(a, w)$  is connected, and at  $w = 1$  it takes the value  $\sum_{k < p} k^{-a} \geq 1 \geq p^{-a}$ . If  $m_{p-1}(a) \leq p^{-a}$ , it contains a point  $z$  with  $|z| = p^{-a}$ , and the free coordinate  $w_p$  with  $p^{-a}w_p = -z$  gives a zero of  $P_p(a, \cdot)$ , because  $p$  divides no  $k < p$ . If  $m_{p-1}(a) > p^{-a}$ , then  $|P_{p-1}(a, w)| > |p^{-a}w_p|$  on the torus and  $P_p(a, \cdot)$  has no zero. Since  $m_{p-1} = 0$  on  $\mathcal{A}_{p-1}$ ,  $e(p) \geq e(p-1)$ .  $\square$

*Status:* proved.

For a composite  $N$  the direction of  $N^{-s}$  is fixed by those of its prime factors, and no such statement holds. We computed  $e(N)$  for  $3 \leq N \leq 60$  by minimizing  $|P_N(a, \cdot)|$  on the torus with many random starts and bisecting in  $a$  (`sections/edges.py`, `sections/torus.py`); each value is a lower bound for  $e(N)$  up to the bisection tolerance, since the zero found is genuine, and an upper bound only to the extent that the minimization finds the global minimum. In all 15 cases where

$N$  is prime,  $e(N) > e(N-1)$ , as Proposition 13 requires. In all 8 cases where  $N$  is a prime power (4, 8, 9, 16, 25, 27, 32, 49),  $e(N) < e(N-1)$ ; for the other composites the edge rose 13 times and fell 21 times (Figure 4). The edge exceeds 1 for the first time at  $N = 19$  (1.0070), falls below 1 again at  $N = 20, 21$  and 28, and lies between 1.006 and 1.033 for  $29 \leq N \leq 60$ , in agreement with the theorem of Platt and Trudgian [PT16] that  $S_N$  has no zeros in  $\sigma > 1$  exactly when  $N \leq 18$  or  $N \in \{20, 21, 28\}$ . Turán showed that  $S_N(s) \neq 0$  for  $\sigma > 1$  and all large  $N$  would imply RH [Tu]; Montgomery showed that  $S_N$  has zeros with  $\sigma > 1 + (\frac{4}{\pi} - 1 - \varepsilon) \log \log N / \log N$  for large  $N$  [Mo83], and Montgomery and Vaughan gave the matching bound from the other side [MV]. The configuration in which all primes have direction  $\pi$ , which turns  $P_N(a, \cdot)$  into the Liouville sum  $\sum_{n \leq N} \lambda(n) n^{-a}$  of Turán's problem, is not extremal: its largest root is 0.901 at  $N = 19$ , against  $e(19) = 1.007$  (`sections/liou.py`). *Status*: numerical.

The dimension of the problem can be reduced. Two primes larger than  $\sqrt{N}$  have product larger than  $N$ , so every  $n \leq N$  contains at most one of them, to the first power.

**Proposition 14** (polygon reduction). *For  $N \geq 2$  and  $a \geq 0$  write*

$$P_N(a, w) = R_N(w') + \sum_{\sqrt{N} < p \leq N} w_p p^{-a} C_p(w'),$$

where  $w' = (w_p)_{p \leq \sqrt{N}}$ ,  $R_N$  collects the terms  $n^{-a} \prod_p w_p^{v_p(n)}$  with  $n$  free of primes larger than  $\sqrt{N}$ , and  $C_p = \sum m^{-a} \prod_q w_q^{v_q(m)}$  over the  $m \leq N/p$  free of such primes (so  $C_p = 1$  for  $p > N/2$ ). Then  $a \in \mathcal{A}_N$  if and only if there is  $w' \in \mathbb{T}^{\pi(\sqrt{N})}$  for which the largest of the numbers  $|R_N(w')|$  and  $p^{-a} |C_p(w')|$ ,  $\sqrt{N} < p \leq N$ , is at most the sum of the others.

*Proof.* The coordinates  $w_p$  with  $p > \sqrt{N}$  appear in no  $R_N$  and no  $C_q$ , and each appears in exactly one term. For fixed  $w'$ , the sums  $\sum_p w_p \ell_p$  with free  $|w_p| = 1$  and lengths  $\ell_p = p^{-a} |C_p(w')|$  fill the annulus  $\max(0, 2 \max_p \ell_p - \sum_p \ell_p) \leq |z| \leq \sum_p \ell_p$ . Hence  $P_N(a, \cdot)$  vanishes for some choice of these coordinates if and only if  $|R_N(w')|$  lies in that annulus, which is the polygon condition.  $\square$

*Status*: proved. By Bertrand's postulate there is always a prime in  $(N/2, N]$ , a side of length  $p^{-a}$  independent of  $w'$ ; Proposition 13 is the case in which this is the only change from  $N-1$  to  $N$ . For  $4 \leq N \leq 8$  the problem has one variable,  $w_2$ , and for  $9 \leq N \leq 24$  two. For  $N = 4$  the sides are  $|1 + z + z^2|$ ,  $z = 2^{-a} w_2$ , and  $3^{-a}$ ; the minimum of  $|1 + z + z^2|$  on  $|w_2| = 1$  is  $\frac{\sqrt{3}}{2}(1 - 4^{-a})$  when  $2^{-a} \geq 2 - \sqrt{3}$ , and  $3^{-a} - \frac{\sqrt{3}}{2}(1 - 4^{-a})$  is decreasing, so  $e(4) = 0.62629$  is the root of  $3^{-a} = \frac{\sqrt{3}}{2}(1 - 4^{-a})$ . At  $a = e(3)$ , where  $3^{-a} = 1 - 2^{-a}$ , the inequality  $3^{-a} < \frac{\sqrt{3}}{2}(1 - 4^{-a})$  reduces to  $2^{-e(3)} > 2/\sqrt{3} - 1$ , which holds ( $0.5792 > 0.1547$ ); hence  $e(4) < e(3)$ . *Status*: proved.

**Proposition 15** (powers of two).  $e(2^k) < e(2^k - 1)$  for  $k = 2, 3, 4$ . More precisely  $e(8) < 0.93 < 0.97 \leq e(7)$  and  $e(16) < 0.975 < 0.99 \leq e(15)$ .

*Proof.*  $k = 2$  is the computation above. Write the polygon defect of Proposition 14 as  $\text{Def} = \max_i (\ell_i - \sum_{j \neq i} \ell_j)$ , where  $\ell_i$  are the side lengths; the polygon closes if and only if  $\text{Def} \leq 0$ . All the following inequalities are evaluated in the ball arithmetic of Arb [Jo], which encloses every rounding error. *Lower bounds.* For  $N = 7$  at  $a = 0.97$  and  $N = 15$  at  $a = 0.99$  we exhibit points  $w'$  (binary floating-point numbers, taken as exact) at which the Arb enclosure of  $\text{Def}$  is negative ( $\leq -0.0083$  and  $\leq -0.0028$ ), so the polygon closes,  $0.97 \in \mathcal{A}_7$  and  $0.99 \in \mathcal{A}_{15}$ . *Upper bounds.* For  $N = 8$  on  $0.93 \leq a \leq 3$  (variables  $\theta_2, a$ ) and  $N = 16$  on  $0.975 \leq a \leq 3$  (variables  $\theta_2, \theta_3, a$ ) we cover the domain by boxes and evaluate the side lengths on each box in ball arithmetic; the result encloses their values on the whole box, so no Lipschitz bound is needed. A box is discarded when the lower bound of  $\text{Def}$  is positive, and bisected otherwise. Since  $P_N(a, \bar{w}) = \overline{P_N(a, w)}$ , the first angle can be restricted to  $[0, \pi]$ . All boxes are discarded: 2010 boxes for  $N = 8$  and 271344 for  $N = 16$ , after at

most 14 and 24 bisections. For  $a \geq 3$  the term 1 dominates: with  $Q = R_N(w')$ ,  $|Q| \geq 1 - \sum n^{-a}$  over its terms  $n > 1$ , which exceeds the sum of the lengths of the remaining sides (0.857 against 0.053 for  $N = 8$ , 0.814 against 0.014 for  $N = 16$ , certified at  $a = 3$ ; the left side increases and the right side decreases with  $a$ ). Since  $\mathcal{A}_N$  is closed,  $e(8) < 0.93$  and  $e(16) < 0.975$ .  $\square$

*Status:* computer-assisted proof in interval arithmetic (`sections/arbcert.py`, 64-bit balls, under a minute). The earlier double-precision certificates `sections/cert8.py` and `sections/cert16.py`, with explicit Lipschitz bounds but without control of rounding, give the same conclusions and are kept for comparison.

