

# CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 17: CONTROLS FOR THE RIEMANN HYPOTHESIS: DAVENPORT–HEILBRONN, BEURLING SYSTEMS AND FUNCTION FIELDS

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ABSTRACT. A statement meant to imply the Riemann hypothesis must fail for every function that shares part of the structure of  $\zeta$  and has zeros off the critical line. We collect such controls and measure them. In the strip a zero is defined only by continuation, so every representation of  $\zeta$  defines it; we list thirteen representations and consequences of a zero, separate equivalences from global criteria, and note that known theorems combine two. At the eight off-line zeros found for the Davenport–Heilbronn function  $f$  with  $t < 400$ , the two Euler products composing  $f$  have ordinary size (14th to 38th percentile on the same line): the zeros come from two products meeting in phase. A window inequality from Littlewood’s lemma, Jensen and Minkowski bounds the off-line mass in a rectangle by a mollified energy; on short windows its Littlewood sum reproduces the mass of each of the seven zeros of  $f$  with  $\beta > 0.6$  to four decimals. A simplified Diamond–Montgomery–Vorhauer system with a zero at  $0.75 + 20i$ , primes rounded to a common scale, and the systems of Broucke–Debruyne–Révész give Beurling controls with an Euler product; with  $f$  they show that a proof must use the functional equation and the Euler product together. Over  $\mathbb{F}_q[T]$ , where RH is Weil’s theorem, the cross-energy criteria are restatements:  $C_2 = -2$  at every zero, Jensen’s criterion is an identity with the class number, and a Davenport–Heilbronn analogue fails at base change (RH at 7, 4, 2, 0 of 12 moduli on levels 1–4). Elliptic curves, Ramanujan graphs (Petersen against a ring of four Petersen graphs) and amplification locate the deciding ingredient in a positivity carried by an operator. Three filters and Landau’s positivity–pole dichotomy sort candidate rigidity statements. We do not prove the Riemann hypothesis.

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## 1. INTRODUCTION

**1.1. Context.** This is Part 17 of a series on the Riemann hypothesis (RH) through the cross-energy of consecutive Dirichlet partial sums. With  $X_N(s) = \sum_{k \leq N} k^{-s}$ ,  $s = \sigma + it$ , the cross-energy is

$$C_2(n, s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma} = 2 \operatorname{Re} (X_{n-1}(s) \overline{n^{-s}}),$$

and its theory (the additive representation  $\zeta = \lim_N [X_N - N^{1-s}/(1-s)]$  in the strip, the trichotomy of its limits, the Jensen formula of the  $C_2$  wave, the energy law of the primes, the energy slope and Weil’s functional, and the structural limits) is in [CE1, CE2, CE5, CE10, CE16]; the reflected cross-energy is in [CE3]; the Euler–Caceres bound, with its function-field model, Chebyshev’s identity at a zero and a Beurling system with the zeros of  $\zeta$ , is in [CE11]. Results quoted from these papers are stated where they are used, and only the definitions of Section 2 are needed to read this one. We write  $\rho = \beta + i\gamma$  for a nontrivial zero,  $\lambda = s(1-s)$ ,  $C_B = 2 + \gamma_0 - \log 4\pi$ , and [PT] for the verification of RH up to height  $3 \cdot 10^{12}$ .

Every criterion of the series, and every mechanism proposed to explain RH, must be tested on functions that share part of the structure of  $\zeta$  and have zeros off the critical line. This paper collects these *controls* and what they show. Two are already in [CE16, Sect. 2]: the Epstein zeta function of  $m^2 + 100n^2$ , which has the pole, nonnegative coefficients, a functional equation and the whole trichotomy of  $C_2$ , and a real zero at 0.8134; and the superpositions  $\xi + \varepsilon \Lambda(\cdot, \chi_5)$ . The others are here: the Davenport–Heilbronn function, which has a functional equation of the same shape as  $\zeta$  and no Euler product; Beurling systems, which have an Euler product,  $\Lambda \geq 0$  and  $\log = \Lambda * 1$ , but not the integers  $1, 2, 3, \dots$ ; and the function-field and graph analogues, where RH is a theorem or a decidable statement and one can see which ingredient decides it.

**1.2. What is classical and what is new.** The Davenport–Heilbronn function [DH], its off-line zeros [Spi, BS], the Beurling systems of Diamond, Montgomery and Vorhauer [DMV] and of Broucke, Debruyne and Révész [BDR], Weil’s theorem [We] and its elementary proof by Stepanov and Bombieri [St, Bo73], the Ihara zeta function and the Ramanujan property [Ih, Ha, LPS], Landau’s theorem on Dirichlet series with nonnegative coefficients [La05] and the heuristic of Selberg and of Bohr–Landau that linear combinations of  $L$ -functions have many off-line zeros [Sel92, T] are classical. What is new is a set of explicit, reproducible comparisons in the notation of the cross-energy: the quantitative anatomy of the off-line zeros of the Davenport–Heilbronn function (two Euler products of generic size meeting in phase); a short window inequality with its floor and certificate and the corresponding measurements; a simplified Beurling control with a prescribed zero; primes rounded to a common scale; an exact function-field bench in which the  $C_2$  criteria become restatements and the  $C_2$ –Jensen criterion an identity with a class number; a Davenport–Heilbronn analogue over  $\mathbb{F}_q[T]$  that fails exactly at base change; the wave over an elliptic curve and over a graph; and a list of filters for candidate rigidity statements.

### 1.3. Results and their status.

- (1) *Anatomy of a zero* (Section 3). Thirteen representations and consequences of  $\zeta(\rho) = 0$  (eight equivalences, two necessary conditions, three global criteria), their classification against the Epstein control, the table of known theorems as combinations of two of them, and the size balance of the Riemann–Siegel halves (Proposition 1: classical; Lemma 2: proved; the classification: heuristic).
- (2) *The off-line zeros of the Davenport–Heilbronn function* (Section 4). At the eight off-line zeros found with  $10 \leq t \leq 400$  (not a certified count) the components  $c_1 L(\cdot, \chi)$  and  $c_2 L(\cdot, \bar{\chi})$  take values between the 14th and 38th percentile of their own size distribution on the same vertical line (numerical); the level-set mechanism behind it (heuristic); the prime side resonates as at a zero of  $\zeta$  (numerical); the inverse coefficients grow (numerical).
- (3) *A window inequality* (Section 5). Proposition 3 and Corollaries 4–5 (proved), and the measurements of off-line mass, mollifiability and local sharpness (numerical).
- (4) *Beurling controls* (Section 6). A simplified [DMV] system with a zero at  $0.75 + 20i$  (numerical); a summary of the [BDR]-type system with every zero of  $\zeta$  and a real zero at 0.6, whose analysis is in [CE11, Sect. 13.5]; primes rounded to a common scale (Proposition 7: proved; folding and absence of a finite recurrence: numerical); the three controls (Table 8).
- (5) *Function fields and graphs* (Section 7). Propositions 8, 9 and 10 (proved); the Davenport–Heilbronn analogue and its failure at base change (numerical); amplification and parity (Proposition 11: classical; sieve data: numerical); the wave over an elliptic curve (Proposition 12: proved) and the degree form (classical); graphs (classical identities, numerical examples).
- (6) *Structural obstructions* (Section 8). The filters of [CE2, Def. 2.22] and Landau’s positivity–pole dichotomy [CE2, Prop. 2.23] applied to the candidates of the series (heuristic); the blindness of the Euler state, proved in [CE8, Prop. 15], is recalled.

No result of this paper implies RH, and none is claimed to.

**1.4. Organization.** Section 2 fixes notation and the Davenport–Heilbronn function. Sections 3–8 contain the results listed above, and Section 9 the discussion and open problems.

## 2. SETTING

**2.1. Cross-energy.** We use the notation of [CE1]:  $X_N$ ,  $C_2$  and the wave  $W_n(s) = 2X_{n-1}(s)\overline{n^{-s}} = C_2 + iS_2$ . For a general Dirichlet series  $\sum a_n n^{-s}$ , or a power series  $\sum A_k u^k$  over  $\mathbb{F}_q[T]$ ,  $X_N$  and  $C_2$  are defined with the coefficients inserted.

**2.2. The Davenport–Heilbronn function.** Let  $\chi$  be the Dirichlet character modulo 5 with  $\chi(2) = i$ , and

$$\kappa = \frac{\sqrt{10 - 2\sqrt{5}} - 2}{\sqrt{5} - 1} = 0.2840790438\dots, \quad c_{1,2} = \frac{1}{2}(1 \mp i\kappa).$$

The Davenport–Heilbronn function [DH], [T, Sect. 10.25] is

$$f(s) = c_1 L(s, \chi) + c_2 L(s, \bar{\chi}) = \sum_{n \geq 1} \frac{a_n}{n^s}, \quad (a_n)_{n \bmod 5} = (0, 1, \kappa, -\kappa, -1),$$

and satisfies  $f(s) = \mathcal{X}(s)f(1-s)$  with  $\mathcal{X}(s) = (5/\pi)^{1/2-s} \Gamma(\frac{2-s}{2})/\Gamma(\frac{1+s}{2})$ , which we checked numerically to  $10^{-14}$  in the strip. The coefficients are real and periodic but not multiplicative:  $a_n = c_1 \chi(n) + c_2 \bar{\chi}(n)$  is a sum of two completely multiplicative sequences. Searching  $0.52 \leq \sigma \leq 0.98$ ,  $10 \leq t \leq 400$  for local minima of  $|f|$  and refining by Newton’s method to  $|f| < 10^{-13}$  we found the eight off-line zeros of Table 1; the first four agree with Spira’s values [Spi] (see also [BS]). The search is not a certified count: we did not apply the argument principle with rigorous error bounds, and zeros with  $\frac{1}{2} < \beta < 0.52$  are outside the region searched. As a consistency check, on  $[10, 300]$  Littlewood’s identity gives the off-line mass 0.6551 in  $\sigma > 0.6$ , against 0.6531 from the four zeros found there (Section 5). Below, “the eight off-line zeros” means these eight. We evaluate  $f$  through Hurwitz zeta functions (Euler–Maclaurin with  $t/2 + 40$  terms and seven Bernoulli corrections, `anc/dh.py`).

|           |            |            |            |            |            |            |            |
|-----------|------------|------------|------------|------------|------------|------------|------------|
| 0.808517  | 0.650830   | 0.574356   | 0.724258   | 0.869531   | 0.819550   | 0.768223   | 0.628508   |
| 85.699348 | 114.163343 | 166.479306 | 176.702461 | 240.404672 | 320.876490 | 331.050259 | 366.640908 |

TABLE 1. The off-line zeros  $\beta + i\gamma$  of the Davenport–Heilbronn function found by the search in  $0.52 \leq \beta \leq 0.98$ ,  $10 \leq \gamma \leq 400$  (first row  $\beta$ , second row  $\gamma$ ); not a certified count, and  $f$  also has zeros with  $\beta > 1$  [DH]. File `anc/anatomy/offline.npy`.

### 3. ANATOMY OF A ZERO

In  $0 < \sigma < 1$  the series  $\sum n^{-s}$  diverges, so  $\zeta(\rho) = 0$  is defined by analytic continuation, and every representation of  $\zeta$  in the strip gives an equivalent definition. A zero must satisfy all of them at once; RH asks whether they are compatible only on  $\text{Re } \rho = \frac{1}{2}$ .

**Proposition 1** (thirteen representations and consequences of the zero condition). *Let  $\rho = \beta + i\gamma$  with  $0 < \beta < 1$ . The statements (1)–(6), (8) and (10) below are each equivalent to  $\zeta(\rho) = 0$ . Statements (7) and (9) are consequences of  $\zeta(\rho) = 0$ , necessary conditions. Statements (11)–(13) concern the whole set of zeros: they describe how a zero, or a zero with  $\beta > \frac{1}{2}$ , enters a global criterion for RH, and they are not characterizations of an individual zero.*

- (1) Partial sums.  $X_N(\rho) - N^{1-\rho}/(1-\rho) \rightarrow 0$  as  $N \rightarrow \infty$  [CE1, Sect. 2]; equivalently the  $C_2$  wave energy  $\Omega_\beta(\gamma) = 2|\zeta(\rho)|^2$  vanishes [CE10, Sect. 2].
- (2) Alternating series.  $\eta(\rho) = \sum_{n \geq 1} (-1)^{n-1} n^{-\rho} = 0$ , since  $\eta = (1 - 2^{1-s})\zeta$  and the factor has no zeros in the strip.
- (3) Sawtooth.  $\int_0^\infty \{x\} x^{-\rho-1} dx = 0$ , since  $\zeta(s) = -s \int_0^\infty \{x\} x^{-s-1} dx$  for  $0 < \sigma < 1$  [T, (2.1.5)].
- (4) Functional equation.  $\zeta(1-\rho) = 0$ ; zeros come in quadruples  $\rho, \bar{\rho}, 1-\rho, 1-\bar{\rho}$ . By Hamburger’s theorem the functional equation with conductor 1 characterizes  $\zeta$  among Dirichlet series of a suitable class [T, Sect. 2.13].
- (5) Theta function. With  $\psi(x) = \sum_{n \geq 1} e^{-\pi n^2 x}$ ,  $\int_1^\infty \psi(x) (x^{\rho/2-1} + x^{(1-\rho)/2-1}) dx = -1/(\rho(\rho-1))$ ; this uses the modularity of  $\theta$  [T, Sect. 2.6].

- (6) Hadamard product.  $\rho$  is a zero of the entire function  $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ , of order 1, whose Hadamard product is  $\xi(s) = \xi(0) \prod_{\rho'} (1 - s/\rho')$  with  $\rho'$  and  $\bar{\rho}'$  paired.
- (7) Riemann–Siegel (necessary condition). There is a constant  $A$ , depending only on a compact range of  $\sigma$  containing  $\beta$ , such that for all  $x, y \geq 1$  with  $2\pi xy = \gamma$ ,

$$\left| \sum_{n \leq x} n^{-\rho} + \chi(\rho) \sum_{n \leq y} n^{\rho-1} \right| \leq A(x^{-\beta} + \gamma^{1/2-\beta} y^{\beta-1}),$$

where  $\zeta(s) = \chi(s)\zeta(1-s)$  [T, Thm. 4.13], whose error terms are uniform in this sense.

