

# An Engineer's Approach To The Riemann Hypothesis And Why It Is True

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Abstract: Prime numbers are the atoms of mathematics and mathematics is needed to make sense of the real world. Finding the Prime number structure and eventually being able to crack their code is the ultimate goal in what is called Number Theory. From the evolution of species to cryptography, Nature finds help in Prime numbers.

One of the most important advances in the study of Prime numbers was the paper by Bernhard Riemann in November 1859 called "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (On the number of primes less than a given quantity).

In that paper, Riemann gave a formula for the number of primes less than  $x$  in terms the integral of  $1/\log(x)$  and the roots (zeros) of the zeta function defined by:

$$[\text{RZF}] \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Where  $\zeta(z)$  is a function of a complex variable  $z$  that analytically continues the Dirichlet series.

Riemann also formulated a conjecture about the location of the zeros of RZF, which fall into two classes: the "trivial zeros"  $-2, -4, -6$ , etc., and those whose real part lies between 0 and 1. Riemann's conjecture Riemann hypothesis [RH] was formulated as this:

[RH] The real part of every nontrivial zero  $z^*$  of the RZF is  $1/2$ .

Proving the RH is, as of today, one of the most important problems in mathematics. In this paper we will provide a proof of the RH. The proof of the RH will be built following these five parts:

- PART 1: Description of the RZF  $\zeta(z)$
- PART 2: The C-transformation
- PART 3: Application of the C-transformation to  $f(z) = \frac{1}{x^z}$  in  $\text{Re}(z) \geq 0$  to obtain  $\zeta(z) = X(z) - Y(z)$  (for)
- PART 4:
  - o Analysis of the values of  $z$  such that  $X(z) = Y(z)$ , and  $|X(z)| = |Y(z)|$ , that equates to  $\zeta(z) = 0$
  - o Proof that  $|X(z)| = |Y(z)|$  only if  $\text{Re}(z) = 1/2$
  - o Conclude that  $\zeta(z) = 0$  only if  $\text{Re}(z) = 1/2$  for  $\text{Re}(z) \geq 0$
- PART 5: We will also prove that all non-trivial zeros of  $\zeta(z)$  in the critical line of the form  $z = 1/2 + \beta i$  are not distributed randomly. There is a relationship between the values of those zeros and the Harmonic function that leads to an algebraic relationship between any two zeros.

We will use mathematical and computational methods available to Engineers.

### Nomenclature and conventions

- $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$  is the Riemann Zeta Function (RZF).
- $z^*$  is any nontrivial zero (NTZ) of the RZF verifying that  $\zeta(z^*) = 0$ .
- $\beta^*(n)$  is the  $n^{\text{th}}$  zero of the Riemann function in the critical line  $\text{Re}(z)=1/2$  in  $\mathcal{C}$
- $\alpha = \text{Re}(z)$  is the real part of  $z$
- $\beta = \text{Im}(z)$  is the imaginary part of  $z$
- If  $z = \alpha + i\beta$ ,  $\text{Modulus}(z) = |z| = \sqrt{\alpha^2 + \beta^2}$  and  $\text{SquareAbsolute}(z) = |z|^2$

## **PART 1:**

### **The Riemann Zeta function $\zeta(s)$ [RZF]**

1.  $\zeta(s)$  in  $R$

As defined in literature (Sondow et al, Weisstein, Edwards, Jekel)

1.1.  $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$  converges for  $s \neq 1$

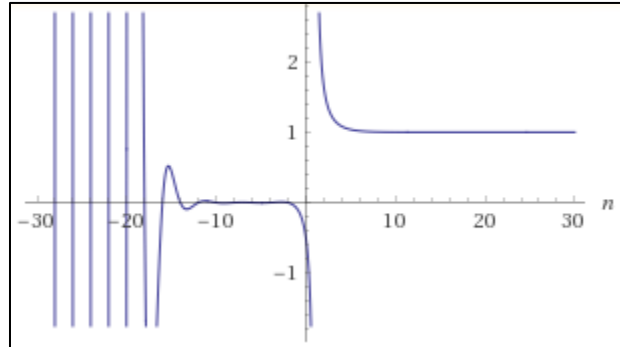


Figure. 1. Riemann Zeta function in  $R$

1.2. Euler Product Formula that links  $\zeta(s)$  with the distribution of prime numbers

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \prod_{p=\text{prime}} \frac{1}{1 - p^{-s}} \quad [1]$$

Example for  $k=2$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{1}{1 - 2^{-2}} \times \frac{1}{1 - 3^{-2}} \times \frac{1}{1 - 5^{-2}} \times \frac{1}{1 - 7^{-2}} \times \dots$$

1.3. Integral definition of  $\zeta(s)$ :

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{1}{e^x - 1} x^s \frac{dx}{x}$$

Where  $\Gamma(s)$ , is the Gamma function

1.4. Analytical continuation of  $\zeta(s)$  for :

$\text{Re}(s) > 0$ : [Dirichlet]

$$\zeta(s) = \frac{1}{s-1} \sum_{k=1}^{\infty} \left( \frac{n}{(n+1)^s} - \frac{n-s}{n^s} \right)$$

$0 < \text{Re}(s) < 1$ :

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}$$

$-k < \text{Re}(s)$  [Kopp, Konrad. 1945]:

$$\zeta(s) = \frac{1}{s-1} \sum_{k=1}^{\infty} \frac{k(k+1)}{2} \left( \frac{2k+3+s}{(k+1)^{s+2}} - \frac{2k-1-s}{k^{s+2}} \right)$$

1.5. Laurent series of  $\zeta(s)$ :

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k}{k!} (s-1)^k$$

where  $\gamma_n$  are the Stieltjes constants.

1.6. Hurwitz function  $\zeta(k, z)$ :

$$\zeta(k, z) = \sum_{j=0}^{\infty} (j+z)^{-k} = \sum_{j=z}^{\infty} j^{-k} \quad \text{converges for } k > 1$$

1.7. Generalized Harmonic Function  $H_n^{(k)}$ :

$$H_n^{(k)} = \sum_{j=1}^n j^{-k} = \left( \frac{1}{1^k} + \frac{1}{2^k} + \dots + \frac{1}{n^k} \right) \quad \text{converges for } k > 1$$

1.8.  $\zeta(s)$  converges for  $s > 1$  to the following values (Sloane):

s	$\zeta(s)$	Known $\zeta(s)$ representations over $\pi$
2	1.6449179	$\pi^2/6$
4	1.0823232	$\pi^4/90$
6	1.0173431	$\pi^6/945$
8	1.0040774	$\pi^8/9450$

*Table 1. Values of  $\zeta(s)$*

1.9. An approximation for the values of  $\zeta(s)$  for  $s > 1$  in  $R$

One can calculate that:

$$\lim_{s \rightarrow \infty} \left( \frac{\zeta(s)}{\zeta(s)+1} \right)^{\frac{1}{s}} = 1$$

And:

$$\lim_{s \rightarrow \infty} \left( \frac{\zeta(s)}{\zeta(s)-1} \right)^{\frac{1}{s}} = 2$$

Based on this expression, for  $s$  sufficiently large, one can represent  $\zeta(s)$  as a multiple of  $\pi^s$ :

$$\zeta(s) = \frac{\pi^s}{K_s} \quad \text{with } K_s = (2^s - 1) * \frac{\pi^s}{2^s}$$

with a very good approximation given by:

$$K_s^* = \text{int} \left( (2^s - 1) * \frac{\pi^s}{2^s} \right) - 1 \quad \text{where } \text{int}(k) \text{ is the integer part of } k. \quad [2]$$

The error between the  $K_s^*$  calculated and  $K_s$  actual is very small for  $s > 4$ .

Some calculated values of  $K_s^*$  calculated and  $K_s$  actual:

s	Calculated	Actual
2	6.0	6.0
3	26.0	25.8
4	90.0	90.0
5	295.0	295.1
6	945.0	945.0
7	2,995.0	2,995.3
8	9,450.0	9,450.0
9	29,749.0	29,749.4
10	93,555.0	93,555.0
11	294,059.0	294,058.7

Table 2. Values of  $K_s^*$  calculated and  $K_s$  actual

One can use [2] to propose the following approximation for  $\zeta(s)$ :

$$CZ(s) = \frac{1}{1 - \pi^{-s} - 2^{-s}} \quad [3]$$

	s=3	s=4	s=10	s=14
$\zeta(s)$	1.20206	1.0823	1.000994	1.0000612
CZ(s)	1.18659	1.0784	1.000988	1.0000611

Table 3. Comparing  $\zeta(s)$  and CZ(s)

Graphically:

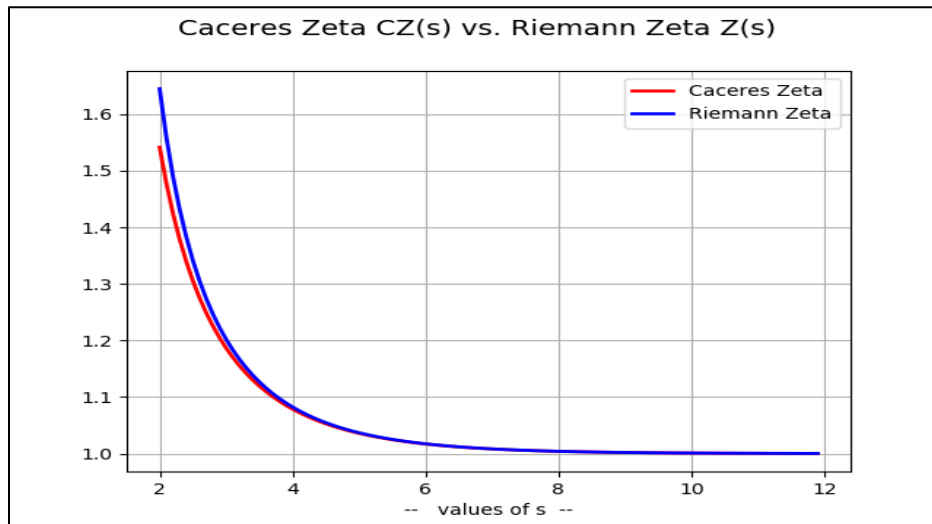


Figure. 2. Caceres' approximation for the Riemann Zeta function in R

## 1.10. Zeros of $\zeta(z)$ (Weisstein)

### 1.10.1. $Re(z) > 1$

There are no zeros of  $\zeta(s)$ .  $\zeta(s)$  converges for  $s > 1$ .

### 1.10.2. $z = 1$

$\zeta(1)$  is the Harmonic function and diverges, and:

$$\lim_{s \rightarrow 1} (\zeta(s) - \frac{1}{1-s}) = \gamma \text{ (the Euler-Mascheroni constant)}$$

### 1.10.3. $Re(z) < 1$

$\zeta(z)$  has trivial zeros at  $z = -2k$ , with  $k$  natural numbers

$\zeta(z)$  has nontrivial zeros at  $z = \frac{1}{2} \pm \beta i$  at the critical line  $Re(z) = \frac{1}{2}$  (Brentt)

The Riemann Hypothesis [RH] establishes that all nontrivial zeros of zeta have  $Re(z) = \frac{1}{2}$

List of imaginary part of the first nontrivial zeros of zeta (Odlyzko):

$$\begin{aligned} \beta(1) &= 14.134725142 \\ \beta(3) &= 25.010857580 \\ \beta(4) &= 30.424876126 \\ \beta(6) &= 37.586178159 \\ \beta(7) &= 40.918719012 \\ \beta(9) &= 48.005150881 \\ \beta(10) &= 49.773832478 \end{aligned}$$

Representation of  $|\zeta(z)|$ :

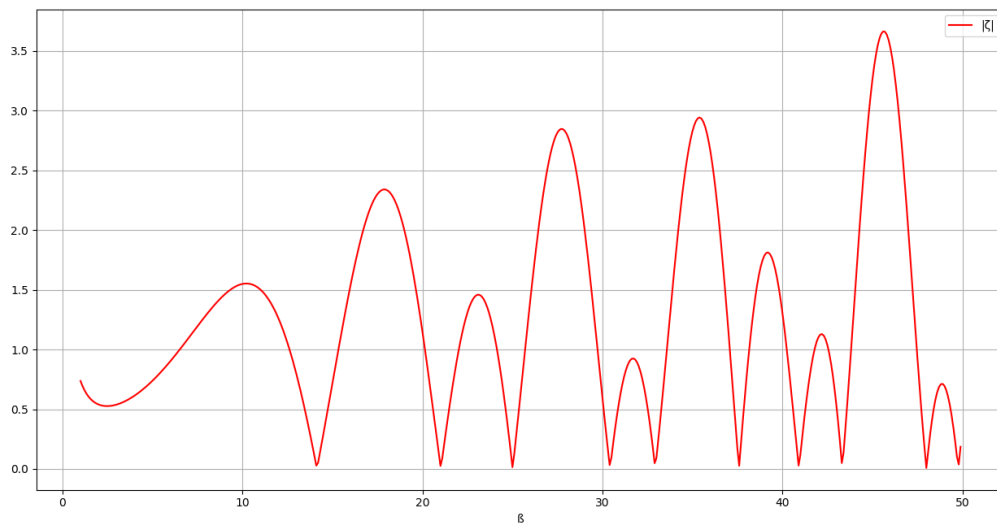


Figure. 3. Modulus of the Riemann Zeta function  $|\zeta(z)|$

**PART 2:**  
**The C-Transformation**



1. C-Transformation of  $f(x)$

Let's define the C-transformation of an integrable function  $f(x)$  by:

$$C_n\{f(x)\} = \sum_{k=1}^n f(k) - \int f(n) \, dn \quad [4]$$

And the C-values is the limit, if it exists, of the C-transformation when  $n \rightarrow \infty$ :

$$C\{f(x)\} = \lim_{n \rightarrow \infty} C_n\{f(x)\} \quad [5]$$

1.1. C-Transformation of  $f(x) = \frac{1}{x}$  for  $x \in R$ :

$$C_n\left\{\frac{1}{x}\right\} = \sum_{k=1}^n \frac{1}{k} - \int \frac{dn}{n}$$

And the C-Value of  $f(x) = \frac{1}{x}$  is  $\gamma = 0.5772$  (Euler-Mascheroni constant)

$$C\left\{\frac{1}{x}\right\} = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln(n) \right) = \gamma$$