With the reduction we extended the computation of  $e(N)$  to  $N \leq 128$  (`sections/red.py`, `sections/careful.py`, `sections/polycert.py`). The powers 64, 81, 125 and 128 again lower the edge, from 1.0320, 1.0328, 1.0392, 1.0383 to 1.0277, 1.0300, 1.0380, 1.0366, but  $121 = 11^2$  raises it slightly, from 1.03714 to 1.03720. At  $a = 1.03719$  the polygon for  $N = 121$  closes (defect  $-2 \cdot 10^{-5}$ ), while 1600 independent minimizations for  $N = 120$  return the same positive defect  $1.8 \cdot 10^{-4}$ . The regularity of prime powers is therefore not a general law. *Status:* numerical.

The powers of 2 behave differently because of a mechanism that can be read off the extremal configurations. In  $P_{2^k-1}$  the powers of 2 form the geometric sum  $1 + z + \dots + z^{k-1} = (1 - z^k)/(1 - z)$ , and at the edge the direction of  $w_2$  sits near a  $k$ -th root of unity, where this sum nearly cancels:  $k\theta_2 \bmod 2\pi$  is  $0.008\pi$ ,  $0.067\pi$ ,  $0.128\pi$ ,  $0.227\pi$  for  $k = 3, \dots, 6$ . The new term  $z^k$  completes the geometric sum and undoes the cancellation. In the form of Proposition 14 the edge of  $N - 1$  is the point where the minimum of the polygon defect reaches 0. If this minimum is attained at a single point  $w^*$ , where the side  $R_{N-1}$  is the largest, adding  $\epsilon q^{-a} w_p^k$ ,  $q = p^k$ , changes the defect by  $\epsilon q^{-a} \operatorname{Re}(\overline{R_{N-1}(w^*)} w_p^{*k})/|R_{N-1}(w^*)| + O(\epsilon^2)$ , so the edge moves to the left when this projection is positive. For  $q = 2^k$  the normalized projection is 0.93, 0.92, 0.94, 0.99 for  $k = 3, \dots, 6$ . For  $11^2$  the powers of 11 give only  $1 + z_{11}$ , the projection is 0.10, and the edge moves to a second local minimum. *Status:* heuristic, with numerical inputs. We expect  $e(2^k) < e(2^k - 1)$  for every  $k \geq 2$ ; it holds for  $k \leq 4$  by Proposition 15 and numerically for  $k \leq 7$ , and the powers 9, 27, 81 of 3 behave in the same way.

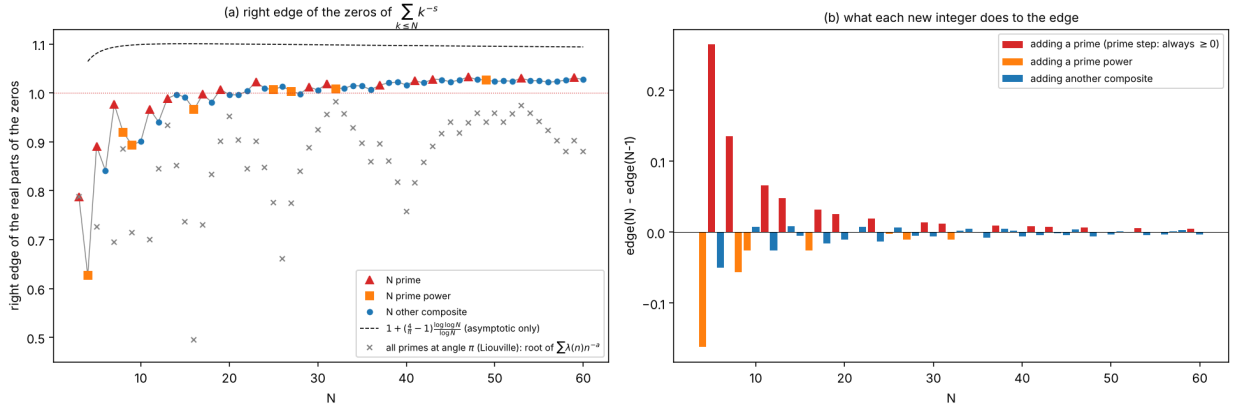


FIGURE 4. Zeros of the sections  $S_N(s) = \sum_{k \leq N} k^{-s}$ . (a) The right edge  $e(N)$  of the real parts of the zeros,  $3 \leq N \leq 60$ , marked by the type of  $N$ ; dashed: the asymptotic  $1 + (\frac{4}{\pi} - 1) \log \log N / \log N$  of [Mo83, MV]; crosses: the largest real root of the Liouville sum. (b) The change  $e(N) - e(N - 1)$ : positive for every prime (Proposition 13), negative for every prime power up to 60.

5. THE CIRCLE OF  $e$ : AN ERRATUM TO THE 2024 ELLIPSE

In 2024 the author proposed to study  $\zeta$  through a plane curve [E24]. For fixed  $\alpha > 0$  the values of the exponential series  $DC(\alpha + i\beta) = \sum_{k \geq 1} e^{-k(\alpha + i\beta)}$  trace a closed curve, which was fitted by an ellipse, and the zeros of  $\zeta$  were placed on it. This section is the erratum [Er] to [E24]: the curve is an exact circle, it is the Euler factor of a single “prime” of size  $e$ , and the positions of the zeros on it reflect only the equidistribution of their ordinates. Here  $\beta$  denotes, as in [E24], the imaginary part of the variable; it is not the real part of a zero.

**Proposition 16** (the curve is a circle). *For  $\alpha > 0$  put  $r = e^{-\alpha}$ . The curve  $\beta \mapsto DC(\alpha + i\beta)$  is the circle with*

$$\text{centre } h = \frac{r^2}{1 - r^2}, \quad \text{radius } R = \frac{r}{1 - r^2}, \quad \frac{R}{h} = e^\alpha.$$

For  $\alpha = \frac{1}{2}$ ,  $h = 0.5819767$  and  $R = 0.9595174$ . Moreover  $1 + DC(s) = (1 - e^{-s})^{-1}$ , the Euler factor  $(1 - p^{-s})^{-1}$  at  $p = e$ .

*Proof.* Summing the geometric series,  $DC(s) = e^{-s}/(1 - e^{-s}) = 1/(e^s - 1)$ . Put  $w = e^{-s}$ ; for fixed  $\alpha$  the point  $w = re^{-i\beta}$  runs over the circle  $|w| = r$ , and  $w \mapsto w/(1 - w)$  is a Möbius transformation, so the image is a circle. It is symmetric about the real axis and meets it at  $r/(1 - r)$  and  $-r/(1 + r)$ ; its centre is the midpoint  $r^2/(1 - r^2)$  and its radius is half the distance,  $r/(1 - r^2)$ .  $\square$

*Status:* proved. The preprint [E24] reported the centre 0.581974, semi-axes 0.959513 and 0.959276, and the ratio  $1.6487206 = e^{1/2}$  of the major semi-axis to the centre. The difference between the two semi-axes was a fitting artefact; the exact values are those above.

For a true prime  $p$  the same construction with  $w_p = p^{-s}$  sends the vertical line  $\text{Re } s = \sigma$  to the circle  $|w_p| = p^{-\sigma}$ , and the critical line to  $|w_p| = p^{-1/2}$ ; for  $p = e$  this is  $|w| = 1/\sqrt{e}$  (Figure 5, left). A single Euler factor has no zeros, and its poles  $s = 2\pi ik/\log p$  lie on one vertical line: the “Riemann hypothesis for one prime” holds trivially and says nothing about  $\zeta$ .