- (8) Euler product.  $\log \zeta$  has a logarithmic branch point at  $\rho$ , and  $-\zeta'/\zeta = \sum \Lambda(n)n^{-s}$  has a simple pole there with residue  $-m$ ,  $m$  the multiplicity.
- (9) Explicit formula (consequence).  $\psi(x) - x$  contains the mode  $-x^\rho/\rho$ , of size  $x^\beta$ , and the prime sum resonates at  $\rho$ : for  $N$  not a prime power, with symmetric summation over the other zeros,

$$\sum_{n \leq N} \frac{\Lambda(n)}{n^\rho} = \frac{N^{1-\rho}}{1-\rho} - m \log N + c_\rho - \sum_{\rho' \neq \rho} \frac{N^{\rho'-\rho}}{\rho' - \rho} + o(1)$$

[In, Ch. IV]. The sum over the other zeros is not bounded in general (it is of size  $N^{\Theta-\beta}$  up to logarithms,  $\Theta = \sup \operatorname{Re} \rho'$ , and even under RH it contains  $N^{-\rho}(\psi(N) - N)$ ), so the term  $-m \log N$  is not isolated without further information; with a smooth cutoff and no zero to the right of  $\rho$  it is.

- (10) Fourier transform.  $\Xi(z) = \xi(\frac{1}{2} + iz) = \int_{-\infty}^{\infty} \Phi(u) e^{izu} du$  vanishes at  $z = \gamma - i(\beta - \frac{1}{2})$ , where  $\Phi$  is the positive, even, super-exponentially decreasing kernel of Riemann and Pólya;  $z$  is real if and only if  $\beta = \frac{1}{2}$ .
- (11) Harmonic measure (global). If  $\beta > \frac{1}{2}$ ,  $\rho$  contributes  $\log |\rho/(1-\rho)| > 0$  to the Jensen functional  $\mathcal{J} = \frac{1}{2\pi} \int \log |\zeta(\frac{1}{2} + it)| \frac{dt}{\frac{1}{4} + t^2}$  of Balazard, Saias and Yor [BSY]; in the language of the cross-energy,  $\mathcal{J}$  is half the mean of  $\log(\Omega/2)$  against the harmonic measure of the critical line seen from the pole [CE10, Sect. 2].
- (12) Nyman–Beurling (global). For  $0 < \theta \leq 1$ ,  $\int_0^1 (\{\theta/x\} - \theta\{1/x\}) x^{s-1} dx = (\theta - \theta^s)\zeta(s)/s$  for  $0 < \sigma < 1$ ; hence if  $\beta > \frac{1}{2}$  the vector  $x^{\bar{\rho}-1} \in L^2(0, 1)$  is orthogonal to every  $\{\theta/x\} - \theta\{1/x\}$ , an explicit obstruction to the density that is equivalent to RH [Be55, BD].
- (13) Weil’s explicit formula (global).  $\rho$  is a term of a quadratic form built from the primes, the pole and the archimedean place; RH is equivalent to the positivity of that form [We2], [CE10, Sect. 3].

*Proof.* Items (1)–(6), (8) and (10) are classical representations of  $\zeta$ , or of a function with the same zeros in the strip, valid for  $0 < \sigma < 1$ , and vanishing of the represented function at  $\rho$  is the statement; references are given in the statement. Item (7) is the approximate functional equation evaluated at a zero. Items (11)–(13) are the stated properties of the global criteria; they are consequences, for the set of zeros, of the existence of a zero with the stated real part, and conversely each criterion is equivalent to RH only as a statement about all zeros. For (9): the explicit formula  $\sum_{n \leq N} \Lambda(n)n^{-s} = -\frac{\zeta'}{\zeta}(s) + \frac{N^{1-s}}{1-s} - \sum_{\rho'} \frac{N^{\rho'-s}}{\rho'-s} + O_s(1)$  holds for  $s$  not a zero (symmetric summation over  $\rho'$ ,  $N$  not a prime power); letting  $s \rightarrow \rho$ , the singular parts  $-\frac{\zeta'}{\zeta}(s) = -\frac{m}{s-\rho} + O(1)$  and  $-m \frac{N^{\rho-s}}{\rho-s} = \frac{m}{s-\rho} - m \log N + O(s-\rho)$  cancel and leave  $-m \log N$ , the remaining terms being those displayed.  $\square$

Status: classical; the formulation of (1) and (11) in cross-energy terms is from [CE1, CE10]. In short: eight individual equivalences, two necessary conditions and three global criteria.

**3.1. Classification.** Rows 1 and 6 hold verbatim for the Epstein function  $Z_Q$  of  $Q = m^2 + 100n^2$  [CE16, Sect. 2], which has an off-line zero (with  $X_N$  replaced by the weighted partial sums and the  $C_2$  trichotomy unchanged); no proof can rest on them alone. Rows 4, 5, 7, 8, 9, 10, 12 and 13 use properties specific to  $\zeta$  (the functional equation with conductor 1, the Euler product, or both), and a proof must use at least one of them. Row 3 is specific only through the integer sawtooth. Every known theorem on the location of the zeros combines two rows (Table 2). Status: the classification is a heuristic reading of the literature.

| combination of rows   | result                                                                               |
|-----------------------|--------------------------------------------------------------------------------------|
| 8 + positivity + pole | $\zeta(1 + it) \neq 0$ , the prime number theorem, zero-free regions [CE10, Sect. 3] |
| 4 + 5 (or 10)         | Hardy: infinitely many zeros on the line                                             |
| 7 + mollifier         | Levinson, Conrey: a positive proportion (more than 40%) on the line                  |
| 9 + pair correlation  | Montgomery: under RH, a proportion of simple zeros                                   |
| 8 + large values      | zero-density estimates                                                               |

TABLE 2. Known theorems as combinations of two of the items of Proposition 1.

**Lemma 2** (size balance of the Riemann–Siegel halves). *Let  $0 < \beta < 1$ ,  $\gamma \rightarrow \infty$  and  $x = y = \sqrt{\gamma/2\pi}$  in row 7. Then  $|\chi(\beta + i\gamma)| = (\gamma/2\pi)^{1/2-\beta}(1 + O(1/\gamma))$ , and the trivial bounds of the two halves,  $\sum_{n \leq x} n^{-\beta}$  and  $|\chi(\rho)| \sum_{n \leq y} n^{\beta-1}$ , are respectively  $(\gamma/2\pi)^{(1-\beta)/2}(\frac{1}{1-\beta} + o(1))$  and  $(\gamma/2\pi)^{(1-\beta)/2}(\frac{1}{\beta} + o(1))$ , of the same order.*

*Proof.* The modulus of  $\chi$  is Stirling’s formula [T, (4.12.3)]. Then  $\sum_{n \leq x} n^{-\beta} \sim x^{1-\beta}/(1-\beta)$  and  $\sum_{n \leq y} n^{\beta-1} \sim y^{\beta}/\beta$ , and  $(\gamma/2\pi)^{1/2-\beta}y^{\beta} = (\gamma/2\pi)^{(1-\beta)/2} = x^{1-\beta}$ .  $\square$

Status: proved (elementary). One might hope that for  $\beta \neq \frac{1}{2}$  the two halves have different sizes and cannot cancel; the lemma says they have the same natural scale, so no comparison of sizes at the single scale  $x = y$  can exclude an off-line zero. This is the Riemann–Siegel form of the locality principle of [CE10, Sect. 2]. Status of this conclusion: heuristic.

**3.2. The open combination.** An off-line zero with  $\beta > \frac{1}{2}$  would force at once (row 7) a cancellation, up to the error terms, between  $\sum_{n \leq x} n^{-\rho}$  and  $\chi(\rho) \sum_{n \leq x} n^{\rho-1}$  at the length  $x = \sqrt{\gamma/2\pi}$ , the transition region of the Fresnel law of [CE2], and (row 9) a fluctuation  $x^{\beta} \cos(\gamma \log x + \phi)$  of the primes that persists in every prime sum. The question left open by Table 2 is whether the multiplicativity of the coefficients (row 8) makes the cancellation of row 7 at an off-line point incompatible with row 9. Any argument must fail for the Davenport–Heilbronn function, which has an approximate functional equation of the same shape (conductor 5), off-line zeros and no Euler product. The next section measures what happens at its zeros.

#### 4. THE OFF-LINE ZEROS OF THE DAVENPORT–HEILBRONN FUNCTION

**4.1. The Riemann–Siegel halves.** For  $f$  the approximate functional equation reads  $f(s) \approx H_1 + H_2$  with  $H_1 = \sum_{n \leq x} a_n n^{-s}$ ,  $H_2 = \mathcal{X}(s) \sum_{n \leq y} a_n n^{s-1}$  and  $2\pi xy = 5t$ . At each off-line zero, with  $x = y = \sqrt{5\gamma/2\pi}$ , the two halves are nearly opposite and the residual is of the size of the sharp-cutoff error (Table 3). Changing the split with  $xy$  fixed does not change this: at Spira’s first zero  $|H_1 + H_2| \leq 0.09$  for  $x = 3, 5, 8.26, 12, 20$ . For  $\zeta$  at nearby heights, with  $x = y = \sqrt{\gamma/2\pi}$ ,  $|H_1| = |H_2|$  holds identically on the line ( $|\chi(\frac{1}{2} + it)| = 1$ ), and at off-line points that are not zeros the halves are also nearly opposite (angles between  $166^\circ$  and  $170^\circ$ ); only  $|H_1 + H_2| \approx |\zeta(s)|$  distinguishes them. At these heights the halves show no trace of the Euler product. Status: numerical (anc/anatomy/steps.py, steps2.py).

| zero                | $x$   | $ H_1 $ | $ H_2 $ | $\arg(H_1/H_2)$ | $ H_1 + H_2 $ |
|---------------------|-------|---------|---------|-----------------|---------------|
| $0.8085 + 85.699i$  | 8.26  | 0.343   | 0.276   | $-168.7^\circ$  | 0.090         |
| $0.6508 + 114.163i$ | 9.53  | 0.304   | 0.372   | $-159.8^\circ$  | 0.136         |
| $0.7243 + 176.702i$ | 11.86 | 0.478   | 0.505   | $171.5^\circ$   | 0.078         |
| $0.8695 + 240.405i$ | 13.83 | 0.375   | 0.329   | $168.3^\circ$   | 0.086         |
| $0.6285 + 366.641i$ | 17.08 | 0.400   | 0.396   | $178.0^\circ$   | 0.015         |

TABLE 3. The Riemann–Siegel halves of  $f$  at five off-line zeros. Script `anc/anatomy/steps.py`.

**4.2. Two Euler products of generic size.** At a zero of  $f$ ,  $c_1 L(\rho, \chi) = -c_2 L(\rho, \bar{\chi})$  by definition, so the two components have equal modulus and opposite phase; this carries no information. The informative question is how large they are: is the zero produced by one Euler product becoming small, or by two of ordinary size meeting in phase? We compare the common modulus  $|c_1 L(\rho, \chi)|$  with the distribution of  $|c_1 L(\beta + it, \chi)|$  and  $|c_2 L(\beta + it, \bar{\chi})|$  over  $|t - \gamma| \leq 25$  on the same vertical line (1001 points per component). Table 4 gives the percentile of the value at the zero, its deviation in standard units on the logarithmic scale, the median, and the smallest value in the window.

| zero                | $ c_1 L(\rho, \chi) $ | percentile | log-deviation | median | window min |
|---------------------|-----------------------|------------|---------------|--------|------------|
| $0.8085 + 85.699i$  | 0.3650                | 32         | $-0.62$       | 0.503  | 0.145      |
| $0.6508 + 114.163i$ | 0.2193                | 14         | $-1.17$       | 0.509  | 0.096      |
| $0.5744 + 166.479i$ | 0.2245                | 18         | $-0.94$       | 0.520  | 0.051      |
| $0.7243 + 176.702i$ | 0.2785                | 17         | $-0.97$       | 0.509  | 0.112      |
| $0.8695 + 240.405i$ | 0.4131                | 38         | $-0.44$       | 0.496  | 0.130      |
| $0.8195 + 320.876i$ | 0.3987                | 37         | $-0.47$       | 0.511  | 0.168      |
| $0.7682 + 331.050i$ | 0.3680                | 33         | $-0.57$       | 0.503  | 0.148      |
| $0.6285 + 366.641i$ | 0.2545                | 18         | $-0.93$       | 0.498  | 0.079      |

TABLE 4. Size of the Euler products composing  $f$  at its eight off-line zeros, against their distribution on  $|t - \gamma| \leq 25$  on the same vertical line. Script `anc/anatomy/typical.py`.

At every off-line zero both Euler products take ordinary values, between the 14th and the 38th percentile and within about one standard deviation of the mean on the logarithmic scale; neither is close to vanishing. (At Spira’s zero,  $|c_1| = \frac{1}{2}\sqrt{1 + \kappa^2} = 0.5198$  gives  $|L(\rho, \chi)| = |L(\rho, \bar{\chi})| = 0.702$ .) The zero is produced by the phase of  $L(s, \chi)/L(s, \bar{\chi})$  reaching  $\arg(-c_2/c_1)$  at a point where the moduli already match. Status: numerical.

**The level-set mechanism.** Write  $R(s) = c_1 L(s, \chi)/(c_2 L(s, \bar{\chi}))$ . A zero of  $f$  is a point where  $|R| = 1$  and  $\arg R = \pi$ : two real conditions on two real variables. Both  $\log |R|$  and  $\arg R$  fluctuate at the typical scale of the Euler products, so both level sets cross  $\beta > \frac{1}{2}$  again and again and meet at isolated points with a positive density in  $t$ , consistent with the linear growth of the off-line mass (Table 5) and with the theorem that such combinations have  $\gg T$  zeros in  $\frac{1}{2} < \sigma < 1$  up to height  $T$  [KK]. For  $\zeta$  there is one Euler product and nothing to interfere with: an off-line zero would require  $\log |\zeta(\beta + it)| \rightarrow -\infty$  for a single product, an extreme event rather than the generic crossing of two level sets. This is the heuristic of Selberg [Sel92] and of the density theorems of Bohr–Landau and Selberg [T, Ch. IX]; Table 4 measures it. Status: heuristic.