1.2. C-Transformation of  $f(x) = \frac{\ln(x)^m}{x}$  for  $x \in R, m \in Z$ :

$$C_n\left\{\frac{\ln(x)^m}{x}\right\} = \sum_{k=1}^n \frac{\ln(k)^m}{k} - \int \frac{\ln(n)^m \, dn}{n}$$

And the C-Value of  $f(x) = \frac{\ln(x)^m}{x}$  are the Stieltjes constants that occur in the Laurent series expansion of the Riemann zeta function:

$$C\left\{\frac{\ln(x)^m}{x}\right\} = \lim_{m \rightarrow \infty} \left( \sum_{k=1}^m \frac{(\ln k)^n}{k} - \frac{(\ln n)^{m+1}}{m+1} \right) = \gamma_m$$

<b>m</b>	<b>approximate value of <math>\gamma_m</math></b>
0	+0.57721566490153286060651209
1	-0.07281584548367672486058637
2	-0.00969036319287231848453038
3	+0.00205383442030334586616004

1.3. C-Transformation of  $f(x) = m$ , for  $m \in R$  constant:

$$C_n\{m\} = \sum_{k=1}^n m - \int m \, dn$$

$$C_n\{m\} = m * n - m * n = 0$$

and the C-values of  $f(x) = m$  constant is:

$$C\{m\} = 0$$

1.4. C-Transformation of  $f(x) = \sin(x)$  for  $x \in R$ :

$$C_n\{\sin(x)\} = \sum_{k=1}^n \sin(k) - \int \sin(n) dn$$

$$C_n\{\sin(x)\} = \frac{1}{2 \left( \sin(n) - \cot\left(\frac{1}{2}\right) \cos(n) + \cot\left(\frac{1}{2}\right) + \cos(n) \right)}$$

And the C-values of  $f(x) = \sin(x)$  are in the interval:

$$C\{\sin(x)\} \in \left[ \frac{1}{2} \left( 2 \cot\left(\frac{1}{2}\right) - 3 \right), \frac{3}{2} \right]$$

One can also calculate that:

$$C\{\cos(x)\} \in \left[ \frac{1}{2} \left( \cot\left(\frac{1}{2}\right) - 4 \right), \frac{1}{2} \left( 2 - \cot\left(\frac{1}{2}\right) \right) \right]$$

1.5. C-Transformation of  $f(x) = e^{-x}$  for  $x \in R$ :

$$C_n\{e^{-x}\} = \sum_{k=1}^n e^{-k} - \int e^{-n} dn$$

$$C_n\{\sin(x)\} = \sum_{k=1}^n e^{-k} + \frac{e^{-n}}{n}$$

And the C-values of  $f(x) = e^{-x}$  are:

$$C\{e^{-x}\} = \frac{1}{e-1}$$

1.6. C-Transformation of  $f(x) = x^{-s}$  for  $x, s \in R, s > 1$ :

$$C_n\left\{\frac{1}{x^s}\right\} = \sum_{k=1}^n \frac{1}{k^s} - \int \frac{dn}{n^s}$$

$$C_n\left\{\frac{1}{x^s}\right\} = \sum_{k=1}^n \frac{1}{k^s} - \frac{n^{1-s}}{1-s}$$

and the C-value of  $f(x) = \frac{1}{x^s}$  is the Riemann Zeta function for  $s > 1$ :

$$C\left\{\frac{1}{x^s}\right\} = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k^s} - \frac{n^{1-s}}{1-s} \right) = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k^s} \right) - \lim_{n \rightarrow \infty} \left( \frac{n^{1-s}}{1-s} \right) = \zeta(s)$$

1.7. C-Transformation of  $f(z) = \frac{1}{x^z}$  for  $z \in C, Re(z) \geq 0, z \neq 1$

$$C_n\left\{\frac{1}{x^z}\right\} = \sum_{k=1}^n \frac{1}{k^z} - \int \frac{dn}{n^z}$$

One can use Euler's identity:

$$e^x = \cos(x) + i * \sin(x)$$

To calculate for  $z=\alpha+\beta i$ :

$$k^{-z} = k^{-\alpha} [\cos(\beta * \ln k) - i (\sin (\beta * \ln k))]$$

And:

$$\int \frac{dn}{n^z} = n^{(1-\alpha)} [\cos(\beta * \ln(n)) - i \sin(\beta * \ln(n))] * \frac{[(1-\alpha) + i\beta]}{[(1-\alpha)^2 + \beta^2]}$$

One can now express the real and imaginary components of  $C_n\{f\}$  as:

$$Re(C_n\{f\}) = \sum_{k=1}^n k^{-\alpha} (\cos(\beta * \ln(k)) + \frac{1}{[(1-\alpha)^2 + \beta^2]} (n^{(1-\alpha)} [(1-\alpha)*\cos(\beta*\ln(n))+\beta* \sin(\beta*\ln(n))])) \quad [6]$$

$$Im(C_n\{f\}) = -\sum_{k=1}^n k^{-\alpha} (\sin(\beta * \ln(k)) + \frac{1}{[(1-\alpha)^2 + \beta^2]} (n^{(1-\alpha)} [\beta*\cos(\beta*\ln(n)) - (1-\alpha)*\sin(\beta*\ln(n))])) \quad [7]$$

One can calculate that, for  $\alpha=Re(z)>1$ , and for any  $\epsilon$  arbitrarily small, there is a value of  $n=N$  such that for  $n>N$ ,  $C_N\{f\} - \zeta(z) < \epsilon$ , as the following table shows:

$\alpha$	$\beta$	$C_N\{f\}$ for $N=500$	$\zeta(z)$	$ C_N\{f\} - \zeta(z) $
2	0	1.644934068	1.654934067	$< 10^{-8}$
2	1	1.150355702 + 0.437530865 i	1.150355703+0.437530866 i	$< 10^{-8}$
3	0	1.202056903	1.202056903	$< 10^{-9}$

Table 4. Values of  $C_n\{f(n) = k^{-z}\}$  for  $\alpha=Re(z)>1$  for  $N=500$

That shows that the C-values of  $f(z) = \frac{1}{x^z}$  for  $Re(z)>1$  is  $\zeta(z)$ .

**PART 3 :**

**A decomposition of  $\zeta(z)$  based on the C-transformation of  $f(x) = \frac{1}{x^z}$   
for  $z \in \mathbb{C}, 0 \leq \text{Re}(z) < 1$**

1. C-Transformation of  $f(z) = \frac{1}{x^z}$  for  $z \in \mathbb{C}, 0 \leq \text{Re}(z) < 1$

The C-values of  $f(z) = \frac{1}{x^z}$  from [6] and [7] are equal to the  $\zeta(z)$  when  $\text{Re}(z) > 1$ , this error  $|C_n\{f\} - \zeta(z)|$  grows significantly in the critical strip for  $0 \leq \text{Re}(z) < 1$  as observed in the following table:

$A$	$\beta$	$C_n\{f\}$	$\zeta(z)$	$ C_n\{f\} - \zeta(z) $
0.0	0	$C_n\{f\}$ for $N=500$	-0.5	0.5
0.2	2	0.399824505+0.322650799 i	0.360103 + 0.266246 i	> 0.05
0.7	0	-2.777900606	-2.7783884455	> $10^{-4}$

Table 5. Values of  $C_n\{f(n) = k^{-z}\}$  for  $0 \leq \text{Re}(z) < 1$  for  $N=500$

To understand better the value of the difference  $C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)$ , one can plot the difference for  $\alpha \in [0,1)$  and  $\beta = 0$ : (Similar exponential charts occur for all values of  $\alpha \in [0,1)$  for any given value of  $\beta$ )

### Cn{1/z^n} - Zeta(n)

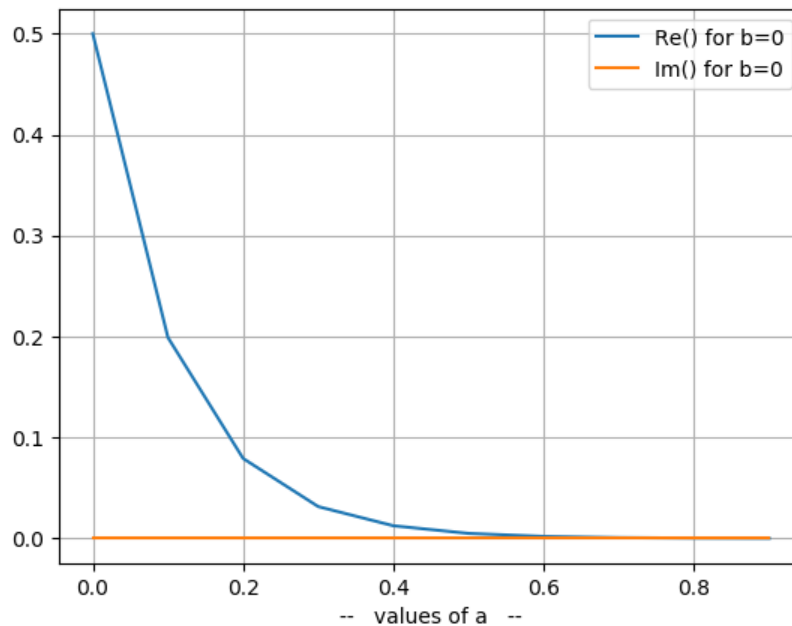


Figure. 4a. Where  $a=\text{Re}(z)$  and  $b=\text{Im}(z)$

And plot the difference for variable values of  $\beta \in [0,1)$  and  $\alpha = 0$ : (Similar sine charts occur for all values of  $\beta \in [0,1)$  for any given value of  $\alpha$ )

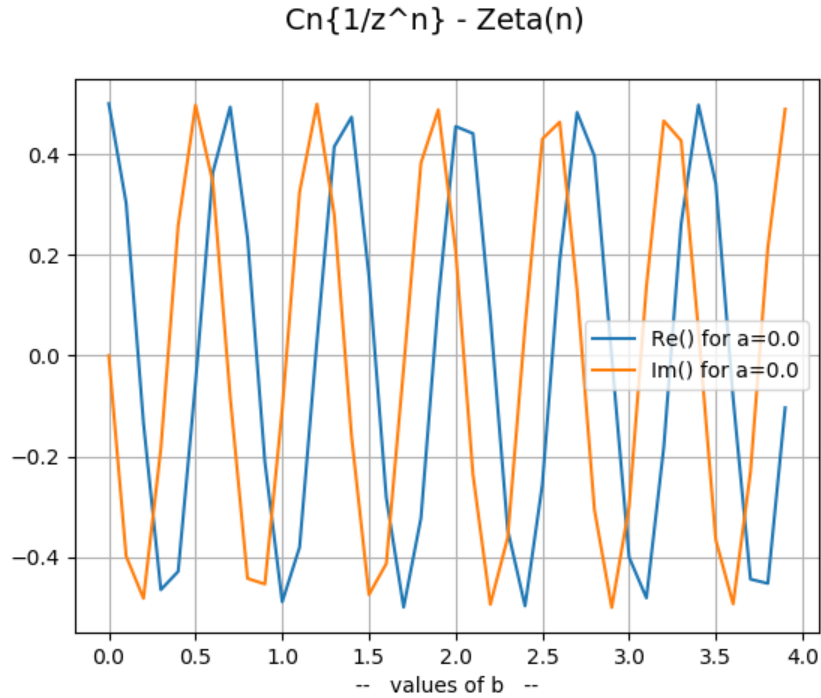


Figure. 4b. Where  $a=Re(z)$  and  $b=Im(z)$

These charts lead to the following calculation of the difference  $C_n \left\{ \frac{1}{k^z} \right\} - \zeta(z)$ :

$$Re[C_n \left\{ \frac{1}{k^z} \right\} - \zeta(z)] = \frac{1}{2} n^{-a} * \cos(\beta * \ln(n)) + O\left(\frac{1}{n}\right)$$

$$Im[C_n \left\{ \frac{1}{k^z} \right\} - \zeta(z)] = \frac{1}{2} n^{-a} * \sin(\beta * \ln(n)) + O\left(\frac{1}{n}\right)$$

With  $O(1/n) \rightarrow 0$  when  $n \rightarrow \infty$ .

