

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 5: THE ENERGY OF THE PRIMES

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ABSTRACT. This is Part 5 of a series on the cross-energy $C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma}$ of the partial sums of ζ . Let $E(u) = e^{-u/2}(\psi(e^u) - e^u + \log 2\pi + \frac{1}{2} \log(1 - e^{-2u}))$ be the normalized error of the prime number theorem. Theorem B: the mean energy of E equals $\sum_{\rho} \lim_N C_2(N, \rho)$ when the zeros are simple (in general both are finite or both infinite); RH holds if and only if it is finite, and it is at least $C_B = 2 + \gamma_0 - \log 4\pi$, twice Li's first coefficient, with equality if and only if RH holds and the zeros are simple. The explicit formula reads $1/\rho = \frac{1}{2}(C_2(\infty, \rho) - iS_2(\infty, \rho))$; the energy in a Gaussian window is an absolutely convergent double sum over the zeros; and the Nyman–Beurling distances and the Gram data of the Báez-Duarte criterion fit the same picture. The arithmetic cross-energy of the prime partial sums satisfies a 3-4-1 positivity at every finite cut, which fails for an Epstein zeta function, and RH is equivalent to $\sum_{n \leq N} C_2^{\Lambda}(n, \frac{1}{2}) = 4N + O(\sqrt{N})$. We do not prove the Riemann hypothesis.

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1. INTRODUCTION

Let $X_N(s) = \sum_{k \leq N} k^{-s}$ be the partial sums of the Dirichlet series of ζ , with $s = \sigma + it$. The *cross-energy* of consecutive partial sums is

$$C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} (X_{N-1}(s) N^{-\bar{s}}) = 2N^{-\sigma} \sum_{n < N} n^{-\sigma} \cos \left(t \log \frac{N}{n} \right).$$

It records how the newest term N^{-s} interacts with all the previous ones. This part relates C_2 to the mean square of the error term of the prime number theorem, to Li's coefficients, to the explicit formula and to the Nyman–Beurling criterion, and it introduces the arithmetic cross-energy of the prime partial sums.

Throughout, $\rho = \beta + i\gamma$ is a nontrivial zero of ζ , $\Theta = \sup \beta$, $\lambda = s(1-s)$, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, and

$$C_B = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots,$$

the sum taken in symmetric order [Da]; under RH, $\rho(1-\rho) = |\rho|^2 = \frac{1}{4} + \gamma^2$. We write $Y_N(s) = N^{1-s}/(1-s)$ for the continuous envelope of X_N and $X_N^C = X_N - \frac{1}{2}N^{-s}$.

This paper is Part 5 of the series *Cross-energy of Dirichlet partial sums* (seventeen parts; the plan of the series is in Part 1 [CE1]). It uses the notation and the basic results of Part 1, in particular [CE1, Theorem A], which are cited with the prefix [CE1].

1.1. Main results. The energy of the primes (Section 2). Let $E(u) = e^{-u/2}(\psi(e^u) - e^u + \log 2\pi + \frac{1}{2} \log(1 - e^{-2u}))$ be the normalized error of the prime number theorem. Theorem B states that, in $[0, \infty]$, the mean energy of E equals $\sum_{\rho} \lim_N C_2(N, \rho)$ when the zeros are simple; RH holds if and only if it is finite, and then it equals C_B , which is also Li's first coefficient times two (Proposition 2.4). The energy is at least C_B , with equality if and only if RH holds and all zeros are simple (Corollary 2.3). The explicit formula reads naturally in cross-energy coordinates, $1/\rho = \frac{1}{2}(C_2(\infty, \rho) - iS_2(\infty, \rho))$ (Theorem 2.7); the energy in a Gaussian window is an absolutely convergent double sum over the zeros (Theorem 2.11); and the Nyman–Beurling distances, their conjectured constant C_B [BBLS] and the Gram data of the Báez-Duarte criterion fit the same picture (Proposition 2.14). The arithmetic cross-energy C_2^{Λ} of the prime partial sums satisfies a 3-4-1 positivity at every finite cut, which fails for an Epstein zeta function (Theorem 2.17, Proposition 2.18), and RH is equivalent to $\sum_{n \leq N} C_2^{\Lambda}(n, \frac{1}{2}) = 4N + O(\sqrt{N})$ (Theorem 2.23).

1.2. What is new and what is classical. Theorem B in this form, its local version, the cross-energy coordinates of the explicit formula and the arithmetic energy criterion are, as far as we know, new. Cramér's theorem, the equivalence of bounded prime energy with RH, and the criteria of Li and of Nyman–Beurling–Báez-Duarte are classical and cited as such. We do not prove the Riemann hypothesis.

2. THE ENERGY OF THE PRIMES

Weil positivity is a sign problem. There is a classical way to make the sign automatic: exhibit the quadratic form as the energy of a real signal. For ζ the natural signal is the normalized error term of the prime number theorem. With $u = \log x$ put

$$E(u) = e^{-u/2} \left(\psi(e^u) - e^u + \log 2\pi + \frac{1}{2} \log(1 - e^{-2u}) \right).$$

By the explicit formula [In, T], $E(u) = -\sum_{\rho} e^{(\rho - \frac{1}{2})u} / \rho$ (symmetric summation), which is $-\sum_{\rho} e^{i\gamma_{\rho}u} / \rho$ when all γ_{ρ} are real. The signal is built from the primes alone, and if RH holds its frequencies are the ordinates of the zeros. This section shows that its energy is the sum over the

zeros of the limits of the cross-energy, writes the explicit formula itself in cross-energy coordinates, relates the same constant to Li's criterion and to the Nyman–Beurling distance, and studies the cross-energy C_2^Λ of the prime sums.

2.1. The energy law. Proposition 2.1 (Cramér). (a) Assume RH and that the zeros are simple. Then the autocorrelation

$$R(a) = \lim_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)E(u+a) du = \sum_{\rho} \frac{e^{i\gamma_{\rho}a}}{|\rho|^2} = 2 \sum_{\gamma > 0} \frac{\cos \gamma a}{\frac{1}{4} + \gamma^2}$$

exists and is positive-definite. For every test function φ ,

$$\lim_{U \rightarrow \infty} \frac{1}{U} \int_0^U |(\varphi * E)(u)|^2 du = \sum_{\rho} \frac{|\hat{\varphi}(\gamma_{\rho})|^2}{\frac{1}{4} + \gamma_{\rho}^2} \geq 0.$$

In particular $R(0) = \sum_{\rho} |\rho|^{-2} = C_B = 2 + \gamma_0 - \log 4\pi$.

(b) Conversely, if $\frac{1}{U} \int_0^U E(u)^2 du$ is bounded as $U \rightarrow \infty$, then RH holds.

Proof. (a) Under RH, E is a Besicovitch almost periodic function with Fourier–Bohr coefficients $-1/\rho$ at the frequencies γ_{ρ} (Cramér [Cr]; see [In, Ch. V]). Parseval's identity for almost periodic functions gives R and the filtered energy. For simple zeros, $\sum_{\rho} |\rho|^{-2} = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi$ [Da, Ch. 12].

(b) Suppose the running energy is bounded. By summation by parts, $\int_0^\infty E(u)^2 e^{-2\varepsilon u} du < \infty$ for every $\varepsilon > 0$. By Cauchy–Schwarz the Laplace transform $\int_0^\infty E(u) e^{-(s-1/2)u} du$ converges absolutely for $\operatorname{Re} s > \frac{1}{2}$ and is analytic there. For $\operatorname{Re} s > 1$ it equals

$$\int_1^\infty (\psi(x) - x) x^{-s-1} dx + (\text{analytic in } \operatorname{Re} s > 0) = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} + (\text{analytic in } \operatorname{Re} s > 0).$$

So ζ'/ζ continues analytically to $\operatorname{Re} s > \frac{1}{2}$, and ζ has no zeros there. $\square$

Status: classical [Cr, In].

So a Weil-type positivity with the weight $1/(\frac{1}{4} + \gamma^2)$ holds by construction as soon as the prime signal has finite energy: the sign problem becomes a size problem, and the size problem is exactly RH. The weight $1/(\frac{1}{4} + \gamma^2)$ is the limit of C_2 at the zeros on the line ([CE1, Theorem A]) and 2π times the harmonic measure of the critical line seen from the pole ([CE10, Proposition 2.6]). The C_2 weight is precisely what converts Weil's distribution into the autocorrelation of a real arithmetic signal.

The growth rate of the energy. Let $\Theta = \sup_{\rho} \operatorname{Re} \rho \in [\frac{1}{2}, 1]$ and $F(U) = \int_0^U E(u)^2 du$.

Theorem 2.2 (growth rate). If $\Theta > \frac{1}{2}$, then

$$\limsup_{U \rightarrow \infty} \frac{1}{U} \log F(U) = 2\Theta - 1.$$

If $\Theta = \frac{1}{2}$, then $F(U) \sim C_B U$ when the zeros are simple, and $F(U) = O(U)$ in general. In particular RH holds if and only if $F(U) = O(U)$, and if and only if F grows subexponentially.

Proof. The case $\Theta = \frac{1}{2}$ is Proposition 2.1(a) and, for multiple zeros, Cramér's theorem with the coefficients $-m_{\rho}/\rho$. Let $\Theta > \frac{1}{2}$ and let δ_c be the abscissa of convergence of $I(\delta) = \int_0^\infty E(u)^2 e^{-2\delta u} du$. Since F is nondecreasing and unbounded, the Laplace–Stieltjes abscissa formula gives $\limsup_U \frac{1}{U} \log F(U) = 2\delta_c$, and it remains to show $\delta_c = \Theta - \frac{1}{2}$. The classical bound $\psi(x) - x \ll x^\Theta \log^2 x$ [In, Ch. IV] gives $E(u) \ll u^2 e^{(\Theta-1/2)u}$, so $I(\delta) < \infty$ for $\delta > \Theta - \frac{1}{2}$. Conversely, if $I(\delta) < \infty$ for some $\delta < \Theta - \frac{1}{2}$, the Cauchy–Schwarz argument of Proposition 2.1(b) shows that ζ'/ζ continues analytically to $\operatorname{Re} s > \frac{1}{2} + \delta$, which contradicts the existence of a zero with $\beta > \frac{1}{2} + \delta$. $\square$

Status: proved.

In the language of the cross-energy, the energy of the prime signal is a sum of C_2 slopes. At a zero on the critical line the partial sums satisfy the straight-line law $|X_n(\rho)|^2 = (n + \frac{1}{2})/|\rho|^2 + O(n^{-1})$ (Part 1 [CE1]), so $1/|\rho|^2 = 1/(\frac{1}{4} + \gamma^2)$ is the slope of the energy $|X_n(\rho)|^2$. Proposition 2.1(a) states that the mean energy of the primes is the sum of these slopes over all zeros, and Theorem 2.2 that the exponential growth rate of the same energy is $2\Theta - 1$, twice the distance from the critical line of the zero farthest from it. The same argument applies to the Davenport–Heilbronn signal E_f , whose Laplace transform is $-f'(s)/(sf(s))$ without a pole term: the growth rate of its energy is $2\Theta_f - 1$. Davenport and Heilbronn proved that f has zeros with real part greater than 1 [DH], so $2\Theta_f - 1 > 1$; these zeros lie high in the strip and have small weight $1/|\rho|^2$, so in any computable range the growth of E_f is governed by its low off-line zeros.

Theorem B (C_2 energy law). In $[0, \infty]$,

$$\limsup_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)^2 du = \sum_{\rho} \lim_{n \rightarrow \infty} C_2(n, \rho) \quad \text{if all zeros are simple,} \quad (2.1)$$

and in general both sides are finite or both are infinite. RH holds if and only if they are finite, and then the right side equals $\sum_{\rho} |\rho|^{-2} = C_B$.

Proof. Suppose RH fails, and let ρ be a zero with $\beta > \frac{1}{2}$. Then $\Theta > \frac{1}{2}$, so the left side is $+\infty$ by Theorem 2.2. The reflected zero $1 - \bar{\rho}$ has real part $1 - \beta < \frac{1}{2}$, so by [CE1, Theorem A] its term on the right is $+\infty$, and all other terms are nonnegative. Suppose RH holds. By [CE1, Theorem A] the right side is $\sum_{\rho} (\frac{1}{4} + \gamma^2)^{-1} = C_B$, counted with multiplicity. By Cramér’s theorem the left side is a limit, equal to $\sum_{\rho \text{ distinct}} m_{\rho}^2/|\rho|^2$, where m_{ρ} is the multiplicity; this is finite because $m_{\rho} \ll \log |\gamma|$, and it equals C_B when all zeros are simple. $\square$

Status: proved.

Thus the mean energy of the primes is the sum over the zeros of the limiting values of C_2 , and RH is the statement that no zero has an infinite C_2 limit. This is the trichotomy of [CE1, Theorem A] read through the primes. The identity is a reformulation, not a new constraint: proving that the right side is finite is RH.

Corollary 2.3 (the energy attains its lower bound exactly under RH). Let

$$\mathcal{E} = \limsup_{U \rightarrow \infty} \frac{1}{U} \int_0^U E(u)^2 du \in [0, \infty].$$

Then

$$\mathcal{E} \geq 2 + \gamma_0 - \log 4\pi = 0.0461914 \dots,$$

with equality if and only if RH holds and all zeros are simple.

Proof. The identity $\sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi$, with multiplicity and in symmetric order, holds unconditionally [Da, Ch. 12]. If RH fails, $\mathcal{E} = \infty$ by Theorem B. If RH holds, $\rho(1-\rho) = |\rho|^2$ and $E(u) = -\sum_{\gamma} m_{\gamma} e^{i\gamma u}/\rho$ over the distinct zeros; the mean of the cross terms vanishes, and $\mathcal{E} = \sum_{\gamma} m_{\gamma}^2/|\rho|^2 \geq \sum_{\gamma} m_{\gamma}/|\rho|^2 = 2 + \gamma_0 - \log 4\pi$, with equality exactly when every $m_{\gamma} = 1$. $\square$

Status: proved.

One side of the criterion is an explicit constant and the other is computed from the primes alone. We computed $E(u)$ from all prime powers up to 10^8 and took time averages over $1 < u < \log 10^8$ (Figure 1). The measured energy 0.0453 lies slightly below the limit, by an amount consistent with the finite range of u ; a finite computation cannot decide the equality, since an off-line zero at height γ affects the energy only for $u \gtrsim 2 \log \gamma/(2\beta - 1)$. The autocorrelation obtained from the primes alone matches $2 \sum_{0 < \gamma < 600} \cos(\gamma a)/(\frac{1}{4} + \gamma^2)$ with correlation 0.9997 over $0 \leq a \leq 3$:

a	0	0.10	0.50	0.69	1.00	1.10	2.00	3.00
primes $\leq 10^8$	0.04533	-0.00694	0.00627	-0.01681	0.00227	-0.01991	-0.00713	0.00648
zeros $ \gamma < 600$	0.04324	-0.00693	0.00603	-0.01672	0.00220	-0.02051	-0.00713	0.00649

At $a = 0$ the empirical energy 0.0453 lies between the truncated zero sum 0.0432 and $C_B = 0.0462$. Status: numerical (**anc/signal/**).