*Remark 17* (the critical circle is the fixed circle of the symmetry). The zeros of  $\zeta$  are symmetric under the involution  $\iota(s) = 1 - \bar{s}$ , which combines the functional equation with  $\overline{\zeta(s)} = \zeta(\bar{s})$ . For every  $p > 1$  and  $w_p = p^{-s}$ ,

$$w_p(\iota(s)) = p^{-1+\bar{s}} = \frac{1}{p w_p(s)},$$

so  $\iota$  acts as the inversion  $w \mapsto 1/(p\bar{w})$  in the circle  $|w| = p^{-1/2}$ . The image of the critical line is the fixed circle of this inversion, and the circles  $|w| = p^{-\sigma}$  and  $|w| = p^{-(1-\sigma)}$  are exchanged. The same holds for Li’s variable  $z = 1 - 1/s$ , where  $\iota$  becomes  $z \mapsto 1/\bar{z}$  [CE3], and for the variable  $u$  of the Ihara zeta function of a  $(q+1)$ -regular graph, where it becomes  $u \mapsto 1/(q\bar{u})$  [CE17]. The symmetry determines the circle, but it only forces the zeros to occur in pairs  $\rho, \iota(\rho)$ ; RH is the statement that every zero is fixed by  $\iota$ . The function  $DC$  alone has the different symmetry  $DC(-s) = -1 - DC(s)$ , whose fixed line is  $\alpha = 0$ . *Status:* proved (elementary).

**Proposition 18** (the zeros on the circle of  $e$ ). *Let  $\alpha > 0$ ,  $r = e^{-\alpha}$ , and let the angle  $\phi$  be uniformly distributed modulo  $2\pi$ . Then  $z = DC(\alpha + i\phi)$  is distributed on the circle of Proposition 16 with arc-length density proportional to  $|1 - w|^2$ ,  $w = re^{-i\phi}$ , and*

$$\mathbb{P}(\text{Re } z < h) = 1 - \frac{\text{gd}(\alpha)}{\pi}, \quad \text{gd}(\alpha) = \arccos(\text{sech } \alpha) = \arctan(\sinh \alpha),$$

*the Gudermannian function.* For  $\alpha = \frac{1}{2}$  this is 0.84709. Consequently, the fraction of the zeros  $\beta + i\gamma$  with  $0 < \gamma \leq T$  whose point  $DC(\frac{1}{2} + i\gamma)$  lies left of the centre tends to 0.84709 as  $T \rightarrow \infty$ .

*Proof.* The derivative of  $w \mapsto w/(1 - w)$  is  $(1 - w)^{-2}$ , so arc length on the image is  $|1 - w|^{-2}$  times arc length on  $|w| = r$ , and a uniform angle has image density proportional to  $|1 - w|^2$ .

With  $c = \cos \phi$ ,  $\operatorname{Re} z = (rc - r^2)/(1 - 2rc + r^2)$ , and  $\operatorname{Re} z < h$  is equivalent, after multiplying by  $(1 - 2rc + r^2)(1 - r^2) > 0$ , to  $c(r + r^3) < 2r^2$ , that is,  $\cos \phi < 2r/(1 + r^2) = \operatorname{sech} \alpha$ , which has probability  $1 - \arccos(\operatorname{sech} \alpha)/\pi$ . The last statement follows from Hlawka's theorem that the ordinates  $\gamma$  are uniformly distributed modulo  $2\pi$  [Hl], since the set  $\{\cos \phi < \operatorname{sech} \alpha\}$  is an interval with boundary of measure zero.  $\square$

*Status:* proved (the last statement with Hlawka's theorem, which is unconditional). The 341 zeros with  $0 < \gamma < 600$  give  $290/341 = 85.04\%$  (`ellipse/fig8a.py`, Figure 5, right). The map wraps the critical line around the circle infinitely often, so only  $\gamma \bmod 2\pi$  survives, and the Möbius map pushes uniform angles away from the pole  $w = 1$ . The angular patterns of [E24, Figs. 11–14] are consequences of this and carry no further information about the zeros.

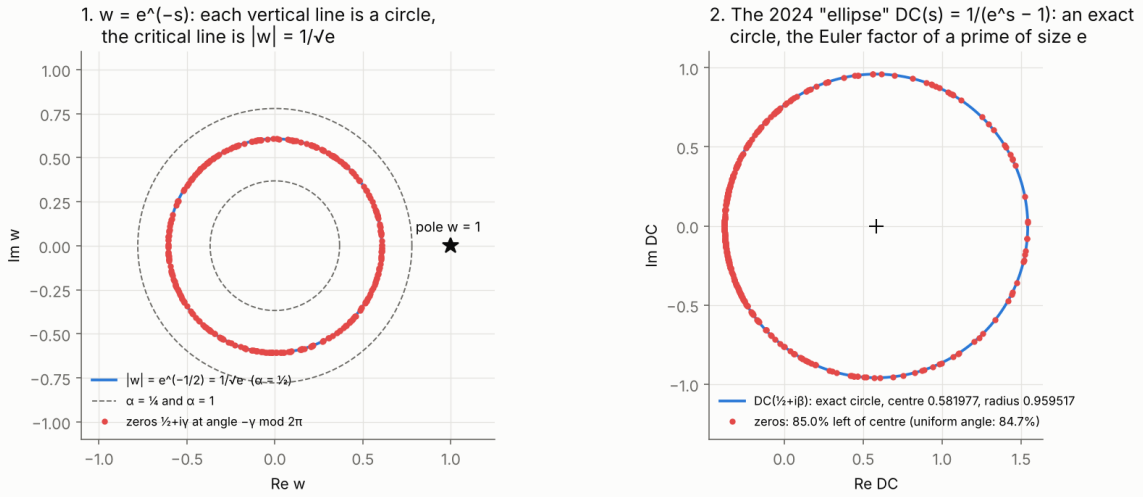


FIGURE 5. The circle of  $e$  (Propositions 16 and 18). Left:  $w = e^{-s}$  maps each vertical line to a circle; the critical line goes to  $|w| = 1/\sqrt{e}$ , and the zeros with  $0 < \gamma < 600$  sit at the angle  $-\gamma \bmod 2\pi$ . Right: the 2024 curve  $DC(\frac{1}{2} + i\beta)$  is the exact circle with centre 0.5819767 and radius 0.9595174; 85.0% of the zeros lie left of the centre, against 84.7% for uniform angles.

**What survives of [E24].** Two other statements of [E24] are corrected elsewhere in this series. The harmonic decomposition of [E24, Sect. 7.2] survives as the straight-line law  $|X_n(\rho)|^2 = (n + \frac{1}{2})/|\rho|^2 + O(n^{-1})$  at a zero on the critical line [CE5, Sect. 2]. The claim of [E24, Sect. 7.2] that the harmonic offset  $H_n - |X_n(\frac{1}{2} + i\gamma)|^2 + n/(\frac{1}{4} + \gamma^2)$  has the same limit at two zeros is not correct: by the straight-line law the offset at a zero on the critical line is  $H_n - \frac{1}{2}(\frac{1}{4} + \gamma^2)^{-1} + O(n^{-1})$ , so the difference between its values at the first and at the second zero tends to  $\frac{1}{2}[(\frac{1}{4} + \gamma_2^2)^{-1} - (\frac{1}{4} + \gamma_1^2)^{-1}] = -0.001369$ , not to 0 [CE1]. Finally, recording the height as a third coordinate lifts the circle  $|w| = e^{-1/2}$  to a helix; applied to the cross-energy instead of to  $e^{-s}$ , this lift is the helix of [CE1, Thm. 3.2].

## 6. THE CIRCLES OF THE PRIMES

With a true prime the circle carries arithmetic information. The tool is Landau's formula.

**Theorem 19** (Landau). *For fixed  $x > 1$ , as  $T \rightarrow \infty$ ,*

$$\sum_{0 < \gamma \leq T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O_x(\log T),$$

where  $\Lambda(x) = 0$  unless  $x$  is a prime power.

*Status:* classical [Lan].