**4.3. The prime side.** Let  $-f'/f = \sum b(n)n^{-s}$ . For  $n \leq 2 \cdot 10^5$ , 72.3% of  $\sum |b(n)|/n$  is carried by integers that are not prime powers, and  $b(p)$  changes sign; for  $\zeta$  the coefficients are  $\Lambda(n) \geq 0$  on the prime powers. Nevertheless, at Spira’s zero the partial sums resonate exactly as row 9 of Proposition 1 predicts at a zero of  $\zeta$  (here without the pole term):  $\operatorname{Re} \sum_{n \leq N} b(n)n^{-\rho} = -6.656$ ,

$-8.938, -11.230, -11.927$  for  $N = 10^3, 10^4, 10^5, 2 \cdot 10^5$ , against  $\log N = 6.908, 9.210, 11.513, 12.206$ , a fitted slope  $-0.995$  against the predicted  $-m = -1$ . The resonance alone is not specific to  $\zeta$ ; what distinguishes  $\zeta$  is a single completely multiplicative coefficient sequence together with  $\Lambda \geq 0$ . Status: numerical (`anc/anatomy/steps3.py`).

**4.4. Inverse coefficients.** For  $\zeta$  the Dirichlet inverse is  $\mu$ , with  $|\mu(n)| \leq 1$ . For  $f$  the largest  $|g(n)|$ ,  $n \leq X$ , of the Dirichlet inverse  $g$  of  $(a_n)$  is  $11.6, 52.2, 267, 383$  for  $X = 10^3, 10^4, 10^5, 2 \cdot 10^5$ . The inverse of a single Euler product has bounded multiplicative coefficients; the inverse of a sum of two Euler products has growing ones. Status: numerical (`anc/anatomy/mol12.py`, divisor sieve).

## 5. A WINDOW INEQUALITY

**Proposition 3** (window inequality). *Let  $F$  be analytic in a neighbourhood of the rectangle  $R = [\sigma_0, \sigma_1] \times [T_1, T_2]$ , let  $M$  be a Dirichlet polynomial and  $G = FM$ . Assume  $G \neq 0$  on  $\partial R$  and  $|G(\sigma_1 + it) - 1| < 1$  for  $T_1 \leq t \leq T_2$ . Let  $H = T_2 - T_1$ ,*

$$I = \frac{1}{H} \int_{T_1}^{T_2} |1 - G(\sigma_0 + it)|^2 dt,$$

$$B = - \int_{T_1}^{T_2} \log |G(\sigma_1 + it)| dt + \int_{\sigma_0}^{\sigma_1} [\arg G(\sigma + iT_2) - \arg G(\sigma + iT_1)] d\sigma,$$

with  $\arg G$  defined by continuous variation along each horizontal segment from its principal value at  $\sigma_1$ . Then, counting zeros with multiplicity,

$$2\pi \sum_{\substack{F(\rho)=0 \\ \rho \in R}} (\beta - \sigma_0) \leq 2\pi \sum_{\substack{G(\rho)=0 \\ \rho \in R}} (\beta - \sigma_0) = \int_{T_1}^{T_2} \log |G(\sigma_0 + it)| dt + B \leq H \log(1 + \sqrt{I}) + B.$$

*Proof. The argument on the boundary.* Since  $|G(\sigma_1 + it) - 1| < 1$ ,  $\operatorname{Re} G > 0$  on the right side of  $R$ , so the principal value of  $\arg G$  is continuous there, with values in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . Littlewood's lemma uses a branch of  $\log G$  on  $R$  cut along the horizontal segments that run from each zero of  $G$  to the left side, continuous from the right side; on the horizontal sides  $t = T_1, T_2$ , which contain no zeros, this branch is the continuous variation of  $\arg G$  from its principal value at  $\sigma_1$ , as in the definition of  $B$ . With this choice the equality below holds with no multiple of  $2\pi$  to be added. *The inequalities.* The equality is Littlewood's lemma [T, Sect. 9.9] applied to  $G$ :  $2\pi \int_{\sigma_0}^{\sigma_1} \nu(\sigma) d\sigma$ , where  $\nu(\sigma)$  is the number of zeros of  $G$  in  $R$  with real part greater than  $\sigma$ , equals  $\sum (\beta - \sigma_0)$  over the zeros of  $G$  in  $R$ . Every zero of  $F$  in  $R$  is a zero of  $G$  of at least the same multiplicity, because  $M$  is entire, and every term  $\beta - \sigma_0$  is nonnegative; this gives the first inequality. For the last, Jensen's inequality for the concave logarithm gives  $\frac{1}{H} \int \log |G| dt = \frac{1}{2H} \int \log |G|^2 dt \leq \frac{1}{2} \log \left( \frac{1}{H} \int |G|^2 dt \right)$ , and Minkowski's inequality in  $L^2(dt/H)$  gives  $\|G\| \leq \|1\| + \|1 - G\| = 1 + \sqrt{I}$ .  $\square$

Status: proved.

**Corollary 4** (floor). *Let  $m = \sum_{F(\rho)=0, \rho \in R} (\beta - \sigma_0)$ . For every Dirichlet polynomial  $M$  satisfying the hypotheses, with  $B = B(M)$ ,*

$$I \geq \left( \max \left\{ \exp \left( \frac{2\pi m - B}{H} \right) - 1, 0 \right\} \right)^2.$$

**Corollary 5** (certificate). *Conversely,  $\sum_{F(\rho)=0, \rho \in R} (\beta - \sigma_0) \leq (H \log(1 + \sqrt{I}) + B)/2\pi$ : a small mollified energy, with a small boundary term, certifies a small off-line mass.*

Status of both corollaries: proved (immediate from Proposition 3).

*Remark 6.* (i)  $B$  depends on  $M$  through  $G$ . The term  $\int \log |G(\sigma_1 + it)| dt$  tends to 0 as  $\sigma_1 \rightarrow \infty$ , and the argument terms are bounded by the standard lemma on the variation of the argument [T, Sect. 9.4]; in the local windows of Table 7,  $B/2\pi$  lies between 0.041 and 0.064. In particular the floor of Corollary 4 is a statement about each  $M$  with its own  $B$ , not a uniform lower bound over all mollifiers. (ii) The inequality is the local, vertical-window analogue of the  $C_2$ -Jensen formula of [CE10, Sect. 2], where the harmonic measure of the line seen from the pole plays the role of the normalized Lebesgue measure on the window; the Jensen moment with an Euler-product mollifier is developed in [CE11]. (iii) Global certificates from Corollary 5 are weak: for  $\zeta$  on  $[300, 600]$  with  $\sigma_0 = 0.6$  and  $X = 4 \cdot 10^4$  it gives  $\sum(\beta - \sigma_0) \lesssim 13.5$ , whereas the true value is 0 [PT]. On short windows the Littlewood equality reproduces the off-line mass of each of the seven zeros with  $\beta > 0.6$  to four decimals (Table 7); the final bound  $H \log(1 + \sqrt{I}) + B$  is not sharp.

**5.1. Off-line zero mass.** We evaluated Littlewood's identity with  $M = 1$  and  $\sigma_1 = 3$  on windows of height 300 (Table 5). On  $[10, 300]$  with  $\sigma_0 = 0.6$  the identity gives 0.6551 and the four located zeros with  $\beta > 0.6$  give 0.6531. The off-line mass of  $f$  grows linearly with the height, consistent with [DH, KK]; that of  $\zeta$  vanishes to numerical precision, as it must below  $3 \cdot 10^{12}$  [PT]. Status: numerical (`anc/anatomy/littlewood.py`; step 0.02 on vertical segments, 400–500 points on horizontal segments, argument unwrapped from  $\sigma = 3$ ).

| window       | $f, \sigma_0 = 0.6$ | $f, \sigma_0 = 0.7$ | $f, \sigma_0 = 0.8$ | $\zeta, \sigma_0 = 0.6$ |
|--------------|---------------------|---------------------|---------------------|-------------------------|
| [10, 300]    | 0.655               | 0.304               | 0.079               | 0.001                   |
| [300, 600]   | 1.302               | 0.574               | 0.081               | −0.002                  |
| [600, 900]   | 1.314               | 0.536               | 0.110               | 0.001                   |
| [900, 1200]  | 0.970               | 0.319               | 0.054               | −0.002                  |
| [1200, 1500] | 1.621               | 0.613               | 0.093               | 0.001                   |

TABLE 5. Off-line zero mass  $\sum(\beta - \sigma_0)$  from Littlewood's identity ( $M = 1, \sigma_1 = 3$ ). For  $\zeta$  the entries are quadrature error.

**5.2. Mollifiability.** Let  $M_X(s) = \sum_{n \leq X} g(n) (1 - \frac{\log n}{\log X}) n^{-s}$ , with  $g = \mu$  for  $\zeta$  and  $g$  the Dirichlet inverse of  $(a_n)$  for  $f$ , and let  $I(X)$  be the mean of  $|1 - FM_X|^2$  over  $t \in [300, 600]$  (Table 6). For  $\zeta$ ,  $I(X)$  decays like  $X^{1-2\sigma}$ : from  $X = 100$  to  $4 \cdot 10^4$  it drops by a factor 3.4 at  $\sigma = 0.6$ , against the predicted  $400^{0.2} = 3.3$ . For  $f$  at  $\sigma = 0.6$  it reaches a minimum near  $X \approx 5000$  and then increases. Status: numerical (`anc/anatomy/mol12.py`).

The floor of  $f$  is an observation, not a consequence of Corollary 4: on this window ( $H = 300$ ,  $m = 1.302$  at  $\sigma_0 = 0.6$ ) the corollary gives at most  $(e^{2\pi m/H} - 1)^2 \approx 7.6 \cdot 10^{-4}$  when  $B \geq 0$  (a negative  $B$  would raise the floor), far below the observed values. (The source of this section, Part V of the earlier series, attributed the floor to the corollary; we correct that here.) The obstruction it reflects is the growth of the inverse coefficients of  $f$  (Section 4).

| $X$   | $\zeta, \sigma = 0.6$ | $f, \sigma = 0.6$ | $\zeta, \sigma = 0.75$ | $f, \sigma = 0.75$ |
|-------|-----------------------|-------------------|------------------------|--------------------|
| 100   | 0.368                 | 0.555             | 0.131                  | 0.163              |
| 1000  | 0.211                 | 0.414             | 0.064                  | 0.094              |
| 5000  | 0.154                 | 0.392             | 0.043                  | 0.071              |
| 20000 | 0.120                 | 0.424             | 0.032                  | 0.058              |
| 40000 | 0.107                 | 0.456             | 0.028                  | 0.054              |

TABLE 6. Mollified energy  $I(X)$  on  $t \in [300, 600]$ .

**5.3. The local inequality at each zero.** We take windows of height  $h$  centred at each off-line zero of  $f$  with  $\beta > \sigma_0 = 0.6$ ,  $M = M_X$  with  $X = 5000$ , and  $\sigma_1 = 3$ . In Table 7,  $m_F = \beta - \sigma_0$  is the mass of the zero itself,  $S_G$  the Littlewood sum for  $G$ , “bound” the right side of Corollary 5 and “floor” the right side of Corollary 4, both with the boundary term  $B$  of this  $M$ . With  $h = 1$  the Littlewood sum of  $G$  reproduces the mass of the zero to four decimals; with  $h = 2$ ,  $S_G > m_F$  because the window also contains zeros of  $M_X$ . The bound holds in every case. For the four zeros with  $\beta \geq 0.768$  the floor forces the local mollified energy of  $M_{5000}$  above 1. Status: numerical (`anc/anatomy/localfloor.py`; step  $h = 0.005$  on local windows).

| zero                | $h$ | $m_F$  | $S_G$  | $I_h$ | $B/2\pi$ | bound on $S_G$ | floor on $I_h$ |
|---------------------|-----|--------|--------|-------|----------|----------------|----------------|
| $0.8085 + 85.699i$  | 1   | 0.2085 | 0.2085 | 6.80  | 0.061    | 0.266          | 2.32           |
| $0.6508 + 114.163i$ | 1   | 0.0508 | 0.0508 | 1.52  | 0.064    | 0.192          | 0              |
| $0.7243 + 176.702i$ | 1   | 0.1243 | 0.1243 | 2.83  | 0.057    | 0.214          | 0.27           |
| $0.8695 + 240.405i$ | 1   | 0.2695 | 0.2695 | 17.64 | 0.041    | 0.303          | 10.33          |
| $0.8195 + 320.876i$ | 1   | 0.2195 | 0.2195 | 9.41  | 0.041    | 0.264          | 4.27           |
| $0.7682 + 331.050i$ | 1   | 0.1682 | 0.1682 | 4.77  | 0.049    | 0.233          | 1.25           |
| $0.6285 + 366.641i$ | 1   | 0.0285 | 0.0285 | 1.29  | 0.052    | 0.173          | 0              |
| $0.8085 + 85.699i$  | 2   | 0.2085 | 0.2340 | 4.38  | 0.060    | 0.419          | 0.36           |
| $0.8695 + 240.405i$ | 2   | 0.2695 | 0.3625 | 11.23 | 0.046    | 0.514          | 1.04           |

TABLE 7. The window inequality at the off-line zeros of  $f$  with  $\beta > 0.6$  ( $X = 5000$ ,  $\sigma_1 = 3$ ).

## 6. BEURLING CONTROLS

A Beurling system is a sequence of generalized primes  $1 < p_1 \leq p_2 \leq \dots \rightarrow \infty$  with the multiplicative semigroup they generate (the generalized integers, counted by  $N(x)$ ) and the zeta function  $\zeta_P(s) = \prod_k (1 - p_k^{-s})^{-1}$  [Be37]. It has unique factorization, an Euler product,  $\Lambda_P \geq 0$  and  $\log = \Lambda_P * 1$ , but its integers are not  $1, 2, 3, \dots$ . It is the control opposite to the Davenport–Heilbronn function.

**6.1. A simplified Diamond–Montgomery–Vorhauer system.** Diamond, Montgomery and Vorhauer [DMV] constructed Beurling systems with well-distributed integers whose zeta function has zeros as close to  $\sigma = 1$  as the region of de la Vallée Poussin allows, so unique factorization and a regular count of integers do not by themselves exclude zeros off the line. We built a simplified example with one prescribed zero. The prime-power density is  $d(x) = 1 - 2 \operatorname{Re} x^{\rho-1}$  for  $x \geq 20$ , with  $\rho = 0.75 + 20i$ , so that  $\psi_P(x) = x - 2 \operatorname{Re}(x^\rho/\rho) + \dots$ , and the primes  $p_k$  are the points where  $\Pi(x) = \int d(u) du / \log u$  equals  $k - \frac{1}{2}$ . This gives 663 888 generalized primes below  $10^7$  (against 664 579 primes), beginning 1.739, 2.462, 3.546,  $\dots$ ; their products, 2 269 793 of them below  $10^6$ , are counted by  $N_P(x) = 2.2715x - 1.65\sqrt{x} + O(10^2)$  on  $10^3 \leq x \leq 10^6$ , against  $[x] = x + O(1)$  for the integers.