And one can finally write:

$$\begin{aligned}
 Re(C_n\{f\}) &= \sum_{k=1}^n k^{-\alpha} (\cos(\beta * \ln(k)) + \\
 &\quad + \frac{1}{[(1-\alpha)^2 + \beta^2]} (n^{(1-\alpha)} [(1-\alpha)*\cos(\beta*\ln(n)) + \beta* \sin(\beta*\ln(n))])) \\
 &\quad + \frac{1}{2} n^{-a} * \cos(\beta * \ln(n)) \tag{8}
 \end{aligned}$$

$$\begin{aligned}
 Im(C_n\{f\}) &= -\sum_{k=1}^n k^{-\alpha} (\sin(\beta * \ln(k)) + \\
 &\quad + \frac{1}{[(1-\alpha)^2 + \beta^2]} (n^{(1-\alpha)} [\beta*\cos(\beta*\ln(n)) - (1-\alpha)*\sin(\beta*\ln(n))])) \\
 &\quad + \frac{1}{2} n^{-a} * \sin(\beta * \ln(n)) \tag{9}
 \end{aligned}$$

and the C-value of  $f(x) = \frac{1}{x^z}$  for  $z \in C, Re(z) \geq 0, z \neq 1$  is the Riemann Zeta function  $\zeta(z)$ .

2. Decomposition of  $\zeta(z) = X(z) - Y(z)$

One can rewrite [8] and [9] creating the  $X(z, n)$  and  $Y(z, n)$  functions:

$$\zeta(z) = \lim_{n \rightarrow \infty} [X(z, n) - Y(z, n)], \text{ where:}$$

$$X(z, n) = \left( \sum_{k=1}^n k^{-\alpha} * \cos(\beta * \ln(k)) + \frac{1}{2} n^{-\alpha} \cos(\beta \ln(n)) + i * \left( \sum_{k=1}^n k^{-\alpha} * \sin(\beta * \ln(k)) + \frac{1}{2} n^{-\alpha} \sin(\beta \ln(n)) \right) \right) \quad [10]$$

$$Y(z, n) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + \beta^2]} \left[ ((1-\alpha) * \cos(\beta \ln(n)) + \beta * \sin(\beta \ln(n))) + i (\beta * \cos(\beta \ln(n)) - (1-\alpha) * \sin(\beta \ln(n))) \right] \quad [11]$$

and define:

$$X(z) = \lim_{n \rightarrow \infty} X(z, n) \text{ and}$$

$$Y(z) = \lim_{n \rightarrow \infty} Y(z, n)$$

to write:

$$\zeta(z) = X(z) - Y(z) \quad [12]$$

The following table compared the values of  $\zeta(z)$  and  $X(z) - Y(z)$ :

z= 0 +j* 0 and n=500
Zeta(z) = -0.5 + i* 0.0 X(z)-Y(z) = -0.5 +i* 0.0 ---> Error = 0.0 +i* 0.0
z= 0.2 +j* 2 and n=500
Zeta(z) = 0.360102590022591 + i* -0.266246199765574 X(z)-Y(z) = 0.360102741838091 +i* -0.266246128959438 ---> Error= -1.5181550 e-7 +i* -7.080613 e-8
z= 0.4 +j* 0 and n=500
Zeta(z) = -1.13479778386698 + i* 0.0 X(z)-Y(z) = -1.1347977871726 +i* 0.0 ---> Error= 3.305619 e-9 +i* 0.0

Table 6.  $\zeta(z)$  compared to  $X(z) - Y(z)$

The highest error for  $\alpha \in [0,1)$ ,  $\beta \in [0,100]$ ,  $n=1000$  is  $8x10^{-6}$ .

3. Representation of the function  $\zeta(z) = X(z) - Y(z)$  for  $\text{Re}(z)=1/2$

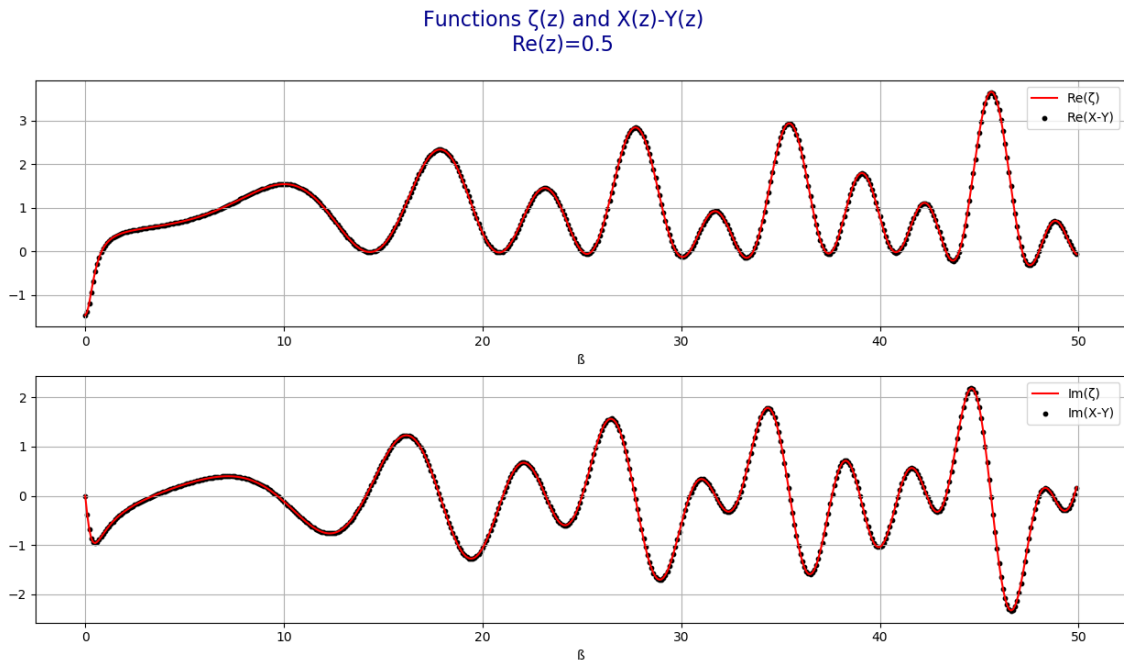


Figure. 5:  $\zeta(z) = X(z) - Y(z)$

4. Representation of the function  $|\zeta(z)| = |X(z) - Y(z)|$  for  $\text{Re}(z)=1/2$

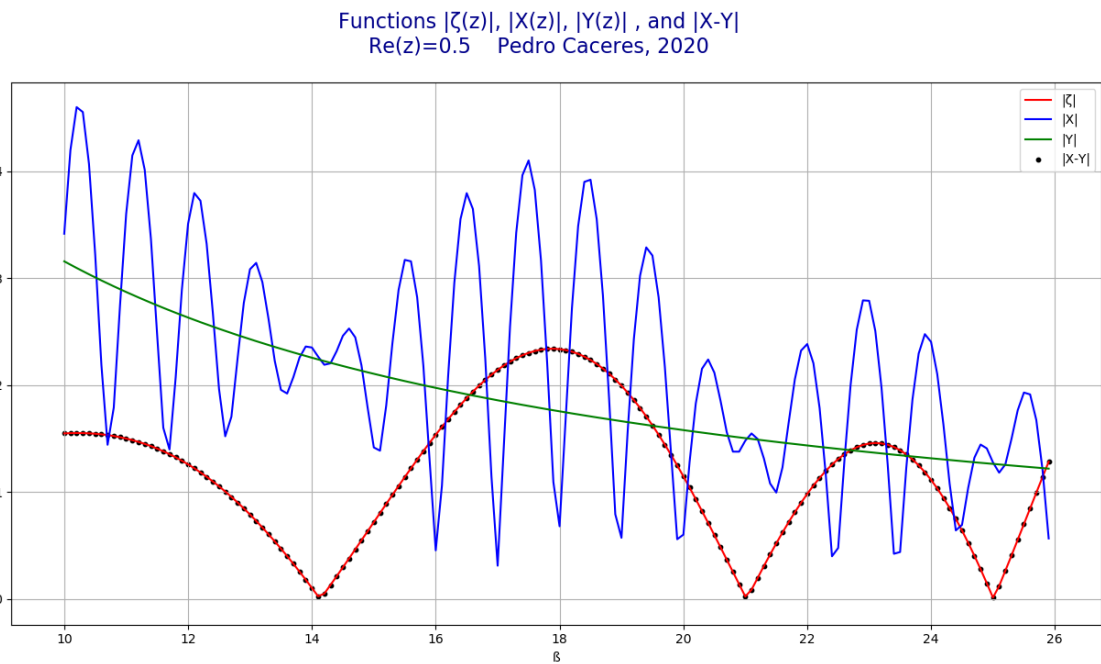


Figure. 6:  $|\zeta(z)| = |X(z) - Y(z)|$



5. Representation of the function  $X(z, n)$

The following chart represents  $X(z, n)$  for  $a = 1/2$  and  $b \in [1,6]$  and  $n = 250$

Function  $X(z)$

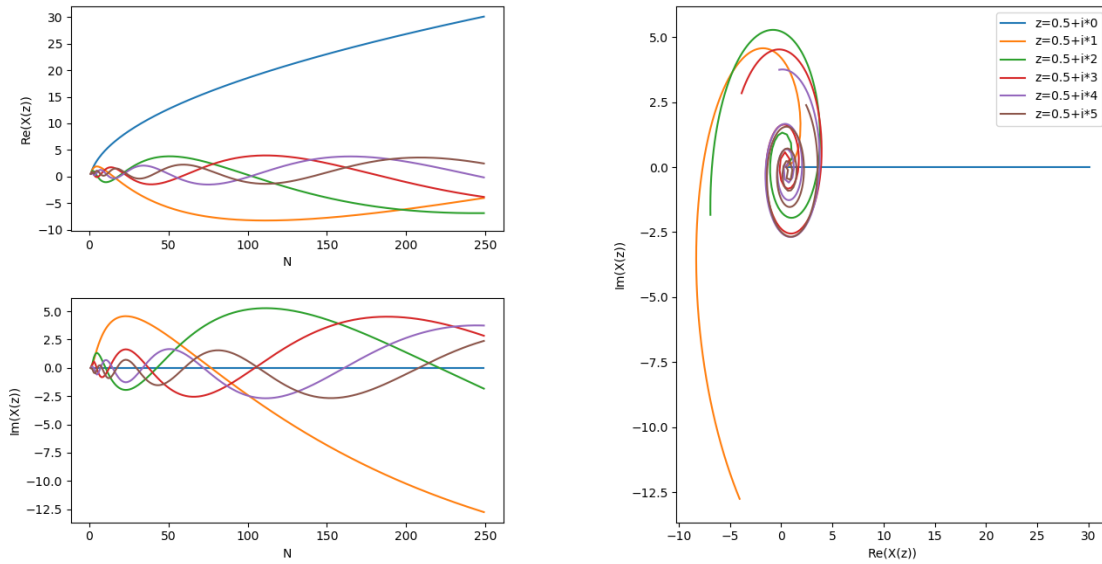


Figure. 7:  $X(z, n)$

The following chart represents  $X(z, n)$  for  $a \in [1,6]$  and  $b = 1$  and  $n = 250$

Function  $X(z)$

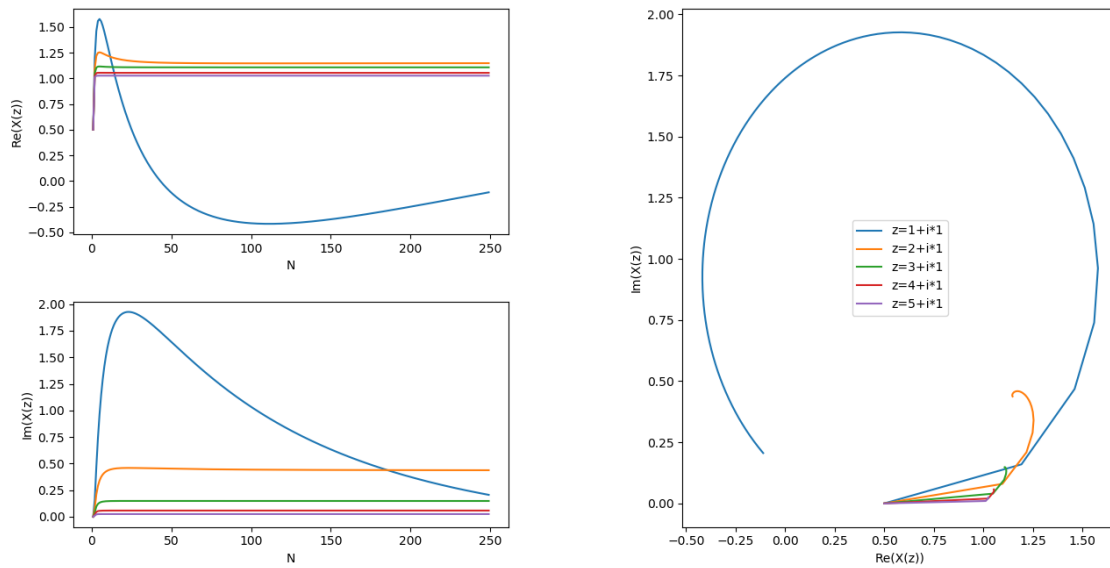


Figure. 8:  $X(z, n)$

6. Representation of the function  $Y(z, n)$

The following chart represents  $Y(z, n)$  for  $a = 1/2$  and  $b \in [1,6]$  and  $n = 250$