**Corollary 20** (the bias on the circle of a prime). *Assume RH. On the circle  $|x^{-s}| = x^{-1/2}$  of a prime power  $x$ , the zero  $\rho$  sits at the angle  $-\gamma \log x$ , and*

$$\frac{1}{N(T)} \sum_{0 < \gamma \leq T} \cos(\gamma \log x) = -\frac{\Lambda(x)}{\sqrt{x} \log \frac{T}{2\pi e}} (1 + o(1)),$$

while, still under RH, for  $x > 1$  not a prime power the mean is  $O_x(1/T)$ . Under RH the angles of the zeros on the circle of a prime power are biased toward  $\pi$ , and those on the circle of any other base are not.

*Proof.* Under RH,  $x^\rho = \sqrt{x} e^{i\gamma \log x}$  for every zero; take real parts in Theorem 19, divide by  $\sqrt{x}$  and by  $N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$ . For  $x$  not a prime power the main term vanishes and the error is  $O_x(\log T)/N(T) = O_x(1/T)$ .  $\square$

*Status:* conditional on RH. Theorem 19 itself is unconditional, but it controls  $\sum x^\rho$ , which equals  $\sqrt{x} \sum x^{\beta - \frac{1}{2}} e^{i\gamma \log x}$  and in which each angle is weighted by  $x^{\beta - \frac{1}{2}}$ ; the two sums agree after division by  $\sqrt{x}$  only if every zero counted has  $\beta = \frac{1}{2}$ . Every statement below about the means of the angles  $\gamma \log x$ , for prime powers and for other bases alike, is therefore conditional on RH (on GRH for  $L(s, \chi)$ ), unless stated otherwise; Open problem 4 asks for an unconditional version. Results on the distribution of  $\gamma \alpha$  modulo 1 in this direction are due to Ford and Zaharescu [FZ]. Table 2 compares the means for the 202 zeros with  $0 < \gamma < 400$  with the prediction  $-(T/2\pi)\Lambda(b)/(\sqrt{b}N)$  of Corollary 20,  $N = 202$  (`landau/features.py`); all these zeros lie on the critical line, so for them the angle sum and Landau's sum coincide.

| base $b$     | 2       | 3       | 5       | 7       | $e$     | 6       | 10      |
|--------------|---------|---------|---------|---------|---------|---------|---------|
| measured     | -0.1465 | -0.1938 | -0.2259 | -0.2224 | -0.0031 | +0.0135 | +0.0094 |
| Corollary 20 | -0.1545 | -0.1999 | -0.2268 | -0.2318 | 0       | 0       | 0       |

TABLE 2. Mean of  $\cos(\gamma \log b)$  over the 202 zeros with  $0 < \gamma < 400$ . The circle of [E24] is the column  $b = e$ .

The full spectrum  $\sum_{0 < \gamma < 400} \cos(\gamma \log n)$ ,  $2 \leq n \leq 40$ , has spikes exactly at the prime powers, all negative, of 92%–100% of the prediction  $-(T/2\pi)\Lambda(n)/\sqrt{n}$  at the primes and 73%–98% at the higher prime powers, and it is at most 7.1 in modulus at the other integers (`landau/landau.py`, Figure 6; details in Appendix A). For the Davenport–Heilbronn function the same sum over its zeros,  $\sum \operatorname{Re} n^{\rho-1/2}$ , reproduces  $-(T/2\pi)b(n)/\sqrt{n}$ , where  $b(n)$  are the Dirichlet coefficients of  $-f'/f$ ; these take both signs and are not supported on prime powers (+46.58 against +45.91 at  $n = 4$ , -50.73 against -50.33 at  $n = 6$ , +49.11 against +48.51 at  $n = 9$ ). The one-signed spectrum is the Euler product seen from the zeros. *Status:* numerical.

**Positions, not statistics.** The arithmetic information lies in the exact positions of the zeros, not in their local statistics. Displacing each zero independently by a Gaussian amount of standard deviation  $\epsilon$  times the local mean spacing  $2\pi/\log(\gamma/2\pi)$  keeps, on average over 200 draws, a fraction 0.961, 0.888, 0.731 and 0.317 of the prime spikes ( $p \leq 13$ , relative to Landau's prediction) for  $\epsilon = 0.02, 0.1, 0.2$  and  $0.5$ ; the undisplaced zeros give 0.964. Points with the statistics of the Gaussian unitary ensemble, unfolded to the same density, keep  $0.04 \pm 0.06$  (`landau/features.py`). Pair correlation [Mo] and the spacing statistics [Od] are therefore blind to the prime spikes, which require the positions to a fraction of the mean spacing. *Status:* numerical.

**Twisted circles.** The same holds for Dirichlet  $L$ -functions, with the character visible in the angle.

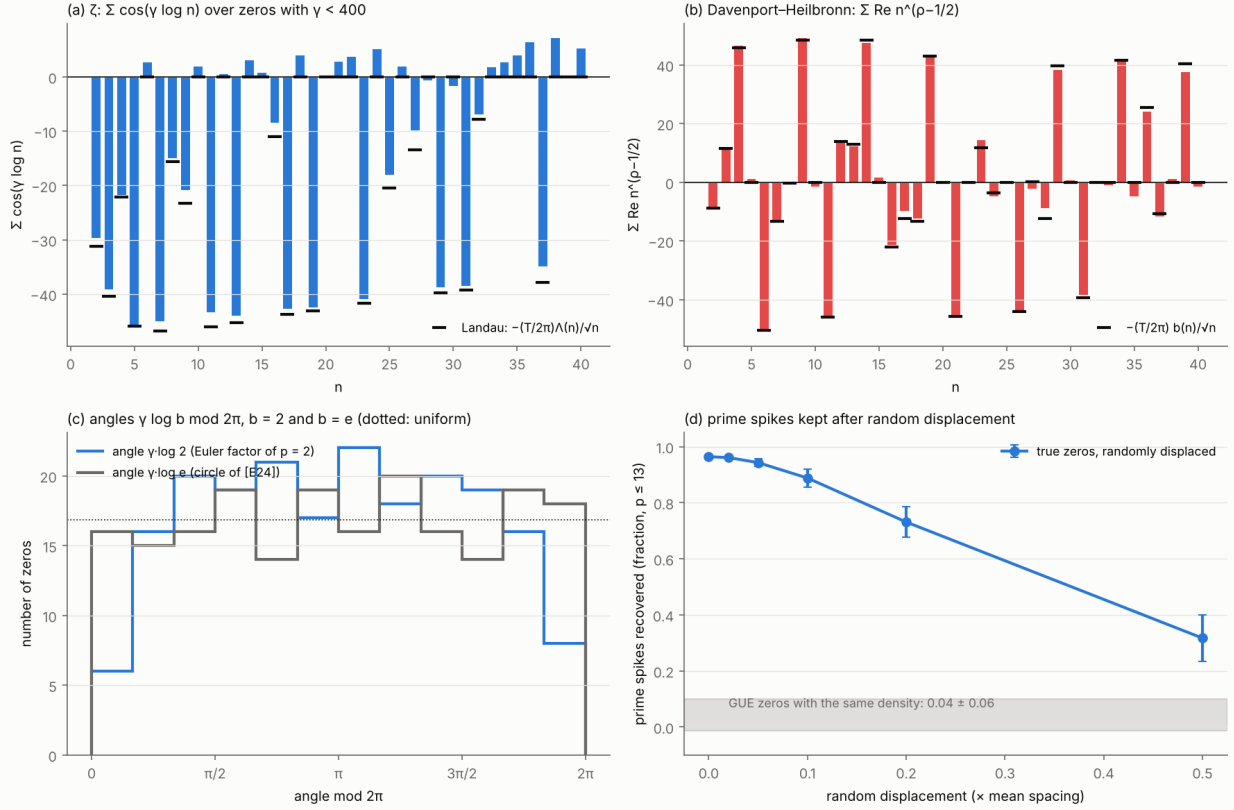


FIGURE 6. (a)  $\sum_{0 < \gamma < 400} \cos(\gamma \log n)$  for  $2 \leq n \leq 40$  (bars) and Landau's prediction  $-(T/2\pi)\Lambda(n)/\sqrt{n}$  (marks). (b) The same sum for the Davenport–Heilbronn function and  $-(T/2\pi)b(n)/\sqrt{n}$ . (c) Histogram of  $\gamma \log b \bmod 2\pi$  for  $b = 2$  and for  $b = e$ , the circle of [E24]. (d) Fraction of the prime spikes ( $p \leq 13$ ) kept after an independent Gaussian displacement of each zero, in units of the mean spacing; shaded band: points with the statistics of the Gaussian unitary ensemble and the same density.