Three diagnostics separate the system from the primes (Figure 1). The error  $|\psi_P(x) - x|$  follows  $2x^{0.75}/|\rho|$  at the upper end of the range; the exponent fitted on dyadic blocks from  $2^8$  to  $2^{23}$  is 0.61, lower than 0.75 because the  $\sqrt{x}$  fluctuations of the rounding dominate at the low end (the two are equal near  $0.1x^{0.75} \approx \sqrt{x}$ ), against 0.50 for the primes. The exponent of the increments of the C-transform  $C_N(s; \Lambda_P) = \sum_{n \leq N} \Lambda_P(n) n^{-s} - \int_1^N x^{-s} dx$  [CE1, Sect. 2] at  $t = 20$  is  $0.75 - \sigma$  to within 0.005 for  $0.55 \leq \sigma \leq 1$ , so the abscissa of convergence is 0.75, while for the primes it stays negative for  $\sigma \geq 0.55$  and lies within 0.05 of  $\frac{1}{2} - \sigma$ . The prime sum  $R_N(\frac{1}{2} + i\gamma_0) = \sum_{n \leq N} (\Lambda_P(n) - 1) n^{-1/2 - i\gamma_0}$  grows like  $N^{0.25}/0.25$  at  $\gamma_0 = 20$  and stays between 1 and 10 at  $\gamma_0 = 30$ , as it does for the primes at both frequencies.

Status: numerical (**anc/beurling/**); the construction is a simplified form of [DMV], and no error bound is claimed for its zeta function beyond the measured exponents. In particular we have not proved that  $\zeta_P$  has a zero at, or near,  $0.75 + 20i$ : the zero is prescribed in the density of the generalized primes, and the diagnostics above are consistent with it. A theorem would need an analytic continuation of  $\zeta_P$  with explicit error bounds and an argument-principle count around the prescribed point; for the rigorous constructions see [DMV] and [BDR].

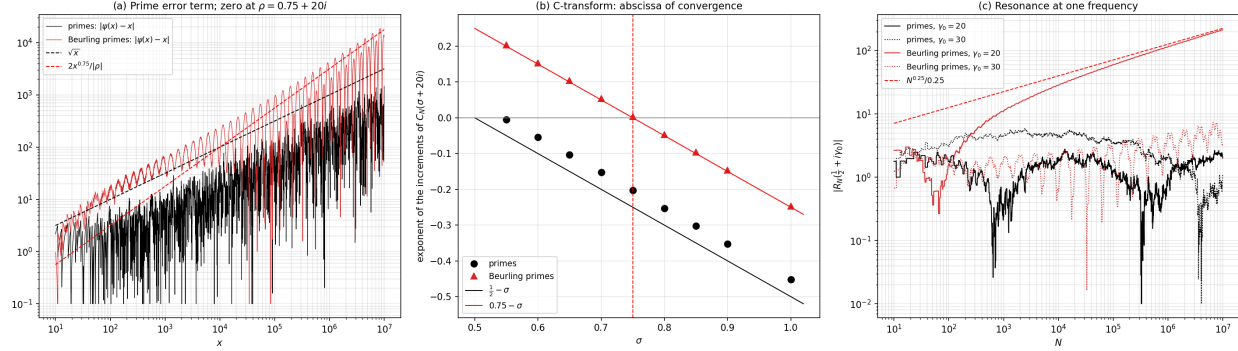


FIGURE 1. A Beurling system with a zero at  $\rho = 0.75 + 20i$ , against the primes. (a)  $|\psi(x) - x|$ , with  $\sqrt{x}$  and  $2x^{0.75}/|\rho|$ . (b) Exponent of the increments of the C-transform at  $t = 20$  on dyadic blocks up to  $10^7$ , with the lines  $\frac{1}{2} - \sigma$  and  $0.75 - \sigma$ . (c)  $|R_N(\frac{1}{2} + i\gamma_0)|$  at  $\gamma_0 = 20$  and  $30$ , with  $N^{0.25}/0.25$ .

**6.2. A Beurling system with the zeros of  $\zeta$ .** The control above has integers with error of order  $\sqrt{x}$ . Broucke, Debruyne and Révész [BDR, Thm. 1.3] construct, under RH, systems with  $N(x) = cx + O(x^{\beta+\varepsilon})$ ,  $\frac{2\alpha}{\alpha+2} \leq \beta < \frac{1}{2}$ , and a real zero at  $\alpha \in (\frac{1}{2}, \frac{2}{3})$ : they add the numbers  $p^{1/\beta}$  to the primes and remove a set  $\mathcal{S}$  of primes with  $\pi_{\mathcal{S}}(x) = \text{Li}(x^\alpha) + O(1)$ , so that  $\zeta_B = \zeta(s)\zeta(s/\beta)/\zeta_{\mathcal{S}}(s)$ . In  $\text{Re } s > \frac{\alpha}{2}$  such a system has every zero of  $\zeta$  there and a real zero at  $\alpha$ ; its integers are better than  $\sqrt{x}$ . The analysis of this system is in [CE11, Sect. 13.5], which we summarize: Chebyshev's identity  $\sum_m \Lambda_B(m) \lfloor N/m \rfloor_B = \log N!_B$ , evaluated at a zero  $\rho$  of  $\zeta_B$ , holds in every discrete Beurling system, and when the integer error is  $O(x^\beta)$  with  $\beta < \frac{1}{2}$  the integers enter it only at order  $N^{\beta-1} \log N$ , so that at a zero on the line it is equivalent to  $\psi_B(x) - x = O(x^{1/2+\varepsilon})$ ; no bound on the size of the integer error can make it force RH. A computation with  $\alpha = 0.6$ ,  $\beta = 0.47$  up to  $3 \cdot 10^7$  (6360 removed primes below  $10^8$ ,  $c = 0.69529746$ ,  $\zeta'_B(\alpha) = -11.5733$ ) confirms the identity to ten digits and shows the system violating, at the zero  $\rho_1 = \frac{1}{2} + 14.1347i$  that it shares with  $\zeta$ , the condition that the integers satisfy. The reflected cross-energy, which detects the real zero 0.6 from the integers of the system alone and drifts with  $N$ , is in [CE3]. Status: proved and numerical in [CE11], [CE3]; we do not reproduce it.

**6.3. Primes rounded to a common scale.** The Euler product of  $\zeta$  lives on the infinitely many circles  $|p^{-s}| = p^{-\sigma}$ . Fix  $0 < \delta < 2 \log 2$  (so that  $n_p \geq 1$  for every  $p$ ) and replace each prime  $p$  by the generalized prime  $e^{\delta n_p}$ ,  $n_p = \text{round}(\log p/\delta)$ .

**Proposition 7** (one circle).  $\zeta_\delta(s) = \prod_p (1 - e^{-\delta n_p s})^{-1}$  is a power series  $\sum_{k \geq 0} r_k w^k$  in  $w = e^{-\delta s}$  with nonnegative integer coefficients, with radius of convergence  $e^{-\delta}$ ; in  $\sigma > 1$  it has no zeros. It is periodic in  $t$  with period  $2\pi/\delta$ , and each vertical line is mapped onto a circle  $|w| = e^{-\delta\sigma}$ . Its logarithmic derivative satisfies  $-\zeta'_\delta/\zeta_\delta(s) \sim c_\delta/(s-1)$  as  $s \rightarrow 1^+$ , with  $c_\delta = 2 \sinh(\delta/2)/\delta \in (1, 2)$ ; so  $\zeta_\delta$  has a branch point at  $w = e^{-\delta}$ , is not a rational function of  $w$ , and has no single-valued meromorphic continuation across the circle  $|w| = e^{-\delta}$ .

*Proof.* Each factor is  $\sum_j w^{jn_p}$ , and the number of primes with  $n_p = n$  is at most  $\pi(e^{\delta(n+1/2)})$ , so the product converges absolutely for  $|w| < e^{-\delta}$ , where  $\sum_p |w|^{n_p} \leq \sum_n e^{\delta(n+1/2)} |w|^n < \infty$ , and it has no zeros there;  $r_k$  counts the representations of  $k$  as  $\sum_p j_p n_p$ . Periodicity is  $e^{-\delta(s+2\pi i/\delta)} = e^{-\delta s}$ . By the prime number theorem, the generalized von Mangoldt mass at level  $n$  is  $\delta n(\pi(e^{\delta(n+1/2)}) - \pi(e^{\delta(n-1/2)}))$  plus the prime powers,  $O(ne^{\delta n/2})$ , and this is  $2 \sinh(\delta/2) e^{\delta n} (1 + O(1/n))$ . Hence  $-\zeta'_\delta/\zeta_\delta(s) = 2 \sinh(\delta/2) \sum_n e^{\delta n(1-s)} + \sum_n O(e^{\delta n(1-\sigma)}/n)$ ; the first sum is  $2 \sinh(\delta/2)/(1 - e^{-\delta(1-s)}) \sim c_\delta/(s-1)$ , which also gives the radius of convergence, and the second is  $O(\log \frac{1}{\sigma-1})$ . So  $(s-1)\zeta'_\delta/\zeta_\delta(s) \rightarrow -c_\delta$  as  $s \rightarrow 1^+$ . If  $\zeta_\delta$  were meromorphic at  $s = 1$ , this limit would be an integer (minus the order of  $\zeta_\delta$  there), and  $c_\delta \in (1, 2)$  is not an integer.  $\square$

Status: proved (elementary). We built the Chebyshev function of this system from the primes up to  $10^8$  on the lattice  $u = \delta m$  and the normalized error  $E_\delta(m) = e^{-\delta m/2}(\psi_\delta(e^{\delta m}) - e^{\delta/2} e^{\delta m})$ , where  $e^{\delta/2}$  corrects the mean shift caused by rounding (Figure 2).

**Folding.** For  $\delta = 0.05$  (period 125.7) the spectral peaks of  $E_\delta$  lie at 14.2, 21.0, 25.0, 30.6, 32.9, 37.6, the first zeros 14.13, 21.02, 25.01, 30.42, 32.94, 37.59 of  $\zeta$ . For  $\delta = 0.2$  (period 31.4) they lie at 14.0,  $10.4 = 31.4 - 21.0$  and  $1.6 \approx 32.9 - 31.4$ : the zeros are folded onto the circle.

**No finite recurrence.** A finite sum of exponentials  $\sum_j c_j \alpha_j^m$  has a Hankel matrix of finite rank. For the Petersen graph of Section 7.7 the normalized singular values of the  $40 \times 40$  Hankel matrix of its prime-cycle error drop to  $10^{-6}$ , the level of rounding error, after index 4, matching its four nontrivial inverse roots. For the rounded primes they decay slowly: at indices 5, 10, 20, 40 they are 0.247, 0.193, 0.102, 0.020 for  $\delta = 0.2$  and 0.188, 0.128, 0.091, 0.008 for  $\delta = 0.05$ . This is consistent with Proposition 7:  $\zeta_\delta$  is not rational in  $w$ , and no finite set of inverse roots describes the error.

**Energy.** The mean of  $E_\delta^2$  over  $u > 2$  is 0.3205, 0.0933, 0.0738 for  $\delta = 0.2, 0.1, 0.05$ , against  $C_B = 0.0462$  for the true primes [CE5, Sect. 2]; rounding adds energy that decreases as  $\delta \rightarrow 0$ . Status: numerical (`anc/rounded_primes/beurling.py`, `figb.py`). A common scale puts the whole Euler product on one circle, but the result is a power series with a branch point on the circle of convergence, not a rational function with finitely many zeros, and nothing finite-dimensional describes it; Beurling systems can satisfy a strong prime number theorem and still violate RH [DMV]. Over  $\mathbb{F}_q[T]$  and for curves the same substitution  $u = q^{-s}$  produces a rational function with finitely many zeros, the eigenvalues of Frobenius, and Weil's theorem places them on the circle (Section 7).

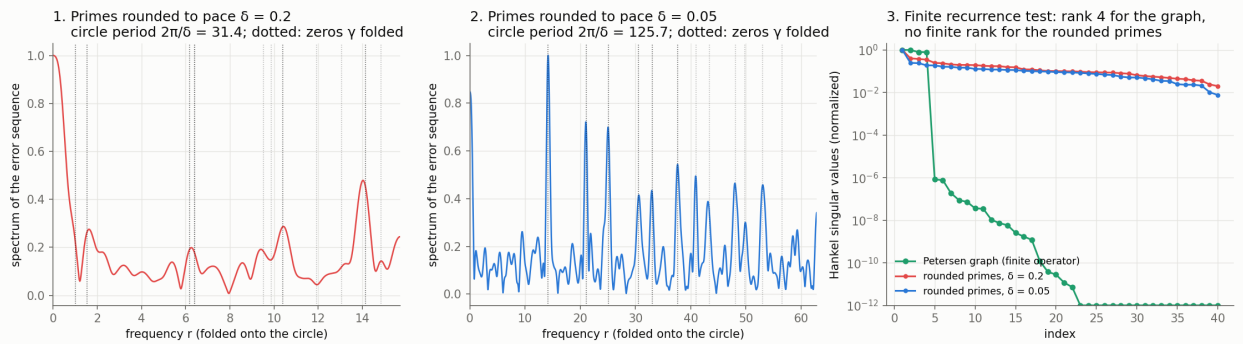


FIGURE 2. Primes rounded to multiples of  $\delta$ . (1), (2): spectrum of the error sequence  $E_\delta$ , with the zeros of  $\zeta$  folded modulo  $2\pi/\delta$  (dotted). (3): normalized Hankel singular values; rank 4 for the Petersen graph, no finite rank for the rounded primes.