Function  $Y(z)$

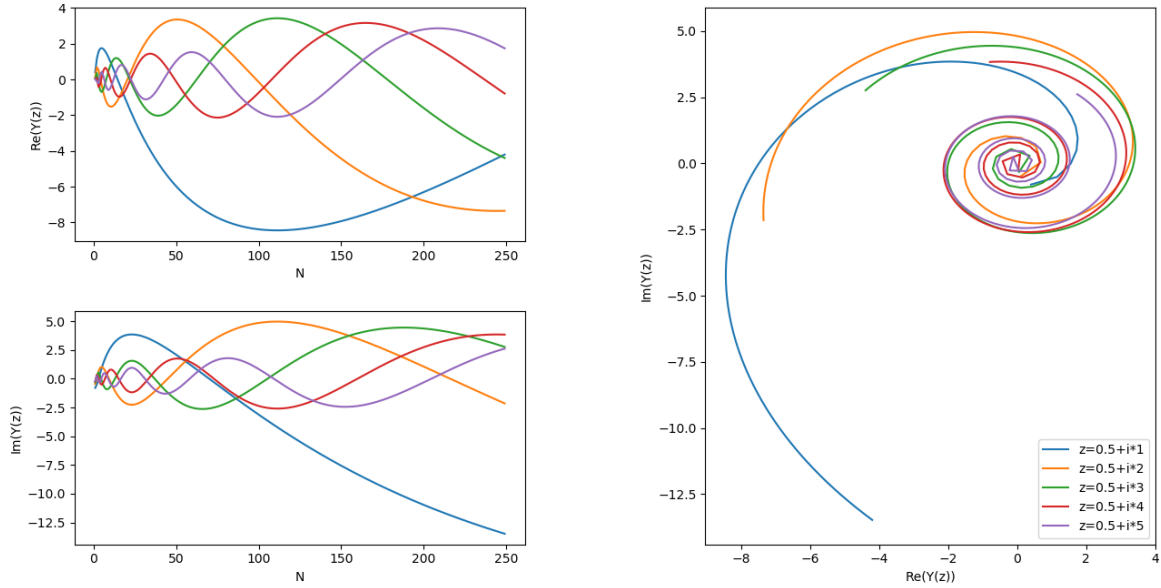


Figure. 9:  $Y(z, n)$

The following chart represents  $Y(z, n)$  for  $a \in [1,6]$  and  $b = 1$  and  $n = 250$

Function  $Y(z)$

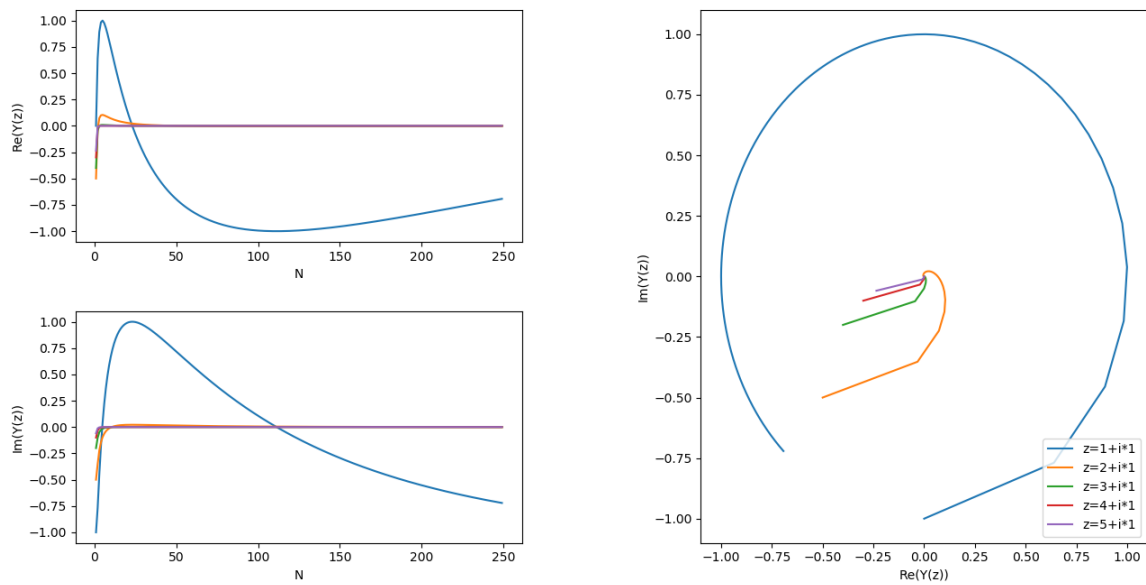
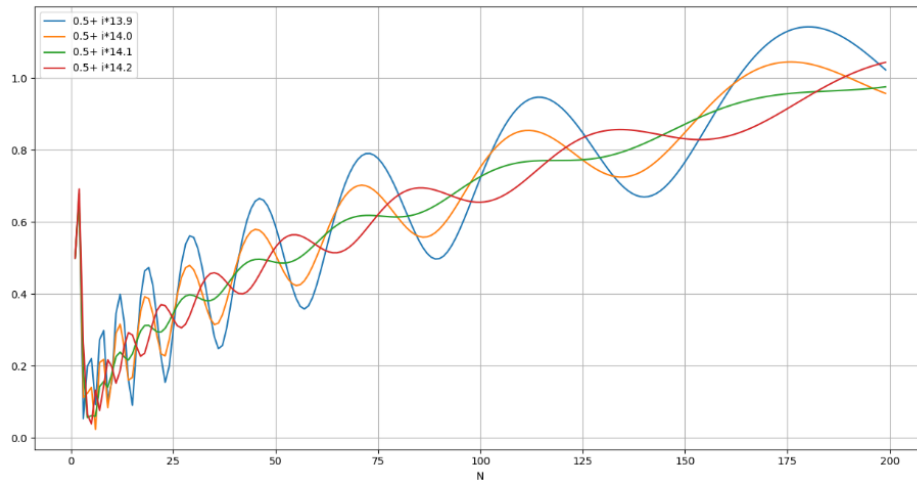


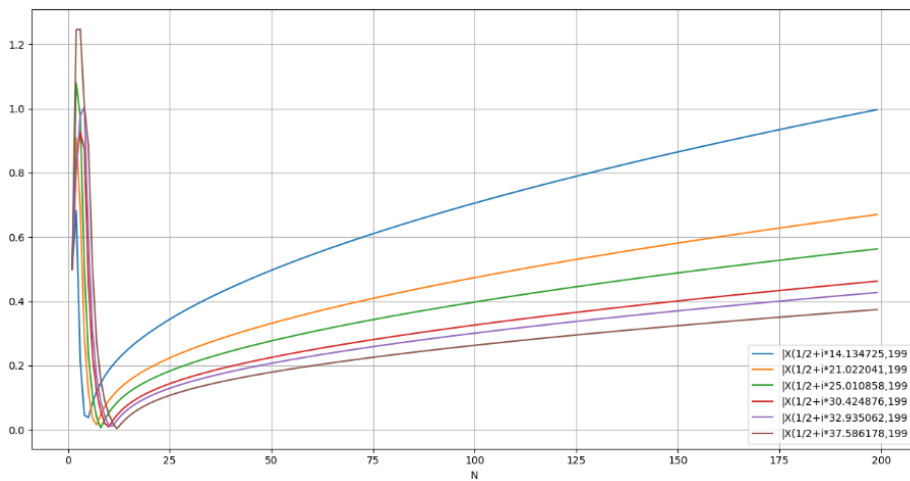
Figure. 10:  $Y(z, n)$

7. Representation of  $|X(z, n)|$

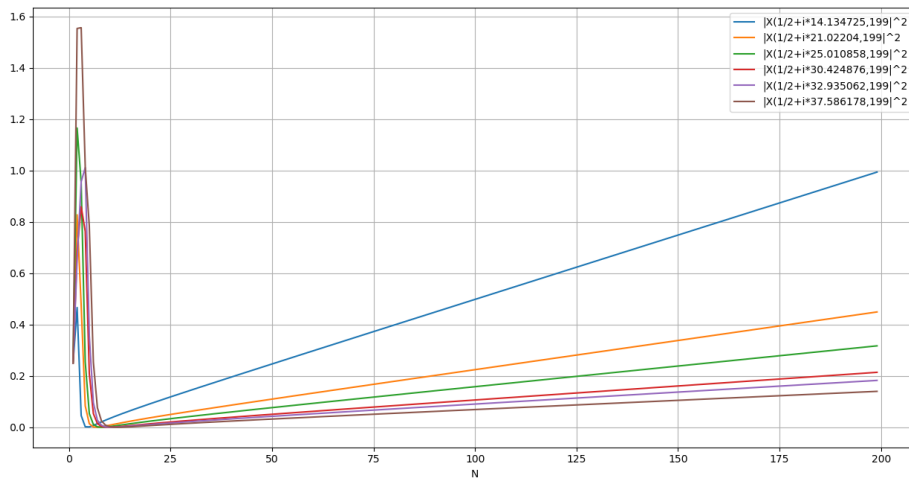
Wave representation for  $|X(z, n)|$  for  $Re(z) = 1/2$  and  $Im(z)$  variable. [Figure. 11]



Parabolic representation for  $|X(z, n)|$  for  $(z)$  a nontrivial zero of Riemann Zeta. [Figure. 12]

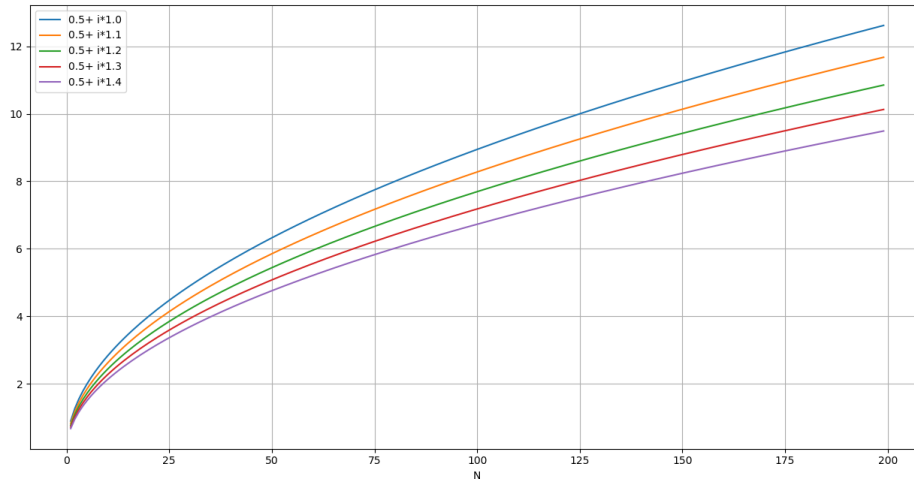


Linear representation for  $|X(z, n)|^2$  for  $(z)$  a nontrivial zero of Riemann Zeta. [Figure. 13]

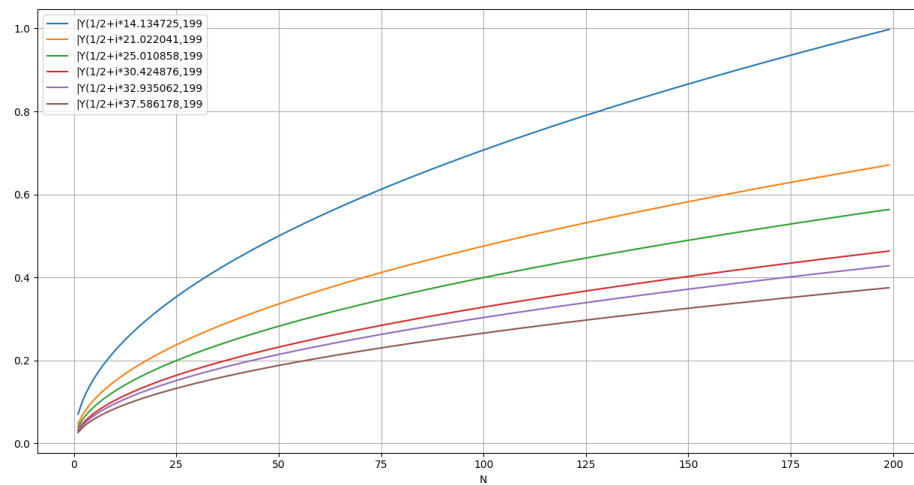


8. Representation of  $|Y(z, n)|$

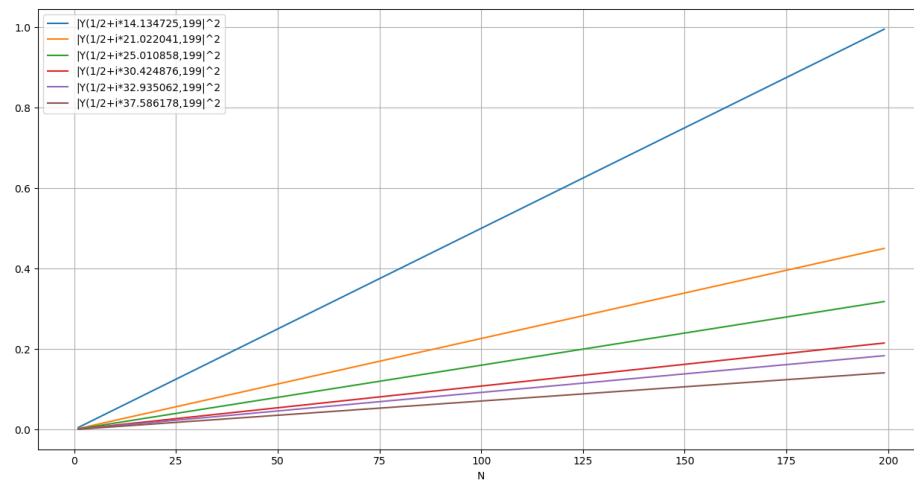
Polynomial representation for  $|Y(z, n)|$  for  $Re(z) = 1/2$  and  $Im(z)$  variable. [Figure. 14]



Parabolic representation for  $|Y(z, n)|$  for  $(z)$  a nontrivial zero of Riemann Zeta. [Figure. 15]



Linear representation for  $|Y(z, n)|^2$  for  $(z)$  a nontrivial zero of Riemann Zeta. [Figure. 16]



## 9. Conclusion PART 3

Using the defined C-transformation, one can write the Riemann Zeta function as the difference of two functions  $X(z)$  and  $Y(z)$  for  $Re(z) \geq 0, z \neq 1$ . This will provide a new way of analyzing the zeros of the Zeta function, and a new approach to the Riemann Hypothesis.