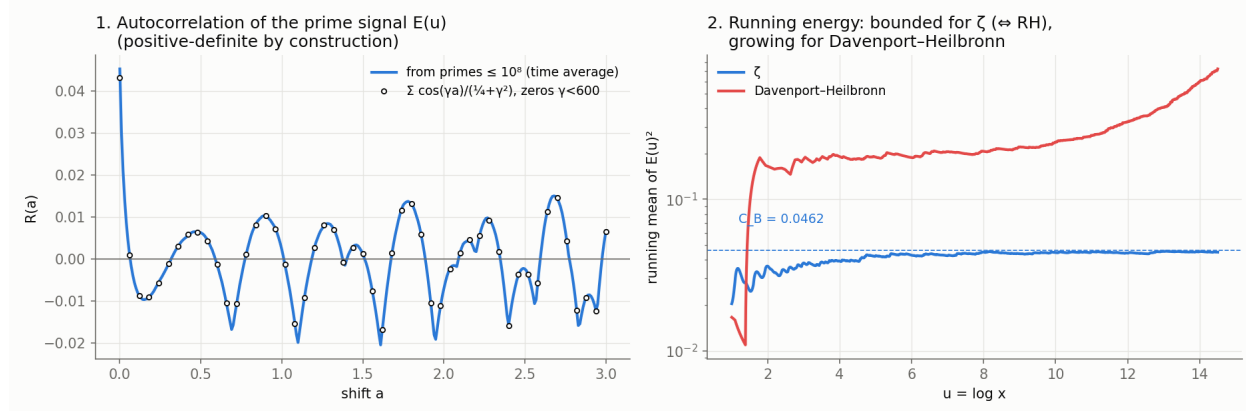


FIGURE 1. Left: autocorrelation of the prime signal $E(u)$ from the primes up to 10^8 (line) against $2 \sum_{0 < \gamma < 600} \cos(\gamma a) / ((\frac{1}{4} + \gamma^2))$ (circles). Right: running mean of $E(u)^2$ for ζ , tending to C_B , and of the analogous signal for the Davenport–Heilbronn function, which grows.

For the Davenport–Heilbronn function the analogous signal $E_f(u) = e^{-u/2} \sum_{n \leq e^u} b(n)$, with $b(n)$ the coefficients of $-f'/f$, has no pole term. Its running energy grows:

$x = e^U$	55	403	2981	22026	162755	$1.2 \cdot 10^6$
running energy, ζ	0.041	0.044	0.045	0.045	0.045	0.045
running energy, f	0.21	0.20	0.22	0.25	0.34	0.61

The growth is driven by the off-line zeros, which contribute modes of size $e^{(\beta-1/2)u}$; by Proposition 2.1(b) applied to f , bounded energy would force f to satisfy RH, which it does not. To test Theorem 2.2 we extended E_f to $x = 2 \cdot 10^7$ and computed the energy over windows $U - 1 < u \leq U$. Each off-line zero contributes on average $2e^{(2\beta-1)u}/|\rho|^2$ to E_f^2 ; adding these contributions for the eight off-line zeros with $\gamma < 400$ to the base level 0.200 observed for $U \leq 8$ gives the third row:

U	6	8	10	11	12	13	14	15	16
window energy of E_f	0.183	0.222	0.395	0.526	0.952	1.449	2.729	5.256	9.225
base + off-line zeros $\gamma < 400$	0.213	0.244	0.356	0.496	0.764	1.281	2.282	4.231	8.038

The prediction follows the measured growth within 10–25%; the remainder is consistent with off-line zeros above height 400. The local growth rate on $12 \leq U \leq 16$ is 0.58, between the exponents $2\beta - 1 = 0.617$ and 0.739 of the dominant zeros $0.8085 + 85.70i$ and $0.8695 + 240.40i$. For ζ the window energy stays between 0.036 and 0.060 throughout. Status: numerical (**anc/signal/**, **anc/c2law/**).

2.2. Li's coefficients. Put $z = 1 - 1/s$, so that $s = 1/(1 - z)$. Since $|z| = |s - 1|/|s|$, the critical line goes bijectively onto the unit circle, the pole $s = 1$ goes to $z = 0$, and the half-plane $\operatorname{Re} s > \frac{1}{2}$ goes to the unit disc; this is the map of [CE10, Corollary 2.8] and [CE10, Proposition 2.16], under which μ becomes normalized arc length. The functional equation becomes $z \mapsto 1/z$ and the reflection $s \mapsto \bar{s}$ becomes $z \mapsto \bar{z}$, so the zeros are symmetric under the reflection $z \mapsto 1/\bar{z}$ in the circle. With $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, expand

$$\log \xi\left(\frac{1}{1-z}\right) = -\log 2 + \sum_{n \geq 1} \frac{\lambda_n}{n} z^n, \quad \lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right],$$

the sum over the zeros taken symmetrically. Li's criterion [Li, BL]: RH holds if and only if $\lambda_n \geq 0$ for all $n \geq 1$; equivalently, if and only if $\log \xi(1/(1-z))$ is analytic in the open disc. Under RH, $\lambda_n = \frac{n}{2}(\log n + \gamma_0 - 1 - \log 2\pi) + O(\sqrt{n} \log n)$ [BL].

Proposition 2.4 (Li's first coefficient is the C_2 constant). Unconditionally,

$$\lambda_1 = \sum_{\rho} \frac{1}{\rho} = \frac{\xi'}{\xi}(1) = 1 + \frac{\gamma_0}{2} - \frac{1}{2} \log 4\pi = \frac{C_B}{2} = 0.0230957\dots,$$

and

$$\frac{1}{2} \sum_{\rho} \lim_{n \rightarrow \infty} C_2(n, \rho) \geq \lambda_1,$$

with equality if and only if RH holds; the left side is $+\infty$ if RH fails.

Proof. Since the zeros are symmetric under $\rho \mapsto 1 - \rho$, $\sum_{\rho} \frac{1}{\rho(1-\rho)} = \sum_{\rho} \left(\frac{1}{\rho} + \frac{1}{1-\rho}\right) = 2 \sum_{\rho} \frac{1}{\rho}$, and $\sum_{\rho} \frac{1}{\rho(1-\rho)} = C_B$ [Da, Ch. 12]. If RH holds, $\lim_n C_2(n, \rho) = 1/|\rho|^2 = 1/(\rho(1-\rho))$ by [CE1, Theorem A], and the sum is $C_B = 2\lambda_1$. If RH fails, a zero $1 - \bar{\rho}$ with real part $< \frac{1}{2}$ has infinite limit. $\square$

Status: classical [Li]; the C_2 form is proved.

The weight $1/(\frac{1}{4} + \gamma^2)$ is thus, at once, the slope of the straight-line law (Part 1 [CE1]), the weight of a zero in the Jensen integral ([CE10, Corollary 2.8]), the mean energy of the prime signal (Theorem B) and twice Li's first coefficient: the same number seen from the pole.

We computed λ_n in two independent ways. The first uses only values of ξ on the circle $|z| = 0.9$, which lies inside the disc where $\log \xi$ is analytic without any hypothesis: 4096 points, a continuous branch of $\log \xi$ and an FFT, the n -th Taylor coefficient multiplied by n . The second uses the 341 zeros with $0 < \gamma < 600$ in the sum over zeros, with a tail integral against the zero density $\frac{1}{2\pi} \log \frac{t}{2\pi}$ for $t > 600$.

n	λ_n (circle, FFT)	λ_n (zeros)	$\frac{n}{2}(\log n + \gamma_0 - 1 - \log 2\pi)$
1	0.02309571	0.02309428	-1.130
2	0.09234574	0.09234001	-1.568
5	0.57554271	0.57550693	-1.628
10	2.27933936	2.27919621	0.210
20	8.76927687	8.76870425	7.351
50	43.53109649	43.52751681	41.284
100	118.60377538	118.58944562	117.225

The two columns agree to 10^{-4} in relative terms, the accuracy of the zero-density tail, and all λ_n with $n \leq 120$ are positive, as is known far beyond this range [Ke] (Figure 2). Status: numerical (`anc/li/li.py`, `anc/li/fig_li_coefficients.py`).

How far an off-line zero is from the circle. Let $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$. Since $|\rho - 1|^2 - |\rho|^2 = 1 - 2\beta$,

$$\left|1 - \frac{1}{\rho}\right|^2 = 1 - \frac{2\beta - 1}{|\rho|^2}, \quad \left|1 - \frac{1}{\rho}\right| = 1 - \frac{\beta - \frac{1}{2}}{\gamma^2} + O(\gamma^{-4}),$$

and the partner $1 - \bar{\rho}$ lies outside the circle at the same distance. Its term in λ_n grows like $(1 + (\beta - \frac{1}{2})/\gamma^2)^n$ and becomes of order one only when $n \approx \gamma^2/(\beta - \frac{1}{2})$. A pair with $\beta = 0.6$ and $\gamma = 30$ sits 10^{-4} from the circle; a pair with $\beta = 0.6$ at height 10^{13} , beyond current verification, would first be seen near $n \approx 10^{27}$. All zeros crowd towards the single point $z = 1$, and the criterion is exactly as hard as RH. The distance is the same quantity that weighs a zero in the Jensen integral: $\log |\rho/(1 - \rho)| = \log(1/|1 - 1/\rho|)$ ([CE10, Corollary 2.8]).

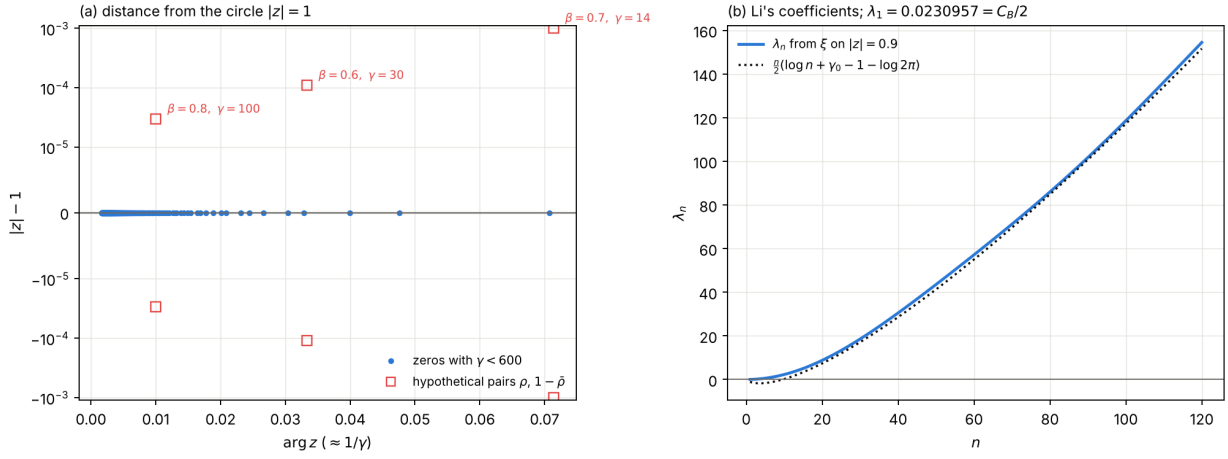


FIGURE 2. Li's variable $z = 1 - 1/s$. Left: distance $|z| - 1$ from the unit circle against $\arg z \approx 1/\gamma$, for the zeros with $\gamma < 600$ (on the circle) and for hypothetical off-line pairs $\rho, 1 - \bar{\rho}$, at $\pm(\beta - \frac{1}{2})/\gamma^2$. Right: λ_n for $n \leq 120$ computed from ξ on $|z| = 0.9$, and the asymptotic $\frac{n}{2}(\log n + \gamma_0 - 1 - \log 2\pi)$; $\lambda_1 = C_B/2$.

Proposition 2.5 (the non-holomorphic gap). With multiplicity and in symmetric order,

$$\sum_{\rho} \frac{1}{|\rho|^2} = 2 + \gamma_0 - \log 4\pi + \mathcal{Q}, \quad \mathcal{Q} = 2 \sum_{\rho} \frac{(\beta - \frac{1}{2})^2}{|\rho|^2 |1 - \rho|^2} = \frac{1}{2} \sum_{\rho} (|w_{\rho}| - |w_{\rho}|^{-1})^2 \geq 0, \quad w_{\rho} = 1 - \frac{1}{\rho}.$$

Both series converge absolutely, and $\mathcal{Q} = 0$ if and only if RH holds.

Proof. Pair ρ with $1 - \bar{\rho}$, a bijection of the zeros preserving multiplicity, and put $a = |\rho|^2$, $b = |1 - \rho|^2 = a + 1 - 2\beta$. Then $\frac{\beta}{a} + \frac{1-\beta}{b} - \frac{1}{2}(\frac{1}{a} + \frac{1}{b}) = -\frac{2(\beta - \frac{1}{2})^2}{ab}$. Summing this over all ρ counts each term of $\sum_{\rho} (\operatorname{Re} \frac{1}{\rho} - \frac{1}{2}|\rho|^{-2})$ twice, once from ρ and once from its partner, so with $\operatorname{Re} \frac{1}{\rho} = \frac{\beta}{a}$ and $\lambda_1 = \sum_{\rho} \operatorname{Re} \frac{1}{\rho} = \frac{1}{2}C_B$ (Proposition 2.4) it gives $\lambda_1 = \frac{1}{2} \sum_{\rho} |\rho|^{-2} - \sum_{\rho} \frac{(\beta - \frac{1}{2})^2}{ab} = \frac{1}{2} \sum_{\rho} |\rho|^{-2} - \frac{1}{2}\mathcal{Q}$. Finally $|w_{\rho}|^2 = b/a$, so $\frac{1}{4}(|w_{\rho}| - |w_{\rho}|^{-1})^2 = \frac{(a-b)^2}{4ab} = \frac{(\beta - \frac{1}{2})^2}{ab}$. $\square$

Status: the identity is elementary; the equivalence of RH with $\sum_{\rho} |\rho|^{-2} = 2 + \gamma_0 - \log 4\pi$ is known.

The functional equation gives $|w_{1-\bar{\rho}}| = |w_{\rho}|^{-1}$: an off-line pair has reciprocal radii $r, 1/r$ and contributes $(r - r^{-1})^2 > 0$, so the symmetry does not cancel the defect. The defect is tiny: for a zero $\rho = 0.6 + 1000i$ each of the four zeros $\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}$ contributes $2(\beta - \frac{1}{2})^2/(|\rho|^2|1 - \rho|^2) \approx 2 \cdot 10^{-14}$ to $\mathcal{Q}$, while at $1 - \bar{\rho} = 0.4 + 1000i$ the cross-energy grows like $n^{0.2}$ ([CE1, Theorem A]). What vanishes under RH is the excess $\mathcal{Q}$, not the energy, and the cross-energy amplifies the same defect

into divergence. By [CE1, Theorem A], $1 - 2\beta = \lim_n \log C_2(n, \rho) / \log n$ for $0 < \beta < 1$, so $\mathcal{Q}$ is also a sum of squared growth exponents of the cross-energy, weighted by $\frac{1}{2}|\rho|^{-2}|1 - \rho|^{-2}$. The identity explains why the criterion cannot be closed from the primes alone. The explicit formula evaluates $\sum_\rho F(\rho)$ only for F holomorphic near the strip, and gives $\sum_\rho \frac{1}{\rho(1-\rho)} = C_B = 2\lambda_1$ without information on β ; $\mathcal{Q}$ is the part of $\sum_\rho |\rho|^{-2}$ that depends on $|\rho|$ and not on ρ , and it is reached only through a positivity statement on the spectral side, such as the mean square of Theorem B. The holomorphic side has its own cross-energy, built from $X_N(s)X_N(1-s)$, whose limit $1/(s(1-s))$ at every point of the strip sums to C_B over the zeros; it is the subject of [CE3].