**6.4. Three controls.** With the Davenport–Heilbronn function the controls cover each half of the structure of  $\zeta$  (Table 8). The function  $f$  has the functional equation and no Euler product. The

Beurling systems have the Euler product,  $\Lambda \geq 0$  and  $\log = \Lambda * 1$ ; those of [BDR] also have integers better than  $\sqrt{x}$  and every zero of  $\zeta$ , but no functional equation of the type of  $\zeta$  is known for them. Each has an off-line zero. A proof of RH must therefore use the functional equation and the Euler product together, and by the analysis of [CE11, Sect. 13.5] the regularity of the integers, measured by the size of their error term, cannot replace the functional equation. What  $\zeta$  has and the controls lack is that its integers are exactly  $1, 2, 3, \dots$ , whose sawtooth gives the functional equation by Poisson summation. The reflected cross-energy separates the two failures: it sees the off-line zero of a Beurling system from its integers alone, drifting with  $N$  because the system has no functional equation, while that of  $f$  is fixed [CE3]. Weil’s form is the classical exact form of the coupling [CE10, Sect. 3], [CE12]. Status: Table 8 summarizes the literature and the results above; the conclusion about proofs is heuristic.

|                                  | functional<br>equation | Euler product,<br>$\Lambda \geq 0$ , $\log = \Lambda * 1$ | integer error<br>$O(x^\beta)$ , $\beta < \frac{1}{2}$ | zeros of $\zeta$ on<br>the line | off-line<br>zero |
|----------------------------------|------------------------|-----------------------------------------------------------|-------------------------------------------------------|---------------------------------|------------------|
| Davenport–Heilbronn              | yes                    | no                                                        | —                                                     | no                              | yes              |
| Beurling [DMV],<br>Section 6.1   | none known             | yes                                                       | no                                                    | no                              | yes              |
| rounded primes,<br>Section 6.3   | none known             | yes                                                       | no                                                    | folded                          | not<br>known     |
| Beurling [BDR,<br>Thm. 1.3]      | none known             | yes                                                       | yes                                                   | yes (under<br>RH)               | yes              |
| Epstein $Z_Q$ [CE16,<br>Sect. 2] | yes                    | no ( $b_Q$ not $\geq 0$ )                                 | yes ( $\theta = \frac{1}{3}$ ,<br>lattice)            | no                              | yes              |
| $\zeta$                          | yes                    | yes                                                       | yes ( $\beta = 0$ )                                   | —                               | RH               |

TABLE 8. The controls and  $\zeta$ . The rows of [DMV], [BDR] and the Davenport–Heilbronn function are those of [CE11, Table 12]; the rounded primes and the Epstein function are added here.

The energy slope in a Beurling system. RH is equivalent to the positivity in  $\sigma > \frac{1}{2}$  of the energy slope  $S = \partial_\sigma \log |\xi| = G + \operatorname{Re} \zeta'/\zeta$  of [CE10, Sect. 3], where  $G(\sigma, t) = \operatorname{Re} \left( \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{s}{2} \right) \right)$ ; for the Davenport–Heilbronn function the slope fails only near its off-line zeros [CE10, Sect. 3]. A Beurling system has no Gamma factor; borrowing that of  $\zeta$ , put  $S_P = G + \operatorname{Re} \zeta'_P/\zeta_P$  for the continuous model of Section 6.1, in which  $-\zeta'_P/\zeta_P(s) = \frac{1}{s-1} - \frac{20^{\rho-s}}{s-\rho} - \frac{20^{\bar{\rho}-s}}{s-\bar{\rho}}$  with  $\rho = 0.75 + 20i$ . On a grid of  $100 \times 2400$  points in  $\frac{1}{2} < \sigma \leq 1$ ,  $10 \leq t \leq 30$ ,  $S_P \geq 0.05$  for  $|t - 20| > 4$ , and  $S_P < 0$  on a region around  $\rho$  in which it tends to  $-\infty$  at  $\rho$  (the minimum on the grid,  $-120$  at  $0.745 + 20.004i$ , is set by the mesh). Since the system has no functional equation there is no reflected zero  $1 - \bar{\rho}$ , and the region reaches the critical line, where the slope of  $\zeta$  vanishes identically: on  $\sigma = 0.505$ ,  $S_P < 0$  for  $17.2 \leq t \leq 22.8$ . The slope therefore fails for both controls exactly at their off-line zeros, although the Beurling system has the Euler product,  $\Lambda_P \geq 0$  and  $\log = \Lambda_P * 1$ . Status: numerical, for a model with a borrowed Gamma factor (`anc/beurling/beurling_margin.py`); the companion rows for  $\zeta$  and  $f$  are in [CE10, Sect. 3].

## 7. FUNCTION FIELDS AND GRAPHS

Over  $\mathbb{F}_q[T]$  RH is Weil’s theorem [We]. It is therefore the place to see which ingredient a proof uses and whether the  $C_2$  observables see it. The Euler–Caceres bound over  $\mathbb{F}_q[T]$ , with its exact factorization and the equivalence “bound  $\Leftrightarrow$  no zero in the disc  $\Leftrightarrow$  bounded normalized prime sums”, is in [CE11, Sect. 13.1]; here we use the same setting as a test bench for the cross-energy and for the Davenport–Heilbronn construction.

**7.1. Setting and dictionary.** Let  $q$  be an odd prime,  $D \in \mathbb{F}_q[T]$  monic and squarefree of odd degree  $d$ ,  $\omega$  a character of  $\mathbb{F}_q^\times$  of order  $r \in \{2, 4\}$ , and  $\chi(g) = \omega(\text{Res}(D, g))$ , a Dirichlet character modulo  $D$  nontrivial on constants. With  $u = q^{-s}$ ,

$$L(u, \chi) = \sum_{g \text{ monic}} \chi(g) u^{\deg g} = \sum_{k=0}^{\delta} A_k u^k, \quad \delta = d - 1,$$

because  $\sum_{\deg g=k} \chi(g) = 0$  for  $k \geq d$  [Ro, Ch. 4]. By Weil's theorem  $L(u, \chi) = \prod_{i=1}^{\delta} (1 - \alpha_i u)$  with  $|\alpha_i| = \sqrt{q}$ , and the functional equation reads  $A_{\delta-k} = \varepsilon q^{\delta/2-k} \overline{A_k}$  with  $|\varepsilon| = 1$ . For  $r = 2$ ,  $L$  is the numerator of the zeta function of the hyperelliptic curve  $y^2 = -D(x)$ , of genus  $g = \delta/2$ . The dictionary is:  $X_N(s)$  and  $X_n(u) = \sum_{k \leq n} A_k u^k$  (the degree plays the role of  $\log N$ ); the critical line and the circle  $|u| = q^{-1/2}$ ; the pole  $s = 1$  and the pole  $u = 1/q$  of  $\zeta_{\mathbb{F}_q[T]}(u) = 1/(1-qu)$ ; the asymptotic explicit formula and the exact finite one,  $\sum_{\deg g=m} \Lambda(g) \chi(g) = -\sum_i \alpha_i^m$ . We computed  $L$  for 45 characters by direct enumeration ( $q = 3, d = 7, 9, r = 2$ ;  $q = 5, d = 5, r = 4$ ); all satisfy  $|\alpha_i| = \sqrt{q}$  to  $10^{-14}$ , and for one quartic character the explicit formula was checked term by term for  $m \leq 4$  (`anc/anatomy/ff/`). The partial sums are exact,  $X_n = L$  for  $n \geq \delta$ : there is no tail and no Fresnel transition.

**7.2. The  $C_2$  criteria are restatements.** Put  $C_2(n, u) = |X_n(u)|^2 - |X_{n-1}(u)|^2 - |A_n u^n|^2 = 2 \operatorname{Re} (X_{n-1}(u) \overline{A_n u^n})$ .

**Proposition 8.** *Let  $L(u) = \sum_{k \leq \delta} A_k u^k$  with  $A_0 = 1$  and  $|A_\delta| = q^{\delta/2}$  (as for every  $L$  above). At every zero  $\rho$  of  $L$ ,  $C_2(\delta, \rho) = -2(q|\rho|^2)^\delta$ . Hence RH for  $L$  holds if and only if  $C_2(\delta, \rho) = -2$  at every zero.*

*Proof.*  $X_\delta(\rho) = L(\rho) = 0$  gives  $X_{\delta-1}(\rho) = -A_\delta \rho^\delta$ , so  $C_2(\delta, \rho) = 2 \operatorname{Re} (-A_\delta \rho^\delta \overline{A_\delta \rho^\delta}) = -2|A_\delta|^2 |\rho|^{2\delta} = -2q^\delta |\rho|^{2\delta}$ ; this equals  $-2$  if and only if  $|\rho| = q^{-1/2}$ .  $\square$

Status: proved. This is the analogue of the pointwise criterion  $0 < \lim_N C_2(N, \rho) < \infty$  of [CE1], and equally a restatement.

**Proposition 9** (the  $C_2$ -Jensen criterion as a class-number identity). *Let  $R = q^{-1/2}$  and  $\mu_{1/q}$  the harmonic measure of the circle  $|u| = R$  seen from  $u = 1/q$ . Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \log |L(Re^{i\theta})| d\theta = \sum_{|\rho| < R} \log \frac{R}{|\rho|} \geq 0,$$

$$\int \log |L| d\mu_{1/q} = \log |L(1/q)| + \sum_{|\rho| < R} \log \frac{|q^{-1} - \bar{\rho} q^{-1}|}{R |q^{-1} - \rho|},$$

*each term of both sums being positive, and both sums vanish if and only if RH holds for  $L$ . For  $r = 2$ ,  $L(1/q) = h q^{-g}$  with  $h = \#J(\mathbb{F}_q)$  the order of the Jacobian of the curve, and*

$$RH \iff \int \log |L| d\mu_{1/q} = \log \frac{h}{q^g}.$$

*Proof.* The first identity is Jensen's formula ( $L(0) = 1$ ), the second the Poisson-Jensen formula at  $u_0 = 1/q$ ,  $|u_0| < R$ , with the Blaschke factors  $\frac{R^2 - \bar{\rho} u_0}{R(u_0 - \rho)}$ , of modulus greater than 1 for  $|\rho| < R$ . By the functional equation the zeros are symmetric under  $\rho \mapsto 1/(q\bar{\rho})$ , so no zero inside the disc means all zeros on the circle. For  $r = 2$ ,  $L = P$  is the numerator of the zeta function of the curve,  $P(1) = h$  [Ro, Ch. 5], and the functional equation  $P(u) = q^g u^{2g} P(1/(qu))$  at  $u = 1/q$  gives  $P(1/q) = q^{-g} h$ .  $\square$

Status: proved. We verified the identity to  $10^{-5}$  for curves of genus 3 with  $h = 19, 27, 32, 46, 85$  (`anc/anatomy/ff/analysis.py`). The  $C_2$ -Jensen criterion of [CE10, Sect. 2] thus becomes an arithmetic identity, and is again a restatement of RH.

**Proposition 10** (partial Euler products). *Let  $E_m(u) = \prod_{\deg P \leq m} (1 - \chi(P)u^{\deg P})^{-1}$  over monic irreducible  $P$ . Then*

$$\log E_m(u) = \log L(u) + \sum_{n > m} \left( \sum_i \alpha_i^n \right) \frac{u^n}{n} + \sum_{\deg P \leq m} \sum_{k \deg P > m} \frac{\chi(P)^k u^{k \deg P}}{k}$$

near  $u = 0$ , and  $E_m \rightarrow L$  uniformly on compact subsets of  $|u| < q^{-1/2}$  if and only if  $L$  satisfies RH.

*Proof.* The identity compares coefficients: in degree  $n \leq m$  both  $\log E_m$  and  $\log L$  have the coefficient  $c_n/n$ ,  $c_n = \sum_{\deg g=n} \Lambda(g)\chi(g) = -\sum_i \alpha_i^n$ , and in degree  $n > m$ ,  $\log E_m$  keeps only the prime powers  $P^k$  with  $\deg P \leq m$ . If RH holds,  $L$  has no zeros in  $|u| < q^{-1/2}$ , so  $\log L$  is analytic there and the identity, which holds near  $u = 0$ , holds on the whole disc, both sides being analytic in it ( $E_m$  is analytic and zero-free in  $|u| < 1$ ). Fix  $\tau < 1$  and  $|u| \leq \tau/\sqrt{q}$ . With  $\delta$  the number of inverse roots, all of modulus  $\sqrt{q}$  under RH, the second term is bounded by  $\delta \sum_{n > m} (\sqrt{q}|u|)^n/n \leq \delta \tau^{m+1}/((m+1)(1-\tau))$ ; in the third, the number of  $P$  of degree  $j$  is at most  $q^j/j$  and  $k \geq 2$ , so  $q^j|u|^{kj}/j \leq \tau^{kj}/j$ , and the sum is at most  $\sum_{j \leq m} \frac{1}{j} \sum_{kj > m} \tau^{kj} \leq (1 + \log m)\tau^m/(1-\tau)$ . Both tend to 0 uniformly on  $|u| \leq \tau/\sqrt{q}$ , and so does  $E_m/L - 1 = \exp(\text{second} + \text{third}) - 1$ ; since  $L$  is bounded on the disc,  $E_m \rightarrow L$  uniformly there. Conversely, each  $E_m$  is analytic and zero-free in  $|u| < 1$ ; if  $E_m \rightarrow G$  uniformly on compacta of  $|u| < q^{-1/2}$ , then  $G = L$  (they agree for  $|u| < 1/q$ , where the Euler product converges absolutely), and by Hurwitz's theorem  $L$ , which is not identically zero, has no zeros in the disc; with the functional equation, all zeros lie on the circle.  $\square$

Status: proved. The modes are  $(\alpha_i u)^n$ , and over  $\mathbb{F}_q[T]$  the convergence of the partial Euler products in the half-plane  $\sigma > \frac{1}{2}$  is exactly Weil's theorem; over  $\mathbb{Q}$  the corresponding statement, with the pole correction, is the subject of [CE8].

**7.3. A Davenport–Heilbronn control over  $\mathbb{F}_q[T]$ .** For  $r = 4$ ,  $q = 5$ , put  $f(u) = L(u, \chi) + \varepsilon L(u, \bar{\chi})$ ,  $\varepsilon$  the root number of  $L(u, \chi)$ . From  $A_{\delta-k} = \varepsilon q^{\delta/2-k} \bar{A}_k$  and the corresponding relation for  $\bar{\chi}$  (root number  $\bar{\varepsilon}$ ) one gets  $f_{\delta-k} = q^{\delta/2-k} f_k$ :  $f$  has a functional equation of the type of  $L$ , with root number 1, but no Euler product. This is the exact analogue of the construction of Section 2.2. Of 12 random moduli of degree 5, the combination violates RH in 5 cases (Table 9). In each case the Jensen mean equals the sum over the zeros inside the circle, the Poisson–Jensen identity from the pole holds with a positive excess, and  $C_2(\delta, \rho) = -2(q|\rho|^2)^\delta$  for the normalized polynomial  $f/f(0)$ , as in Proposition 8. In case 1 an inverse root has modulus  $2.481\sqrt{5} = 5.55 > q$ , so  $f$  vanishes at a point with  $|u| < 1/q$ , where the Euler products of  $L(\chi)$  and  $L(\bar{\chi})$  converge absolutely; an Euler product cannot vanish there, and the Davenport–Heilbronn function over  $\mathbb{Q}$  likewise has zeros with  $\sigma > 1$  [DH]. The control has all the level-one structure: an exact functional equation, exact finite partial sums, and  $C_2$  and Jensen observables that detect its off-line zeros as over  $\mathbb{Q}$ . Status: numerical, exact up to  $10^{-12}$  (`anc/anatomy/ff/controls.py`, `analysis.py`).

