

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS.

PART 10: JENSEN’S FORMULA, THE ENERGY SLOPE AND WEIL’S FUNCTIONAL

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ABSTRACT. This is Part 10 of a series on the cross-energy $C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma}$ of the partial sums of ζ . The wave of C_2 satisfies a Jensen formula for every $0 < \sigma < 1$; on the critical line its weight is the harmonic measure seen from the pole, and the resulting criterion is that of Balazard, Saias and Yor: $\sum_{\beta > 1/2} \log |\rho/(1 - \rho)| \geq 0$, with equality if and only if RH holds. The family in σ recovers the supremum Θ of the real parts of the zeros, and the second moment of the wave is exact at every N . RH is also equivalent to the positivity of the energy slope $S(\sigma, t) = \partial_\sigma \log |\xi(\sigma + it)|$ for $\sigma > \frac{1}{2}$ (Lagarias). An off-line pair lowers S exactly inside a disc, the failure set has measure at most $2\sqrt{\sigma(1 - \sigma)}N(\sigma, T + 1)$, and S is Weil’s functional at a Poisson kernel, whose truncation is an exact finite expression in the prime powers up to e^L . Every criterion is tested on the Davenport–Heilbronn function, which violates each one as predicted. We do not prove the Riemann hypothesis.

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1. INTRODUCTION

Let $X_N(s) = \sum_{k \leq N} k^{-s}$ be the partial sums of the Dirichlet series of ζ , with $s = \sigma + it$. The *cross-energy* of consecutive partial sums is

$$C_2(N, s) = |X_N(s)|^2 - |X_{N-1}(s)|^2 - N^{-2\sigma} = 2 \operatorname{Re} (X_{N-1}(s) N^{-\bar{s}}) = 2N^{-\sigma} \sum_{n < N} n^{-\sigma} \cos \left(t \log \frac{N}{n} \right).$$

It records how the newest term N^{-s} interacts with all the previous ones. This part gathers the global positivity criteria in cross-energy form: a Jensen formula for the wave of C_2 , which gives the criterion of Balazard, Saias and Yor, and the energy slope $\partial_\sigma \log |\xi|$, which is Lagarias' criterion and Weil's functional at a Poisson kernel.

Throughout, $\rho = \beta + i\gamma$ is a nontrivial zero of ζ , $\Theta = \sup \beta$, $\lambda = s(1-s)$, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, and

$$C_B = \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma_0 - \log 4\pi = 0.0461914\dots,$$

the sum taken in symmetric order [Da]; under RH, $\rho(1-\rho) = |\rho|^2 = \frac{1}{4} + \gamma^2$. We write $Y_N(s) = N^{1-s}/(1-s)$ for the continuous envelope of X_N and $X_N^C = X_N - \frac{1}{2}N^{-s}$.

This paper is Part 10 of the series *Cross-energy of Dirichlet partial sums* (seventeen parts; the plan of the series is in Part 1 [CE1]). It uses the notation and the basic results of Part 1, in particular [CE1, Theorem A], which are cited with the prefix [CE1].

1.1. Main results. Jensen's formula (Section 2). The wave of C_2 satisfies a Jensen formula for every $0 < \sigma < 1$; on the line its weight is the harmonic measure seen from the pole, and the resulting criterion is that of Balazard, Saias and Yor [BSY]: $\mathcal{J} = \sum_{\beta > 1/2} \log |\rho/(1-\rho)| \geq 0$ and RH holds if and only if $\mathcal{J} = 0$ (Theorem 2.7, Corollary 2.8). The family in σ recovers Θ , and the second moment of the wave is exact at every N .

The energy slope and Weil's functional (Section 3). RH is equivalent to $S(\sigma, t) = \partial_\sigma \log |\xi(\sigma + it)| > 0$ for $\sigma > \frac{1}{2}$ [La, SD]. An off-line pair lowers S exactly inside a disc (Theorem 3.2), the failure set has measure at most $2\sqrt{\sigma(1-\sigma)}N(\sigma, T+1)$ (Theorem 3.7), and S is Weil's functional at a Poisson kernel, whose truncation is an exact finite expression in the prime powers up to e^L (Theorem 3.13).

1.2. What is new and what is classical. The C_2 form of the Jensen formula and of the Balazard–Saias–Yor criterion, the disc lemma, the exceptional-set bound and the exact truncation of Weil's functional are, as far as we know, new in this form. The criteria of Balazard, Saias and Yor, of Lagarias and of Weil, and the zero-density theorems, are classical and cited as such. We do not prove the Riemann hypothesis.

2. THE JENSEN FORMULA OF THE CROSS-ENERGY WAVE

[CE1, Theorem A] describes the cross-energy at one point s as $n \rightarrow \infty$. This section shows that every single-point observable built from C_2 carries no more than the three numbers σ , t and $\zeta(s)$, and then works globally in t . Two facts make the passage possible: the logarithmic energy of the oscillation of C_2 around its envelope is $2|\zeta(s)|^2$, and the coefficient of the envelope is 2π times the harmonic measure of the vertical line seen from the pole $s = 1$. Together they give a Jensen formula for C_2 that holds on every vertical line of the strip and determines $\Theta = \sup \operatorname{Re} \rho$. On the critical line it is the criterion of Balazard, Saias and Yor [BSY], written in the language of the

cross-energy. The material comes from [C4]; it is reorganized here, with the overlaps with Part 1 [CE1] and Part 5 [CE5] removed.