2.3. The explicit formula in cross-energy coordinates. The explicit formula sees the zeros through the primes. In the coordinates of the cross-energy its coefficients are the limits of C_2 and S_2 . These coordinates hold at every point of the critical line; what distinguishes a zero is only the speed at which they are reached. Write $C_2(\infty, s)$ and $S_2(\infty, s)$ for the limits of $C_2(n, s)$ and $S_2(n, s)$ when they exist.

Proposition 2.6 (the dictionary). (i) For $0 < \sigma < 1$, $W_\infty(s) := \lim_n n^{2\sigma-1} W_n(s) = 2/(1-s)$; hence $\frac{1}{s} = \frac{1}{2} \overline{W_\infty(1-\bar{s})}$.

(ii) For every $s = \frac{1}{2} + it$, $W_n(s) \rightarrow 2/\bar{s}$, and

$$\frac{1}{s} = \frac{C_2(\infty, s) - iS_2(\infty, s)}{2}, \quad C_2(\infty, s) = \frac{1}{|s|^2} = \frac{1}{\frac{1}{4} + t^2}, \quad S_2(\infty, s) = 2t C_2(\infty, s).$$

(iii) At a zero $\rho = \frac{1}{2} + i\gamma$ the oscillating term $2\zeta(s)n^{-\bar{s}}$ of [CE10, (2.1)], of size $n^{-1/2}$, is absent, and for every $n \geq 2$

$$\frac{1}{\rho} = \frac{1}{2} \overline{W_n(\rho)} + \frac{1}{2n} + \frac{\bar{\rho}}{12n^2} + O(|\rho|^4 n^{-4}), \quad C_2(n, \rho) = \frac{1}{|\rho|^2} - \frac{1}{n} - \frac{1}{12n^2} + O(|\rho|^4 n^{-4}).$$

Proof. Euler–Maclaurin with two Bernoulli terms gives $X_{n-1}(s) = \zeta(s) + \frac{n^{1-s}}{1-s} - \frac{1}{2}n^{-s} - \frac{s}{12}n^{-s-1} + O(|s(s+1)(s+2)(s+3)|n^{-\sigma-3})$; multiply by $2n^{-\bar{s}}$. (i) Multiply by $n^{2\sigma-1}$; the wave is $O(n^{\sigma-1})$. Apply it at $1-\bar{s}$, for which $1-(1-\bar{s}) = \bar{s}$. (ii) is (i) with $\sigma = \frac{1}{2}$, and $\frac{2}{s} = \frac{1+2it}{|s|^2}$. (iii) Put $\zeta(\rho) = 0$, use $1-\rho = \bar{\rho}$, and take complex conjugates. $\square$

Status: proved.

So $C_2(\infty, \cdot)$ does not detect the zeros: $C_2(\infty, \frac{1}{2} + it) = 1/(\frac{1}{4} + t^2)$ for every t , and the cross-energy sees a zero on the line only through the rate of convergence, $1/n$ instead of $n^{-1/2}$ (off the line, through the trichotomy of [CE1, Theorem A]). Up to the factor 2π it is the density of the harmonic measure μ ([CE10, Proposition 2.6]). Table 5 shows the convergence at three zeros.

n	$C_2(n, \rho_k)/C_2(\infty, \rho_k)$			relative error in Proposition 2.6(iii)		
	$k=1$	$k=100$	$k=1000$	$k=1$	$k=100$	$k=1000$
10^2	-1.0021	-559.49	-97158	$6 \cdot 10^{-7}$	$5 \cdot 10^{-2}$	98
10^3	0.7999	-54.949	-2014.0	$6 \cdot 10^{-11}$	$4 \cdot 10^{-6}$	$6 \cdot 10^{-3}$
10^4	0.9800	-4.5944	-200.48	$< 10^{-13}$	$4 \cdot 10^{-10}$	$6 \cdot 10^{-7}$
10^5	0.9980	0.4406	-19.148	$< 10^{-13}$	$< 10^{-12}$	$6 \cdot 10^{-11}$
10^6	0.9998	0.9441	-1.0148	$< 10^{-13}$	$< 10^{-12}$	$< 10^{-11}$
10^7	1.0000	0.9944	0.7985	$< 10^{-13}$	$< 10^{-12}$	$< 10^{-11}$

TABLE 5. The cross-energy at the zeros $\rho_1 = \frac{1}{2} + 14.1347i$, $\rho_{100} = \frac{1}{2} + 236.524i$, $\rho_{1000} = \frac{1}{2} + 1419.42i$, computed from the partial sums, and the relative error $|\rho| \left| \frac{1}{2} \overline{W_n(\rho)} + \frac{1}{2n} + \frac{\bar{\rho}}{12n^2} - \frac{1}{\rho} \right|$ of Proposition 2.6(iii). For $n \geq |\rho|$ the error is the B_4 term, about $|\rho|^4/(720n^4)$, down to rounding at about 10^{-11} . $C_2(n, \rho)$ changes sign near $n^*(\rho) \approx \gamma^2$ (200, $5.6 \cdot 10^4$, $2.0 \cdot 10^6$; Part 1 [CE1]).

Theorem 2.7 (the explicit formula in cross-energy coordinates). Let $x > 1$, let $\psi_0(x)$ be $\psi(x)$ with the value at a jump replaced by the mean of the two one-sided limits, and let sums over the zeros be $\lim_{T \rightarrow \infty} \sum_{|\gamma| < T}$, with multiplicity.

- (a) Unconditionally, $\psi_0(x) = x - \frac{1}{2} \sum_{\rho} x^{\rho} \overline{W_{\infty}(1 - \bar{\rho})} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2})$.
- (b) Under RH,

$$\begin{aligned} \psi_0(x) &= x - \sqrt{x} \sum_{\gamma > 0} \left[C_2(\infty, \rho) \cos(\gamma \log x) + S_2(\infty, \rho) \sin(\gamma \log x) \right] - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}) \\ &= x - 2\sqrt{x} \sum_{\gamma > 0} \sqrt{C_2(\infty, \rho)} \cos(\gamma \log x - \arctan 2\gamma) - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}). \end{aligned}$$

A distinct ordinate $\gamma > 0$ of multiplicity $m(\gamma)$ contributes one cosine in $\log x$ with amplitude $2m(\gamma)\sqrt{C_2(\infty, \rho)} = 2m(\gamma)/|\rho|$, frequency γ and phase $\arctan 2\gamma = \arg \rho$.

Proof. Von Mangoldt's explicit formula $\psi_0(x) = x - \sum_{\rho} x^{\rho}/\rho - \log 2\pi - \frac{1}{2} \log(1 - x^{-2})$ [Da, Ch. 17], with $1/\rho$ written by Proposition 2.6. For (b), $\bar{\rho}$ is a zero with the multiplicity of ρ , and $\frac{x^{\rho}}{\rho} + \frac{x^{\bar{\rho}}}{\bar{\rho}} = 2 \operatorname{Re} \frac{x^{\rho}}{\rho} = \sqrt{x} \operatorname{Re} [e^{i\gamma \log x} (C_2 - iS_2)] = \sqrt{x} (C_2 \cos + S_2 \sin)(\gamma \log x)$. The amplitude is $(C_2^2 + S_2^2)^{1/2} = C_2(1 + 4\gamma^2)^{1/2} = 2|\rho|C_2 = 2\sqrt{C_2}$ and the phase is $\arctan(S_2/C_2) = \arctan 2\gamma$. The symmetric partial sums over $|\gamma| < T$ become partial sums over $0 < \gamma < T$. $\square$

Status: (a) is proved; (b) is conditional on RH.

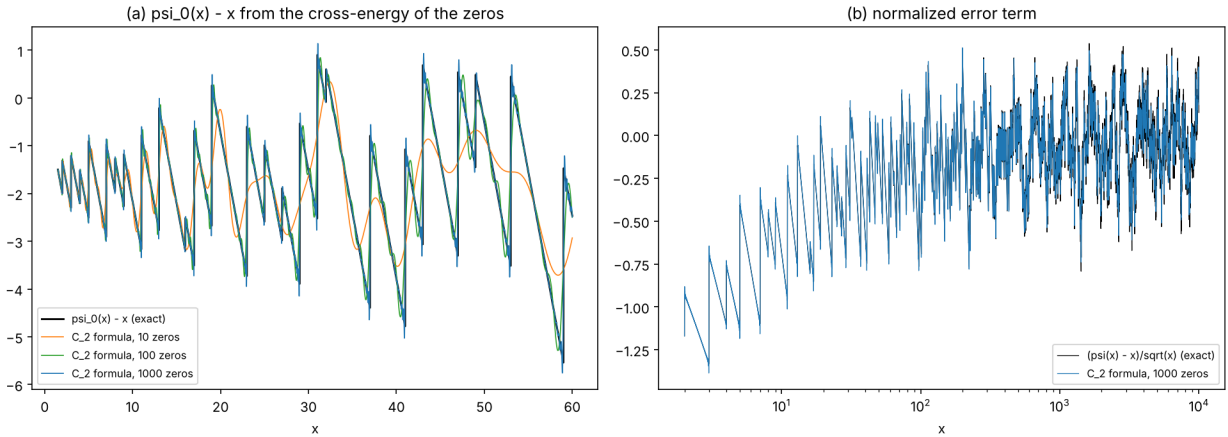


FIGURE 3. Theorem 2.7(b) with the coefficients $C_2(\infty, \rho)$, $S_2(\infty, \rho)$ of the first 10, 100 and 1000 zeros. (a) $\psi_0(x) - x$ for $x \leq 60$: the cosines rebuild the jumps at the prime powers. (b) $(\psi(x) - x)/\sqrt{x}$ for $x \leq 10^4$ (black) and the formula with 1000 zeros (blue).

Remark 2.8 (Cramér's mean square in cross-energy coordinates). Under RH, with $\Delta(u) = e^{-u/2}(\psi(e^u) - e^u)$, which differs from $E(u)$ by $O(e^{-u/2})$, the cosine of $\gamma > 0$ in Theorem 2.7(b) has mean square $\frac{1}{2}(2m(\gamma)\sqrt{C_2})^2 = 2m(\gamma)^2 C_2(\infty, \rho)$, and distinct frequencies do not interact in the mean (Cramér's theorem justifies squaring the conditionally convergent series term by term). Hence

$$\lim_{U \rightarrow \infty} \frac{1}{U} \int_0^U \Delta(u)^2 du = \sum_{\gamma} m(\gamma)^2 C_2(\infty, \frac{1}{2} + i\gamma) \geq \sum_{\rho} C_2(\infty, \rho) = 2\lambda_1 = C_B,$$

the first sum over the distinct ordinates of both signs, the second over the zeros with multiplicity, with equality if and only if all zeros are simple. This is Corollary 2.3 under RH, and Figure 3 shows the cosines at work. Status: conditional on RH; classical [Cr].

Remark 2.9 (Schoenfeld's constant as a triangle inequality). Under RH, Schoenfeld proved $|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x}\log^2 x$ for $x \geq 73.2$ [Sch], and Goldston proved $\psi(x) - x \leq \frac{1}{8\pi}\sqrt{x}\log^2 x + O(\sqrt{x}\log x \log \log x)$ [Go], by applying the triangle inequality to the zeros up to a height of order $\sqrt{x}$. In the coordinates of Theorem 2.7(b) the constant is the sum of the amplitudes,

$$2 \sum_{0 < \gamma \leq T} \sqrt{C_2(\infty, \rho)} = 2 \sum_{0 < \gamma \leq T} \frac{1}{|\rho|} = \frac{1}{2\pi} \log^2 T + O(\log T),$$

by the Riemann–von Mangoldt formula, and $T = \sqrt{x}$ gives $\frac{1}{8\pi} \log^2 x$. The zeros below height $\sqrt{x}$ are those whose cross-energy has changed sign by $n = x$, since the crossing of $C_2(\cdot, \rho)$ is at $n^*(\rho) \approx \gamma^2$ (Part 1 [CE1]). Computing ψ with a sieve up to 10^9 at every jump, the maximum of $|\psi(x) - x|/(\sqrt{x}\log^2 x)$ over $x \geq 74$ in $[10^k, 10^{k+1}]$ is 0.0365 (at $x = 97$), 0.0325, 0.0152, 0.0081, 0.0045, 0.0033, 0.0025, 0.0023 for $k = 1, \dots, 8$, against $\frac{1}{8\pi} = 0.0398$. Near $x = 100$ the bound is nearly attained; further out the cosines align only to the extent that Littlewood's theorem $\psi(x) - x = \Omega_{\pm}(\sqrt{x}\log \log \log x)$ forces [In]. Status: the rewriting is proved; the maxima are numerical (anc/c2_explicit/psi_check.py).

Remark 2.10 (phases, and the off-line case). The phases $\arctan 2\gamma = \frac{\pi}{2} - \frac{1}{2\gamma} + O(\gamma^{-3})$ are almost constant, so under RH the size of $\psi(x) - x$ at a given x is decided by the simultaneous alignment of the numbers $\gamma \log x$ modulo 2π , a Diophantine question (Kronecker's theorem, as in Littlewood's $\Omega_{\pm}$ result [In]); the energy of Theorem B does not see this alignment. Off the line, if $\beta > \frac{1}{2}$, Proposition 2.6(i) places the coefficient $1/\rho$ in the normalized cross-energy at the reflected zero $1 - \bar{\rho}$, and $W_n(1 - \bar{\rho}) \sim 2n^{2\beta-1}/\bar{\rho}$ grows like the square of the factor $x^{\beta-1/2}$ by which the term of ρ outgrows $\sqrt{x}$, as an energy should. This is the mechanism of Theorem B: the infinite C_2 limit at $1 - \bar{\rho}$ is the exponential growth $e^{(2\beta-1)u}$ of the energy of the primes. Status: proved (asymptotic statement).

2.4. The local energy law. Theorem B concerns an average over all u . A version over a finite window holds unconditionally. For $U \in \mathbb{R}$ and $H > 0$ put

$$\mathcal{E}(U, H) = \frac{1}{H} \int_{-\infty}^{\infty} E(u)^2 e^{-\pi(u-U)^2/H^2} du.$$

For $u \leq 0$, where the expression defining E is not available, $E(u)$ denotes the zero series $-\sum_{\rho} e^{(\rho-\frac{1}{2})u}/\rho$ (symmetric summation; it converges in L^2 on bounded intervals because $\sum_{\rho} |\rho|^{-2} < \infty$); for U large the contribution of $u \leq 0$ is $O(e^{-\pi U^2/H^2})$.