The decomposition is as follows:

$$\zeta(z) = X(z) - Y(z), \text{ where:}$$

$$X(z, n) = \sum_{k=1}^n k^{-\alpha} * \cos(\beta \ln(k)) + \frac{1}{2} n^{-\alpha} \cos(\beta \ln(n)) + \\ + i * (\sum_{k=1}^n k^{-\alpha} * \sin(\beta \ln(k)) + \frac{1}{2} n^{-\alpha} \sin(\beta \ln(n)))$$

$$\text{and: } X(z) = \lim_{n \rightarrow \infty} X(z, n)$$

$$Y(z, n) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + \beta^2]} [((1-\alpha) * \cos(\beta \ln(n)) + \beta * \sin(\beta \ln(n))) + \\ + i (\beta \cos(\beta \ln(n)) - (1-\alpha) * \sin(\beta \ln(n)))]$$

$$\text{and: } Y(z) = \lim_{n \rightarrow \infty} Y(z, n)$$

Observations:

- a.  $|X(z, n)|$  has a wave representation
- b.  $|X(z, n)|$  becomes a parable when  $z$  is a nontrivial zero of Riemann Zeta
- c.  $|X(z, n)|^2$  becomes a line when  $z$  is a nontrivial zero of RZF with slope equal  $\frac{1}{\beta^2 + 1/4}$
- d.  $|Y(z, n)|$  has a polynomial representation
- e.  $|Y(z, n)|$  becomes a parable when  $z$  is a nontrivial zero of Riemann Zeta
- f.  $|Y(z, n)|^2$  becomes a line when  $Re(z)=1/2$  with slope equal  $1/(\beta^2 + 1/4)$

So, the only common representation for  $|X(z)|$  and  $|Y(z)|$  occurs when  $Re(z)=1/2$ , so

$$X(z) - Y(z) = 0 \text{ if and only if } Re(z)=1/2$$

## **PART 4:**

### **Proof of the Riemann Hypothesis using the decomposition**

$$\zeta(z) = X(z) - Y(z)$$

1. Analysis of Absolute Square  $|Y(z, n)|^2$

$$|Y(z, n)|^2 = \left[ \left( n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + \beta^2]} [(1-\alpha) * \cos(\beta \ln(n)) + \beta * \sin(\beta \ln(n))] \right)^2 \right. \\ \left. + \left( n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + \beta^2]} [\beta * \cos(\beta \ln(n)) - (1-\alpha) * \sin(\beta \ln(n))] \right)^2 \right]$$

$$|Y(z, n)|^2 = n^{2(1-\alpha)} * \frac{1}{[\beta^2 + (1-\alpha)^2]} \text{ Polynomial representation} \quad [13]$$

This could be observed in Figure. 14, 15, 16.

1.1.  $|Y(z, n)|^2$  is a straight line if and only if  $\alpha = \frac{1}{2}$

The slope of  $|Y(z, n)|^2$  with respect to  $n$  is given by:

$$\text{slope}(|Y(z, n)|^2) = d(|Y(z, n)|^2)/dn$$

Which equals to:

$$d(|Y(z, n)|^2)/dn = 2(1-\alpha) n^{1-2\alpha} * \frac{1}{[\beta^2 + (1-\alpha)^2]}$$

$|Y(z, n)|^2$  can only be a line when the slope is constant, which can only happen if and only if:

$$(1 - 2\alpha) = 0$$

therefore:

$$|Y(z, n)|^2 \text{ is a straight line if and only if } \alpha = \frac{1}{2} \quad [14]$$

1.2. Summary for  $|Y(z, n)|^2$  for  $\alpha = \frac{1}{2}$ :

$$\Rightarrow \text{the slope } |Y(z, n)|^2 \text{ is constant if and only if } \alpha = \frac{1}{2}$$

$$\Rightarrow \text{When } \alpha=1/2, |Y(z, n)|^2 = \frac{n}{[\beta^2 + \frac{1}{4}]}$$

$$\Rightarrow \text{The slope for } |Y(z, n)|^2 \text{ is } \frac{1}{[\beta^2 + \frac{1}{4}]} \text{ for } \alpha = \frac{1}{2}$$

2. Analysis of Absolute Square  $|X(z, n)|^2$  [15]

$$|X(z, n)|^2 = \left( \frac{1}{2} n^{-\alpha} \cos(\beta \ln(n)) + \sum k^{-\alpha} \cos(\beta \ln(k)) \right)^2 + \\ \left( \frac{1}{2} n^{-\alpha} \sin(\beta \ln(n)) + \sum k^{-\alpha} \sin(\beta \ln(k)) \right)^2$$

Applying properties of infinite series (Kopp):

$$|X(z, n)|^2 = \frac{1}{4} n^{-2\alpha} (\cos^2(\beta \ln(n)) + \sin^2(\beta \ln(n))) + \\ \left( \sum k^{-\alpha} \cos(\beta \ln(k)) \right)^2 + \left( \sum k^{-\alpha} \sin(\beta \ln(k)) \right)^2 + \\ + n^{-\alpha} [\cos(\beta \ln(n)) * \sum k^{-\alpha} \cos(\beta \ln(k))] + \\ + n^{-\alpha} [\sin(\beta \ln(n)) * \sum k^{-\alpha} \sin(\beta \ln(k))]$$

$$|X(z, n)|^2 = \sum_{k=1}^n \sum_{j=1}^n k^{-\alpha} * j^{-\alpha} * \cos\left(\beta \ln\left(\frac{k}{j}\right)\right) + \sum_{k=1}^n k^{-2\alpha} + \frac{1}{4}n^{-2\alpha} + n^{-\alpha}[\cos(\beta \ln(n)) * \sum k^{-\alpha} \cos(\beta \ln(k))] + n^{-\alpha}[\sin(\beta \ln(n)) * \sum k^{-\alpha} \sin(\beta \ln(k))]$$

One can express the previous expression replacing:

$$R(n) = \frac{1}{4}n^{-2\alpha} + n^{-\alpha} [\cos(\beta \ln(n)) * \sum k^{-\alpha} \cos(\beta \ln(k)) + \sin(\beta \ln(n)) * \sum k^{-\alpha} \sin(\beta \ln(k))]$$

With:

$$\lim_{n \rightarrow \infty} R(n) = 0 \text{ if } \alpha > 0, \text{ therefore,} \tag{16}$$

$$|X(z, n)|^2 = \sum_{k=1}^n \sum_{j=1}^n k^{-\alpha} * j^{-\alpha} * \cos\left(\beta * \ln\left(\frac{k}{j}\right)\right) + \sum_{k=1}^n k^{-2\alpha} + R(n) \tag{17}$$

When  $|X(z, n)|^2$  is represented graphically, one can observe that:

- $|X(z, n)|^2$  is a wave that converges when  $n \rightarrow \infty$  and  $\alpha > 1$  (Figure. 17)
- $|X(z, n)|^2$  is a wave that does not converge when  $n \rightarrow \infty$  and  $\alpha < 1$  (Figure. 18)
- $|X(z, n)|^2$  is a wave that collapses to a line when  $n \rightarrow \infty$  and  $\alpha = 1/2$  and  $\beta = \text{Im}(\zeta(z^*))$  (Figure. 19) where  $z^*$  is a nontrivial zero of RZF.

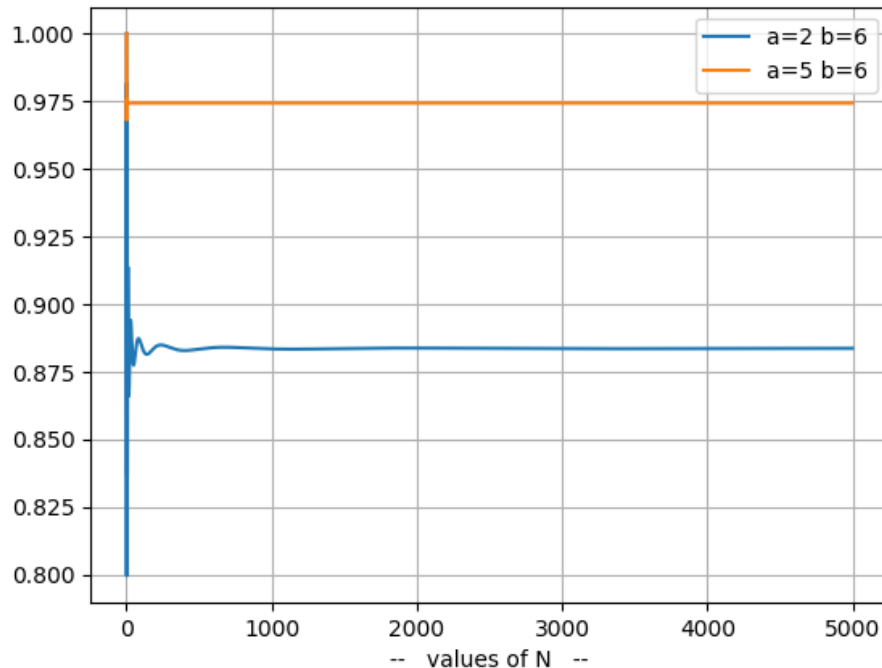


Figure. 17.  $|X(z, n)|^2$  for  $\alpha > 1$



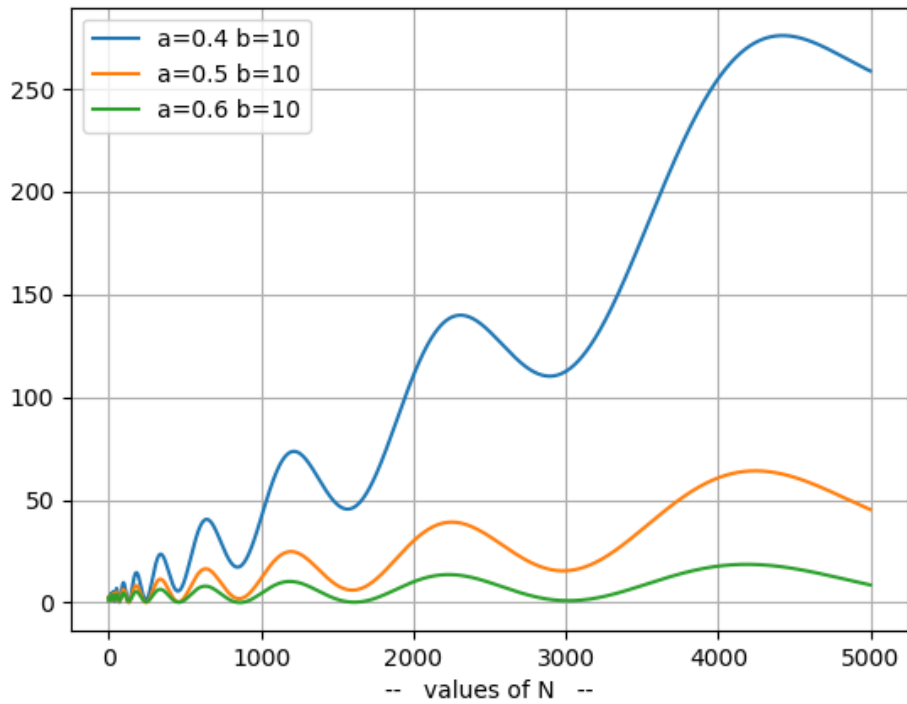


Figure. 18.  $|X(z, n)|^2$  for  $\alpha < 1$

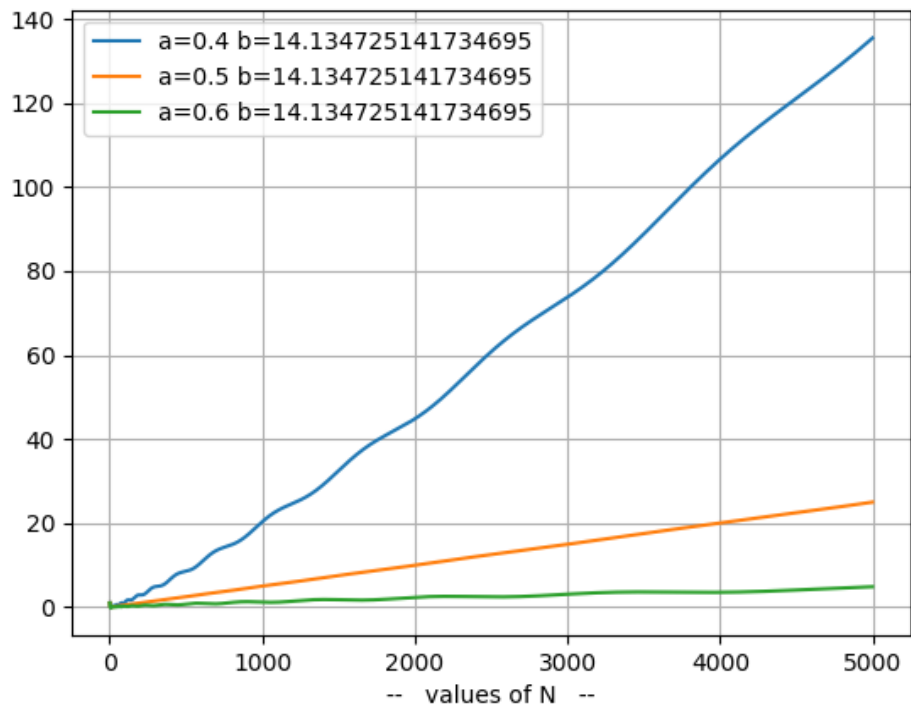


Figure. 19. For  $a=0.5$ ,  $b=\beta_1$ ,  $|X(z, n)|^2$  collapses to a line

2.1.  $|X(z, n)|^2$  converges when  $n \rightarrow \infty$  and  $\alpha > 1$  to  $|\zeta(\alpha, \beta)|^2$