Throughout, $s = \sigma + it$, $W_n(s) = C_2(n, s) + iS_2(n, s) = 2X_{n-1}(s)n^{-\bar{s}}$, and

$$A(\sigma, t) = \operatorname{Re} \frac{2}{1-s} = \frac{2(1-\sigma)}{(1-\sigma)^2 + t^2}, \quad L(t) = A(\tfrac{1}{2}, t) = \frac{1}{\frac{1}{4} + t^2}.$$

By [CE1, Theorem A], for $0 < \sigma < 1$,

$$W_n(s) = \frac{2n^{1-2\sigma}}{1-s} + 2\zeta(s)n^{-\bar{s}} - n^{-2\sigma} + O(|s|n^{-2\sigma-1}), \quad (2.1)$$

and taking real parts,

$$C_2(n, s) = A(\sigma, t)n^{1-2\sigma} + 2n^{-\sigma} \operatorname{Re} [\zeta(s)n^{it}] - n^{-2\sigma} + O(|s|n^{-2\sigma-1}). \quad (2.2)$$

The first term is the envelope, the trace of the pole; the second is the wave, which carries $\zeta(s)$.

2.1. The locality principle. Proposition 2.1 (locality principle). Let $0 < \sigma < 1$. Any quantity built from $C_2(n, s)$ and $S_2(n, s)$ has asymptotics, as $n \rightarrow \infty$, determined by (σ, t) and the single complex number $\zeta(s)$. The leading term depends on (σ, t) only, whether or not $\zeta(s) = 0$; the zero condition removes only the wave, which is of relative order $n^{\sigma-1}$. In particular:

- (a) *Angle and tangent.* Let $\varphi_n = \arg W_n(s)$. Then $\tan \varphi_n = S_2/C_2 \rightarrow t/(1-\sigma)$ at every point of the strip. Extend C_2 to real $x \geq 1$ as in Proposition 2.2 below and let $\vartheta_n = -xC_2'(x)/C_2(x)$ at $x = n$ be the logarithmic tangent. Then $\vartheta_n = \sigma + t \tan \varphi_n$ exactly, so $\vartheta_n \rightarrow \vartheta_\infty = \sigma + t^2/(1-\sigma)$ everywhere, and

$$\vartheta_\infty - 2t^2 - \frac{1}{2} = \frac{(2\sigma-1)(2t^2+1-\sigma)}{2(1-\sigma)}$$

has the sign of $\sigma - \frac{1}{2}$ at every point, zero or not.

- (b) *Oriented area.* With $Y_N(s) = N^{1-s}/(1-s)$, the oriented area $\mathcal{A}_N = -\operatorname{Im}(X_N \overline{Y_N})$ is the fixed linear combination

$$\mathcal{A}_N = \frac{N}{|1-s|^2} \left[t \left(\frac{1}{2} C_2(N, s) + N^{-2\sigma} \right) - \frac{1}{2} (1-\sigma) S_2(N, s) \right].$$

At a zero, $\mathcal{A}_N \sim tN^{1-2\sigma}/(2|1-s|^2)$, which diverges, stays bounded away from 0, or collapses according as $\sigma < \frac{1}{2}$, $\sigma = \frac{1}{2}$ or $\sigma > \frac{1}{2}$. At a point that is not a zero, $\mathcal{A}_N$ oscillates with amplitude of order $|\zeta(s)|N^{1-\sigma}$.

- (c) *Angle of X_N against Y_N .* At a zero, $\arg X_N - \arg Y_N = -\frac{t}{2N} + \frac{t(2-\sigma)}{12N^2} + O_s(N^{-3})$.

Proof. The first assertion is (2.1). (a) By (2.1), $W_n = \frac{2n^{1-2\sigma}}{1-s}(1 + O(n^{\sigma-1}))$, and $\arg \frac{1}{1-s} = \arg((1-\sigma) + it)$. The identity for ϑ_n follows from Proposition 2.2: between integers $\frac{d}{du}W = (-\sigma + it)W$ with $u = \log x$, so $\frac{d}{du}C_2 = -\sigma C_2 - tS_2$ and $\vartheta = \sigma + tS_2/C_2$. The displayed identity is algebra: $\sigma - \frac{1}{2} + t^2(\frac{1}{1-\sigma} - 2) = (2\sigma-1)(\frac{1}{2} + \frac{t^2}{1-\sigma})$. (b) Since $X_N = X_{N-1} + N^{-s}$, $X_N \overline{Y_N} = \frac{N(1-s)}{|1-s|^2}(\frac{1}{2}W_N + N^{-2\sigma})$; write $\frac{1}{2}W_N + N^{-2\sigma} = a + ib$ with $a = \frac{1}{2}C_2 + N^{-2\sigma}$, $b = \frac{1}{2}S_2$, and use $\operatorname{Im}((a+ib)(1-\sigma-it)) = b(1-\sigma) - at$. At a zero, by (2.1), the terms of order $N^{1-2\sigma}$ in $at - b(1-\sigma)$ cancel, and the next term is $\frac{1}{2}tN^{-2\sigma}$. Off a zero the wave contributes $O(|\zeta(s)|N^{-\sigma})$ to a and b . (c) From the Euler–Maclaurin expansion, $X_N = \zeta(s) + Y_N + \frac{1}{2}N^{-s} - \frac{s}{12}N^{-s-1} + O(|s|^3N^{-\sigma-3})$, so at a zero $X_N/Y_N = 1 + \frac{1-s}{2N} - \frac{s(1-s)}{12N^2} + O(N^{-3})$. Taking the imaginary part of the logarithm, with $\operatorname{Im}(1-s) = -t$, $\operatorname{Im} s(1-s) = t(1-2\sigma)$ and $\operatorname{Im}(1-s)^2 = -2t(1-\sigma)$, gives $-\frac{t}{2N} + \frac{t}{N^2}(\frac{1-\sigma}{4} - \frac{1-2\sigma}{12}) = -\frac{t}{2N} + \frac{t(2-\sigma)}{12N^2}$. $\square$