Theorem 2.11 (local energy law). For all U and $H > 0$,

$$\mathcal{E}(U, H) = \sum_{\rho, \rho'} \frac{e^{zU + z^2 H^2/(4\pi)}}{\rho \bar{\rho}'}, \quad z = \rho + \bar{\rho}' - 1, \quad (2.2)$$

where the sum runs over all pairs of nontrivial zeros, counted with multiplicity, and converges absolutely.

Proof. By the truncated explicit formula [In, T], $E_T(u) = -\sum_{|\gamma| < T} e^{(\rho-\frac{1}{2})u}/\rho$ converges to E in L^2 on every bounded interval as $T \rightarrow \infty$, and the Gaussian weight makes the contribution of $|u - U| > R$ negligible uniformly in T . For finite T , $\frac{1}{H} \int e^{zu} e^{-\pi(u-U)^2/H^2} du = e^{zU + z^2 H^2/(4\pi)}$ gives (2.2) with the sum restricted to $|\gamma|, |\gamma'| < T$. Since $\operatorname{Re} z \leq 1$ and $\operatorname{Re} z^2 \leq 1 - (\gamma - \gamma')^2$, each term is at most $e^{U + H^2/(4\pi)} e^{-(\gamma - \gamma')^2 H^2/(4\pi)} / |\rho \rho'|$, and $\sum_{\rho, \rho'} e^{-(\gamma - \gamma')^2 H^2/(4\pi)} / |\rho \rho'| < \infty$ because $N(T+1) - N(T) \ll \log T$. Letting $T \rightarrow \infty$ gives the result. $\square$

Status: proved.

Corollary 2.12. Let $H > 0$ and $B(H) = \sum_{\rho \neq \rho'} e^{-(\gamma - \gamma')^2 H^2 / (4\pi)} / |\rho \rho'|$, the sum over pairs of distinct indices.

- (a) If RH holds, then $|\mathcal{E}(U, H) - C_B| \leq B(H)$ for all U .
- (b) RH holds if and only if $\sup_U \mathcal{E}(U, H) < \infty$.
- (c) (Energy cost of a spiral.) Let $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$ with $\delta > 0$, let $\Gamma > 0$, and suppose that the zeros with $|\gamma - \gamma_0| \leq \Gamma$ other than ρ_0 and $1 - \bar{\rho}_0$ lie on the critical line. Then for all $U \geq 0$ and $H > 0$,

$$\sqrt{\mathcal{E}(U, H)} \geq \frac{e^{\delta U + \delta^2 H^2 / (2\pi)}}{|\rho_0|} - D_{\gamma_0}(H) - T_{\gamma_0}(U, H, \Gamma),$$

where $D_{\gamma_0}(H) = \sum_{\rho \neq \rho_0} e^{-(\gamma - \gamma_0)^2 H^2 / (4\pi)} / |\rho|$, with the zeros other than ρ_0 counted as if on the line, and $T_{\gamma_0}(U, H, \Gamma) = e^{U/2 + 3H^2 / (16\pi)} \sum_{|\gamma - \gamma_0| > \Gamma} e^{-(\gamma - \gamma_0)^2 H^2 / (4\pi)} / |\rho|$.

Proof. (a) Under RH, $z = i(\gamma - \gamma')$; the terms with $\rho = \rho'$ give $\sum_{\rho} |\rho|^{-2} = C_B$, and the others have modulus at most $e^{-(\gamma - \gamma')^2 H^2 / (4\pi)} / |\rho \rho'|$. (b) If $\sup_U \mathcal{E}(U, H) \leq M$, then $\int_{U-H/2}^{U+H/2} E^2 \leq e^{\pi/4} H M$ for all U , so $\int_0^U E^2 = O(U)$, and Theorem 2.2 gives $\Theta = \frac{1}{2}$. The converse is (a). (c) $\mathcal{E}(U, H) = \|E\|^2$ in $L^2(w)$, $w(u) = H^{-1} e^{-\pi(u-U)^2 / H^2}$, so $\mathcal{E} \geq |\langle E, h \rangle|^2 / \|h\|^2$ for $h(u) = e^{(\rho_0 - \frac{1}{2})u}$. By the computation in the proof of Theorem 2.11, $\langle E, h \rangle = -\sum_{\rho} G_{\rho \rho_0} / \rho$ with $G_{\rho \rho_0} = e^{zU + z^2 H^2 / (4\pi)}$, $z = \rho + \bar{\rho}_0 - 1$, and $\|h\|^2 = G_0 = e^{2\delta U + \delta^2 H^2 / \pi}$. The term $\rho = \rho_0$ gives $\sqrt{G_0} / |\rho_0|$. For a zero on the line, $|G_{\rho \rho_0}| / \sqrt{G_0} = e^{-(\delta^2 + (\gamma - \gamma_0)^2) H^2 / (4\pi)}$; for $1 - \bar{\rho}_0$, $z = 0$ and the ratio is $G_0^{-1/2} \leq 1$; for a zero with $|\gamma - \gamma_0| > \Gamma$ the ratio is at most $e^{U/2 + 3H^2 / (16\pi) - (\gamma - \gamma_0)^2 H^2 / (4\pi)}$, since $\beta - \frac{1}{2} \leq \frac{1}{2}$. The triangle inequality gives the claim. $\square$

Status: proved.

A zero with $\beta > \frac{1}{2}$ enters $\mathcal{E}$ through its diagonal term $e^{(2\beta-1)U + (2\beta-1)^2 H^2 / (4\pi)} / |\rho|^2 > 0$, which no other term can cancel for large U , since all other terms involving ρ grow at most like $e^{(\beta - \frac{1}{2})U}$. It becomes visible only for $U \gtrsim 2 \log |\gamma| / (2\beta - 1)$, so the law does not by itself exclude zeros above the computed range. We computed $\mathcal{E}(U, H)$ from the prime powers up to 10^8 and from the double sum over the 682 zeros with $|\gamma| < 600$, adding the diagonal contribution $C_B - \sum_{|\gamma| < 600} |\rho|^{-2} = 0.00295$ of the higher zeros; the two agree to within $3.3 \cdot 10^{-4}$ for $H = 1, 2$ and $6 \leq U \leq 14$. On $5 \leq U \leq 14.5$, $|\mathcal{E}(U, H) - C_B|$ is at most 0.0074 for $H = 1$ and 0.0023 for $H = 2$, against $B(1) = 0.044$ and $B(2) = 0.0123$ computed from the same zeros. For the Davenport–Heilbronn signal ($H = 1$) the window energy grows from 0.22 at $U = 6$ to 2.06 at $U = 13$; the double sum over its zeros with $\gamma < 400$ accounts for 84–96% of it, the diagonal terms of its eight off-line zeros grow from 0.02 to 1.52, and the remainder increases with U , as expected from off-line zeros above height 400. The Dirichlet L -function $L(s, \chi)$ of conductor 5 that shares the conductor and gamma factor of f , but has an Euler product, behaves like ζ : the mean of $|E_{\chi}(u)|^2$, with $E_{\chi}(u) = e^{-u/2} \sum_{n \leq e^u} \Lambda(n) \chi(n)$, is 0.2052 on $3 \leq u \leq 18.4$, against $\sum_{\rho} |\rho|^{-2} = 0.2032$ over its zeros (those with $|\gamma| < 400$ plus a density estimate for the rest), and its window energy with $H = 2$ stays between 0.189 and 0.221, inside the band 0.203 ± 0.077 of Corollary 2.12(a). Status: numerical (`anc/local`).

In Corollary 2.12(c) the off-diagonal terms do not grow with U , so a spiral of drift δ at height γ_0 costs at least $e^{2\delta U} / |\rho_0|^2$ in window energy, up to a bounded interference. With the bound $\mathcal{E}(U, 1) \leq C_B + 0.0074$ measured above for $U \leq 14.5$, $\Gamma = 20$ and the zeros below 2000, the corollary excludes an isolated off-line pair at height γ_0 unless $\beta \leq \frac{1}{2} + \log(|\rho_0|(\sqrt{\mathcal{E}} + D + T)) / U$: this gives $\beta \leq 0.600$ at $\gamma_0 = 14.13$, 0.727 at 100, 0.877 at 1000 and 0.921 at 1900, with $D = 0.072$, 0.036, 0.006, 0.004 and $T < 10^{-13}$. To exclude $\delta = 0.01$ at the same heights one needs $U = 146$, 329, 547 and 610. For the Davenport–Heilbronn zero $0.8085 + 85.699i$ the diagonal term alone forces $\mathcal{E}(13, 1) \geq 0.43$, against the measured 2.06. In general the primes up to x exclude $\beta > \frac{1}{2} + \delta$ only up

to heights of order x^δ , a region of the Landau–Turán type, far weaker than the direct verification of RH [PT]; this quantifies Corollary 2.12(b): every finite window of primes controls a finite range of heights, and RH requires the bound for all U . Status: numerical (`anc/spiral_bound/spiral_bound.py`).

2.5. The law on vertical lines. The Laplace transform of E gives an equivalent form in terms of ζ'/ζ , the logarithmic gradient of the energy $|\zeta|^2$ whose C_2 reading is [CE10, Lemma 2.5]. Put $s = \frac{1}{2} + \delta + it$ and

$$J(\delta) = \delta \int_{-\infty}^{\infty} \left| \frac{\zeta'}{\zeta}(s) + \frac{s}{s-1} \right|^2 \frac{dt}{|s|^2}.$$

Proposition 2.13. If RH holds and the zeros are simple, then $J(\delta) \rightarrow \pi C_B$ as $\delta \rightarrow 0^+$. If $\rho = \beta + i\gamma$ is a zero with $\frac{1}{2} < \beta < 1$, then $J(\beta - \frac{1}{2}) = \infty$.

Proof. For $\text{Re } s > \Theta$ the Laplace transform $\mathcal{L}(s) = \int_0^\infty E(u) e^{-(s-1/2)u} du$ equals $-\frac{1}{s}(\frac{\zeta'}{\zeta}(s) + \frac{s}{s-1}) + H(s)$, where $H(s) = O(\log |s|/|s|)$ comes from the lower-order terms of E . By Plancherel, $I(\delta) = \int_0^\infty E(u)^2 e^{-2\delta u} du = \frac{1}{2\pi} \int |\mathcal{L}(\frac{1}{2} + \delta + it)|^2 dt$. Under RH, $F(U) \sim C_B U$, and an Abelian argument gives $I(\delta) \sim C_B/(2\delta)$. The contribution of H changes J by $O(\delta^{1/2})$, so $J(\delta) = 2\pi\delta I(\delta) + o(1) \rightarrow \pi C_B$. If the line $\text{Re } s = \beta$ contains a zero, the integrand behaves like $|t - \gamma|^{-2}$ near $t = \gamma$ and the integral diverges. $\square$

Status: proved in outline (the Abelian step and the estimate for H are standard).

Near each zero on the critical line, $\zeta'/\zeta(s) \approx (s - \rho)^{-1}$ and $\delta \int |s - \rho|^{-2} |s|^{-2} dt \rightarrow \pi/|\rho|^2$: each zero contributes π times its C_2 limit through a Poisson kernel of width δ . The mean square of ζ'/ζ near the critical line is also the object that Goldston, Gonek and Montgomery relate to primes in short intervals [GGM]. We computed $J(\delta)$ from ζ on $|t| \leq T$ ($T = 600$, and 400, 300 for the two smallest δ), adding the estimate $(\log \frac{T}{2\pi} + 1)/T$ for the zeros above T :

δ	0.2	0.1	0.05	0.025	0.0125	0.00625
$J(\delta)$	2.220	1.407	0.849	0.518	0.337	0.243

The differences halve with δ , consistent with $J(\delta) = \pi C_B + O(\delta)$; Richardson extrapolation from the last two values gives 0.148, against $\pi C_B = 0.1451$. Status: numerical (`anc/signal/`, `anc/c2law/`).

2.6. The Nyman–Beurling distance. The same constant appears in an approximation problem whose positivity is that of a Gram matrix built from the integers. Let $\rho_k(x) = \{1/(kx)\}$, let χ be the indicator of $(0, 1)$, and

$$d_N^2 = \inf_{c_1, \dots, c_N} \int_0^\infty \left| \chi(x) - \sum_{k \leq N} c_k \rho_k(x) \right|^2 dx = 1 - b^\top G^{-1} b, \quad b_k = \langle \chi, \rho_k \rangle, \quad G_{jk} = \langle \rho_j, \rho_k \rangle.$$

Proposition 2.14 (Gram data). Let μ be the harmonic measure of [CE10, Proposition 2.6] and $s = \frac{1}{2} + it$. For integers $j, k \geq 1$,

$$G_{jk} = \int \zeta(s) j^{-s} \overline{\zeta(s) k^{-s}} d\mu(t) = \int_0^\infty \left\{ \frac{u}{j} \right\} \left\{ \frac{u}{k} \right\} \frac{du}{u^2}, \quad b_k = - \int \zeta(s) k^{-s} d\mu(t) = \frac{\log k + 1 - \gamma_0}{k}.$$

Consequently, for every Dirichlet polynomial $C_N(s) = \sum_{k \leq N} c_k k^{-s}$ with real coefficients,

$$\begin{aligned} \int_0^\infty \left| \chi - \sum_{k \leq N} c_k \rho_k \right|^2 dx &= 1 - 2 \sum_k c_k b_k + \sum_{j,k} c_j c_k G_{jk} \\ &= \int |1 + \zeta(s) C_N(s)|^2 d\mu(t) = \frac{1}{2\pi} \int_{-\infty}^\infty |1 + \zeta(s) C_N(s)|^2 \frac{dt}{|s|^2}. \end{aligned}$$

Proof. The substitution $x = u/k$ in $\zeta(s)/s = -\int_0^\infty \{x\}x^{-s-1}dx$ ($0 < \sigma < 1$) gives $\zeta(s)k^{-s}/s = -\int_0^\infty \{u/k\}u^{-s-1}du$, and $1/s = \int_1^\infty u^{-s-1}du$. On the critical line $d\mu = dt/(2\pi|s|^2)$, so Mellin–Plancherel on $\text{Re } s = \frac{1}{2}$ gives both formulas, with $\int_1^\infty \{v\}v^{-2}dv = 1 - \gamma_0$ for b_k ; the substitution $x = 1/u$ identifies $\int \{u/j\}\{u/k\}u^{-2}du$ with $\langle \rho_j, \rho_k \rangle$ and $\int_1^\infty \{u/k\}u^{-2}du$ with $\langle \chi, \rho_k \rangle$. $\square$

Status: classical [Va, BBLs]; the form with μ is proved.

The case $j = k = 1$ is $\mathcal{M}(1) = \log 2\pi - \gamma_0$ ([CE10, Proposition 2.11]). The entries G_{jk} are the classical Nyman–Beurling Gram entries, which Vasyunin computed in closed form [Va]; the weight $|\zeta|^2 d\mu$ acts on Dirichlet polynomials through exact arithmetic, with no value of ζ needed. In the notation of Part 10 [CE10], $d_N^2 = \min D^2(A)$ over the Dirichlet polynomials $A = -C_N$ of length N . The following are known, written with [CE1, Theorem A].