**Proposition 21** (Landau's formula for  $L(s, \chi)$ ). *Let  $\chi$  be a primitive Dirichlet character and  $x > 1$  fixed. Summing over the nontrivial zeros  $\rho = \beta + i\gamma$  of  $L(s, \chi)$ ,*

$$\sum_{0 < \gamma \leq T} x^\rho = -\frac{T}{2\pi} \Lambda(x) \chi(x) + O_{x,\chi}(\log T), \quad \sum_{|\gamma| \leq T} x^\rho = -\frac{T}{\pi} \Lambda(x) \chi(x) + O_{x,\chi}(\log T).$$

*Under the generalized Riemann hypothesis the zeros lean, on the circle of  $p$ , toward the angle  $\pi + \arg \chi(p)$ .*

*Proof in outline.* The first formula follows from the contour argument for Theorem 19 applied to  $-\frac{L'}{L}(s, \chi)x^s = \sum_n \Lambda(n) \chi(n) (x/n)^s$ : the term  $n = x$  of the integral on  $\operatorname{Re} s = c > 1$  from  $c$  to  $c + iT$  gives  $\frac{T}{2\pi} \Lambda(x) \chi(x)$ , the residues at the zeros give  $-\sum x^\rho$ , and the remaining sides contribute  $O_{x,\chi}(\log T)$ . The zeros of  $L(s, \chi)$  with  $-T \leq \gamma < 0$  are the conjugates of the zeros of  $L(s, \bar{\chi})$  with  $0 < \gamma \leq T$ , and the first formula for  $\bar{\chi}$ , conjugated, gives the same main term; adding gives the second. Under GRH,  $\sum_{|\gamma| \leq T} x^{i\gamma} = -\frac{T}{\pi} \Lambda(x) \chi(x) / \sqrt{x} + O(\log T)$ , whose argument is  $\pi + \arg \chi(x)$ .  $\square$

*Status:* proved in outline (classical in substance). For the character  $\chi$  modulo 5 with  $\chi(2) = i$  and the 606 zeros of  $L(s, \chi)$  found on the critical line with  $|\gamma| < 400$ , the fit of  $\sum_{|\gamma| < 400} n^{i\gamma}$  by

$c\Lambda(n)\chi(n)/\sqrt{n}$  gives  $c = -126.6$  against  $-T/\pi = -127.3$ , and the mean directions on the circles of  $p = 2, 3, 7, 11$  are within  $0.004\pi$  of  $\pi + \arg \chi(p)$  (`twin/landau.chi.py`; details in Appendix A). *Status*: numerical.

*Remark 22* (the angles do not see  $\beta$ ). The angle of a zero  $\rho = \beta + i\gamma$  on the circle of  $p$  is  $-\gamma \log p$ , which does not depend on  $\beta$ : moving a zero horizontally at fixed height leaves every angle statistic unchanged, and changes only the modulus  $x^{\beta-1/2}$  of its term in Theorem 19. Statistics of the angles alone therefore cannot exclude off-line zeros; under RH they are equivalent to Theorem 19, and without RH they are a different quantity. The joint distribution of the angles on the circles of several primes has the Fourier coefficients  $N(T)^{-1} \sum_{\gamma} (m/n)^{i\gamma}$ , which Theorem 19 determines under RH. The quadratic version of the question, how small a Dirichlet polynomial  $\sum a_n n^{i\gamma}$  can be at all zeros simultaneously, is Weil's form in a basis of Dirichlet polynomials, which is studied in Part 12 [CE12]. *Status*: remark.

**Landau's formula under the heat flow.** The circles of the primes can be followed along the de Bruijn–Newman deformation  $H_t(z) = \int_0^\infty e^{tu^2} \Phi(u) \cos(zu) du$ ,  $H_0(z) = \frac{1}{8}\xi(\frac{1}{2} + \frac{iz}{2})$ . Write  $x_k(t)$  for its zeros and  $\gamma_k(t) = x_k(t)/2$ , so that  $\gamma_k(0)$  are the ordinates of the zeros of  $\zeta$ , and replace  $\gamma$  by  $\gamma_k(t)$  in Corollary 20. This is studied in Part 15 [CE15, Sect. 10], and we only summarize it. For  $0 < t \leq \frac{1}{2}$  the bias is erased: the mean of  $n^{i\gamma_k(t)}$  over  $T < \gamma_k(t) \leq 2T$  is  $O(\log n \cdot T^{-ct})$  [CE15, Prop. 10.1] (unconditional, with  $c$  not explicit). Part 15 proves the precise law [CE15, Thm. 11.1], which it first conjectured [CE15, Conj. 10.2]: for fixed  $n \geq 2$  and  $0 < t \leq \frac{1}{2}$ , and  $1 \leq H \leq T$  with  $T \geq T_0(n, t)$ ,

$$\sum_{T < \gamma_k(t) \leq T+H} n^{i\gamma_k(t)} = -\frac{1}{2\pi} \int_T^{T+H} \frac{\lambda_t(n; \gamma)}{\sqrt{n}} d\gamma + O_{n,t} \left( \log T + \frac{H}{T} \log^2 T \right),$$

so that in a window of heights around  $\gamma$ ,

$$\langle n^{i\gamma_k(t)} \rangle \approx -\frac{\lambda_t(n; \gamma)}{\sqrt{n} \log \frac{\gamma}{2\pi}},$$

where  $\lambda_t(n; \gamma)$  are the coefficients of  $-D'/D$  for  $D(s) = \sum_m c_m m^{-s}$ ,  $c_m = \exp(\frac{t}{4} \log^2 m - \frac{t}{2} \alpha_* \log m)$  and  $\alpha_* = \frac{1}{2} \log \frac{\gamma}{2\pi} + \frac{i\pi}{4}$ ; they satisfy  $\lambda_t(n; \gamma) = c_n \log n - \sum_{d|n, d < n} \lambda_t(d; \gamma) c_{n/d}$ . At  $t = 0$ ,  $c_m = 1$ ,  $\lambda_0 = \Lambda$ , and this is Corollary 20 (under RH). For  $t > 0$  it says three things about the circles. (a) *Decay*:  $\lambda_t(p; \gamma) = \log p \cdot p^{-(t/4)(\log(\gamma/2\pi p) + i\pi/2)}$ , so the bias on the circle of a prime is multiplied by  $p^{-(t/4) \log(\gamma/2\pi p)}$ ; the exponent  $\frac{t}{4} \log p \log \frac{\gamma}{2\pi p}$  is largest at  $p = (\gamma/2\pi)^{1/2}$ , so the primes near  $(\gamma/2\pi)^{1/2}$  forget fastest. (b) *Rotation*: the mean of  $p^{i\gamma_k(t)}$  no longer points to  $\pi$  but to  $\pi - \frac{\pi}{8} t \log p$ . (c) *Memory of composites*: the weights  $e^{(t/4) \log^2 m}$  are not multiplicative,  $c_{ab} = c_a c_b e^{(t/2) \log a \log b}$ , so for instance  $\lambda_t(6; \gamma) = c_2 c_3 \log 6 (e^{(t/2) \log 2 \log 3} - 1) \approx \frac{t}{2} \log 2 \log 3 \log 6$ , which is not 0 although  $\Lambda(6) = 0$ . Under the flow the zeros acquire a bias on the circles of composite numbers, which the zeros of  $\zeta$  do not have under RH (Corollary 20; numerically, the columns 6 and 10 of Table 2). This is proved for the squarefree  $n$ , where  $\Lambda_t(n) > 0$  by a connected-graph expansion [CE15, Rem. 11.7], and is numerical for the other composites.