The limit of  $|X(z, n)|^2$  outside the critical strip  $[0, 1]$  can be calculated using [16]:

$$\lim_{n \rightarrow \infty} |X(z, n)|^2 = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sum_{j=1}^n k^{-\alpha} * j^{-\alpha} * \cos\left(\beta \left(\ln\left(\frac{k}{j}\right)\right)\right)$$

As one can see in some examples in the following table where  $z = \alpha + i\beta$ :

$\alpha$	$\beta$	$\lim_{n \rightarrow \infty}  X(z, n) ^2$	$ \zeta(\alpha, \beta) ^2$
1.0	7	1.074711506185445	1.074756
1.0	10	1.4413521753699579	1.441430
2.5	7	1.0093487944300192	1.009349
2.5	10	1.0507402208589398	1.050740

Table 7

$$\lim_{n \rightarrow \infty} |X(z, n)|^2 = |\zeta(z)|^2 = \zeta(\alpha + \beta i) * \zeta(\alpha - \beta i) \text{ for } \alpha > 1$$

And also, in the following Figure. 20:

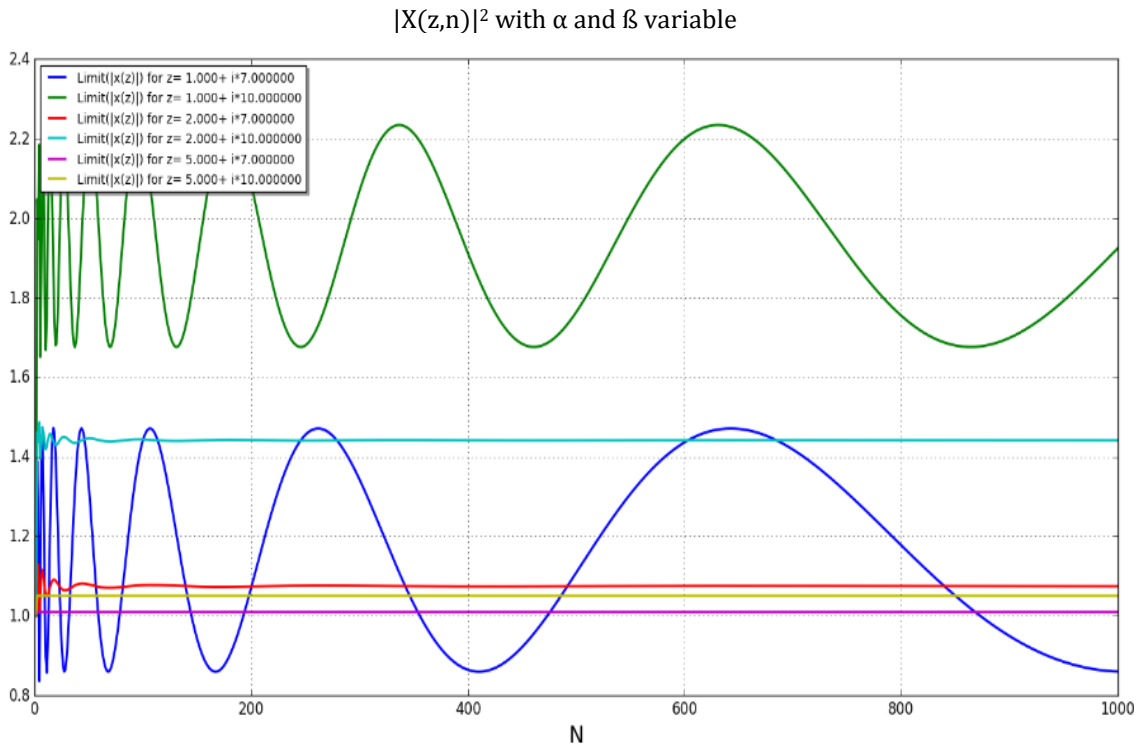


Figure. 20.  $|X(z, n)|^2$  converges when  $n \rightarrow \infty$  and  $\alpha > 1$

One can observe that the graphs for  $\alpha = 1$  do not converge while graphs for  $\alpha > 1$  they all converge. This observation can be used to prove that there are no zero values of  $\zeta(z)$  for  $z$  with  $\text{Re}(z) > 1$ .

2.2.  $|X(z, n)|^2$  diverges when  $n \rightarrow \infty$  for  $\alpha \leq 1$

$|X(z, n)|^2$  diverges when  $n \rightarrow \infty$  for  $\alpha < 1$  because of [16] and [17]:

$$|\cos\left(\beta \ln\left(\frac{k}{j}\right)\right)| < 1$$

And:

$$\sum_{k=1}^n \sum_{j=1}^n k^{-\alpha} * j^{-\alpha} \text{ diverges for } \alpha < 1$$

Therefore:

$$\lim_{n \rightarrow \infty} |X(z, n)|^2 = \sum_{k=1}^n \sum_{j=1}^n k^{-\alpha} * j^{-\alpha} * \cos\left(\beta \ln\left(\frac{k}{j}\right)\right) \text{ diverges for } \alpha < 1$$

2.3.  $|X(z, n)|^2$  does not collapse to any polynomial function  $|X(z, n)|^2 = C * n^t$  for  $t > 1$ , and  $C$  constant

One can prove it with a reduction to absurd.

Let's assume that  $|X(z, n)|^2 = C * n^t$  for  $t > 1$  where  $C$  and  $t$  integers  $C > 0$  and  $t > 0$

If  $|X(z, n)|^2 = C * n^t$  then:

$$\lim_{n \rightarrow \infty} |X(z, n)|^2 / n^t = C$$

But:

$$\lim_{n \rightarrow \infty} |X(z, n)|^2 / n^t = \frac{1}{n^t} * \lim_{n \rightarrow \infty} \sum_{k=1}^n k^{-2\alpha} + \frac{1}{n^t} * \sum_{k=1}^n \sum_{j \neq k}^n k^{-\alpha} * j^{-\alpha} * \cos\left(\beta \ln\left(\frac{k}{j}\right)\right)$$

And:

$$\frac{1}{n^t} * \lim_{n \rightarrow \infty} \sum_{k=1}^n k^{-2\alpha} = 0 \text{ for } t > 1$$

$$\frac{1}{n^t} * \sum_{k=1}^n \sum_{j \neq k}^n k^{-\alpha} * j^{-\alpha} * \cos\left(\beta \ln\left(\frac{k}{j}\right)\right) = 0 \text{ for } t > 1$$

So,  $C$  must be 0 which is an absurd.

2.4.  $|X(z, n)|^2$  collapses to a straight-line  $|X(z, n)|^2 = Cn$  if  $\text{Re}(z) = 1/2$

The proposition says that the following limit exists only for  $\text{Re}(z) = 1/2$

$$\lim_{n \rightarrow \infty} (|X(z, n)|^2 / n) = S$$

Using the expression:

$$\lim_{n \rightarrow \infty} (|X(z, n)|^2 / n) = \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n k^{-2\alpha} + \sum_{k=1}^n \sum_{j \neq k}^n k^{-\alpha} * j^{-\alpha} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right)$$

2.4.1. For  $\alpha > 1/2$ , one can see that  $\lim_{n \rightarrow \infty} (|x(z, n)|^2 / n) = 0$ :

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n k^{-2\alpha} \right) = 0 \quad \text{because } 2\alpha > 1 \text{ and the series is convergent}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{j \neq k}^n k^{-\alpha} * j^{-\alpha} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) < \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{j \neq k}^n (k^{-\alpha} * j^{-\alpha}) < \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n k^{-2\alpha} \right)$$

So:

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{k=1}^n \sum_{j \neq k}^n k^{-\alpha} * j^{-\alpha} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right) = 0$$

2.4.2. For  $\alpha < 1/2$ , one can see that  $\lim_{n \rightarrow \infty} (|X(z, n)|^2 / n) = \infty$  as:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n k^{-2\alpha} \right) < \lim_{n \rightarrow \infty} \frac{1}{n} \left( n * \frac{1}{n} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

And:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{j \neq k}^n k^{-\alpha} * j^{-\alpha} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) > \lim_{n \rightarrow \infty} \left( \frac{1}{n} * n^2 * \frac{1}{n^{2\alpha}} \right) = \infty$$

Where the summations are replaced by the number of elements in the matrix ( $n \times n$ ) times the smallest value in each row ( $1/n$ ) then  $1 > (2 - 1 - 2\alpha) > 0$  when  $\alpha < 1/2$

2.4.3. Limit for  $\alpha = 1/2$ .

When  $\alpha = 1/2$ , one can express  $(|X(z, n)|^2 / n)$  as:

$$\begin{aligned} \lim_{n \rightarrow \infty} (|X(z, n)|^2 / n) &= \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n k^{-1} + \sum_{k=1}^n \sum_{j \neq k}^n k^{-1/2} * j^{-1/2} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n k^{-1} \right) + \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n \sum_{j \neq k}^n k^{-1/2} * j^{-1/2} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right) = \end{aligned}$$

$$\begin{aligned}
&= 0 + \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n \sum_{j \neq k}^n k^{-1/2} * j^{-1/2} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right) = \\
&= \lim_{n \rightarrow \infty} \frac{2n}{n} \left( \sum_{j=1}^{n-1} n^{-1/2} * j^{-1/2} * \cos \left( \beta \left( \ln \left( \frac{n}{j} \right) \right) \right) \right) = \\
&= \lim_{n \rightarrow \infty} 2 \left( n^{-\frac{1}{2}} \sum_{j=1}^{n-1} * j^{-\frac{1}{2}} * \cos \left( \beta \left( \ln \left( \frac{n}{j} \right) \right) \right) \right) =
\end{aligned}$$

Using the integral approximation of the infinite series

$$\begin{aligned}
&= 2 * \lim_{n \rightarrow \infty} \frac{2 * \sqrt{n} * \cos \left( \beta * \ln \left( \frac{n}{n} \right) \right) - 2 * \beta * \sin \left( \beta * \ln \left( \frac{n}{n} \right) \right)}{4 * \beta^2 + 1} * n^{-\frac{1}{2}} \\
&= 2 * \frac{2 * \sqrt{n}}{4 * \beta^2 + 1} n^{-\frac{1}{2}} = 2 * \frac{2}{4 * \beta^2 + 1} = \frac{1}{\beta^2 + 1/4}
\end{aligned}$$

So, if  $\lim_{n \rightarrow \infty} (|X(z, n)|^2 / n)$  exists will be equal to:

$$\lim_{n \rightarrow \infty} (|X(z, n)|^2 / n) = \frac{1}{\beta^2 + 1/4} \quad [18]$$

$$\text{if } z=1/2+i\beta$$

And this limit can only exist when  $|X(z, n)|^2$  is monotonous which means that the curve will cross the x-axis only once.

$$\begin{aligned}
|X(z, n)|^2 &= \left( \sum_{k=1}^n \sum_{j=k}^n k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right) \\
&= 2 * n^{-a} * \left( \sum_{j=1}^{n-1} j^{-a} * \cos \left( \beta * \left( \ln \left( \frac{x}{j} \right) \right) \right) \right)
\end{aligned}$$

### 3. Calculating the zeros of $|X(z, n)|^2$

Let's define the function  $C_2(n, a, b) = |X(z, n)|^2$  in R (where  $z=a+bi$ ) such that:

$$C_2(n, a, b) = 2 * n^{-a} * \left( \sum_{j=1}^{n-1} j^{-a} * \cos \left( b * \left( \ln \left( \frac{n}{j} \right) \right) \right) \right) \quad [19]$$

With the following wave representation for  $C_2(n, a, b)$ :