At $\rho_1 = \frac{1}{2} + 14.1347i$ we computed $N^2(\arg X_N - \arg Y_N + t/(2N)) \rightarrow 1.76679$, against $t(2 - \sigma)/12 = 1.76684$.

Status: proved; the check of (c) is numerical (`anc/c2_jensen/c2_part4_numerics.py`, option `angle`).

Proposition 2.2 (rotation form of the dynamics). Extend W to real $x \geq 1$ by $W(x) = 2x^{-s} \sum_{j < x} j^{-s}$ and put $u = \log x$.

- (i) On each open interval $(N, N+1)$, $\frac{dW}{du} = (-\sigma + it)W$: the point W rotates at angular speed t and contracts at rate σ .
- (ii) At $x = N^+$ the term $j = N$ enters and W jumps by the real positive amount $2N^{-2\sigma}$.

Consequently the logarithmic tangent $\vartheta = -\frac{dC_2/du}{C_2} = \sigma + t \tan \arg W$ satisfies the Riccati equation

$$\frac{d\vartheta}{du} = (\vartheta - \sigma)^2 + t^2$$

between integers, and the product $(\vartheta - \sigma)C_2 = tS_2$ is continuous across the jumps.

Proof. Each term $2x^{-\sigma+it}j^{-\sigma-it}$ satisfies $dW_j/du = (-\sigma + it)W_j$, and at $x = j$ it equals $2j^{-2\sigma}$. For the Riccati equation, differentiate $\vartheta - \sigma = t \tan \varphi$ with $d\varphi/du = t$. Since the jump of W is real, S_2 is continuous. $\square$

Status: proved.

Remark 2.3 (why pointwise criteria cannot work). Proposition 2.1 produces many criteria of the form “ $\sigma = \frac{1}{2}$ at every zero”, each equivalent to RH: the tangent bound $\vartheta_\infty \leq 2t^2 + \frac{1}{2}$ at every zero, the non-collapse of $\mathcal{A}_N$ at every zero, and the limit criterion of Part 1 [CE1] ($0 < \lim_n C_2(n, \rho) < \infty$ at every zero; either inequality alone suffices, since $\rho \mapsto 1 - \bar{\rho}$ exchanges the cases $\beta > \frac{1}{2}$ and $\beta < \frac{1}{2}$). Each is a statement about the position (β, γ) of a zero and nothing else, and each follows from (2.1), which holds verbatim for any Dirichlet series with a simple pole of residue 1 at $s = 1$ and coefficients of mean 1. Proposition 2.2 shows why: between integers the dynamics of W is a universal rotation at speed t , driven by the real kicks $2n^{-2\sigma}$, so the geometry of one trajectory contains no more than $(\sigma, t, \zeta(s))$. For the Epstein zeta function Z_Q of $Q = m^2 + 100n^2$, normalized to residue 1, the same computation at its real zero $\beta_0 = 0.813367836786$ and at the partner $1 - \beta_0$ (here $t = 0$ and $A = 2/(1 - \sigma)$) gives

N	C_2 at β_0	$A N^{1-2\beta_0}$	C_2 at $1 - \beta_0$	$A N^{2\beta_0-1}$
10^3	0.14136	0.14120	187.9	186.6
10^5	0.00788	0.00788	3345	3345
$2 \cdot 10^6$	0.00120	0.00120	21871	21869

One limit collapses to 0 and its partner diverges, exactly as [CE1, Theorem A] describes for a hypothetical off-line zero of ζ . A proof of any of these criteria must therefore use a property of ζ that Z_Q lacks. The rest of the section looks for a statement that can be computed without knowing where the zeros are. Status: the criteria are proved equivalent to RH; the table is numerical (`anc/c2_jensen/c2_part4_numerics.py`, option `epstein_limit`).