**7.4. What the proof uses: the tower.** The elementary proof of Weil's theorem (Stepanov [St], Bombieri [Bo73]; see [IK, Ch. 11]) has three steps. (1) *Base change*: over  $\mathbb{F}_{q^n}$  the character  $\chi \circ N_{\mathbb{F}_{q^n}/\mathbb{F}_q}$  has  $L$ -function  $\prod_i (1 - \alpha_i^n u)$  (Hasse–Davenport), so one spectrum governs every level. (2) *A square-root bound at each level*: for a curve,  $\#C(\mathbb{F}_{q^n}) = q^n + 1 - \sum_i \alpha_i^n$  is a count of points, and an auxiliary polynomial vanishing to high order at the fixed points of Frobenius gives  $|\sum_i \alpha_i^n| \leq Cq^{n/2}$  for  $q^n$  large, with  $C$  independent of  $n$ . (3) *Amplification*: if  $|\sum_i \alpha_i^n| \leq Cq^{n/2}$  for all large  $n$ , the series  $\sum_n (\sum_i \alpha_i^n) u^n$  has radius of convergence  $1/\max |\alpha_i|$ , so  $\max |\alpha_i| \leq \sqrt{q}$ . In the language of [CE5],  $\sum_i \alpha_i^n$  is the prime sum at level  $n$ , and step (3) says that a bounded prime-sum energy

| case | $ \alpha /\sqrt{q}$ for the four inverse roots | $C_2(\delta, \rho)$ at the four zeros | Jensen mean |
|------|------------------------------------------------|---------------------------------------|-------------|
| 1    | 0.403, 1, 1, 2.481                             | -2873.7, -2, -2, -0.0014              | 0.909       |
| 2    | 0.518, 0.518, 1.932, 1.932                     | -388.0, -388.0, -0.010, -0.010        | 1.317       |
| 3-5  | 0.618, 1, 1, 1.618                             | -94.0, -2, -2, -0.043                 | 0.481       |

TABLE 9. The Davenport–Heilbronn analogue  $f = L(\chi) + \varepsilon L(\bar{\chi})$  over  $\mathbb{F}_5[T]$ ,  $\deg D = 5$ : the five cases (of twelve) that violate RH. The Jensen mean is  $\sum_{|\rho| < R} \log(R/|\rho|)$ , e.g.  $\log 2.481 = 0.909$ .

across the tower is equivalent to RH. For the twelve quartic  $L$ -functions,  $\max_{n \leq 40} |\sum_i \alpha_i^n|/q^{n/2} \leq 3.99 < \delta = 4$ ; for the control of case 1 the ratio is  $6 \cdot 10^{15}$  at  $n = 40$ .

The decisive test is step (1) for the control. At level  $n$  the natural Davenport–Heilbronn combination is  $f^{(n)} = L_n(\chi) + \varepsilon_n L_n(\bar{\chi})$ ,  $L_n$  the base change and  $\varepsilon_n$  its root number; its zeros are not the  $n$ -th powers of those of  $f$ . The number of the 12 moduli for which  $f^{(n)}$  satisfies RH is 7, 4, 2, 0 at the levels  $n = 1, 2, 3, 4$ , against 12 at every level for  $L_n(\chi)$ . No control satisfies RH at all four levels, and whether it does at one level says nothing about the next: the control has no tower, so step (3) has nothing to amplify. Status: numerical (`anc/anatomy/ff/lv12.py`, `analysis.py`).

**The lesson.** Over  $\mathbb{F}_q[T]$  the  $C_2$  observables at a single level are exact and finite, and still only restate RH, exactly as over  $\mathbb{Q}$ ; partial sums, pointwise limits, Jensen’s formula and harmonic measure are all of this kind. The proof lives in the tower: one spectrum seen at infinitely many levels, a nonnegative integer count at each level, and a geometric source (Frobenius, Riemann–Roch) for a square-root bound. The control shares the functional equation, the partial-sum structure and the level-one explicit formula, and fails exactly at step (1). Over  $\mathbb{Q}$  the Euler product and  $\Lambda \geq 0$  are present; what has no known counterpart is the tower, a family of levels at which the zeros are raised to powers with a count at each level that would be a prime sum. The test bench says that this ingredient is missing over  $\mathbb{Q}$ , not that it fails. The nearest object is the prime-sum energy across scales,  $\frac{1}{Y} \int_0^Y |\psi(e^y) - e^y|^2 e^{-y} dy$ , which under RH tends to  $\sum_\rho m_\rho^2/|\rho|^2$ , equal to  $C_B$  for simple zeros [Cr], [CE5, Sect. 2]: the scales  $e^y$  play the role of the levels  $q^n$ , and nothing plays the role of Frobenius in step (2). Status: heuristic.

**7.5. Amplification and parity.** For a curve  $C$  of genus  $g$  over  $\mathbb{F}_q$ ,  $N_k - q^k - 1 = -\sum_{i \leq 2g} \alpha_i^k$ , and a bound  $|N_k - q^k - 1| \leq Cq^{k/2}$  for all  $k$  forces  $|\alpha_i| = \sqrt{q}$ , because a root with  $|\alpha| > \sqrt{q}$  produces a deviation that grows geometrically in  $k$ . Figure 3a shows this for  $q = 101$ : the numerator  $1 - 20T + 101T^2$  of an elliptic curve stays within the Weil bound, while  $1 - 25T + 101T^2$ , whose counts  $N_k$  and numbers of closed points are nonnegative integers but which is not the numerator of a curve, leaves it by a factor  $4.5 \cdot 10^5$  at  $k = 20$ . Stepanov’s method [St, Bo73] proves the upper bound  $N_k \leq q^k + O(q^{k/2})$  with an auxiliary polynomial; the lower bound needs an auxiliary family. In the example the upper bound holds for every  $k$  while the analogue of RH fails: the large root is real, the analogue of a real zero. Deligne extended amplification to all varieties through tensor powers [De].

Over  $\mathbb{Q}$  the powers  $\alpha^k$  become  $x^\rho$ ,  $N_k - q^k - 1$  becomes  $x - \psi(x)$ , and one side suffices.

**Proposition 11** (one-sided bounds). *If  $\psi(x) \leq x + Cx^\theta$  for all large  $x$ , with  $0 < \theta < 1$ , then  $\zeta$  has no zeros with  $\beta > \theta$ . Hence RH holds if and only if  $\psi(x) \leq x + O(x^{1/2} \log^2 x)$ .*

*Proof.* The function  $g(x) = x + Cx^\theta - \psi(x)$  is nonnegative for large  $x$ , and for  $\sigma > 1$

$$\int_1^\infty g(x) x^{-s-1} dx = \frac{1}{s-1} + \frac{C}{s-\theta} + \frac{\zeta'(s)}{s\zeta(s)},$$

which continues analytically to every real  $s > \theta$ : the poles at  $s = 1$  cancel, and  $\zeta$  has no zeros in  $(0, 1)$ . By Landau’s theorem on Mellin transforms of nonnegative functions [La05], [In, Ch. V]

the integral converges for  $\sigma > \theta$ , so  $\zeta'/\zeta$  is analytic there. The converse direction is von Koch's estimate  $\psi(x) = x + O(x^{1/2} \log^2 x)$  under RH [In].  $\square$

Status: classical. RH is therefore equivalent to an upper bound for the primes, which is the form amplification would take over  $\mathbb{Q}$ . Upper bounds for primes are what sieves provide, and Selberg's  $\Lambda^2$  sieve is a positivity method: it minimizes  $\sum_n (\sum_{d|n, d \leq R} \lambda_d)^2$  with  $\lambda_1 = 1$ , as Stepanov's method minimizes the size of an auxiliary polynomial. On  $(10^8, 10^8 + y]$  with  $R = \sqrt{y}$  and optimal weights the bound exceeds the number of primes by the factors 3.07, 2.61, 2.24 for  $y = 10^4, 10^5, 10^6$  (Figure 3b). Splitting by Liouville's function gives  $859 + 832$ ,  $7159 + 6966$  and  $62341 + 59012$  for  $\lambda(n) = -1$  and  $+1$ , and the sieve data  $\#\{n \equiv 0 \pmod d : \lambda(n) = \pm 1\}$  agree within the statistical fluctuation for every  $d \leq R$ : the sieve cannot distinguish the half that contains the primes from the half that contains none. The sieve bound is about  $y/G(R)$  with  $G(R) \approx \frac{1}{2} \log y + 1.33$ , against  $y/\log x$  primes, so the ratio is  $2 (\log x / \log y) (1 + 2.665 / \log y)^{-1}$ : at  $y = 10^6$ ,  $2 \times 1.333 \times 0.838 = 2.24$ . The factor 2 is the parity barrier [FI10]; the factor  $\log x / \log y$  comes from the level of distribution, and  $2y/\log y$  is the Brun–Titchmarsh bound [MoV]. The barrier is not a numerical gap: with no zeros at all,  $\psi(x+y) - \psi(x) \approx y$  is correct to 0.1%, 0.3%, 1.5% for  $y = 10^6, 10^5, 10^4$ , and the 341 zeros below 600 do not change this materially (Figure 3c). What is missing is an inequality: the cancellation of  $\lambda$  in short intervals, proved for almost all intervals by Matomäki and Radziwiłł [MR] with bilinear information of type II, the input that also gave primes of the forms  $x^2 + y^4$  and  $x^3 + 2y^3$  [FI98, HB]. The cross term  $C_2(n, s)$  is bilinear only in the additive sense, with all coefficients 1; it does not split  $n$  into factors and does not supply type II input. Status: the reductions are classical; the computations are numerical (anc/parity/).

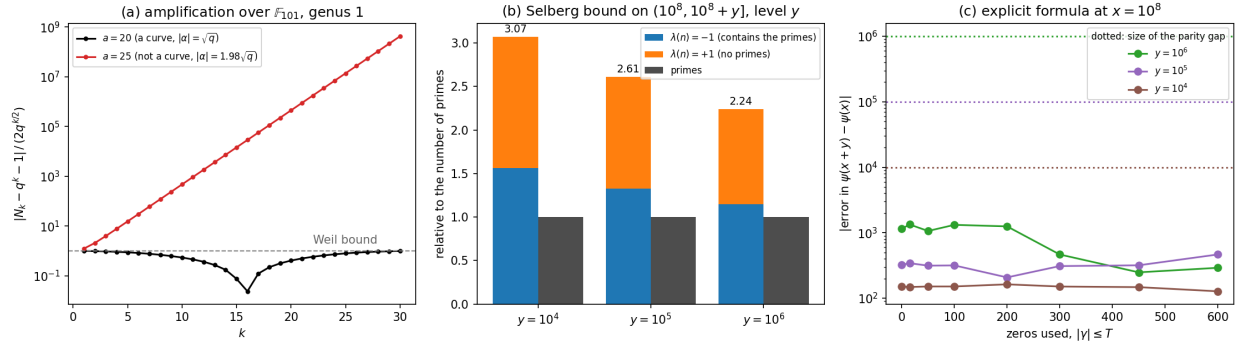


FIGURE 3. Amplification and parity. (a)  $|N_k - q^k - 1|/(2q^{k/2})$  for  $q = 101$  and the numerators  $1 - 20T + 101T^2$  (an elliptic curve) and  $1 - 25T + 101T^2$  (not a curve). (b) Selberg's  $\Lambda^2$  upper bound for the primes in  $(10^8, 10^8 + y]$ ,  $R = \sqrt{y}$ , split by  $\lambda(n) = \mp 1$ , relative to the number of primes. (c) Error of the explicit formula for  $\psi(x+y) - \psi(x)$  at  $x = 10^8$  with the zeros  $|\gamma| \leq T$ ; dotted lines mark  $y$ , the size of the parity gap in this normalization.

**7.6. The wave over an elliptic curve and the degree form.** Let  $E/\mathbb{F}_q$  be an elliptic curve,  $T = q^{-s}$ ,  $a = q + 1 - N_1$ , and  $\zeta_E(s) = Z(T) = \frac{1-aT+qT^2}{(1-T)(1-qT)} = \sum_{n \geq 0} b_n T^n$ , where  $b_0 = 1$  and  $b_n = N_1(q^n - 1)/(q - 1)$  for  $n \geq 1$  count the effective divisors of degree  $n$ . Define  $X_{N-1}(s) = \sum_{n < N} b_n T^n$  and the *auxiliary wave*  $W(N, s) = 2X_{N-1}(s)\overline{T^N}$ . It is not the cross-energy wave of [CE1] with the coefficient inserted, which would be  $2X_{N-1}(s)\overline{b_N T^N}$ ; since  $b_N > 0$ , the two differ by the factor  $b_N$ , and the cross-energy is  $C_2(N, s) = 2 \operatorname{Re} (X_{N-1}(s)\overline{b_N T^N}) = b_N \operatorname{Re} W(N, s)$ . We work with  $W$  because  $b_N$  grows like  $q^N$  and would hide the geometry. Only  $\operatorname{Re} W$  transfers exactly; the modulus identity below becomes the inequality  $|C_2(N, s) - b_N \operatorname{Re}(\text{centre})| \leq b_N |W(N, s) - \text{centre}|$ .