- (i) RH holds if and only if $d_N \rightarrow 0$ [Ny, Be, BD]. If RH fails, d_N is bounded below by a positive constant, so $d_N^2 \log N \rightarrow \infty = \sum_\rho \lim_n C_2(n, \rho)$.
- (ii) Unconditionally, $\liminf_N d_N^2 \log N \geq \sum_{\rho: \beta=1/2} m_\rho^2 \lim_n C_2(n, \rho)$, where m_ρ is the multiplicity [Bu02].
- (iii) Báez-Duarte, Balazard, Landreau and Saias conjectured that $d_N^2 \log N \rightarrow \sum_\rho |\rho|^{-2} = C_B$ under RH with simple zeros [BBLs]; in the notation of [CE1, Theorem A] this reads $\lim_N d_N^2 \log N = \sum_\rho m_\rho^2 \lim_n C_2(n, \rho)$, which by (i) also holds when RH fails.

We computed G from Vasyunin’s formula, checked against direct integration to 10^{-9} , and d_N^2 in double precision for $N \leq 1200$; the condition number of G is $5 \cdot 10^6$ at $N = 1200$. The values of $d_N^2 \log N$ are 0.0464, 0.0469, 0.0462, 0.0453 at $N = 50, 100, 400, 1000$, and lie in $[0.0449, 0.0464]$ with mean 0.0456 for $300 \leq N \leq 1200$ (Figure 4). The sum of the limits of C_2 over the 682 zeros with $|\gamma| < 600$ is 0.04324, and with the tail from the zero density 0.046193, against $C_B = 0.046191$; the mean energy of $E(u)$ from the prime powers up to 10^8 is 0.0453 (the autocorrelation table after Corollary 2.3). The three quantities coincide because each is $\sum_\rho |\rho|^{-2}$, by Parseval applied to the explicit formula, so the coincidence is not a new constraint. It shows, however, that $\lim_n C_2(n, \rho)$ is the contribution of ρ to the Nyman–Beurling distance, in a problem where the positivity is that of a Gram matrix on the integers and the open part, $d_N \rightarrow 0$, is an upper bound. Status: (i)–(iii) are known; the computations are numerical (anc/nyman/).

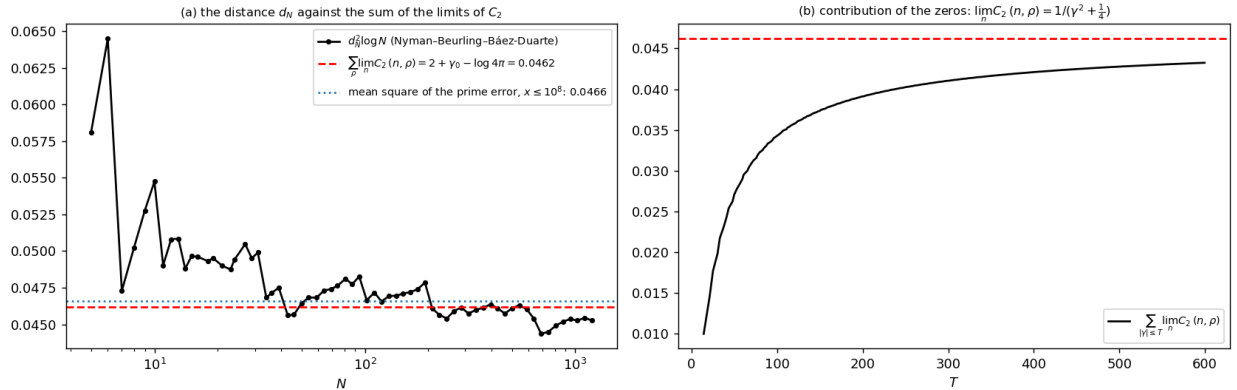


FIGURE 4. The Nyman–Beurling distance and the limits of C_2 . (a) $d_N^2 \log N$ for $5 \leq N \leq 1200$; dashed: $\sum_\rho \lim_n C_2(n, \rho) = C_B$; dotted: mean square of the prime error for $x \leq 10^8$. (b) $\sum_{|\gamma| \leq T} \lim_n C_2(n, \rho)$ over the zeros with $\gamma < 600$.

The optimal coefficients. The minimizing coefficients $c = G^{-1}b$ have a simple structure (Figure 5a, b). For $N = 1200$ their sign is $-\mu(k)$ at every squarefree k , 95.6% of $\sum c_k^2$ lies on the squarefree k , and the profile $w(u) = -c_k \mu(k)$, $u = \log k / \log N$, decreases from about 1 at $u = 0$ to

about 0.2 at $u = 1$; its means over the ten intervals of u agree within 0.02 for $N = 200, 600, 1200$. Restricting c to the squarefree k raises d_N^2 by 2 to 5%. With $c_k = \mu(k)P(\log k/\log N)$ and P a polynomial of degree 10 optimized for each N , $d_N^2 \log N$ lies between 0.051 and 0.058 for $50 \leq N \leq 1200$, within a factor 1.13 to 1.24 of the optimum, while $c_k = \mu(k)$ and $c_k = \mu(k)(1 - \log k/\log N)$, each with the best multiple, are worse by factors 3.5 to 16 (Figure 5c). A profile fitted pointwise to w and used without reoptimization is worse by a factor 5.6 to 10: the quadratic form is sensitive to the fine structure of the coefficients.

The reason is the Mellin form of Proposition 2.14, which we checked numerically: for $N = 50$ the integral over $|t| < 3000$ is 0.011686 against $d_N^2 = 0.011869$, the difference being the omitted tail. The approximation problem is a mollification problem: C_N approximates $-1/\zeta$ on the critical line, and the structure of the optimal coefficients is that of the mollifiers of Levinson and Conrey. At each zero $\zeta C_N = 0$, so the integrand rises to $1/|s|^2$ (Figure 5d); a zero contributes about $1/|\rho|^2 = \lim_n C_2(n, \rho)$ times a width of order $1/\log N$, which is the content of (ii) and (iii). An estimate $d_N^2 \ll 1/\log N$ for $c_k = \mu(k)P(\log k/\log N)$ amounts to a mean value theorem for $\int |1 + \zeta C_N|^2$ over $|t| \leq N^{1/\theta}$ with mollifiers of length t^θ for arbitrarily large θ . Asymptotic formulas of this kind are known for $\theta < \frac{4}{7}$ [Co89]; for large θ they would require the zeros to lie on the line, since a zero off the line leaves $|1 + \zeta C_N| = 1$ on a window of fixed width. The Bettin–Conrey–Farmer polynomial and the explicit mollifiers $A_{N,a}$ are discussed in Part 10 [CE10]. Status: the Mellin identity is classical; the rest is numerical (`anc/nyman/`).

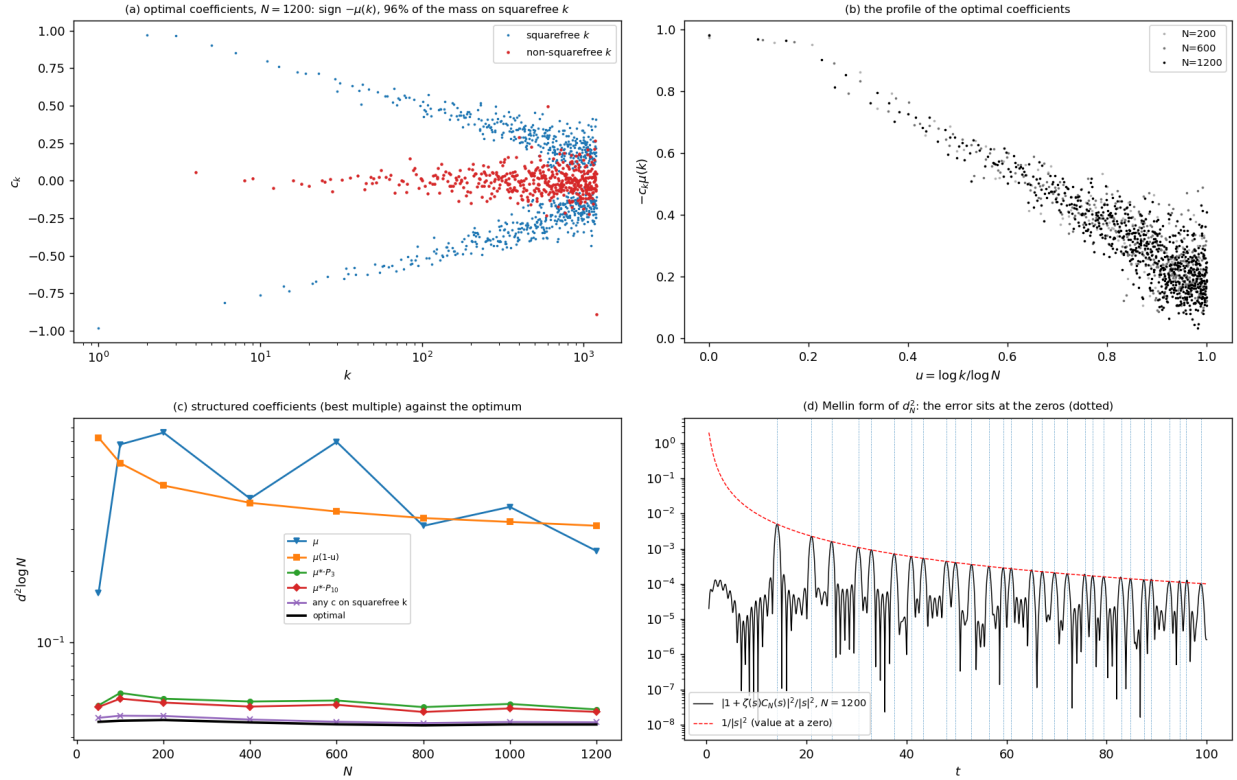


FIGURE 5. The coefficients of the Nyman–Beurling approximation. (a) Optimal c_k for $N = 1200$, on squarefree and non-squarefree k . (b) The profile $-c_k \mu(k)$ against $\log k / \log N$ for $N = 200, 600, 1200$. (c) $d_N^2 \log N$ for coefficients $\mu(k)$, $\mu(k)(1 - u)$, $\mu(k)P_3(u)$, $\mu(k)P_{10}(u)$, arbitrary coefficients on the squarefree k , and the optimum. (d) The integrand $|1 + \zeta(s)C_N(s)|^2/|s|^2$ of the Mellin form for $N = 1200$, and $1/|s|^2$; dotted: zeros.

2.7. The Möbius energy, the energy matrix and the harmonic offset. The Möbius energy. The Mertens function $M(x) = \sum_{n \leq x} \mu(n)$ has the explicit formula with coefficients the residues $1/(\rho\zeta'(\rho))$ of $1/(s\zeta(s))$. Under RH, with simple zeros and a hypothesis on the sums $\sum_{0 < \gamma < T} |\zeta'(\rho)|^{-2}$, the logarithmic mean of $(M(x) + 2)^2/x$ is $2 \sum_{\gamma > 0} |\rho\zeta'(\rho)|^{-2}$ [Ng]. In the variable of Theorem B this constant is

$$C_M = \sum_{\rho} \frac{\lim_n C_2(n, \rho)}{|\zeta'(\rho)|^2},$$

the C_2 energy law with each zero weighted by $|\zeta'(\rho)|^{-2}$. We obtain 0.0290 from $\mu(n)$ up to 10^8 and 0.0287 from the zeros with $\gamma < 600$; with the 1517 zeros with $0 < \gamma < 2000$ and the tail $\sum_{|\gamma| > T} |\rho\zeta'(\rho)|^{-2} \approx 6/(\pi^3 T)$ (twice the one-sided $3/(\pi^3 T)$) suggested by the conjecture of Gonek and Hejhal, $C_M = 0.02914$, and a least-squares fit of $\int_1^X M(x)^2 x^{-2} dx = C_M \log X + K_M$ on $10^4 \leq X \leq 10^8$ gives the slope 0.02891 and, with $C_M = 0.02914$, the constant $K_M = 1.2603 \pm 0.0023$ (mean and spread of the oscillation); we do not derive a closed formula for K_M . For $L(s, \chi)$ with the character mod 5 the slope on $10^3 \leq X \leq 10^6$ is 0.089, against $\sum_{|\gamma| < 400} |\rho L'(\rho)|^{-2} = 0.090$. The same signal gives Littlewood's criterion in convergence form: $\sum_{n \leq N} \mu(n) n^{-s} \rightarrow 1/\zeta(s)$ for $\operatorname{Re} s > \sigma_0$ if and only if there are no zeros with $\beta > \sigma_0$. For the Dirichlet inverse of the Davenport–Heilbronn coefficients at height 85.7, the distance to $1/f(s)$ decreases at $\sigma = 0.95$ and 0.85 and grows at $\sigma = 0.75$ and 0.65 , by the factors 1.40 and 2.47 between $N = 10^3$ and $3 \cdot 10^5$, which are the factors $N^{\beta-\sigma}$ predicted by the off-line zero $\beta = 0.8085$; for ζ at $\sigma = 0.55, 0.6, 0.7, 0.9$ and height 20 it decreases, slowly near the line. The summatory function of that inverse, computed up to 10^7 , has window energies that grow at the rate 0.61 in u , against $2\beta - 1 = 0.617$. Status: the formula for C_M is conditional on RH and the hypothesis of [Ng]; the rest is numerical (`anc/mobius_energy/`, `anc/mobius/`).

The energy matrix of the zeros. The quadratic structure behind Theorem B is the Gram matrix of the orbits $e^{i\gamma u}/\rho$ on $[0, U]$,

$$M(U)_{\rho\rho'} = \frac{1}{\rho\bar{\rho}'} \cdot \frac{1}{U} \int_0^U e^{i(\gamma-\gamma')u} du, \quad M(U)_{\rho\rho} = \frac{1}{|\rho|^2} = \lim_n C_2(n, \rho) \text{ (under RH)},$$

a Hermitian positive semidefinite matrix whose total sum $\mathbf{1}^\top M(U) \mathbf{1}$ is, under RH, the mean energy of $-\sum_{\rho} e^{i\gamma u}/\rho$ on $[0, U]$. Its trace is C_B ; the off-diagonal entries, the interference, decay like $1/U$. As $U \rightarrow 0$, $M(U)$ tends to the rank-one matrix vv^* , $v_\rho = 1/\rho$, with the single eigenvalue C_B ; as $U \rightarrow \infty$ its eigenvalues tend to the diagonal, each C_2 limit appearing twice (ρ and $\bar{\rho}$). With the 682 zeros $|\gamma| < 600$ the eigenvalues agree with $\lim_n C_2(n, \rho)$ to four digits from $U = 10$, and the von Neumann entropy of $M/\operatorname{tr} M$ rises from 0 to 4.391, the entropy of the distribution $\{\lim_n C_2(n, \rho)/C_B\}$ (Figure 6). The asymptotic spectrum of the energy of the primes is therefore the set of C_2 limits at the zeros. The matrix is positive semidefinite whatever the location of the zeros, as every Gram matrix is; an off-line zero would show only in the size of its diagonal entry, which diverges with its C_2 limit ([CE1, Theorem A]). Status: elementary; the computations are numerical (`anc/K_constant/eig.py`, `anc/K_constant/figeig.py`).