The proof in [CE15, Sect. 11] is Landau's contour argument, applied to  $H'_t/H_t$  on the rectangle  $[2T, 2T+2H] \times [-1, 1]$ . On the lines  $\text{Im } z = \pm 1$  the effective approximation of the Polymath project writes  $H_t$  as an explicit factor times  $\sum_m b_m m^{-s_*}$ ,  $b_m = e^{(t/4) \log^2 m}$ , plus a mirror term of relative size  $O(x^{-1/2})$  and an error of relative size  $O(x^{-1} \log^2 x)$ , where  $x = \text{Re } z$ . The effective abscissa  $\text{Re } s_* \approx 1 + \frac{t}{4} \log \frac{x}{4\pi}$  tends to infinity with the height, so the logarithm of the deformed sum can be expanded on the contour itself. Its coefficient at  $n$  is  $\Lambda_t(n)/\log n$ , where  $b_n \log n = \sum_{d|n} \Lambda_t(d) b_{n/d}$ , and the term  $n^{-s_*}$  carries the factor  $n^{-(t/2)\alpha_*}$ : the decay and the rotation of the memory come from the drift of this abscissa. Integrating by parts, the error enters only through  $\log(H_t/B_t)$  and

not through its derivative. Jensen’s formula handles the vertical sides. The constant  $T_0(n, t)$  is effective but astronomically large,  $T_0 \geq 2\pi n^4 e^{8+64/t}$ , for the sum over the real parts of the zeros; the form with  $\gamma_k(t)$  uses the reality of the zeros for  $2T \geq e^{C/t}$ , with the unspecified constant  $C$  of the Polymath project.

Part 15 tests the law on zeros of  $\zeta$  moved forward by an approximate dynamics and on exactly computed zeros of  $H_t$ ; in the first test the means agree with the law within  $10^{-3}$ , in the second the sums, of modulus up to 837, differ from its main term by at most 6.7, and the rotation (b) is needed to fit the data (Appendix A). *Status*: proved in [CE15] for fixed  $n$  and  $0 < t \leq \frac{1}{2}$ ; the comparisons are numerical, and we have recomputed only the predictions (`landau/heat.rotation.py`).

Backwards in time the same factor amplifies the bias. At  $p \approx (\gamma/2\pi)^{1/2}$  the predicted value is  $\frac{1}{2} \exp(|t| \frac{L^2}{16} - \frac{L}{4})$ ,  $L = \log \frac{\gamma}{2\pi}$ , which exceeds 1 once  $L \gtrsim 4/|t|$ . A mean of numbers of modulus 1 cannot exceed 1, so this is impossible if all the zeros of  $H_t$  are real. It is consistent with the theorem of Rodgers and Tao that  $H_t$  has non-real zeros for every  $t < 0$  [RT]. *Status*: heuristic, since the theorem is proved only for  $0 < t \leq \frac{1}{2}$  [CE15, Sect. 11].

## 7. ONE CIRCLE IS NOT ENOUGH

The circles of this paper and of the rest of the series can be compared in one table. In each setting a vertical line, or the critical line, is mapped to a circle by a variable in which the functional equation becomes an inversion (Remark 17).

| setting                         | circle                            | zeros on it              | operator         | RH               |
|---------------------------------|-----------------------------------|--------------------------|------------------|------------------|
| one prime $e$ , Sect. 5         | $ e^{-s}  = e^{-1/2}$             | none                     | —                | trivial          |
| $\zeta$ via $e^{-s}$ , Sect. 5  | wrapped $ w  = e^{-1/2}$          | angles uniform           | —                | uninformative    |
| $\zeta$ via $p^{-s}$ , Sect. 6  | wrapped $ w  = p^{-1/2}$          | biased toward $\pi$ (RH) | —                | blind to $\beta$ |
| $\zeta$ via Li [CE5]            | $ 1 - 1/s  = 1$                   | infinitely many          | none known       | equivalent       |
| Ihara, $(q+1)$ -regular [CE17]  | $ u  = q^{-1/2}$                  | finitely many            | adjacency matrix | decidable        |
| primes at pace $\delta$ [CE17]  | $ e^{-\delta s}  = e^{-\delta/2}$ | infinite per period      | none             | can fail [DMV]   |
| curves over $\mathbb{F}_q$ [We] | $ u  = q^{-1/2}$                  | $2g$                     | Frobenius        | proved           |

TABLE 3. Circles for the critical line.

Li’s criterion [Li] uses the Möbius variable  $z = 1 - 1/s$ , which maps the critical line bijectively onto  $|z| = 1$  without wrapping; its coefficients, and the identity of the first one with half the constant  $C_B = 2 + \gamma_0 - \log 4\pi$  of [CE5] under RH, are in [CE5]. The Ihara zeta function of a regular graph, for which the analogue of RH is the Ramanujan property, and the Beurling primes obtained by rounding every  $\log p$  to a multiple of  $\delta$ , which put the whole Euler product on one circle without making it rational, are in [CE17].

Where RH is known or decided, two things hold together: the circle carries finitely many zeros, and those zeros are the spectrum of an operator whose symmetry forces them onto the circle. The 2024 circle carries a single prime. The circle of a true prime sees the zeros through Landau’s formula, with a one-signed spectrum that the Davenport–Heilbronn function does not have; Landau’s sum  $\sum x^\rho$  carries the factors  $x^\beta$ , but it does not constrain  $\beta$ , and the angle statistics, which are what the circle shows, do not depend on  $\beta$  (Remark 22). Li’s circle carries all zeros of  $\zeta$  but no operator.  $\zeta$  itself does not live on one circle: its Euler product uses one circle  $|p^{-s}| = p^{-\sigma}$  for each prime, and by the linear independence of the  $\log p$  the critical line winds densely around the infinite torus of these circles [BJ]. The finite part of this torus, with one circle per prime  $p \leq N$ , is the torus of Section 4, on which the zeros of the sections are polygons.

The truncations of Sections 2 and 3 meet the circles of the integers rather than those of the primes: the leading error is a multiple of  $n^{-s}$ , and the zero of the truncation turns on a scaled copy of  $n \mapsto n^{-\rho}$ . Its rate is  $\beta$ , and this is the common limitation of everything in this paper:

each construction reads  $\beta$ , or reads angles that do not depend on  $\beta$ , and none of them constrains  $\beta$  by itself. The Davenport–Heilbronn function shows this in every case, except on the circles of the primes, where the one-signed spectrum of Theorem 19 distinguishes  $\zeta$  from it but does not see  $\beta$ .

## 8. OPEN PROBLEMS: A RESEARCH PROGRAMME

The results above point in three directions, which we list with their open problems. The first is effective: to turn the asymptotic statements into verification tools. The second concerns the sections, where the polygon reduction makes the right edge computable. The third is arithmetic: the circles of the primes are the only construction of this paper that distinguishes  $\zeta$  from the Davenport–Heilbronn function, and they do it through the Euler product, not through the geometry.

**A. Effective spirals. 1. An explicit and uniform crossing count.** Make the thresholds  $X(\rho)$  and  $N_0(\rho)$  of Theorems 5 and 6 explicit and uniform in terms of  $|\zeta'(\rho)|$ ,  $\gamma$  and the distance from  $\rho$  to the neighbouring zeros. In Step 3 of the proof the bound  $|\vartheta| \leq 1$  uses the fractional part of  $\gamma \log 2/\pi$ ; the weaker bound  $|\vartheta| \leq \frac{5}{4}$  holds as soon as  $Ne^{-\pi/(8\gamma)} \geq X(\rho)$ , without it. Summed over the zeros, a uniform version would turn the crossing count into a verification method of the type of Turing’s [Tg], with an error term that can be compared with his.

**2. The explicit spiral at all heights.** The hypotheses of Proposition 10 are now certified in interval arithmetic for the first 100 zeros, with  $N_0^{\text{cert}} \leq 0.64|\rho|$ . Extend the certificate to all zeros up to a given height, and decide whether  $N_0 = O(|\rho|)$  in general; the natural route is a lower bound for  $|\zeta'(\rho)|$  and an upper bound for  $\zeta''$  near  $\rho$  valid on average over the zeros. What does it cost, compared with the approximate functional equation, to locate zeros this way?