Remark 2.4 (the Bender–Brody–Müller dictionary). Let $\zeta(s, a) = \sum_{n \geq 0} (n + a)^{-s}$ be the Hurwitz zeta function. For every $s \neq 1$ and every N , $X_N(s) = \zeta(s) - \zeta(s, N+1)$. Bender, Brody and Müller [BBM] proposed an operator $\hat{H}$, with classical limit $2xp$, whose formal eigenfunctions are $\psi_s(x) = -\zeta(s, x+1)$; the boundary condition $\psi_s(0) = 0$ is the equation $\zeta(s) = 0$, and the eigenvalue is real exactly when $\operatorname{Re} s = \frac{1}{2}$. At a zero ρ the partial sums are the eigenfunction sampled at the integers, $X_N(\rho) = \psi_\rho(N)$, and

$$C_2(N, \rho) = |\psi_\rho(N)|^2 - |\psi_\rho(N-1)|^2 - N^{-2\beta}.$$

So C_2 measures the growth of $|\psi_\rho|^2$, and the trichotomy of [CE1, Theorem A] at a zero is a statement about normalizability. Since $|\psi_\rho(x)|^2 = x^{2-2\beta}|1-\rho|^{-2}(1+o(1))$, ψ_ρ lies in $L^2((0, \infty), (1+x)^{-2}dx)$ if and only if $\beta > \frac{1}{2}$, and at $\beta = \frac{1}{2}$ it diverges only logarithmically, like a generalized eigenfunction. In this dictionary $\lim C_2 = 0$ corresponds to a square-integrable eigenfunction with a non-real eigenvalue, $\lim C_2 = 1/(\frac{1}{4} + \gamma^2)$ to a generalized eigenfunction with a real eigenvalue, the limit being its normalization constant, and $\lim C_2 = \infty$ to the mirror case $\beta < \frac{1}{2}$. A self-adjoint realization of $\hat{H}$ on this space would exclude the first case, and RH would follow; this is the conditional statement of [BBM]. We do not claim that any such realization exists. Bellissard [Bel] showed that the construction fails as stated: the momentum operator on the half-line with $\psi(0) = 0$ has no self-adjoint extension, and weighting the space changes the operator. The Davenport–Heilbronn function has analogous tail functions, and its off-line zeros would give square-integrable eigenfunctions with non-real eigenvalues for the analogous operator, so a proof that an operator of this type is self-adjoint must use the Euler product. The remark translates the limit criterion of Part 1 [CE1] into operator language and does not remove the difficulty. Status: the identity $X_N = \zeta - \zeta(\cdot, N+1)$ and the integrability statement are classical; the dictionary is heuristic.

2.2. The wave energy and the harmonic measure of the pole. Lemma 2.5 (wave energy). Let $0 < \sigma < 1$, $t \neq 0$, and

$$V_N(\sigma, t) = N^\sigma \left(C_2(N, \sigma + it) - A(\sigma, t)N^{1-2\sigma} + N^{-2\sigma} \right).$$

Then

$$\Omega_\sigma(t) := \lim_{M \rightarrow \infty} \frac{1}{\log M} \sum_{N \leq M} \frac{V_N^2}{N} = 2|\zeta(\sigma + it)|^2. \quad (2.3)$$

On the critical line this reads $\Omega(t) := \Omega_{1/2}(t) = \lim_M \frac{1}{\log M} \sum_{N \leq M} (C_2(N, \frac{1}{2} + it) - L(t) + \frac{1}{N})^2$, where $L(t)$ is the limit of C_2 at the zeros on the line.

Proof. By (2.2), $V_N = 2\operatorname{Re}[\zeta(s)N^{it}] + O(|s|N^{-\sigma-1})$, so $V_N^2 = 2|\zeta(s)|^2 + 2\operatorname{Re}[\zeta(s)^2N^{2it}] + O(N^{-\sigma-1})$. For $t \neq 0$ the sums $\sum_{N \leq M} N^{-1+2it}$ are bounded, since they equal $\zeta(1-2it) + M^{2it}/(2it) + o(1)$. Hence $\sum_{N \leq M} V_N^2/N = 2|\zeta(s)|^2 \log M + O_s(1)$. $\square$

Status: proved.

The trajectory of C_2 thus carries $|\zeta(s)|$ in the logarithmic energy of its oscillation around the envelope, at every point and not only at zeros. Estimating Ω as the slope of $\sum V_N^2/N$ against $\log M$ for $10^4 < N \leq 10^6$:

σ	t	Ω from C_2	$2 \zeta(\sigma + it) ^2$
0.5	20	2.6143	2.6351
0.5	17	9.1553	9.1824
0.7	20	2.1609	2.1805

The residual discrepancy is the $O(1/\log M)$ term of the proof (`anc/c2_jensen/c2_part4_numerics.py`, option `omega`).

For $\sigma < 1$ let $\mathbb{H}_\sigma = \{\operatorname{Re} s > \sigma\}$. The Poisson kernel of $\mathbb{H}_\sigma$ at the point $s = 1$ is

$$P_\sigma(t) = \frac{1}{\pi} \frac{1-\sigma}{(1-\sigma)^2 + t^2}.$$

Proposition 2.6 (the envelope is harmonic measure). $A(\sigma, t) = 2\pi P_\sigma(t)$. In particular $L(t) = A(\frac{1}{2}, t) = 2\pi P_{1/2}(t)$, and

$$d\mu(t) = \frac{L(t)}{2\pi} dt = \frac{dt}{2\pi(\frac{1}{4} + t^2)}$$

is a probability measure on the critical line: the harmonic measure of $\operatorname{Re} s = \frac{1}{2}$ seen from $s = 1$.