Remark 2.15 (the harmonic offset of [E24]). The construction of [E24, §7.2] compares, at two zeros, the quantity $H(\frac{1}{2}, \gamma, n) = H_n - |X_n(\frac{1}{2} + i\gamma)|^2 + n/(\frac{1}{4} + \gamma^2)$, where H_n is the harmonic number, and claimed that its difference at two zeros tends to 0. By the straight-line law (Part 1 [CE1]), at each zero on the critical line

$$H_n - H(\frac{1}{2}, \gamma, n) \longrightarrow \frac{1}{2} \cdot \frac{1}{\frac{1}{4} + \gamma^2} = \frac{1}{2} \lim_{n \rightarrow \infty} C_2(n, \rho),$$

so the difference at two zeros tends to $\frac{1}{2}[L(\gamma_2) - L(\gamma_1)]$, which is -0.001369 for the first two zeros [Er]. The harmonic offset of each zero is half its C_2 limit and depends only on its own height, so it does not relate one zero to another. The offsets are related collectively: by Theorem B and

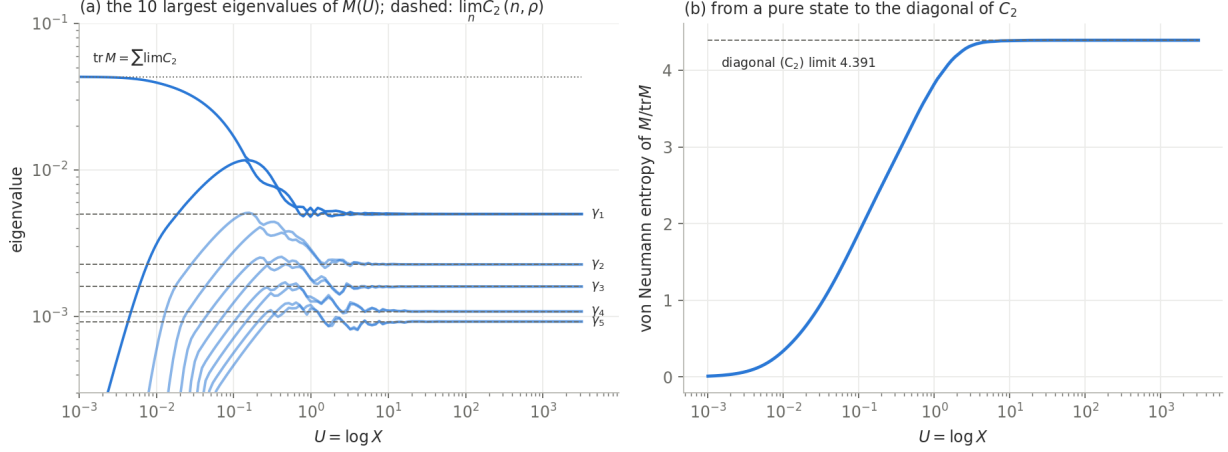


FIGURE 6. (a) The ten largest eigenvalues of $M(U)$ for the zeros with $|\gamma| < 600$; dashed: $\lim_n C_2(n, \rho)$ for the first five zeros; dotted: $\text{tr } M = \sum \lim_n C_2$. (b) The von Neumann entropy of $M(U)/\text{tr } M(U)$.

Proposition 2.4 their sum over all zeros is half the energy of the prime signal, $C_B/2 = \lambda_1 = 0.0231$ under RH, and an off-line pair would make the sum infinite. We checked the limit with $n = 2 \cdot 10^5$, writing the offset for any s as $H(s, n) = \sum_{k \leq n} k^{-2\sigma} - |X_n(s)|^2 + n/(\sigma^2 + t^2)$. At the zeros $\frac{1}{2} + 14.1347i$ and $\frac{1}{2} + 21.0220i$, $H - \log n = 0.5747$ and 0.5761 , against $\gamma_0 - \frac{1}{2}(\frac{1}{4} + \gamma^2)^{-1} = 0.57472$ and 0.57608 (Figure 7). At a point of the line that is not a zero the cross term between $\zeta(s)$ and the pole survives, and H oscillates with amplitude $2\sqrt{n}|\zeta(s)|/|1-s|$ (± 44 at $\frac{1}{2} + 20i$); off the line the correction $n/(\sigma^2 + t^2)$ no longer matches the pole and H grows like n (489 at $0.7 + 20i$), zero or not. For the Davenport–Heilbronn function, which has no pole, we drop the correction and subtract $c \log n$ with c the mean of $|a_k|^2$: the partial sums converge, so $H - c \log n$ tends to a constant at every point of the line, zero or not (0.62 at the zero $\frac{1}{2} + 43.0813i$, -0.15 at $\frac{1}{2} + 60i$), and decreases like $-c \log n$ at the off-line zero $0.8085 + 85.699i$, where the weighted harmonic sum converges. The offset recognises the zeros of ζ on the line through the vanishing of the cross term with the pole, and off the line it records σ , not the zero. Status: the limit follows from the straight-line law (an erratum to [E24]); the numbers are numerical (`anc/harmonic/harmonic_offset.py`).

The energy itself is shown in Figure 8: the normalized error oscillates around zero with stable amplitude, its accumulated energy $I(X) = \int_1^X (\psi(x) - x)^2 x^{-2} dx$ follows a straight line after the initial transient, and the slope settles at $C_B = \sum_\rho \lim_n C_2(n, \rho)$.

Fluctuations and orbits. The finer structure of the energy is the subject of [CE6]. Under RH, simple zeros and an additional spacing hypothesis, $I(X) = C_B \log X + K + \Phi(\log X) + o(1)$, with an explicit constant $K = 2.51248\dots$, almost entirely due to the trivial part of the explicit formula, and a Bohr almost periodic term Φ of mean zero coming from the interference of pairs of zeros at the frequencies $\gamma - \gamma'$; the constant term of $I(X)$ contains the second elementary symmetric function of the C_2 limits. Random models carry more energy than the primes: Cramér’s model 46 to 150 times as much, independent random signs in place of μ about 20 times (6 times for the smallest realization), and random multiplicative functions a median of 8.4 times (1.7 times for the smallest realization) [CE6], because bounded energy requires the discrete spectrum supplied by the zeros. The principal mode of each zero in the prime signal (a closed ellipse if $\beta = \frac{1}{2}$, an expanding spiral if $\beta > \frac{1}{2}$) and its phase, and the product of the spiral of the truncated zeros with the harmonic of the primes, which recovers $1/(\rho \zeta'(\rho))$, are treated there as well; the large values of Dirichlet polynomials are in [CE7].

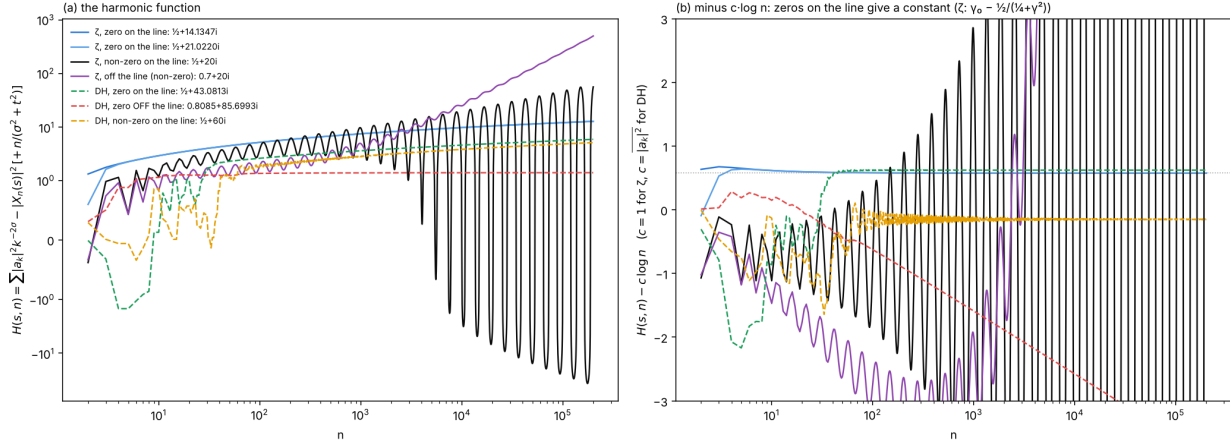


FIGURE 7. The harmonic offset (Remark 2.15) for $n \leq 2 \cdot 10^5$. (a) $H(s, n)$ for zeros of ζ on the line, a point of the line that is not a zero, a point off the line, and for the Davenport–Heilbronn function (dashed) at a zero on the line, at its off-line zero $0.8085 + 85.699i$ and at a point of the line that is not a zero. (b) $H(s, n) - c \log n$; dotted: γ_0 .

2.8. The arithmetic cross-energy. Replace the weights 1 by $\Lambda(n)$. Put $X_x^\Lambda(s) = \sum_{n \leq x} \Lambda(n) n^{-s}$ and, for real $x > 1$,

$$W^\Lambda(x, s) = C_2^\Lambda(x, s) + iS_2^\Lambda(x, s) = 2x^{-\bar{s}} \sum_{n < x} \Lambda(n) n^{-s}, \quad (2.3)$$

so that

$$C_2^\Lambda(x, s) = 2x^{-\sigma} \sum_{n < x} \Lambda(n) n^{-\sigma} \cos\left(t \log \frac{x}{n}\right).$$

At an integer N this is $C_2^\Lambda(N, s) = 2 \operatorname{Re}(N^{-\bar{s}} X_{N-1}^\Lambda(s))$, the definition used throughout this paper, without a factor $\Lambda(N)$. The earlier papers [C1, C2] and the manuscript on the asymptotic balance used the cross-energy increment of $|X_N^\Lambda|^2$,

$$\begin{aligned} |X_N^\Lambda(s)|^2 - |X_{N-1}^\Lambda(s)|^2 - \Lambda(N)^2 N^{-2\sigma} &= \Lambda(N) C_2^\Lambda(N, s), \\ \sum_{n \leq N} \Lambda(n) C_2^\Lambda(n, s) &= |X_N^\Lambda(s)|^2 - \sum_{n \leq N} \Lambda(n)^2 n^{-2\sigma}; \end{aligned} \quad (2.4)$$

and the cross term $K^\Lambda(x, s) = 2 \operatorname{Re}[x^{-\bar{s}} X_x^\Lambda(s)]$ of [C2] coincides with $C_2^\Lambda(x, s)$ when x is not a prime power. All statements below are converted to the normalization (2.3); where the conversion changes a statement, we say so.

Proposition 2.16 (anatomy of C_2^Λ). (i) Let s be neither a zero nor a pole of ζ , and let $x > 1$ not be a prime power. Then

$$C_2^\Lambda(x, s) = A(\sigma, t) x^{1-2\sigma} + 2x^{-\sigma} \operatorname{Re} \left[-\frac{\zeta'}{\zeta}(s) x^{it} \right] - 2 \operatorname{Re} \sum_{\rho} \frac{x^{\rho-2\sigma}}{\rho-s} + 2 \operatorname{Re} \sum_{k \geq 1} \frac{x^{-2k-2\sigma}}{2k+s},$$

with $A(\sigma, t) = 2(1-\sigma)/|1-s|^2$ as in Part 10 [CE10] (the constant A_s of [CE1, Proposition 3.6]) and the sum over the zeros taken symmetrically.

(ii) Unconditionally, for fixed s with $\sigma < 1$, $W^\Lambda(x, s) = \frac{2x^{1-2\sigma}}{1-s} + o(x^{1-2\sigma})$ as $x \rightarrow \infty$. Hence $\lim_x C_2^\Lambda(x, s)$ is $1/(\frac{1}{4} + t^2)$, 0 or $+\infty$ according as $\sigma = \frac{1}{2}$, $\sigma > \frac{1}{2}$ or $\sigma < \frac{1}{2}$, at every point, as for C_2 in [CE1, Theorem A]; on the line $W^\Lambda(x, s) \rightarrow 2/\bar{s} = W_\infty(s)$.

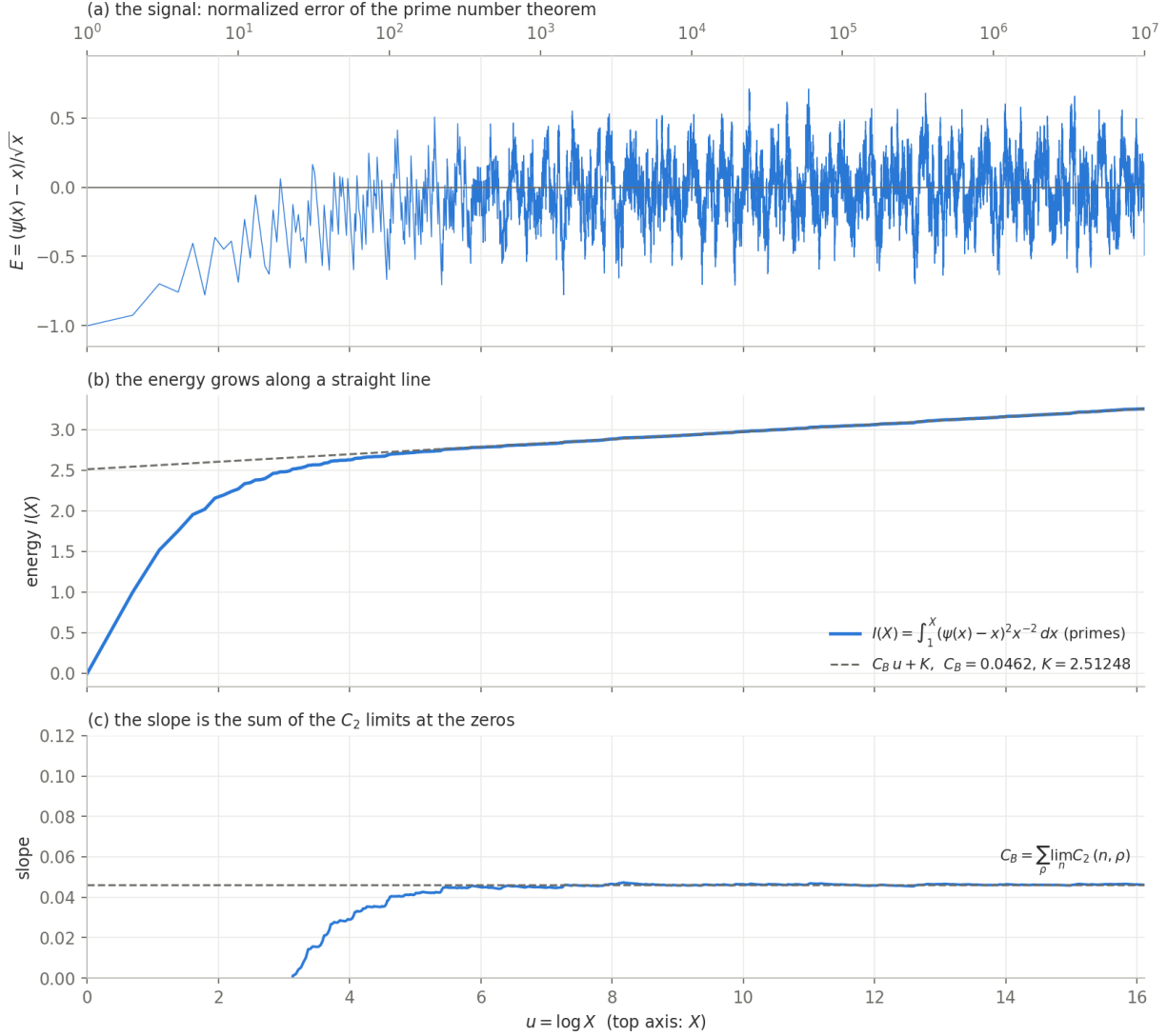


FIGURE 8. The energy of the primes up to 10^7 . (a) $(\psi(x) - x)/\sqrt{x}$ against $u = \log x$. (b) $I(X) = \int_1^X (\psi(x) - x)^2 x^{-2} dx$ and the line $C_B \log X + K$, with the constant $K = 2.51248\dots$ of [CE6]. (c) $(I(X) - K)/\log X$ and C_B .