**B. Sections of the zeta function. 3. Powers in the sections.** Prove  $e(2^k) < e(2^k - 1)$  for every  $k \geq 2$ , and decide which prime powers lower the edge: the projection mechanism of Section 4 predicts it for  $2^k$  and fails for  $11^2$ . Is there a prime power  $q$  above which  $e(q) < e(q - 1)$  never fails, or are there infinitely many exceptions? A first step is to certify the numerical values of  $e(N)$ ,  $N \leq 128$ , with the method of `sections/arbcert.py`; the cost grows with the number  $\pi(\sqrt{N})$  of free angles.

**C. Arithmetic on the circles. 4. The bias without RH.** Show unconditionally that

$$\sum_{0 < \gamma \leq T} \cos(\gamma \log p) = -\frac{T}{2\pi} \frac{\log p}{\sqrt{p}} + o(T),$$

that is, that replacing  $x^{i\gamma}$  by  $x^{\rho-1/2}$  in Theorem 19 costs only  $o(T)$ . Selberg’s density theorem controls the zeros with  $|\beta - \frac{1}{2}| \geq C \log \log T / \log T$ ,  $C > 4$ , and the trivial bound those with  $|\beta - \frac{1}{2}| = o(1/\log T)$ ; the zeros in between are the obstacle.

**5. Two circles at once.** The angles on the circle of one prime do not see  $\beta$  (Remark 22). Is there a functional of the joint positions of the zeros on the circles of two or more primes, computable from Landau’s formula, that does?

**6. Beyond the heat-flow Landau formula.** Part 15 proves the heat-flow Landau law for fixed  $n$  and  $0 < t \leq \frac{1}{2}$  [CE15, Thm. 11.1]. Three questions remain. (a) *Uniformity.* Prove a version uniform in  $n$ , as Gonek’s [Go] is for  $\zeta$ . The proof in [CE15] needs  $\log \frac{T}{2\pi} \geq 4 \log(nK) + 8$  with  $K = \max(n, \lceil e^{16/t} \rceil)$ , so  $T \geq 2\pi n^4 e^{8+64/t}$ , and its constants depend on  $n$ . (b) *Larger  $t$ .* The effective approximation used there is available only for  $t \leq \frac{1}{2}$ , while the numerics of [CE15] suggest that the law also holds at  $t = 0.8$ . (c) *One sign for every  $n$ .* After the rotation, the bias on the circle of  $n$  has the sign of  $-\Lambda_t(n)$ . Part 15 proves  $\Lambda_t(n) > 0$  for prime powers (by Kaluza’s theorem, since  $k \mapsto e^{(t/4)k^2 \log^2 p}$  is log-convex) and for all squarefree  $n = p_1 \cdots p_k$ , by the connected-graph (Mayer) expansion  $\Lambda_t(n) = \log n \prod_i b_{p_i} \sum_G \prod_{ij \in G} w_{ij}$ , summed over the connected graphs  $G$  on  $\{1, \dots, k\}$ , with  $b_m = e^{(t/4) \log^2 m}$  and  $w_{ij} = e^{(t/2) \log p_i \log p_j} - 1 > 0$  [CE15, Rem. 11.7]. The open cases are the

$n$  that are neither squarefree nor prime powers, such as 12 and 18. Numerically  $\Lambda_t(n) > 0$  for all  $n \leq 2 \cdot 10^4$  and  $0.01 \leq t \leq 1$  [CE15, Rem. 11.7]. Is  $\Lambda_t(n) > 0$  for every such  $n$  and every  $t > 0$ ? If so, the zeros of  $H_t$  lean toward  $\pi - \frac{\pi}{8}t \log n$  on the circle of every integer  $n \geq 2$ . The one-signed spectrum of Theorem 19, which is supported on the prime powers, would then be replaced by a one-signed spectrum supported on all integers: the deformation spreads the memory of the primes over all integers.

**What the programme cannot do by itself.** Each construction of Sections 2–3 reads off  $\beta$  from the rate of convergence of a zero that has been located, and the normalized displacement of Remark 3 removes  $\beta$  altogether; the Davenport–Heilbronn function has spirals and helices around its off-line zeros with exactly the predicted rates. The geometry therefore says where a zero is, not why it must lie on the line. A proof of  $\beta = \frac{1}{2}$  along these lines would need an arithmetic inequality, using the Euler product in an essential way, that is incompatible with the rate  $N^{-\beta}$  for  $\beta \neq \frac{1}{2}$ ; problems 4–6, where the Euler product enters through Landau’s formula, are where such an inequality could come from. We do not know of one.

## DATA AND CODE

The ancillary scripts reproduce all tables, figures and numbers; the file `anc/README.txt` lists each script, what it reproduces and its output file, and `anc/run_all.sh` runs them in order. The folders are `zerospiral/` (Proposition 1, Figure 1), `spiral_crossings/` (Theorems 5 and 6, Remark 3, Figure 2), `helix_explicit/` (Lemma 9, Proposition 10, Remark 11, Figure 3), `sections/` (Section 4, Figure 4), `ellipse/` (Section 5, Figure 5), `landau/` (Table 2, Figure 6, and the predictions quoted in the paragraph on the heat flow) and `twin/` (twisted circles). The ordinates of the 341 zeros of  $\zeta$  with  $0 < \gamma < 600$  are in `zeros600.npz`; the off-line zeros of the Davenport–Heilbronn function with  $\gamma < 400$  are in `landau/offline.npy`. Two scripts are computer-assisted proofs in interval arithmetic, with Arb [Jo] through `python-flint`: `sections/arbcert.py` (Proposition 15) and `helix_explicit/arb_hypotheses.py` (the hypotheses of Proposition 10 for the first 100 zeros). All other scripts use NumPy, SciPy, Matplotlib and mpmath in double or moderate precision, without interval arithmetic: their numbers illustrate the propositions, and the proofs do not use them. Some comments in `helix_explicit/` are in Spanish.

Use of AI tools. The proofs, computations and drafting of this paper were carried out with the assistance of an AI system (Claude, Anthropic), under the author’s direction; an independent AI review agent refereed the proofs. The author checked the results and takes responsibility for them.

## APPENDIX A. NUMERICAL RECORD

This appendix collects numerical details removed from the main text; all of them are numerical, and none is used in a proof.

The spiral for the Davenport–Heilbronn function (Section 2). At  $n = 2 \cdot 10^5$  the zero right of the critical line is within  $2.7 \cdot 10^{-5}$  of its limit  $0.8085 + 85.699i$ , and the left one still at distance  $1.25 \cdot 10^{-2}$  from  $0.1915 + 85.699i$ . Version 3 of [CP] reported the exponents  $-0.797$  and  $-0.175$ , which were fitted on all  $n$  (not only multiples of 5, so that the coefficient of the tail oscillates with  $n$  modulo 5) and against a reference point computed from a truncation; their difference 0.622 is the same within the noise (`zerospiral/spz.py`).

The spectrum of Figure 6. For  $n = 2$  the sum  $\sum_{0 < \gamma < 400} \cos(\gamma \log n)$  is  $-29.60$  against  $-31.20$ . The lowest ratios to the prediction, 73%, occur at 16 and 27, where the spike is smallest and the error term  $O(\log T)$  weighs most; the largest value at an integer that is not a prime power is 7.1, at  $n = 38$ .

Twisted circles (Proposition 21). Let  $\chi$  be the character modulo 5 with  $\chi(2) = i$ . On the critical line its  $L$ -function has 303 zeros with  $0 < \gamma < 400$  and 303 with  $-400 < \gamma < 0$  (sign changes of the real rotated completed function), against 303.2 for each sign from the main term  $\frac{T}{2\pi} \log \frac{5T}{2\pi e}$  of the Riemann–von Mangoldt formula, consistent with all zeros up to this height lying on the line; we did not run a Turing-type check (`twin/zeros.py`). Fitting  $\sum_{|\gamma| < 400} n^{i\gamma}$ ,  $2 \leq n \leq 40$ , by  $c \Lambda(n) \chi(n) / \sqrt{n}$  gives  $c = -126.6$ , against  $-T/\pi = -127.3$ , with relative residual 3%; at the integers that are not prime powers the sum is at most 7.9 in modulus, against a median of 76 at the prime powers. The measured mean directions on the circles of  $p = 2, 3, 7, 11$  are  $-0.496\pi$ ,  $0.504\pi$ ,  $-0.503\pi$  and  $-1.000\pi$ , against  $\pi + \arg \chi(p) = -\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}, \pi$  (`twin/landau_chi.py`).