Proof. Direct comparison of the two formulas. $\square$

Status: proved.

So the coefficient of the envelope of C_2 , which on the critical line is also its limit at the zeros, is 2π times the harmonic measure of the line $\operatorname{Re} s = \sigma$ seen from the pole. The pole, which is an obstruction for positivity arguments (Part 5 [CE5], the pole barrier), is here the natural point of observation. The same weight $L(\gamma) = \lim_n C_2(n, \rho)$ is the weight of a zero in the energy of the primes ([CE5, Theorem B]) and in Li's first coefficient (Part 5 [CE5]).

2.3. The Jensen formula. Theorem 2.7 (C_2 -Jensen formula). For $0 < \sigma < 1$,

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} \log \frac{\Omega_{\sigma}(t)}{2} A(\sigma, t) dt = \sum_{\beta > \sigma} \log \left| \frac{1 - 2\sigma + \bar{\rho}}{1 - \rho} \right| - \log(2(1 - \sigma)). \quad (2.4)$$

The sum runs over the nontrivial zeros $\rho = \beta + i\gamma$ with $\beta > \sigma$, counted with multiplicity and for both signs of γ , and every term of the sum is positive.

Proof. By Lemma 2.5 and Proposition 2.6 the left side equals $\int \log |\zeta(\sigma + it)| P_{\sigma}(t) dt$. Let $f(s) = (s-1)\zeta(s)/s$. It is analytic in a neighbourhood of $\mathbb{H}_{\sigma}$, satisfies $f(1) = 1$, and is $O(|s|)$ there by the convexity bound, so $g(s) = f(s)/(s+1)$ is bounded and analytic in $\mathbb{H}_{\sigma}$. Take the canonical factorization of g in the half-plane [Ga, Ch. II], $g = BSO$:

- B is the Blaschke product over the zeros of ζ in $\mathbb{H}_{\sigma}$, with factors $(s - \rho)/(s - \rho^*)$, $\rho^* = 2\sigma - \bar{\rho}$ the reflection of ρ in the line $\operatorname{Re} s = \sigma$;
- S is trivial: there is no boundary singular measure, because g is analytic across the line, and no mass at infinity, because $\log |g(x)|/x \rightarrow 0$ as $x \rightarrow +\infty$;
- O is outer, so $\log |O(1)| = \int \log |g(\sigma + it)| P_{\sigma}(t) dt$.

Evaluating $\log |g|$ at $s = 1$ gives

$$-\log 2 = \sum_{\beta > \sigma} \log \left| \frac{1 - \rho}{1 - \rho^*} \right| + \int \log |\zeta(\sigma + it)| P_{\sigma}(t) dt + J_1 - J_2 - J_3,$$

where J_1, J_2, J_3 are the Poisson integrals of $\log |s-1|$, $\log |s|$ and $\log |s+1|$ on the line $\operatorname{Re} s = \sigma$. The functions $\log |s|$ and $\log |s+1|$ are harmonic in $\mathbb{H}_{\sigma}$ with logarithmic growth, so $J_2 = \log 1 = 0$ and $J_3 = \log 2$. On the line, $|s-1| = |s-(2\sigma-1)|$, and $2\sigma-1 \notin \mathbb{H}_{\sigma}$, so $J_1 = \log |1-(2\sigma-1)| = \log(2-2\sigma)$. Since $1-\rho^* = 1-2\sigma+\bar{\rho}$, this is (2.4). Each term of the sum is positive because $1-2\sigma+\beta > 1-\beta > 0$ when $\beta > \sigma$. $\square$

Status: proved.

At $\sigma = \frac{1}{2}$ the constant vanishes, $1 - 2\sigma + \bar{\rho} = \bar{\rho}$, and (2.4) becomes the identity of Balazard, Saias and Yor [BSY]:

Corollary 2.8 (the Balazard-Saias-Yor criterion in C_2 form). Put

$$\mathcal{J} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \log \frac{\Omega(t)}{2} L(t) dt = \int \log |\zeta(\tfrac{1}{2} + it)| d\mu(t).$$

Then

$$\mathcal{J} = \sum_{\beta > 1/2} \log \left| \frac{\rho}{1 - \rho} \right| \geq 0, \quad \text{and} \quad \text{RH} \iff \mathcal{J} = 0. \quad (2.5)$$

In words: weight the log-energy of the C_2 wave on the critical line by the limit of C_2 itself. The result is never negative, and it vanishes exactly when RH holds. This is the criterion of [BSY]; what is added here is that both the integrand and the weight are cross-energy quantities, by Lemma 2.5 and Proposition 2.6. Each term equals $\log(1/|w_{\rho}|)$ with $w_{\rho} = 1 - 1/\rho$, the variable of Li's criterion (Part 5 [CE5]), and is approximately $(\beta - \frac{1}{2})/|\rho|^2$. For a function F of the same type without a

pole, such as the Davenport–Heilbronn function f , the same argument gives $\int \log |F(\frac{1}{2} + it)| d\mu = \log |F(1)| + \sum_{\beta > 1/2} \log |\rho/(1 - \rho)|$.