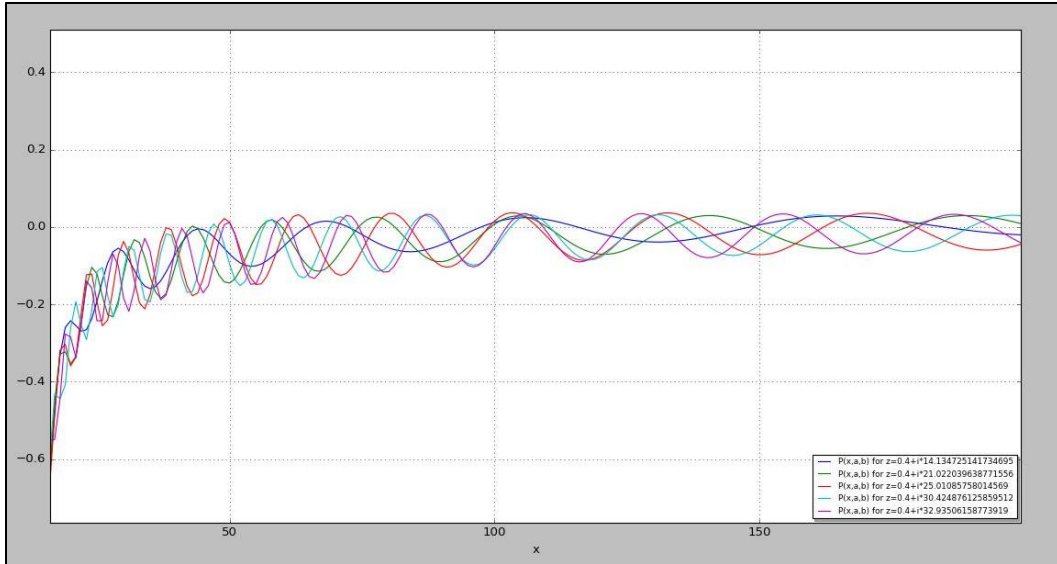


Figure 21.  $C_2(x, a, b)$  for  $a=0.4$  and variable  $b$

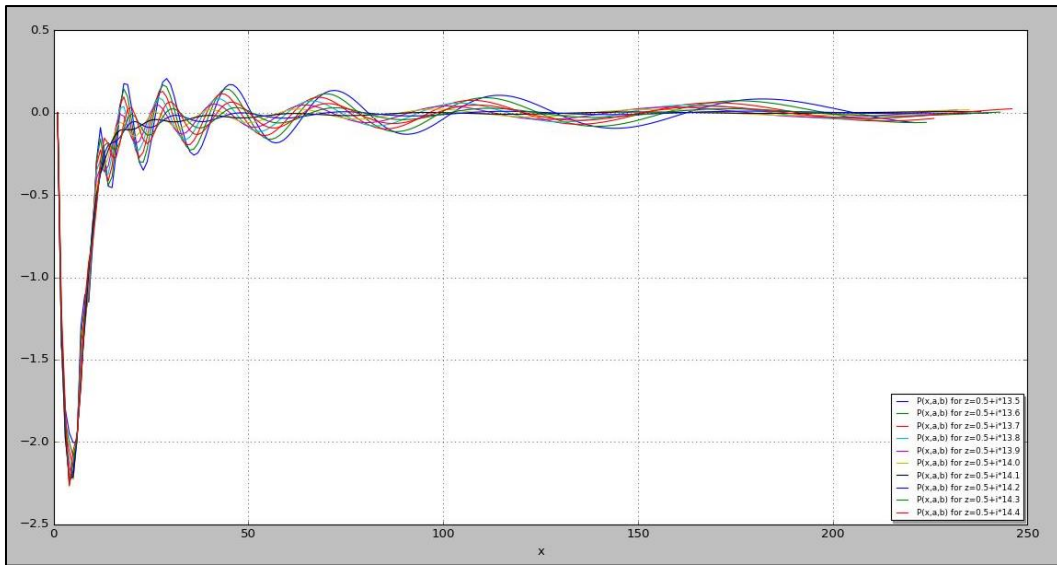


Figure 22.  $C_2(n, a, b)$  for  $a=0.5$  and variable  $b$

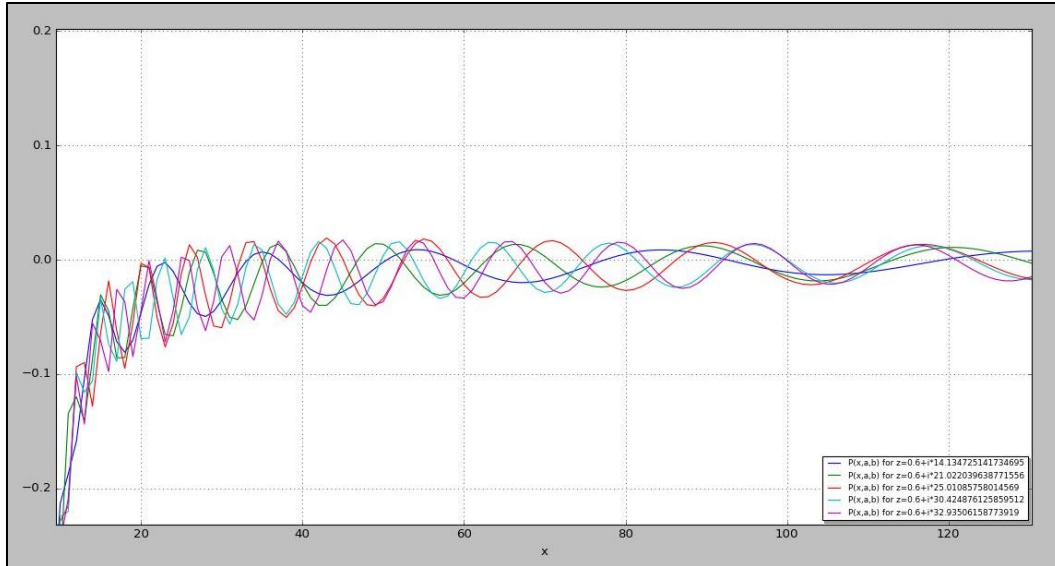


Figure. 23.  $C_2(n, a, b)$  for  $a=0.6$  and variable  $b$

As a wave,  $C_2(n, a, b)$  can have one or more zeros. For  $C_2(n, a, b)$  to have only one zero, it must cross the axis  $y=0$  only once, which means that the wave collapses to a polynomial line. A numeric method has been created and coded (Python) to find the values of  $(n, a, b)$  such that  $C_2(n, a, b)=0$ . The following table shows an example of those calculated values, where  $x=n$ ,  $a=Alfa$ , and  $b=Beta$ :

Alfa	Beta	Number of Zeros	Zero at X=
0.4	14.1	5	
0.4	14.2	5	
0.4	14.3	5	
0.4	14.4	5	
0.5	14.07	5	
0.5	14.08	5	
0.5	14.09	5	
0.5	14.1	4	
0.5	14.11	4	
0.5	14.12	3	
0.5	14.13	1	200
0.5	14.14	3	
0.5	20.97	11	
0.5	20.98	11	
0.5	20.99	11	
0.5	21	9	
0.5	21.01	5	
0.5	21.02	1	442
0.5	21.03	3	
0.5	24.96	16	
0.5	24.97	16	
0.5	24.98	15	
0.5	24.99	11	
0.5	25	7	
0.5	25.01	1	626
0.5	25.02	6	
0.5	25.03	10	

Table 8. Number of Zeros of  $C_2(x, a, b)$  for different values of  $a=Alfa$ , and  $b=Beta$

The calculations for  $a \in (0,1)$  and  $b \in [1, 100]$  only found single zeros for  $C_2(x, a, b)$  for values of  $a = 0.5$  as shown in the following table that summarizes the single zeros found in those intervals:

Values (x,a,b)   C2(x,a,b)=0 SINGLE		
x*	a*	b*
200.1000	0.5000	14.1368
442.2000	0.5000	21.0226
625.8000	0.5000	25.0110
926.0000	0.5000	30.4261
1085.0000	0.5000	32.9355
1413.0000	0.5000	37.5866
1674.6000	0.5000	40.9188
1877.5000	0.5000	43.3272
2304.8000	0.5000	48.0057

Table 9. List of first Zeros of  $C_2(x, a, b)$

One can observe that:

$$\text{if } C_2(x, a, b) = 0 \rightarrow$$

$$a = 1/2$$

$$b = \text{Im}(z) \quad \text{with } \zeta(z) = 0$$

(a, b) are the Nontrivial Zeros of  $\zeta(z)$  in the critical line.

$$x = b^2 + \frac{1}{4}$$

And the calculated values of  $\lim_{x \rightarrow \infty} C_2(x, a, b)$  for the values of (a,b) from Table 9 are:

Values (x,a,b)   C2(x,a,b)=0			Limit (C2(x,a,b))
x	a	b	when x->∞
200.1000	0.5000	14.1368	0.0050
442.2000	0.5000	21.0226	0.0023
625.8000	0.5000	25.0110	0.0016
926.0000	0.5000	30.4261	0.0011
1085.0000	0.5000	32.9355	0.0009
1413.0000	0.5000	37.5866	0.0007
1674.6000	0.5000	40.9188	0.0006
1877.5000	0.5000	43.3272	0.0005

Table 10. Limit of  $C_2(x, a, b)$  for b in Table 10 and  $x \rightarrow \infty$



Values (x,a,b)   C2(x,a,b)=0			Limit (C2(x,a,b))	
x	a	b	when x->∞	Known Zero
200.1000	0.5000	14.1368	0.0050	14.1347
442.2000	0.5000	21.0226	0.0023	21.0220
625.8000	0.5000	25.0110	0.0016	25.0109
926.0000	0.5000	30.4261	0.0011	30.4249
1085.0000	0.5000	32.9355	0.0009	32.9351
1413.0000	0.5000	37.5866	0.0007	37.5862
1674.6000	0.5000	40.9188	0.0006	40.9187
1877.5000	0.5000	43.3272	0.0005	43.3271
2304.8000	0.5000	48.0057	0.0004	48.0052
2477.7000	0.5000	49.7740	0.0004	49.7738

Table 11. Comparing "b" calculated with known zeros of ζ(z)

Therefore,  $|X(z, n)|^2 = C(n, a, b)$  has the following special properties for all (a,b) such that  $\zeta(a+bi)=0$ .

if  $S = \frac{1}{b^2+1/4}$

$C_2(n, a, b) = 0$  when  $x = \frac{1}{S}$ ,  $a = \frac{1}{2}$ ,  $b = \text{Im}(z^*)$  with  $z^*$  a nontrivial zero of  $\zeta(z)$

$\lim_{x \rightarrow \infty} C_2\left(n, \frac{1}{2}, b\right) = S$

Graphically:

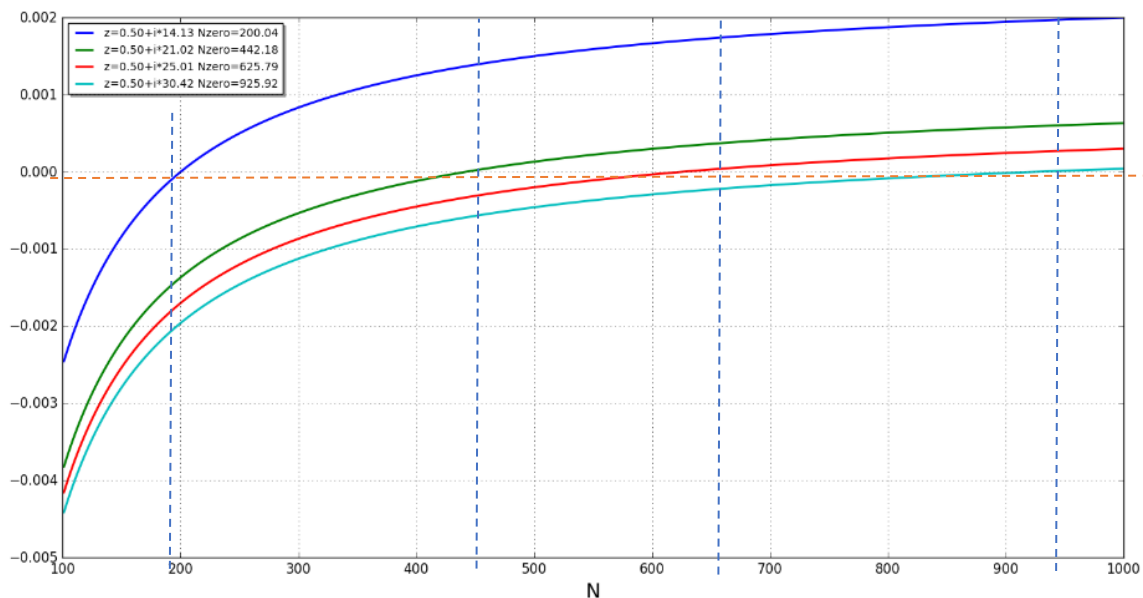


Figure. 24.  $C_2(n, 1/2, b)$  such that  $\zeta(1/2+bi)=0$

4. Theorem: For  $\text{Re}(z) \geq 0$ , if  $z^*$  is a nontrivial zero of  $\zeta(z)$ , then  $\text{Re}(z^*) = 1/2$

Proof:

➤ From [10], [11], [12]:  $\zeta(z) = X(z) - Y(z)$  for  $\text{Re}(z) > 0, z \neq 1$

➤ From [13]:  $|Y(z, n)|^2$  is always a polynomial line.

➤ From [14]:  $|Y(z, n)|^2$  is only straight line if and only if  $\text{Re}(z) = 1/2$

$$|Y(z^*)|^2 = \lim_{n \rightarrow \infty} |Y(z^*, n)|^2 \text{ tends to a straight line with slope } \frac{1}{[\beta^{*2} + 1/4]}$$

➤ From [15]:  $|X(z, n)|^2$  is a wave function that has only one polynomial representation in the form of a straight line if and only if  $\text{Re}(z) = 1/2$  [18] and for certain values of  $\text{Im}(z) = \beta^*$  calculated using [19].