Proof. (i) Multiply the explicit formula [In, Ch. IV]

$$X_x^\Lambda(s) = \frac{x^{1-s}}{1-s} - \frac{\zeta'}{\zeta}(s) - \sum_{\rho} \frac{x^{\rho-s}}{\rho-s} + \sum_{k \geq 1} \frac{x^{-2k-s}}{2k+s}$$

by $2x^{-\bar{s}}$, use $x^{-\bar{s}}x^{-s} = x^{-2\sigma}$, and take real parts. (ii) Partial summation from $\psi(x) = x + o(x)$ gives $X_x^\Lambda(s) = \frac{x^{1-s}}{1-s} + o(x^{1-\sigma})$ for fixed s with $\sigma < 1$; multiply by $2x^{-\bar{s}}$. $\square$

Status: proved.

The comparison with [CE1, Theorem A] is instructive. The envelope $A(\sigma, t)x^{1-2\sigma}$ is the same. The wave carries $-\zeta'/\zeta(s)$ in place of $\zeta(s)$. And the zeros, which C_2 sees only at $s = \rho$, appear in C_2^Λ as one mode $x^{\beta-2\sigma}e^{i\gamma \log x}$ per zero at every s : on $\sigma = \frac{1}{2}$ a zero produces a mode of size $x^{\beta-1}$,

and RH says that all modes have the same exponent $-\frac{1}{2}$. The limit in (ii) needs only the prime number theorem; it is the source of the pole barrier below.

Theorem 2.17 (3-4-1 positivity at every finite x). For all real $x > 1$, σ and t ,

$$3C_2^\Lambda(x, \sigma) + 4C_2^\Lambda(x, \sigma + it) + C_2^\Lambda(x, \sigma + 2it) \geq 0.$$

Proof. By (2.3) the left side equals $2x^{-\sigma} \sum_{n < x} \Lambda(n) n^{-\sigma} k(t \log \frac{x}{n})$ with $k(\theta) = 3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2 \geq 0$, and $\Lambda \geq 0$. $\square$

Status: proved.

The inequality holds for every finite x , and in particular at every integer N , as a pointwise constraint on the trajectory of the arithmetic cross-energy. It needs only that the coefficients be nonnegative, and this is where the Euler product enters. The kernel is Mertens' [T, §3.3], applied here to the endpoint kernel $\cos(t \log(x/n))$ of the cross-energy rather than to the complete series at $\sigma > 1$.

Proposition 2.18 (failure for an Epstein function). Let $b_Q(n)$ be the coefficients of $-Z'_Q/Z_Q$ for $Q = m^2 + 5n^2$, with Z_Q normalized so that its first coefficient is 1 (so $b_Q(1) = 0$), and define C_2^Q by (2.3) with Λ replaced by b_Q . Then the analogue of Theorem 2.17 is false. At $x = 127$ the combination $\frac{x^\sigma}{2} [3C_2^Q(x, \sigma) + 4C_2^Q(x, \sigma + it) + C_2^Q(x, \sigma + 2it)]$ has the following minima over $t \in [74.9, 75.2]$, attained near $t = 75.06$:

σ	0.5	0.8	1.0	1.2
minimum	-26.35	-6.10	-2.18	-0.70

The cause is the negative coefficients, the first being $b_Q(36) = -7.167\dots$. Since $b_Q(127) = 0$, the values are the same whether the endpoint $n = x$ is included, as in [C2], or not, as in (2.3). Status: numerical (`anc/arith_c2/c2_arith_numerics.py`, option `epstein`; `anc/arith_c2/c2lambda_energy.py`).

So Theorem 2.17 passes the filter (F1) of [CE2, Definition 2.22]: it is true for ζ and false for the Epstein function of $m^2 + 5n^2$ (for the form $m^2 + 100n^2$ of Part 16 [CE16] the first negative coefficient of $-Z'_Q/Z_Q$ is $b_Q(10000)$, [CE16, Proposition 2.3]).

Theorem 2.19 (phase-locked Mertens limit). Let $\sigma > 1$, $t \neq 0$ and $x_k = e^{2\pi k/t}$, $k = 1, 2, \dots$, so that $t \log x_k \in 2\pi\mathbb{Z}$. Then

$$\lim_{k \rightarrow \infty} \frac{x_k^\sigma}{2} [3C_2^\Lambda(x_k, \sigma) + 4C_2^\Lambda(x_k, \sigma + it) + C_2^\Lambda(x_k, \sigma + 2it)] = M(\sigma, t) \geq 0,$$

where

$$M(\sigma, t) = 3 \left(-\frac{\zeta'}{\zeta}(\sigma) \right) + 4 \operatorname{Re} \left(-\frac{\zeta'}{\zeta}(\sigma + it) \right) + \operatorname{Re} \left(-\frac{\zeta'}{\zeta}(\sigma + 2it) \right).$$

Proof. By (2.3), $\frac{1}{2} x^\sigma C_2^\Lambda(x, \sigma + i\tau) = \operatorname{Re} [x^{i\tau} \sum_{n < x} \Lambda(n) n^{-\sigma - i\tau}]$. At $x = x_k$, $x_k^{it} = x_k^{2it} = 1$, and for $\sigma > 1$ the prime sums converge absolutely to $-\zeta'/\zeta(\sigma + i\tau)$. Positivity (Theorem 2.17) passes to the limit. $\square$

Status: proved.

Locking the phase $t \log x \equiv 0$ extracts Mertens' combination from the cross-energy, as locking $t \log N \equiv \theta(t)$ extracts Hardy's function from C_2 on the critical line ([CE2, Theorem 3.2]). With $\sigma = 1.5$, $t = 10$ and $M(1.5, 10) = 5.0891$, the phase-locked values at $x_k = 535, 81612$ and 3540026 are $4.82797, 5.06804$ and 5.08590 , the last within 0.0032 of M (`anc/arith_c2/c2_arith_numerics.py`, option `phaselock`).

Corollary 2.20 ($\zeta(1 + it) \neq 0$). For every real $t \neq 0$, $\zeta(1 + it) \neq 0$.

Proof. Suppose $\zeta(1 + it_0) = 0$ with multiplicity $m \geq 1$, $t_0 \neq 0$. As $\sigma \rightarrow 1^+$, $-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1)$, $\operatorname{Re} \left(-\frac{\zeta'}{\zeta}(\sigma + it_0) \right) = -\frac{m}{\sigma-1} + O(1)$, and $\operatorname{Re} \left(-\frac{\zeta'}{\zeta}(\sigma + 2it_0) \right) \leq O(1)$, since a zero at $1 + 2it_0$

of multiplicity $m' \geq 0$ contributes $-m' / (\sigma - 1) \leq 0$. Hence $M(\sigma, t_0) \leq \frac{3-4m}{\sigma-1} + O(1) \rightarrow -\infty$, contradicting Theorem 2.19. $\square$

Status: proved; the conclusion is the theorem of Hadamard and de la Vallée Poussin [T, Ch. III].

What is new is only the route: positivity is a property of the finite trajectory of the arithmetic cross-energy, and the limit $\sigma \rightarrow 1^+$ is taken after phase-locking. Combined with Hadamard's product, the same inequality gives the zero-free region of de la Vallée Poussin [C2].

Proposition 2.21 (the pole barrier). Let $k(\theta) = \sum_{j=0}^J a_j \cos j\theta$ be a nonnegative trigonometric polynomial, not identically zero, and $t \neq 0$. Then for every $x > 1$

$$\frac{1}{2} \sum_{j=0}^J a_j C_2^\Lambda(x, \tfrac{1}{2} + ijt) = x^{-1/2} \sum_{n < x} \frac{\Lambda(n)}{\sqrt{n}} k\left(t \log \frac{x}{n}\right) \geq 0,$$

and, by the prime number theorem alone,

$$\lim_{x \rightarrow \infty} \frac{1}{2} \sum_{j=0}^J a_j C_2^\Lambda(x, \tfrac{1}{2} + ijt) = \sum_{j=0}^J \frac{a_j}{2(\frac{1}{4} + j^2 t^2)} = \int_0^\infty e^{-u/2} k(tu) du > 0, \quad (2.5)$$

whatever the location of the nontrivial zeros.

Proof. The identity is (2.3). By Proposition 2.16(ii), $C_2^\Lambda(x, \frac{1}{2} + ijt) \rightarrow 1/(\frac{1}{4} + j^2 t^2)$ for each j (for $j = 0$ the limit is 4). Finally $\int_0^\infty e^{-u/2} \cos(jtu) du = \frac{1/2}{\frac{1}{4} + j^2 t^2}$, and the integral of the nonnegative, not identically zero, function $e^{-u/2} k(tu)$ is positive. $\square$

Status: proved.

Through Proposition 2.16(i), a zero $\rho = \beta + i\gamma$ contributes to the left side of (2.5) a mode of size $x^{\beta-1}$, which for $\beta < 1$ tends to 0 and is swamped by the positive pole term. The inequality is blind to zeros with $\beta < 1$, except in the limit $\beta \rightarrow 1$, where the mode grows at the rate of the pole term; that limit is where Corollary 2.20 lives. This is the classical reason why positivity methods give zero-free regions only near $\sigma = 1$. In the language of the cross-energy it is the statement that C_2^Λ , like C_2 , has the limit W_∞ of the pole on the critical line: to see a zero at β the pole term must be removed, and removing it destroys positivity. Numerically, with $k = 3 + 4 \cos \theta + \cos 2\theta$ and $t = 10$ (predicted limit 6.0212), the left side of (2.5) at $x_m = e^{2\pi m/10}$ is 5.6773, 5.9454, 6.0036, 6.0144, 6.0180 for $x_m \approx 5.4 \cdot 10^2, 6.6 \cdot 10^3, 8.2 \cdot 10^4, 5.4 \cdot 10^5, 6.6 \cdot 10^6$; the approach is of order $x^{-1/2}$, with leading part $3(-\frac{\zeta'}{\zeta}(\frac{1}{2}))x^{-1/2} \approx -8.06 x^{-1/2}$ on top of the oscillating modes of the zeros (`anc/arith.c2/c2_arith_numerics.py`, option `barrier`).

Remark 2.22 (the arithmetic cross-energy sees a zero). Let $\rho_0 = \frac{1}{2} + i\tau_0$ be a zero of multiplicity m and assume RH. For $x > 1$ not a prime power the explicit formula, in its truncated form, which holds uniformly for s near a zero, gives, as $s \rightarrow \rho_0$,

$$X_x^\Lambda(\rho_0) = \frac{x^{1-\rho_0}}{1-\rho_0} - m \log x + x^{-\rho_0} (\psi(x) - x) + O_{\rho_0}(1),$$

the resonance of Conrad [Co, (6.12)]: the term $-m \log x$ comes from the coalescence of ρ_0 with s , and under RH the remaining sum over $\rho \neq \rho_0$, compared with the sum of x^ρ / ρ , leaves an absolutely convergent series. In the form of a wave,

$$W^\Lambda(n, \rho_0) = \frac{2}{\bar{\rho}_0} - 2m(\log n) n^{-1/2} e^{i\tau_0 \log n} + n^{-1/2} O(|E(\log n)| + 1).$$

Unlike $C_2(\infty, \cdot)$, which equals $1/(\frac{1}{4} + t^2)$ at every point of the line, the arithmetic cross-energy sees a zero, with its multiplicity as the coefficient of a resonance, while at a point of the line that is not a zero the term $-m \log n$ is absent and only the fluctuation E remains. The fluctuation has bounded mean square (Theorem B) but is only $O(\log^2 x)$ pointwise and is $\Omega_\pm(\log \log \log x)$, so the resonance dominates in logarithmic mean, and pointwise if $\psi(x) - x = o(\sqrt{x} \log x)$ [Co, Thms. 6.2–6.3]. Numerically, with the primes up to 10^8 , $\operatorname{Re}(X_N^\Lambda(\rho_1) - N^{1-\rho_1}/(1-\rho_1)) + \log N$ lies in

$[0.03, 0.62]$ at $N = 10^k$, $k = 2, \dots, 8$ (in $[-0.34, 1.13]$ over all $N \in [10^2, 10^8]$), while at $s = \frac{1}{2} + 20i$, which is not a zero, $\operatorname{Re}(X_N^\Lambda(s) - N^{1-s}/(1-s))$ lies in $[-0.6, 1.7]$ at $N = 10^k$ (in $[-1.35, 2.26]$ over all N), with no drift. Status: conditional on RH; proved in outline; the numbers are from `anc/c2_explicit/lambda_c2.py`.

Extrapolation from the known zeros. The integer cross-energy $C_2(N, s)$ contains no zero other than through the value $\zeta(s)$; the arithmetic one contains every zero, as a mode $x^{\rho-2\sigma}/(\rho-s)$ (Proposition 2.16(i)). This raises the question how far in x the zeros that are known, all on the line up to height $3 \cdot 10^{12}$ [PT], determine $W^\Lambda(x, s)$. The truncated explicit formula, proved by the contour argument used for $\psi(x)$ [Da, Ch. 17] with the Perron integral centred at s , gives for $x \geq 2$ not a prime power and $|t|$ small compared with U

$$X_x^\Lambda(s) = \frac{x^{1-s}}{1-s} - \frac{\zeta'}{\zeta}(s) - \sum_{|\gamma| \leq U} \frac{x^{\rho-s}}{\rho-s} + \sum_{k \geq 1} \frac{x^{-2k-s}}{2k+s} + R_U(x, s), \quad (2.6)$$

with, unconditionally,

$$R_U(x, s) \ll_\sigma x^{1-\sigma} \frac{\log^2(xU)}{U} + x^{-\sigma} \log x.$$

In the normalization of the cross-energy, $|W^\Lambda - W_{\text{pred}}^\Lambda| = 2x^{-\sigma}|R_U|$, which on the critical line is of order $\log^2(xU)/U$. Table 8 gives the measured error at $s = \frac{1}{2} + 50i$.

zeros used, $ \gamma \leq U$	10^2	10^3	10^4	10^5	667 907
$\max_x x^{-1/2} R_U(x, s) $, $x \leq 10^{10}$	$6.7 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	$4.3 \cdot 10^{-4}$	$5.8 \cdot 10^{-5}$	$9.8 \cdot 10^{-6}$
maximum divided by $\log^2(xU)/U$	0.005	0.013	0.016	0.013	0.011
re-run, $x \leq 10^8$, first 1000 zeros	$8.8 \cdot 10^{-3}$	$3.6 \cdot 10^{-3}$			

TABLE 8. Error of the prediction (2.6) of the prime sum from the zeros with $|\gamma| \leq U$, at $s = \frac{1}{2} + 50i$ and $10^3 \leq x \leq 10^{10}$ (1 124 029 zeros, all with $0 < \gamma \leq 667\,907.75$; prime powers up to 10^{10} ; 1200 values $x = n + \frac{1}{2}$). Last row: an independent re-run on a denser grid of $x \in [10^3, 10^8]$; there the ratio in the second row reaches 0.019.