Proof. Theorem 2.7 at $\sigma = \frac{1}{2}$. Equivalently: the map $w = 1 - 1/s$ sends $\operatorname{Re} s > \frac{1}{2}$ onto the unit disc and $s = 1$ to its centre, μ is the Poisson measure of the half-plane at $s = 1$, and Poisson–Jensen for $F(s) = (s - 1)\zeta(s)/s$, for which $|F| = |\zeta|$ on the line and $F(1) = 1$, gives the identity. $\square$

Status: classical [BSY]; the C_2 form is proved.

Equivalently, $-\mathcal{J} = \log |B(1)|$ for the Blaschke product $B(s) = \prod_{\beta > 1/2} (s - \rho)/(s - 1 + \bar{\rho})$ of the half-plane, and RH states that $(s - 1)\zeta(s)/s$ is an outer function there. We computed $\mathcal{J}$ by quadrature between consecutive zeros on the line. For ζ , the partial integral $J_\zeta(T) = \int_{-T}^T \log |\zeta(\frac{1}{2} + it)| d\mu$ is $3.1 \cdot 10^{-8}$ at $T = 1000$ (Figure 1a). For the Davenport–Heilbronn function, $\log |f(1)| = -0.0803407$ and $J_f(400) - \log |f(1)| = 1.52477 \cdot 10^{-4}$, against $1.52687 \cdot 10^{-4}$ from its eight off-line zeros below height 400 and their conjugates; the value without off-line zeros, 0, is excluded by a factor 700 of the discrepancy. The partial integral rises in steps at the heights of the off-line zeros (Figure 1b), so the integral localizes them (`anc/jensen/`).

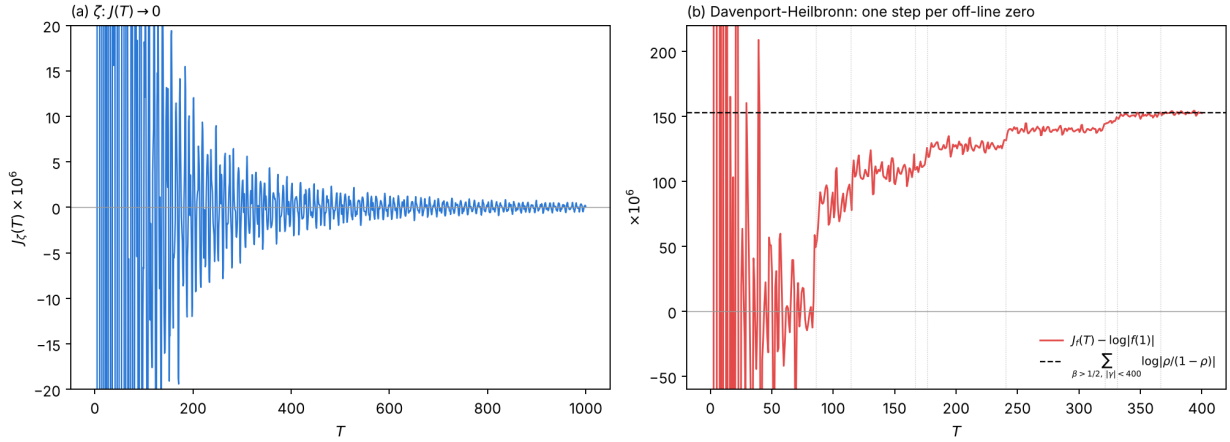


FIGURE 1. The Jensen integral of the C_2 wave (Corollary 2.8). (a) $J_\zeta(T) = \int_{-T}^T \log |\zeta(\frac{1}{2} + it)| d\mu(t)$, which tends to 0. (b) $J_f(T) - \log |f(1)|$ for the Davenport–Heilbronn function; dashed: the sum of $\log |\rho/(1 - \rho)|$ over its off-line zeros below height 400; dotted: their heights.

The Poisson identity at $s = 1$ for partial products and prime sums. The harmonic measure μ of the pole links Corollary 2.8 with the arithmetic of Part 5 [CE5] through two exact identities. (i) The partial Euler product $E_P(s) = \prod_{p \leq P} (1 - p^{-s})^{-1}$ is zero-free and bounded in $\operatorname{Re} s \geq \frac{1}{2}$, so $\log |E_P|$ is harmonic and bounded there and Poisson’s formula at $s = 1$ gives

$$\int \log |E_P(\frac{1}{2} + it)| d\mu(t) = \log \prod_{p \leq P} \left(1 - \frac{1}{p}\right)^{-1} \sim \log(e^{\gamma_0} \log P),$$

$$\int \log \left| \frac{\zeta(\frac{1}{2} + it)}{E_P(\frac{1}{2} + it)} \right| d\mu(t) = \mathcal{J} - \log \prod_{p \leq P} \left(1 - \frac{1}{p}\right)^{-1},$$

by Mertens’ theorem. (ii) Since $\int \cos(t \log p) d\mu(t) = p^{-1/2}$, the harmonic measure turns $p^{-1/2}$ into $1/p$:

$$\int \operatorname{Re} \sum_{p \leq x} p^{-1/2 - it} d\mu(t) = \sum_{p \leq x} \frac{1}{p} = \log \log x + B_1 + o(1),$$