**Proposition 12** (the wave over a curve). *For  $N \geq 1$ ,*

$$W(N, s) = 2Z(T) \overline{T^N} - \frac{2N_1}{q-1} \left[ \frac{q^{N(1-2\sigma)}}{1-qT} - \frac{q^{-2\sigma N}}{1-T} \right].$$

*At a zero  $\rho = \beta + i\gamma$  of  $\zeta_E$ , with centre  $= -2N_1 q^{N(1-2\beta)} / ((q-1)(1-q^{1-\rho}))$ ,*

$$|W(N, \rho) - \text{centre}| = \frac{2N_1}{(q-1)|1-q^{-\rho}|} q^{-2\beta N} \quad \text{for every } N \geq 1.$$

*Proof.*  $Z(T) - X_{N-1} = \sum_{n \geq N} b_n T^n = \frac{N_1}{q-1} \left[ \frac{(qT)^N}{1-qT} - \frac{T^N}{1-T} \right]$  as an identity of rational functions; multiply by  $\overline{2T^N}$  and use  $|T|^{2N} = q^{-2\sigma N}$ . At a zero  $Z(T) = 0$ .  $\square$

Status: proved. This is the exact analogue, for the auxiliary wave, of the decomposition of the  $C_2$  wave in [CE1]: a centre from the pole at  $T = 1/q$ , a term rotating with  $\overline{T^N}$  and proportional to  $\zeta_E(s)$ , and a remainder; the collapse rate  $n^{-2\beta}$  holds here as an identity. The identity uses only the values  $b_n$ , which depend on  $a$  only through  $N_1$ , so it holds equally for the numerator  $1 - 25T + 101T^2$  of Section 7.5. Its zeros are real, with  $\beta = 0.3516$  and  $0.6484$ , and the slopes of  $\log_q |W - \text{centre}|$  are  $-0.7032$  and  $-1.2968$ , against  $-1.0000$  for curves with  $a = \pm 20$  (Figure 4b, 160-digit arithmetic). The wave detects the zeros and measures  $\beta$ , but does not distinguish a curve from a numerator that is not one.

The positivity that does is geometric. If  $F$  is the Frobenius endomorphism, with eigenvalues  $\alpha, \bar{\alpha}$  the roots of  $X^2 - aX + q$ , then for integers  $m, n$ ,  $\deg(m - nF) = (m - n\alpha)(m - n\bar{\alpha}) = m^2 - amn + qn^2$ , a nonnegative integer; the quadratic form is positive semidefinite if and only if  $a^2 \leq 4q$ , which is RH for  $E$  (Hasse). Among the 10100 curves  $y^2 = x^3 + Ax + B$  over  $\mathbb{F}_{101}$ ,  $|a| \leq 20 < 2\sqrt{101} = 20.10$  (Figure 4a). For  $a = 25$  the form is negative at 58 points with  $|m| \leq 25$ ,  $|n| \leq 2$  (Figure 4c), and its minimum on the line  $m = \frac{25}{2}n$  is  $-\frac{221}{4}n^2$ . The identity  $\deg(m - nF) = |m - n\alpha|^2$  shows where the conjugation comes from: the dual isogeny satisfies  $F\bar{F} = q$ , so  $\bar{\alpha} = q/\alpha$  is supplied by the geometry, and positivity of the degree (the case  $g = 1$  of the Rosati positivity and of the Castelnuovo–Severi inequality used by Weil) contains the product  $w\bar{w}$  that a criterion sensitive to  $\beta$  must contain [CE16, Sect. 2]. Over  $\mathbb{Q}$  the functional equation provides the inversion  $w \mapsto 1/w$ , the Dirichlet series provides the wave and its collapse, and no object with degree  $|w|^2$  is known. Status: classical identities; exact counts (`anc/curve/`).

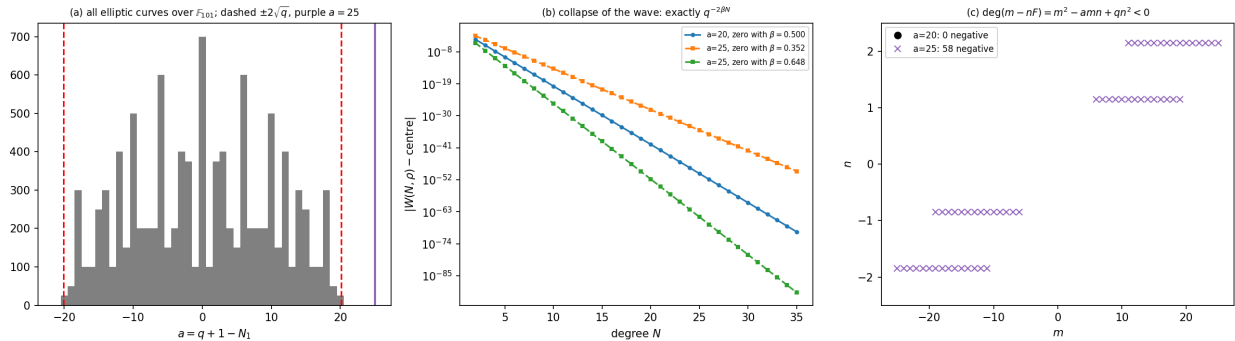


FIGURE 4. The wave over a curve,  $q = 101$ . (a) Traces  $a = q + 1 - N_1$  of the 10100 curves  $y^2 = x^3 + Ax + B$  over  $\mathbb{F}_{101}$ ; dashed  $\pm 2\sqrt{q}$ ; the value  $a = 25$  of the numerator that is not a curve. (b)  $|W(N, \rho) - \text{centre}|$  at the zeros of  $1 - 20T + 101T^2$  ( $\beta = \frac{1}{2}$ ) and of  $1 - 25T + 101T^2$  ( $\beta = 0.352, 0.648$ ), 160 digits. (c) Lattice points with  $m^2 - amn + qn^2 < 0$  for  $a = 20$  (none) and  $a = 25$ .

**7.7. Graphs: the Ihara zeta function.** Let  $G$  be a finite connected  $(q+1)$ -regular graph with adjacency matrix  $A$ ,  $|V|$  vertices and  $|E|$  edges. A prime of  $G$  is an equivalence class of closed non-backtracking tailless walks that are not powers of shorter ones, up to the starting point, and the Ihara zeta function [Ih, Ha, Te] is

$$Z_G(u) = \prod_P (1 - u^{\ell(P)})^{-1} = \frac{(1 - u^2)^{-(|E| - |V|)}}{\det(I - Au + qu^2 I)}.$$

It is an Euler product over the primes of the graph and the reciprocal of a polynomial. Each eigenvalue  $\lambda$  of  $A$  contributes the two roots  $\alpha, q/\alpha$  of  $x^2 - \lambda x + q$  (complex conjugate when  $|\lambda| < 2\sqrt{q}$ ) (the numerator of an elliptic curve with  $a = \lambda$ ). The trivial eigenvalues are  $q+1$ , which gives  $\alpha = q, 1$ , the analogue of the pole, and, when  $G$  is bipartite,  $-(q+1)$ , which gives  $\alpha = -q, -1$ ; all other eigenvalues are nontrivial. For a nontrivial eigenvalue,  $|\lambda| \leq 2\sqrt{q}$  holds exactly when  $|\alpha| = \sqrt{q}$ , that is, when the corresponding zeros of  $1/Z_G$  lie on  $|u| = q^{-1/2}$ : the graph RH is the Ramanujan property [LPS]. Since  $A$  is symmetric,  $\lambda$  is real, so  $\alpha$  lies either on the circle or on the real axis, and zeros can leave the circle only along the real axis, the analogue of the exceptional eigenvalues  $\lambda < \frac{1}{4}$  in Selberg's theory [Se56, Se65] (best bound  $\lambda_1 \geq 975/4096$  [Ki]). With  $B$  the  $2|E| \times 2|E|$  non-backtracking edge matrix [Ha],  $N_m = \text{tr } B^m$  counts closed non-backtracking tailless walks of length  $m$  with a marked starting edge, and

$$N_m = q^m + 1 + (|E| - |V|)(1 + (-1)^m) + \sum_{\lambda \neq q+1} (\alpha^m + \bar{\alpha}^m).$$

For a bipartite graph the eigenvalue  $-(q+1)$  contributes  $(-q)^m + (-1)^m$  to the last sum; it is trivial and is moved to the main term. The normalized error  $E_m = q^{-m/2}(N_m - q^m - 1 - (|E| - |V|)(1 + (-1)^m) - [G \text{ bipartite}]((-q)^m + (-1)^m))$ , the graph version of the prime signal of [CE5, Sect. 2], is bounded if and only if  $G$  is Ramanujan. All the graphs computed below contain odd cycles and are not bipartite. Status: classical.

**Petersen against a ring of four Petersen graphs** ( $q = 2$ , Figure 5). The Petersen graph has eigenvalues 3, 1 (five times) and  $-2$  (four times), so it is Ramanujan ( $2\sqrt{2} = 2.828$ ). Its prime cycles of lengths 5, 6, 8, 9, 10, 11, 12 number 24, 20, 30, 40, 120, 240, 270, with none of length 7, and  $|E_m| = 8.9, 2.1, 1.5, 4.9$  at  $m = 10, 20, 30, 40$ , below 18 up to  $m = 60$ . A ring of four Petersen graphs, obtained by cutting one edge in each copy and reconnecting the copies in a cycle, is still 3-regular, but  $\lambda_2 = 2.9089 > 2\sqrt{2}$  with multiplicity two: two double zeros leave the circle along the real axis to  $u = 0.558$  and  $u = 0.896$ ,  $\max |\alpha|/\sqrt{q} = 1.2687$ , and  $|E_m| = 41.7, 229.7, 2531, 27233$  at  $m = 10, 20, 30, 40$ , reaching  $3 \cdot 10^6$  at  $m = 60$ , following  $1.2687^m$ . The bottleneck of the ring, four weak links between well-connected pieces, pushes  $\lambda_2$  above the Ramanujan bound. Status: numerical, exact up to rounding (`anc/rounded_primes/ihara.py`, `figg.py`).

**Random cubic graph against a dumbbell, and a Davenport–Heilbronn combination over  $\mathbb{F}_3[T]$**  (Figure 6). Unlike the numerator  $1 - 25T + 101T^2$ , a graph that is not Ramanujan is a genuine object with an Euler product and a functional equation. For  $q = 2$ ,  $|V| = 40$ , a random cubic graph has  $\lambda_2 = 2.798$ ,  $\lambda_{\min} = -2.812$ , and a dumbbell made of two random cubic graphs on 20 vertices joined by exchanging one edge has  $\lambda_2 = 2.918 > 2.828$ , with a pair of real poles off the circle at  $\beta = 0.861$  and  $0.139$  in the variable  $u = q^{-s}$ . The counts  $N_m$  from the Hashimoto matrix agree with the formula above, and the normalized deviation stays bounded for the Ramanujan graph and grows geometrically for the dumbbell. The auxiliary wave (no coefficient, as in Section 7.6) built from the partial sums of  $\sum_n \text{tr}(A_n)u^n = \text{tr}((1 - u^2)(I - Au + qu^2 I)^{-1})$ , with  $\text{tr } A_n$  the number of closed non-backtracking walks of length  $n$ , has the centre  $-2(1 + q^{-1})q^N |u|^{2N}/(1 - qu)$  from the trivial pole, of constant modulus on  $\sigma = \frac{1}{2}$  as for  $\zeta$ . In 200-digit arithmetic  $|W - \text{centre}|$  at two points of the critical circle decays with slopes  $-0.50$  and  $-0.47$  in  $\log_q$  per step for the Ramanujan graph (the second affected by the beating of the 78 nontrivial frequencies) and  $-0.139 = \beta - 1$  for

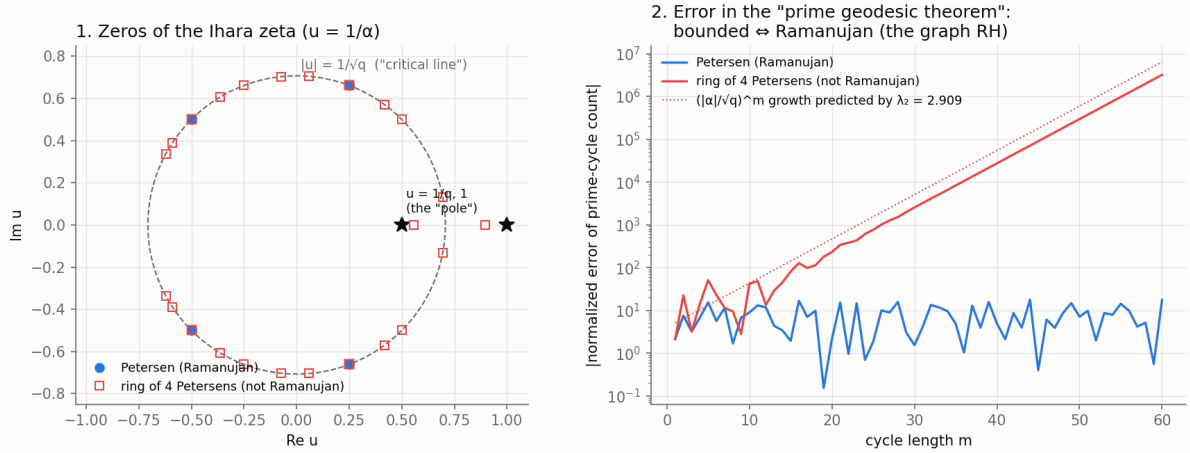


FIGURE 5. Left: zeros  $u = 1/\alpha$  of the Ihara zeta function for the Petersen graph (all on  $|u| = 1/\sqrt{2}$ ) and a ring of four Petersen graphs (two double zeros on the real axis); stars:  $u = 1/q$  and  $u = 1$ . Right: the normalized error  $|E_m|$  of the count of prime cycles, with the growth  $(|\alpha|/\sqrt{q})^m$  predicted by  $\lambda_2 = 2.909$ .

the dumbbell. The Toeplitz matrix  $[s_{j-k}]$  with  $s_m = q^{-m/2}(N_m - q^m - 1 - 2(|E| - |V|)[m \text{ even}])$  and  $s_0 = 2(|V| - 1)$ , the analogue of Weil's matrices [CE12], is positive definite up to size 40 for the Ramanujan graph and has a negative eigenvalue from size 13 for the dumbbell.