The error is largest at the small end of the range of x and decreases as x grows, so $\log^2(xU)/U$ is a worst case; the law $\max x^{-\sigma}|R_U| \leq 0.02 \log^2(xU)/U$ holds in both computations once U is large compared with $|t|$ (it fails for $U \lesssim |t|$, as it should), and the same is found at $t = 10, 200, 1000$ and, with smaller errors, at $\sigma = 0.75$. Two conclusions follow. First, with $U = 3 \cdot 10^{12}$ the known zeros fix $W^\Lambda(x, \frac{1}{2} + it)$, for $|t|$ far below U , to precision ε for $\log x$ up to about $(\varepsilon U / 0.02)^{1/2}$, that is $\log x \approx 4 \cdot 10^5$ for $\varepsilon = 10^{-3}$: far beyond any sieve. Second, this prediction holds whether or not RH holds above U , because (2.6) is unconditional. A zero $\rho^* = \frac{1}{2} + \delta + i\gamma^*$ with $\gamma^* > U$ enters only through R_U , with weight $x^{\delta-1/2}/|\rho^* - s| \leq x^{\delta-1/2}/(\gamma^* - |t|)$ in the same normalization, below the error of the prediction. To see it one must work near $t = \gamma^*$, and there the prediction needs the zeros near γ^* , which are the unknown ones. The known zeros extrapolate the arithmetic cross-energy in length, not in height; the length needed to detect an off-line zero from the primes at its own height is the subject of [CE10, Corollary 3.14] and of [CE12]. Status: (2.6) is classical for $s = 0$ and proved in outline for general s ; the table is numerical (the first two rows from the author's working notes; the re-run is `anc/extrapolation/extrap.py`).

Theorem 2.23 (the arithmetic energy criterion). Let $S^\Lambda(N) = \sum_{n \leq N} C_2^\Lambda(n, \frac{1}{2})$.

(a) Unconditionally,

$$S^\Lambda(N) = 2\sqrt{N} \int_1^N \psi(u) u^{-3/2} du + O(\log N) = 4N - 4\sqrt{N} + 2\sqrt{N} \int_1^N (\psi(u) - u) u^{-3/2} du + O(\log N).$$

(b) If $S^\Lambda(N) = 4N + O(N^\theta)$ for some $\frac{1}{2} \leq \theta < 1$, then ζ has no zeros with $\operatorname{Re} \rho > \theta$.

(c) If RH holds, then

$$S^\Lambda(N) = 4N - 4\frac{\zeta'}{\zeta}(\frac{1}{2})\sqrt{N} - 2\sqrt{N} \sum_{\rho} \frac{N^{i\gamma}}{\rho(\rho - \frac{1}{2})} + O(\log N),$$

where the sum over the zeros converges absolutely; in particular $S^\Lambda(N) = 4N + O(\sqrt{N})$.

(d) Consequently the following are equivalent: RH; $S^\Lambda(N) = 4N + O(\sqrt{N})$; and $S^\Lambda(N) = 4N + O_\varepsilon(N^{1/2+\varepsilon})$ for every $\varepsilon > 0$.

Proof. (a) With $h(x) = \sum_{n \leq x} n^{-1/2} = 2\sqrt{x} + \zeta(\frac{1}{2}) + O(x^{-1/2})$, interchanging the sums gives

$$S^\Lambda(N) = 2 \sum_{m < N} \frac{\Lambda(m)}{\sqrt{m}} (h(N) - h(m)) = 4\sqrt{N} X_{N-1}^\Lambda(\frac{1}{2}) - 4\psi(N-1) + O(\log N),$$

since $h(N) - h(m) = 2\sqrt{N} - 2\sqrt{m} + O(m^{-1/2})$ and $\sum_{m < N} \Lambda(m)/m = O(\log N)$. Partial summation gives $\sqrt{N} X_N^\Lambda(\frac{1}{2}) - \psi(N) = \frac{1}{2}\sqrt{N} \int_1^N \psi(u) u^{-3/2} du$, and the same holds with $N-1$ in place of N on the left, because the two terms change by $\Lambda(N)$ each. This is the first form; the second follows from $\int_1^N u^{-1/2} du = 2\sqrt{N} - 2$.

(b) By (a) the hypothesis gives $G(N) = \int_1^N (\psi(u) - u) u^{-3/2} du = O(N^{\theta-1/2})$. By partial summation, $\int_1^\infty (\psi(u) - u) u^{-s-1} du = \int_1^\infty u^{-(s-1/2)} dG(u)$ converges for $\operatorname{Re} s > \theta$ and is analytic there; for $\operatorname{Re} s > 1$ it equals $-\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1}$. So ζ'/ζ has no poles in $\operatorname{Re} s > \theta$.

(c) Insert the explicit formula $\psi_0(u) = u - \sum_{\rho} u^\rho/\rho - \log 2\pi - \frac{1}{2} \log(1 - u^{-2})$ in $G(N)$ and integrate term by term (justified by the truncated explicit formula, as in the proof of Theorem 2.11). Under RH, $\int_1^N u^{\rho-3/2} du = (N^{i\gamma} - 1)/(\rho - \frac{1}{2})$, so $G(N) = c_0 - \sum_{\rho} N^{i\gamma}/(\rho(\rho - \frac{1}{2})) + O(N^{-1/2})$, with a constant c_0 and a sum that converges absolutely, since $|\rho(\rho - \frac{1}{2})| = |\rho||\gamma|$. The constant is the value at $s = \frac{1}{2}$ of $\int_1^\infty (\psi(u) - u) u^{-s-1} du = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1}$, continued from $\operatorname{Re} s > \frac{1}{2}$ (the oscillating sum has mean zero in $\log N$, so an Abelian argument identifies c_0 with the limit as $s \rightarrow \frac{1}{2}^+$), that is $c_0 = 2 - 2\frac{\zeta'}{\zeta}(\frac{1}{2})$. Inserting this in (a) gives (c).

(d) (c) gives the second statement under RH, the second implies the third, and the third implies RH by (b). $\square$

Status: (a), (b) and (d) are proved; (c) is conditional on RH and proved in outline (the term-by-term integration and the identification of c_0).

In the normalization (2.4) with the factor $\Lambda(n)$, the corresponding statement is that of [C1, Thm. 9.2] and of the manuscript on the asymptotic balance: since $X_N^\Lambda(\frac{1}{2}) > 0$,

$$\sum_{n \leq N} \Lambda(n) C_2^\Lambda(n, \frac{1}{2}) = X_N^\Lambda(\frac{1}{2})^2 - \sum_{n \leq N} \frac{\Lambda(n)^2}{n} = 4N + O(N^\theta) \iff X_N^\Lambda(\frac{1}{2}) = 2\sqrt{N} + O(N^{\theta-1/2}),$$

and partial summation shows that this is equivalent to $\psi(x) - x = O(x^\theta)$, hence to the absence of zeros with $\beta > \theta$ when $\theta < 1$, and to RH when it holds for every $\theta > \frac{1}{2}$. The two normalizations give the same criterion, $4N + O(N^{1/2+\varepsilon})$, but different errors under RH. By partial summation $X_N^\Lambda(\frac{1}{2}) = 2\sqrt{N} - 1 + E(\log N) + \frac{1}{2}G(N) + O(N^{-1/2})$, so under RH the weighted sum is $4N + 4\sqrt{N} E(\log N) + O(\sqrt{N})$, which by Littlewood's theorem $E(u) = \Omega_\pm(\log \log u)$ [Ln] is not $4N + O(\sqrt{N})$; the unweighted sum $S^\Lambda(N)$ integrates the prime error once more, and its error is $O(\sqrt{N})$ exactly when RH holds. The numbers confirm both behaviours, and the constant of (c), $-4\frac{\zeta'}{\zeta}(\frac{1}{2}) = -10.74437$:

N	10^2	10^3	10^4	10^5	10^6	$3 \cdot 10^6$	10^7
$(S^\Lambda(N) - 4N)/\sqrt{N}$	-10.258	-10.669	-10.738	-10.737	-10.730	-10.755	-10.731
$(\sum_{n \leq N} \Lambda(n) C_2^\Lambda(n, \frac{1}{2}) - 4N)/\sqrt{N}$	-12.39	-11.42	-10.48	-10.24	-12.46	-10.81	-12.61

The deviations of the first row from -10.744 after $N = 10^3$ are below 0.015 , within the bound $2 \sum_\rho 1/(|\rho||\gamma|) \approx 0.09$ for the oscillating sum in (c); the second row fluctuates with $4E(\log N)$. Status: numerical (`anc/arith_c2/c2lambda_energy.py`, sieve to 10^7 ; output in `c2lambda_energy_output.txt`).

Theorem 2.23 is a reformulation, not a proof: its content is $\psi(x) - x = O(x^{1/2+\varepsilon})$. Its merit is that it involves the nonnegative weights $\Lambda(n)$, so the Epstein obstruction (Proposition 2.18) does not apply to it as it applies to the limit criterion of Part 1 [CE1], and that it sees all zeros at once, in contrast with the linear growth $|X_N(\frac{1}{2} + it)|^2 \sim N/(\frac{1}{4} + t^2)$ at a single zero. By Proposition 2.21 the positivity of C_2^Λ does not reach it: on the critical line the positive terms converge to the limit of the pole, and the information on the zeros sits in the error.

3. DISCUSSION

The energy of the primes is where the cross-energy meets the arithmetic.

1. *In linear form the pole cancels positivity.* Any inequality $\sum a_j \cos j\theta \geq 0$ has $a_0 > 0$, so its use in the strip pays a correction of size $x^{1-\sigma}$ from the point $t = 0$. The arithmetic cross-energy shows this exactly. The 3-4-1 positivity holds at every finite cut (Theorem 2.17), yet the pole barrier (Proposition 2.21) limits it to the edge $\sigma = 1$, where it gives $\zeta(1 + it) \neq 0$ (Corollary 2.20). No combination of cosines without a constant term is nonnegative at all $\log p$ ([CE16, Proposition 2.9]). This is why the classical method gives zero-free regions only near $\sigma = 1$. The energy-slope form of de la Vallée Poussin's region ([CE10, Proposition 3.6]) has the constants of the elementary method.
2. *In energy form positivity is automatic, and finiteness is the question.* Averaged over t , the square of the cross-energy keeps only its diagonal. This gives the theorem of Bohr and Landau that off-line zeros have density zero ([CE10, Proposition 3.9]), and with higher moments the density theorems, but an average cannot exclude a single zero. Theorem B expresses RH as the finiteness of $\sum_\rho \lim_n C_2(n, \rho)$, that is, of the energy of the prime signal. Under RH and simplicity this energy equals $C_B = 2\lambda_1$, and the primes up to 10^8 give 0.0453 . Sums over the zeros that are holomorphic in ρ come linearly from the primes and do not see β . Non-holomorphic sums, such as $\sum |\rho|^{-2}$ or the crossing points, do see β , but they are reached from the primes only through energies (Proposition 2.5). The cost of a spiral of drift δ in a Gaussian window is $e^{2\delta U}/|\rho|^2$ (Corollary 2.12), and the primes needed to see it grow like a power $T^{1/\delta}$ of the height.

3.1. Open problems.

- Is there a positive quantity built from the partial sums or the partial Euler products that does not pay the pole correction of item 1, for instance an average over twists in which the $t = 0$ term cancels?

We do not prove the Riemann hypothesis.

APPENDIX A. NUMERICAL METHODS AND DATA

All computations use double precision in Python (NumPy, SciPy), except where a number of digits is stated; those use mpmath or Python's `decimal` arithmetic. The zeros of ζ with $0 < \gamma < 600$ (341 ordinates) and the first 1000 ordinates were computed once and are reused throughout; where a statement depends on the precision of an ordinate ([CE2, Remark 2.12]), the ordinates are

refined to 10^{-12} . ζ is evaluated by Euler–Maclaurin summation with Bernoulli corrections, and the Davenport–Heilbronn function f through Hurwitz zeta functions; its eight off-line zeros with $\gamma < 400$ are those listed in [CE17]. $\Lambda(n)$ is computed by sieve up to 10^8 (up to 10^9 for ψ). Every value in the tables was obtained by the script cited in its Status line, and every script prints the numbers quoted in the text.

- *Section 2.* $E(u)$ from the prime powers up to 10^8 and the energy law (`signal/`, `c2law/`, `local/`); Li’s coefficients (`li/`); the explicit formula in cross-energy coordinates and $\psi(x)$ up to 10^9 (`c2_explicit/`); the Nyman–Beurling distances and the Möbius energy (`nyman/`, `mobius/`, `mobius_energy/`); the energy matrix (`K_constant/`); the harmonic offset (`harmonic/`); the arithmetic cross-energy and its energy criterion, by sieve up to 10^7 (`arith_c2/`); the extrapolation of the prime sum from the first 1000 zeros, with the prime powers up to 10^8 (`extrapolation/`). The first two rows of Table 8, which use 1 124 029 zeros and the prime powers up to 10^{10} , are from the author’s working notes; that computation is not included.

How this work was produced. The cross-energy C_2 and the research program are the author’s, beginning with his preprints of 2017 and the series [C1]–[C8]. The author used Claude (Anthropic), an AI assistant, as a research and writing aid: to derive and check proofs, to write and run the Python computations, to test claims against the Davenport–Heilbronn control and against counterexamples, to search the literature, and to draft and edit the text and figures. In preparing this series, the material of the consolidated paper (version 3), of the earlier series [C1]–[C8] and of the author’s manuscript on the asymptotic balance of partial sums was inventoried result by result, regrouped into the seventeen parts of the series (Table 1 of [CE1]), and checked again; the corrections found in the process are stated where they apply and listed in Part 16 [CE16]. Several claims of the author’s earlier preprints were found to be incorrect and are not used. The author directed the work, checked the results and is responsible for the content.

Data availability. The scripts are included in the ancillary directory `anc/` of the source package, one subdirectory per topic as listed above. The earlier preprints [C1]–[C8] and the versions 1–4 of the consolidated paper are deposited on Zenodo under the DOIs listed in the references.

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