with $B_1 = 0.2615 \dots$ the Meissel–Mertens constant: Mertens’ second theorem is the Poisson identity for the partial prime zeta function. Both identities were checked to five decimals for $P, x \leq 1999$; at $P = 1999$ the left side of the second identity in (i) is -2.60814 [C4]. By Corollary 2.8, RH holds if and only if the Poisson identity at $s = 1$, which holds for every E_P and every partial prime sum, persists for $(s - 1)\zeta(s)/s$ itself. Replacing ζ by the prime zeta function $P(s) = \sum_p p^{-s} = \sum_k \frac{\mu(k)}{k} \log \zeta(ks)$ gives no new route: $P(s) - \log \zeta(s)$ is analytic in $\operatorname{Re} s > \frac{1}{2}$, so RH holds if and only if $P(s) + \log(s - 1)$ continues analytically there, but P has no functional equation and $\operatorname{Re} s = 0$ is a natural boundary for it [LW] (its zeros in $\operatorname{Re} s > \frac{1}{2}$ are studied in [CE8, CE9]); its singularities at the zeros are logarithmic, so $\sum_{p \leq x} p^{-\rho}$ resonates at a critical zero only like $m \log \log x$, against $m \log x$ for Λ [C3, Prop. 4.1(c)]. Status: (i) and (ii) are classical identities (Poisson’s formula and Mertens’ theorems); the numbers are from [C4].

Corollary 2.9 (the Jensen family and the abscissa Θ). Let $\mathcal{J}(\sigma)$ be the left side of (2.4) plus $\log(2(1 - \sigma))$, so that $\mathcal{J}(\frac{1}{2}) = \mathcal{J}$. Then $\mathcal{J}$ is continuous and non-increasing on $(0, 1)$, and $\mathcal{J}(\sigma) = 0$ exactly for $\sigma \geq \Theta = \sup \operatorname{Re} \rho$. Hence

$$\Theta = \inf\{\sigma \in (0, 1) : \mathcal{J}(\sigma) = 0\}.$$

Proof. By (2.4), $\mathcal{J}(\sigma) = \sum_{\beta > \sigma} \log |1 - 2\sigma + \beta - i\gamma|/|1 - \rho|$. For $\beta > \sigma$ the modulus $|1 - 2\sigma + \beta - i\gamma|$ decreases in σ , and each term vanishes at $\sigma = \beta$; extending each term by 0 for $\sigma \geq \beta$ gives continuous non-increasing functions, and the series converges locally uniformly, since each term is at most $\frac{1}{2} \log(1 + 4(\beta - \sigma)(1 - \sigma)/|1 - \rho|^2) \leq 2/|1 - \rho|^2$. The sum is 0 exactly when no zero has $\beta > \sigma$. $\square$

Status: proved.

So the cross-energy determines Θ , and RH is the statement $\mathcal{J}(\frac{1}{2}) = 0$. For $\sigma < \frac{1}{2}$ every zero on the critical line contributes, and $\mathcal{J}(\sigma) > 0$ unconditionally. The constant $-\log(2(1 - \sigma))$ in (2.4) is the contribution of the pole seen from its own half-plane. We computed the left side of (2.4) over $|t| \leq 1000$ with an estimated tail, and the right side from the 341 zeros with $|\gamma| \leq 600$ with an estimated tail (`anc/c2-jensen/c2_part4_numerics.py`, option `jensen`):

σ	left side of (2.4)	right side
0.3	-0.32354	-0.32354 (all critical zeros contribute)
0.5	-0.00001	0
0.7	0.51083	$-\log 0.6 = 0.51083$

Corollary 2.10 (the global form of the limit criterion). For a zero $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$, which by [CE1, Theorem A] is exactly a zero with $\lim_n C_2(n, \rho) = 0$, put

$$w(\rho) = \log \left| \frac{\rho}{1 - \rho} \right| = \frac{1}{2} \log \frac{\beta A(\beta, \gamma)}{(1 - \beta) A(1 - \beta, \gamma)} > 0.$$

Then

$$\mathcal{J} = \sum_{\rho: \lim_n C_2(n, \rho) = 0} w(\rho), \quad \text{so} \quad \mathcal{J} = 0 \iff \lim_n C_2(n, \rho) > 0 \text{ for every zero} \iff \text{RH}.$$

Proof. $\beta A(\beta, \gamma) = 2\beta(1 - \beta)/|1 - \rho|^2$ and $(1 - \beta)A(1 - \beta, \gamma) = 2\beta(1 - \beta)/|\rho|^2$. Apply Corollary 2.8. $\square$

Status: proved.

The weight is written only with the envelope coefficients of the reflected pair $\rho, 1 - \bar{\rho}$. The limit criterion of Part 1 [CE1] needs the zeros as input; Corollary 2.10 needs only C_2 on the critical line.

2.4. Small moments and an exact finite- N identity. Define, for $0 < k \leq 1$,

$$\mathcal{M}(k) = \int \left(\frac{\Omega(t)}{2} \right)^k d\mu(t) = \int |\zeta(\tfrac{1}{2} + it)|^{2k} d\mu(t).$$

Proposition 2.11 (small moments).