These values of  $\beta^*$  coincide with the imaginary parts of the nontrivial zeros of Riemann Zeta  $z^*$ , so:

$$|X(z^*)|^2 = \lim_{n \rightarrow \infty} |X(z^*, n)|^2 \text{ tends to a straight line with slope } \frac{1}{[\beta^{*2} + 1/4]}$$

when  $\text{Re}(z) = 1/2$  and  $\beta = \text{NTZ of RZF}$

➤ Therefore  $|X(z^*)|^2 = |Y(z^*)|^2$  and  $|X(z)| = |Y(z)|$  only occur when  $\text{Re}(z^*) = 1/2$

➤ As  $\zeta(z) = X(z) - Y(z)$ , therefore all zeros of  $\zeta(z)$  for  $z >= 0, z \neq 1$  have  $\text{Re}(z) = 1/2$ . [QED]

Functions  $|\zeta(z)|, |X(z)|, |Y(z)|$ , and  $|X-Y|$   
 $\text{Re}(z) = 0.5$  Pedro Caceres, 2020

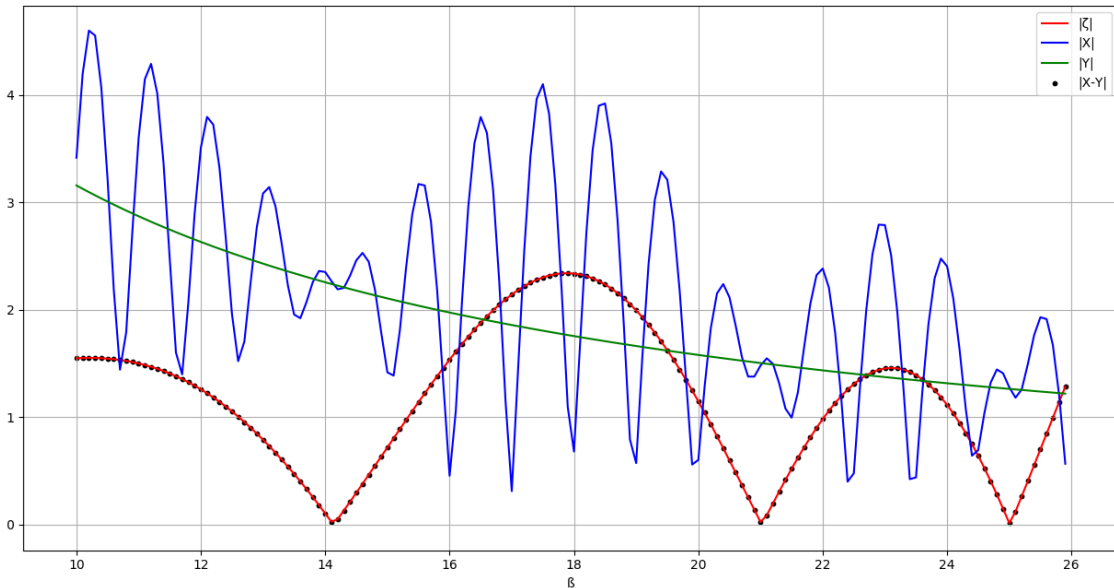


Figure. 25: for  $\zeta(z) = X(z) - Y(z) = 0 \rightarrow |X(z)| = |Y(z)|$  for  $\text{Re}(z) = 1/2$ ,

## **PART 5:**

### **On the distribution of the zeroes of the RZF in the critical line**

1. From [17] one can write:

$$X(z, n) = \left( \sum_{k=1}^n k^{-2\alpha} + \sum_{k=1}^n \sum_{j \neq k}^n k^{-\alpha} * j^{-\alpha} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right)$$

therefore, the limit of  $(|X(z, n)|^2 / n)$

$$\lim_{n \rightarrow \infty} (|X(z, n)|^2 / n) = \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n k^{-2\alpha} + \sum_{k=1}^n \sum_{j \neq k}^n k^{-\alpha} * j^{-\alpha} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right)$$

From [18], if  $\lim_{n \rightarrow \infty} (|X(z, n)|^2 / n)$  exists will be equal to:

$$\lim_{n \rightarrow \infty} (|X(z, n)|^2 / n) = \frac{1}{\beta^2 + 1/4} \quad \text{if } z = \frac{1}{2} + i\beta$$

2. Calculating the nontrivial zeros of  $\zeta(z)$  using the Harmonic function

From the previous equations, and for any  $z^* = \frac{1}{2} + \beta i$ , a nontrivial zero of Zeta in the critical line  $\alpha=1/2$ , one can write:

$$\sum_{k=1}^n k^{-1} \rightarrow \frac{n}{(\beta^2 + \frac{1}{4})} - \sum_{k=1}^n \sum_{j \neq k}^n k^{-1/2} * j^{-1/2} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \quad \text{when } n \rightarrow \infty$$

Where  $H_n = \sum_{k=1}^n k^{-1}$  is the Harmonic function. One can simplify the expression by creating functions  $O(n)$  and  $P(n)$ :

$$O(n) = - \sum_{k=1}^n \sum_{j \neq k}^n k^{-1/2} * j^{-1/2} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right)$$

And

$$P(n) = \frac{n}{(\beta^2 + \frac{1}{4})}$$

From the definition of limit, one can write that for any  $\varepsilon$  arbitrarily small, there exists and  $N$  such that for any  $n > N$ :

$$H_n - (O(n) + P(n)) < \varepsilon \quad [20]$$

If  $H(n) = O(n) + P(n)$ , then [20] can be written as:

$$H_n - H(n) < \varepsilon \quad [21]$$

The following chart shows the representation of  $H(n)$ ,  $O(n)$ , and  $P(n)$  [ $P(n)$  is a straight line with slope  $\frac{1}{(\beta^2 + 1/4)}$ ]:

$P(n) = H_n - O_n \rightarrow$  STRAIGHT LINE for all  $\text{Im}(z^*)$  NT Zero of Zeta  
slope =  $1/(1/4 + \text{Im}(z^*)^2)$

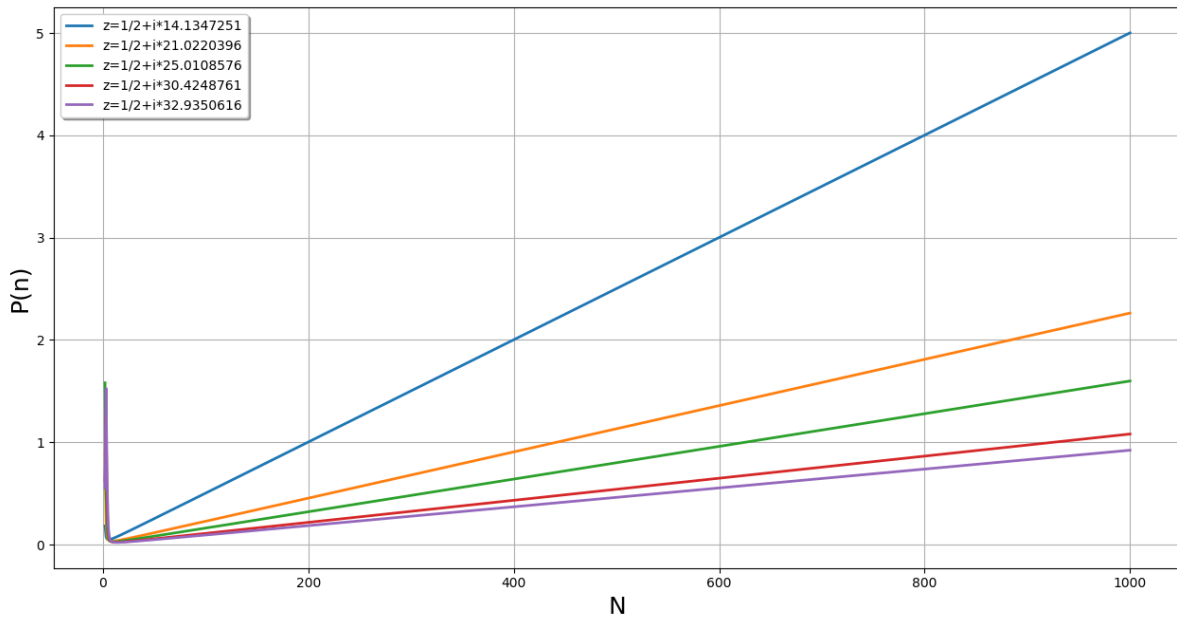


Figure. 26. Straight Lines  $P(n)$

The equation [20] can be used to create an algorithm to find the nontrivial zeros of zeta in the critical line without knowing any of them based on their connection to the Harmonic function.

An example of a Python code to calculate the zeros of zeta in the critical line with 1 decimal places accuracy based on [20]:

```
# __Python 3.7
# __Pedro Caceres__ 2020 Feb 17
#Rough code to find zeros of Riemann Zeta using the Harmonic function
harmo = 0
epsilon = 0.01
nn = 50

for j in range(1,nn):
    harmo += 1 / j
print('Harmonic(',nn,')=', harmo)

for b in range (1,500):
    b = b / 10
    a1 = nn/((1-alfa)**2 + b**2)
    b1 = 0
    for k in range(1,nn):
        for j in range(1,nn):
            if j!=k:
                b1 += (k*j)**(-alfa) * m.cos(b * m.log(k/j))
    h1=a1-b1

    if abs(h1-harmo) < epsilon:
        print('-----> Solution beta=',b, ' ... and->', h1-harmo)
#end_of_code
```

This code tends the following results:

```
Harmonic( 50 )= 4.4792053383294235

-----> Solution beta= 14.1 ... and error -> 0.0067952158225219605
-----> Solution beta= 25.0 ... and error -> -0.008460202279115592
-----> Solution beta= 30.4 ... and error -> 0.0024237587453344034
-----> Solution beta= 37.6 ... and error -> 0.0012958863904977136
-----> Solution beta= 40.9 ... and error -> -0.009083573623293262
-----> Solution beta= 48.0 ... and error -> -0.0027214317425938717
-----> Solution beta= 49.6 ... and error -> 0.0024275253143217768
```

These values compared to (Odlyzko):

```
β(1) = 14.134725142
β(3) = 25.010857580
β(4) = 30.424876126
β(6) = 37.586178159
β(7) = 40.918719012
β(9) = 48.005150881
β(10) = 49.773832478
```

Changing the values of “n” and epsilon, one can increase the accuracy of the results.

The fact that the Harmonic function,  $H_n$ , can be expressed in an infinite number of ways as a function of any  $\beta = \text{Im}(z)$  imaginary part of a nontrivial solution of  $\zeta(z)$ , provides also an algorithm to calculate all nontrivial zeros from any known zero.

Let’s define the function:

$$H(\alpha, \beta, n) \rightarrow \frac{n}{\left(\beta^2 + \frac{1}{4}\right)} - \sum_{k=1}^n \sum_{j \neq k}^n k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos\left(\beta * \left(\ln\left(\frac{k}{j}\right)\right)\right) \text{ when } n \rightarrow \infty$$

For  $\alpha = 1/2$ , and  $\varepsilon$  arbitrarily small, for any two nontrivial zeros of zeta  $(\alpha, \beta_1)$  and  $(\alpha, \beta_2)$ , there exists and  $N$  such that for any  $n > N$ :

$$H\left(\alpha = \frac{1}{2}, \beta_1, n\right) - H\left(\alpha = \frac{1}{2}, \beta_2, n\right) < \varepsilon \quad [21]$$

This proposition means that the nontrivial zeros of the Riemann Zeta are not distributed randomly, and they follow a defined structure.

Sample code to show how [21] can be used to find zeros based on a known zero:

```
# Code to find zeros from any known zero
# __Pedro Caceres__ 2020 Feb 17
nn = 60 #Not really high. Used for a rough calculation

epsilon = 0.00002
# Known Zero β(1)
zero= 14.134725142

#Calculating H(1/2, zero, n) = a - b
a2 = nn/((1-alfa)**2 + zero**2)
b2=0
for k in range(1, nn):
    for j in range(1, nn):
```

```

        if j != k:
            b2 += (k * j) ** (-alfa) * m.cos(zero * m.log(k / j))

h2 = a2-b2 #H2 to compare against

# range to find additional zeros of zeta
for b in range (245000,310000): #adding digits increases accuracy
    b = b / 10000

#Calculating a, b
a1 = nn/((1-alfa)**2 + b**2)
b1 = 0
for k in range(1,nn):
    for j in range(1,nn):
        if j!=k:
            b1 += (k*j)**(-alfa) * m.cos(b * m.log(k/j))

#Calculating H1
h1=a1-b1

#If error < epsilon, then print potential zero
if abs(h1-h2) < epsilon:
    print('-----> Solution beta=',b, ' ... and error ->', h1-h2)

#end_of_code

```

### Results:

```

-----> Solution beta= 25.0155 ... and error -> +1.442262027140373 e-05
-----> Solution beta= 30.4385 ... and error -> -1.140533215249206 e-05

```

These values compared to (Odlyzko):

$$\beta(3) = 25.010857580$$

$$\beta(4) = 30.424876126$$

Changing the values of the variable “nn” and epsilon in the code, the accuracy can be increased to more decimal places.

### 3. Conclusion

The distribution of the nontrivial zeros of the Riemann Zeta function in the critical line is not random. They are located in values of  $z^* = \frac{1}{2} + \beta i$  that verify that for any  $\beta$ , and  $\varepsilon$  arbitrarily small, there exists and  $N$  such that for any  $n > N$ :

$$\sum_{k=1}^n k^{-1} - \left( \frac{n}{\left(\beta^2 + \frac{1}{4}\right)} - \sum_{k=1}^n \sum_{j \neq k}^n k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos \left( \beta \left( \ln \left( \frac{k}{j} \right) \right) \right) \right) < \varepsilon \quad [22]$$

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