The heat-flow Landau law (Section 6). Part 15 tests the law on the 10142 zeros of  $\zeta$  below height  $10^4$ , moved forward to  $t = 0.8$  by an approximate dynamics (the 40 nearest neighbours exact, the far field frozen), with averages over  $1000 < \gamma_k < 9000$ . The real and imaginary parts agree with the conjecture within  $10^{-3}$  for  $n = 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 15, 30$  and  $t \leq 0.8$ ; for example  $-0.0325 + 0.0069i$  measured against  $-0.0326 + 0.0072i$  predicted at  $n = 2$ ,  $t = 0.8$ , and  $-0.0066 + 0.0019i$  against  $-0.0065 + 0.0019i$  at  $n = 6$ ,  $t = 0.4$ , where the bias was exactly 0 at  $t = 0$  [CE15, Table 3, Fig. 4]. Part 15 also computes the zeros of  $H_t$  themselves on  $100 \leq \gamma_k(t) \leq 9100$  for  $t = 0.05, 0.1, 0.2, 0.4$ . Over  $1000 < \gamma_k(t) \leq 9000$  the sums of  $n^{i\gamma_k(t)}$ , which reach 837 in modulus, differ from the main term of the theorem by at most 3 for  $2 \leq n \leq 15$  and by at most 6.7 for  $n = 30$ , whereas Landau’s prediction for  $\zeta$  misses by up to 582 [CE15, Table 4]. For  $n = 30$  at  $t \leq 0.1$  the main term is so small that the error exceeds it, so at these heights the memory of 30 is not resolved for small  $t$ . The rotation is needed to fit the data. The decay factor alone predicts the real parts  $-0.0229$  and  $-0.0148$  for  $p = 5$  and  $p = 13$  at  $t = 0.8$ , whereas the law with its phase predicts  $-0.0200$  and  $-0.0102$  (`landau/heat_rotation.py`); the measured values in the author’s working notes are  $-0.0194$  and  $-0.0099$ . The faster loss of memory at large  $t$  that one sees in the real part alone is therefore the rotation, not a second-order correction to the decay.

## REFERENCES

- [BFFW] P. Borwein, G. Fee, R. Ferguson, A. van der Waall, Zeros of partial sums of the Riemann zeta function, *Experiment. Math.* 16 (2007), 21–39.
- [BJ] H. Bohr, B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, *Acta Math.* 54 (1930), 1–35.
- [C8] P. Caceres, Cross-energy of Dirichlet partial sums VIII: circles for the critical line and a pointwise  $C_2$  energy inequality, preprint, Zenodo, 2026.
- [DH] H. Davenport, H. Heilbronn, On the zeros of certain Dirichlet series I, II, *J. London Math. Soc.* 11 (1936), 181–185 and 307–312.
- [DMV] H. G. Diamond, H. L. Montgomery, U. M. A. Vorhauer, Beurling primes with large oscillation, *Math. Ann.* 334 (2006), 1–36.
- [E24] P. Caceres, Ellipse symmetry in Dirichlet series: a duality between analytic functions (Riemann zeta) and geometric forms, preprint, 16 September 2024.
- [Ed] H. M. Edwards, *Riemann’s Zeta Function*, Academic Press, New York, 1974.
- [Er] P. Caceres, Erratum and addendum to “Ellipse symmetry in Dirichlet series”, 2026; reproduced and extended in Section 5 of the present paper.
- [FZ] K. Ford, A. Zaharescu, On the distribution of imaginary parts of zeros of the Riemann zeta function, *J. reine angew. Math.* 579 (2005), 145–158.
- [GL] S. M. Gonek, A. H. Ledoan, Zeros of partial sums of the Riemann zeta-function, *Int. Math. Res. Not. IMRN* 2010, no. 10, 1775–1791.
- [Go] S. M. Gonek, An explicit formula of Landau and its applications to the theory of the zeta-function, *Contemp. Math.* 143 (1993), 395–413.
- [Hl] E. Hlawka, Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktion zusammenhängen, *Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II* 184 (1975), 459–471.
- [Jo] F. Johansson, Arb: efficient arbitrary-precision midpoint-radius interval arithmetic, *IEEE Trans. Comput.* 66 (2017), 1281–1292.
- [Lan] E. Landau, Über die Nullstellen der Zetafunktion, *Math. Ann.* 71 (1912), 548–564.

- [Li] X.-J. Li, The positivity of a sequence of numbers and the Riemann hypothesis, *J. Number Theory* 65 (1997), 325–333.
- [Mo] H. L. Montgomery, The pair correlation of zeros of the zeta function, *Proc. Sympos. Pure Math.* 24 (1973), 181–193.
- [Mo83] H. L. Montgomery, Zeros of approximations to the zeta function, in *Studies in Pure Mathematics*, Birkhäuser, Basel, 1983, 497–506.
- [MV] H. L. Montgomery, R. C. Vaughan, Mean values of multiplicative functions, *Period. Math. Hungar.* 43 (2001), 199–214.
- [Od] A. M. Odlyzko, On the distribution of spacings between zeros of the zeta function, *Math. Comp.* 48 (1987), 273–308.
- [PT16] D. Platt, T. Trudgian, Zeroes of partial sums of the zeta-function, *LMS J. Comput. Math.* 19 (2016), 37–41.
- [RT] B. Rodgers, T. Tao, The de Bruijn–Newman constant is non-negative, *Forum Math. Pi* 8 (2020), e6.
- [T] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, 1986.
- [Tg] A. M. Turing, Some calculations of the Riemann zeta-function, *Proc. London Math. Soc.* (3) 3 (1953), 99–117.
- [Tu] P. Turán, On some approximative Dirichlet polynomials in the theory of the zeta-function of Riemann, *Danske Vid. Selsk. Mat.-Fys. Medd.* 24 (1948), no. 17.
- [We] A. Weil, *Sur les courbes algébriques et les variétés qui s’en déduisent*, Hermann, Paris, 1948.
- [CP] P. Caceres, The cross-energy of Dirichlet partial sums and the zeros of the Riemann zeta function, versions 1–4, Zenodo, 2026, doi:10.5281/zenodo.23243641; version 4 is divided into Parts 1, 2, 5, 10 and 16 of this series.
- [CE1] P. Caceres, Cross-energy of Dirichlet partial sums. Part 1: The C-transform and the cross-energy, preprint, 2026.
- [CE3] P. Caceres, Cross-energy of Dirichlet partial sums. Part 3: The reflected cross-energy, preprint, 2026.
- [CE5] P. Caceres, Cross-energy of Dirichlet partial sums. Part 5: The energy of the primes, preprint, 2026.
- [CE10] P. Caceres, Cross-energy of Dirichlet partial sums. Part 10: Jensen’s formula, the energy slope and Weil’s functional, preprint, 2026.
- [CE11] P. Caceres, Cross-energy of Dirichlet partial sums. Part 11: The Euler–Caceres bound, preprint, 2026.
- [CE12] P. Caceres, Cross-energy of Dirichlet partial sums. Part 12: Weil positivity at the windows of the zeros, preprint, 2026.
- [CE15] P. Caceres, Cross-energy of Dirichlet partial sums. Part 15: How long can an off-line zero survive the de Bruijn–Newman heat flow?, preprint, 2026.
- [CE17] P. Caceres, Cross-energy of Dirichlet partial sums. Part 17: Controls for the Riemann hypothesis: Davenport–Heilbronn, Beurling systems and function fields, preprint, 2026.

UNIVERSIDAD EUROPEA DE VALENCIA, VALENCIA, SPAIN

Email address: `pedrojesus.caceres@universidadeuropea.es`