Over  $\mathbb{F}_q[T]$ , for  $\chi$  odd (nontrivial on  $\mathbb{F}_q^\times$ ) modulo a monic irreducible  $M$  of degree  $d$ ,  $L(u, \chi)$  is a polynomial of degree  $d - 1$  with zeros on  $|u| = q^{-1/2}$  [Ro] (an even nonprincipal  $\chi$  has in addition the trivial zero  $u = 1$ ) and  $L(u, \chi) = \epsilon(\chi)(\sqrt{q}u)^{d-1}L(1/(qu), \bar{\chi})$ ; the combination  $F = L(\chi) + \epsilon(\chi)L(\bar{\chi})$  satisfies  $F(u) = (\sqrt{q}u)^{d-1}F(1/(qu))$  and has no multiplicative coefficients. For  $q = 3$ ,  $M = T^7 + 2T^6 + T^5 + 1$  and  $\chi(g) = e^{2\pi i/2186}$  on a generator  $g$ , the six zeros of  $L$  lie on the circle, while  $F$  has a real pair off it,  $\beta = -0.120$  and  $1.120$ , outside the strip, as is possible without an Euler product. The prime sums  $\psi_n = \sum_{\deg Pj=n} \deg P \chi(P)^j$ , computed by enumerating irreducibles for  $n \leq 8$ , agree with  $-\sum \alpha^n$  to  $10^{-12}$  and satisfy  $|\psi_n| \leq 6q^{n/2}$ , while  $\sum \alpha^n$  for  $F$  grows geometrically. Here the wave carries no information:  $W(N, \rho) = 2L(\rho)\rho^N = 0$  for  $N \geq d$  at every zero of  $L$  and of  $F$ . The Toeplitz matrix of  $s_m = q^{-m/2} \sum \alpha^m$  is positive semidefinite for  $L$  (smallest eigenvalue 0 to  $10^{-13}$  from size 7, rank  $d - 1 = 6$ ) and indefinite for  $F$  from size 4 ( $-5 \cdot 10^8$  at size 30). For  $L$  the positivity is that of a Gram matrix: the  $\alpha_j/\sqrt{q}$  are the eigenvalues of a unitary operator, Frobenius on the first cohomology of the corresponding curve. Status: the identities are classical; the computations are exact or carried out to  $10^{-12}$  (anc/graphs.fq/).

**What decides.** In the three settings of this section the wave measures  $\beta$  when the Dirichlet series is infinite (curves, graphs) and is blind when it is finite ( $\mathbb{F}_q[T]$ ), and in none of them does it decide the analogue of RH. What decides it is in each case Weil's quadratic positivity, coming from a unitary or self-adjoint operator: the dual isogeny, the symmetric adjacency matrix together with a bound on its spectrum, Frobenius on  $H^1$ . For graphs a self-adjoint operator alone does not suffice, since the real axis remains available to the zeros; over  $\mathbb{Q}$  that alternative is closed ( $\zeta(\sigma) < 0$  on  $(0, 1)$ ), so a Hilbert–Pólya operator would suffice for  $\zeta$ . The explicit Ramanujan graphs of [LPS] obtain  $|\lambda| \leq 2\sqrt{q}$  from the Ramanujan–Petersson conjecture, proved by Deligne, that is, from geometry. Status: heuristic.

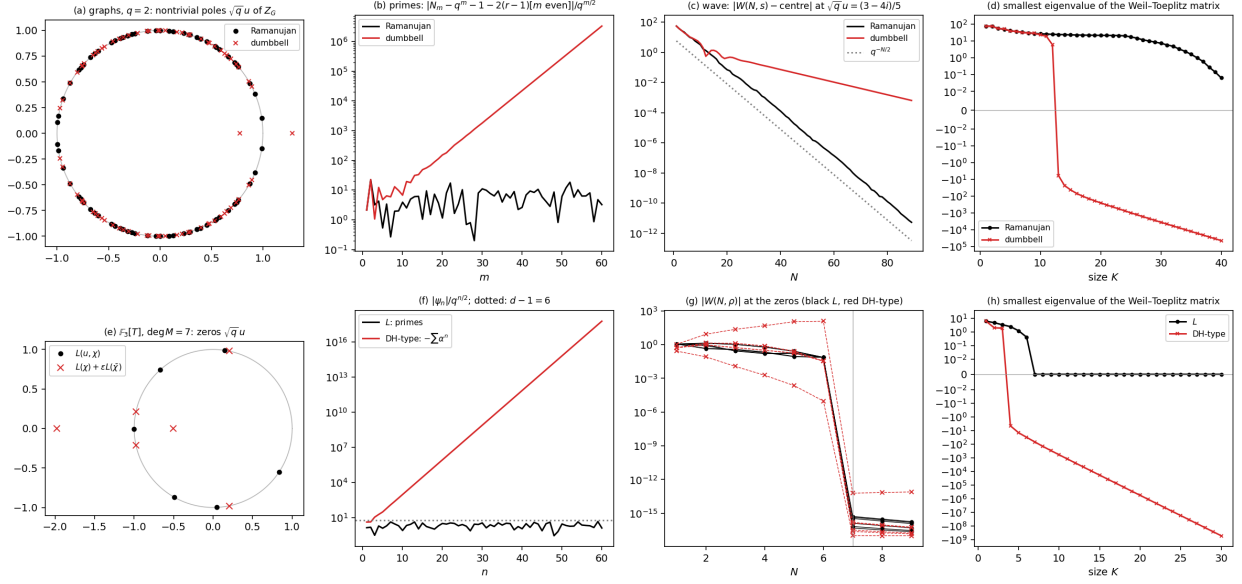


FIGURE 6. Graphs and  $\mathbb{F}_q[T]$ . Top, cubic graphs on 40 vertices ( $q = 2$ ), a Ramanujan graph and a dumbbell with  $\lambda_2 = 2.918$ : (a) nontrivial poles  $\sqrt{q}u$  of  $Z_G$ ; (b) normalized deviation of the counts  $N_m$ ; (c) distance of the wave to its centre at  $\sqrt{q}u = (3 - 4i)/5$ ; (d) smallest eigenvalue of the Toeplitz matrix built from the  $N_m$ . Bottom,  $q = 3$ ,  $\deg M = 7$ : (e) zeros  $\sqrt{q}u$  of  $L(u, \chi)$  and of the Davenport–Heilbron-type combination; (f)  $|\psi_n|/q^{n/2}$ ; (g)  $|W(N, \rho)|$  at the zeros, 0 for  $N \geq 7$ ; (h) smallest eigenvalue of the Toeplitz matrix.

## 8. STRUCTURAL OBSTRUCTIONS

The structural limits of the cross-energy (the Epstein obstruction, identities that do not see  $s$ , positivity on the wrong side of the Bochner duality, the pole barrier and its twists, scale recursions) are proved in [CE16, Sect. 2]. Here we recall the blindness of the Euler state and apply the filters to the candidates of the series. The three filters used throughout the series to sort candidate rigidity statements (the control filter F1, the pole filter F2 and the non-tautology filter F3) are defined in [CE2, Def. 2.22], and Landau’s positivity–pole dichotomy, which says that a Dirichlet series with nonnegative coefficients is singular at its real abscissa of convergence, is [CE2, Prop. 2.23]. A *rigidity statement* is a property  $\mathcal{S}(F)$  of the partial-sum trajectories of  $F$  such that  $\mathcal{S}(\zeta) \Rightarrow \text{RH}$ . The only positivity route through F2 is Landau’s: construct  $a_n \geq 0$  whose Dirichlet series is known to be analytic on the real segment  $(\frac{1}{2}, 1]$  but has a singularity at every off-line zero of  $\zeta$  (as the proof of Proposition 11 does for  $x + Cx^\theta - \psi(x)$ , which is nonnegative only under the hypothesis). No unconditional construction of this kind is known. Here we test the candidate statements of the series against the filters (Table 10).

Status of the classification: heuristic. A candidate passing all three filters would be a new mechanism, and would presumably extend to Euler-product  $L$ -functions, that is, to the generalized Riemann hypothesis.

**8.1. The Euler state.** The Euler product can be read as a quantum state: for  $\sigma > \frac{1}{2}$ ,  $\psi_s = \zeta(2\sigma)^{-1/2} \sum_n n^{-s} |n\rangle \in \ell^2(\mathbb{N})$ , and  $\psi_s = e^{-itL} \psi_\sigma$  with  $L$  the multiplication by  $\log n$ . By unique factorization  $e^{-itL}$  is a tensor product of diagonal unitaries over the primes, so every coherence measure in the basis  $|n\rangle$  and every entanglement entropy for a bipartition of the primes takes the same value at every  $t$ , also for the truncations to  $n \leq N$ ; only overlaps with vectors that are not

| candidate                                                                            | F1     | F2     | F3     | status                            |
|--------------------------------------------------------------------------------------|--------|--------|--------|-----------------------------------|
| non-collapse $\liminf_N C_2(N, \rho) > 0$ at every zero [CE16, (2.1)]                | fails  | —      | fails  | equivalent to RH; false for $Z_Q$ |
| reflected rigidity of the pair $\rho, 1 - \bar{\rho}$ (cf. [CE3])                    | fails  | —      | fails  | equivalent to RH                  |
| monotonicity and Fresnel law at zeros [CE2]                                          | fails  | —      | passes | true, blind off the line          |
| 3-4-1 positivity at $\sigma = \frac{1}{2}$ [CE16, Sect. 2]                           | passes | fails  | —      | vacuous (pole barrier)            |
| arithmetic energy law of $C_2^\Lambda$ [CE5, Sect. 2]                                | passes | passes | fails  | equivalent to RH                  |
| pole-free difference energy                                                          | passes | passes | fails  | equivalent to RH                  |
| $ \sum_{n \leq N} (1 - \Lambda(n))n^{-s} ^2$ bounded for $\sigma > \frac{1}{2}$ [C3] |        |        |        |                                   |

TABLE 10. Candidate rigidity statements of the series against the filters of [CE2, Def. 2.22].

diagonal in the multiplicative structure depend on  $t$ , and the (formal, since the alternating vector is not in  $\ell^2$ ; rigorous for the truncations) overlap with the alternating vector is  $(1 - 2^{1-s})\zeta(s)$  up to normalization, so it vanishes at the zeros. This is [CE8, Prop. 15], where it is proved together with the coherence  $C(\sigma) = \log \zeta(2\sigma) - 2\sigma \zeta'(2\sigma)$  and the numerical checks. In the language of this section the intrinsic properties of the Euler state depend on  $\sigma$  alone and do not see the zeros; this is the quantum-information form of the convolution identities and of the Bochner duality of [CE16, Sect. 2]. Status: proved in [CE8].

## 9. DISCUSSION AND OPEN PROBLEMS

The controls isolate one mechanism and one missing ingredient. The off-line zeros of the Davenport–Heilbronn function are produced by two Euler products of ordinary size meeting in phase, and this interference shows up as a linearly growing off-line mass, a floor of the mollified energy and growing inverse coefficients; the window inequality ties the first two together rigorously on short windows (Table 7). For  $\zeta$  there is a single Euler product,  $\Lambda \geq 0$  and  $|\mu| \leq 1$ , and all three measurements vanish or decay. The statement suggested by these experiments, that a single degree-one Euler product cannot cancel itself off the critical line, is RH itself, in the spirit of Selberg’s conjecture for primitive functions [Sel92]. In the language of Section 3, rows 7 and 9 are satisfied at off-line points of  $f$ ; what  $f$  lacks is the Euler product and the positivity  $\Lambda \geq 0$  behind row 8 (row 8 itself, a pole of the logarithmic derivative, holds for  $f$  too), since its zeros are cancellations between two Euler products. A proof along this route would have to show quantitatively that one Euler product cannot produce the Riemann–Siegel cancellation off the line.

The Beurling controls show the converse: an Euler product,  $\Lambda \geq 0$  and integers better than  $\sqrt{x}$  do not exclude off-line zeros, and Chebyshev’s identity at a zero only restates the prime error [CE11]. With the Davenport–Heilbronn function they leave the coupling of the functional equation, coming from the exact integers  $1, 2, 3, \dots$ , with the Euler product. The function-field and graph analogues show where such a coupling has been found when RH is known: a positivity carried by an operator (dual isogeny, Frobenius on  $H^1$ , a self-adjoint adjacency matrix with a spectral bound), and, for curves, the tower of base changes that turns square-root bounds at each level into RH. The cross-energy observables, at one level, only restate RH there as over  $\mathbb{Q}$ .

### Open problems.

- (1) *One Euler product.* Show that for  $\zeta$  the Riemann–Siegel cancellation of Proposition 1(7) at a point  $\beta + i\gamma$  with  $\beta > \frac{1}{2}$  is incompatible with the complete multiplicativity of the coefficients, in a form that fails for  $f$  (equivalent to RH; the percentile data of Table 4 indicate what such a statement must exclude).
- (2) *The floor.* Explain the floor of the mollified energy of  $f$  (Table 6) quantitatively; Corollary 4 gives only  $7.6 \cdot 10^{-4}$  on a window of height 300.

- (3) *A Davenport–Heilbronn tower.* Is there a family of combinations  $f^{(n)}$  over  $\mathbb{F}_q[T]$ , compatible with base change, that satisfies RH at every level without being an  $L$ -function? The data (7, 4, 2, 0 of 12) suggest not.
- (4) *Integers better than  $x^{2/5}$ .* Under RH, is there a Beurling system with  $N(x) = cx + O(x^\beta)$ ,  $\beta \leq \frac{2}{5}$ , and a zero off the line? This is [CE11, Open problem 5], and by Table 8 it is the sharpest remaining Beurling control.
- (5) *A provable positivity from the primes.* Construct, from the ordinary primes, a quadratic form whose positivity can be proved and which implies  $\text{Re } \rho = \frac{1}{2}$ , as the degree form  $m^2 - amn + qn^2$  does for an elliptic curve. The controls of this paper say what it must not be: a function that only measures where the zeros are (every observable of Sections 3–7 does that), or a positivity that holds for  $f$  or for a Beurling system. Weil’s form is such a quadratic form; what is missing is a proof of its positivity [CE12].
- (6) *A candidate through the filters.* Find a rigidity statement that passes F1–F3 of [CE2, Def. 2.22], or a nonnegative-coefficient series analytic on  $(\frac{1}{2}, 1]$  with a singularity at every off-line zero.

We do not prove the Riemann hypothesis.

#### DATA AND CODE

The ancillary Python scripts (NumPy, SciPy, Matplotlib; mpmath where more digits are stated) reproduce the tables and figures; `anc/README.txt` lists, for each script, the section, table or figure it reproduces and its output. The computations are in double precision unless stated otherwise, without interval arithmetic; they illustrate the statements and are not used in their proofs. The zeros of  $\zeta$  with  $0 < \gamma < 600$  are in `zeros600.npz`; the off-line zeros of  $f$  in `anc/anatomy/offline.npy`.

#### USE OF AI ASSISTANCE

The cross-energy and the research programme of this series are the author’s. Claude (Anthropic) was used as a research and writing aid: to derive and check proofs, to write and run the Python computations, to test statements against the Davenport–Heilbronn function and other controls, to search the literature, and to draft and edit text and figures. Several earlier claims were found to be incorrect in this process and are corrected or excluded (for this paper, the attribution of the mollification floor of  $f$  to Corollary 4). The author checked all results and is responsible for the content.

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