- (i) $\mathcal{M}(1) = \log 2\pi - \gamma_0 = 1.2606614\dots$
- (ii) $\mathcal{M}(k) \geq e^{2k\mathcal{J}} \geq 1$ for every $0 < k \leq 1$, unconditionally.
- (iii) $\frac{1}{k} \log \mathcal{M}(k)$ decreases to $2\mathcal{J}$ as $k \downarrow 0$. Hence

$$\text{RH} \iff \mathcal{M}(k) = 1 + o(k) \quad (k \rightarrow 0^+).$$

Proof. (i) For $0 < \sigma < 1$, $\zeta(s)/s = -\int_0^\infty \{x\}x^{-s-1}dx$ [T, (2.1.5)]. With $x = e^v$ and $s = \frac{1}{2} + it$ this is a Fourier transform in v , and Plancherel gives

$$\int |\zeta(\tfrac{1}{2} + it)|^2 d\mu = \frac{1}{2\pi} \int \left| \frac{\zeta(\tfrac{1}{2} + it)}{\tfrac{1}{2} + it} \right|^2 dt = \int_0^\infty \frac{\{x\}^2}{x^2} dx = 1 + \sum_{n \geq 1} \left[1 + \frac{n}{n+1} - 2n \log \left(1 + \frac{1}{n} \right) \right],$$

which equals $\log 2\pi - \gamma_0$ (Stirling's formula for the partial sums); we checked it numerically to 10^{-12} . (ii) Jensen's inequality for the convex exponential: $\mathcal{M}(k) \geq \exp(k \int \log |\zeta|^2 d\mu) = e^{2k\mathcal{J}}$, and $\mathcal{J} \geq 0$. (iii) For a probability measure, $k \mapsto \frac{1}{k} \log \int F^k d\mu$ is non-decreasing and tends to $\int \log F d\mu$ as $k \rightarrow 0^+$ when $\log F$ is integrable; apply it to $F = |\zeta|^2$. If $\mathcal{J} = 0$ then $\log \mathcal{M}(k) = o(k)$; conversely $\mathcal{M}(k) = 1 + o(k)$ forces $2\mathcal{J} = 0$. $\square$

Status: proved; the evaluation in (i) is classical.

So $\mathcal{M}$ is log-convex with $\mathcal{M}(0) = 1$ and $\mathcal{M} \geq 1$, and RH is the statement that its right derivative at 0 vanishes (`anc/c2-jensen/c2-part4-numeric.py`, options `moments`, `mellin`):

k	0.5	0.2	0.1	0.05	0.02	0.01
$\mathcal{M}(k)$	1.064283	1.010688	1.002734	1.000693	1.000112	1.000028
$\frac{1}{k} \log \mathcal{M}(k)$	0.1246	0.0532	0.0273	0.0139	0.0056	0.0028

The last row tends to 0 linearly in k , as RH predicts: under RH, $\frac{1}{k} \log \mathcal{M}(k) \approx 2k \text{Var}_\mu(\log |\zeta|)$, with $\text{Var}_\mu(\log |\zeta|) \approx 0.14$.

The first moment can also be reached from C_2 alone, at every finite N . Put

$$B_N(t) = C_2(N, \tfrac{1}{2} + it) - L(t) + \frac{1}{N}, \quad \Omega_M(t) = \frac{1}{\log M} \sum_{N \leq M} B_N(t)^2,$$

so that $\Omega_M \rightarrow \Omega$ pointwise by Lemma 2.5.

Proposition 2.12 (exact second moment of the C_2 wave). For every $N \geq 2$,

$$\int B_N(t)^2 d\mu(t) = 4 - \frac{4}{N} \log \frac{N^{N-1}}{(N-1)!} - \frac{2H_{N-1}}{N} - \frac{1}{N^2} = \frac{2(\log 2\pi - \gamma_0)}{N} + O(N^{-2}), \quad (2.6)$$

where H_{N-1} is the harmonic number. Consequently $\lim_{M \rightarrow \infty} \int \frac{\Omega_M}{2} d\mu = \log 2\pi - \gamma_0 = \mathcal{M}(1)$.

Proof. The Fourier transform of μ is $\int e^{itu} d\mu(t) = e^{-|u|/2}$. With $u = \log(N/j)$, $v = \log(N/k)$ and $j, k < N$,

$$\int \cos(tu) \cos(tv) d\mu = \frac{1}{2} \left[\sqrt{\frac{\min(j, k)}{\max(j, k)}} + \frac{\sqrt{jk}}{N} \right].$$

Since $C_2(N, \tfrac{1}{2} + it) = 2N^{-1/2} \sum_{j < N} j^{-1/2} \cos(t \log \frac{N}{j})$,

$$\int C_2^2 d\mu = \frac{2}{N} \sum_{j, k < N} \left[\frac{1}{\max(j, k)} + \frac{1}{N} \right] = 6 - \frac{2H_{N-1}}{N} - \frac{8}{N} + \frac{2}{N^2},$$

using $\sum_{j,k < N} 1/\max(j, k) = 2(N-1) - H_{N-1}$. In the same way $\int C_2 d\mu = 2(N-1)/N$, $\int L d\mu = 2$, $\int L^2 d\mu = 6$, and, since $\int \cos(tu)L(t) d\mu = 2(1 + \frac{|u|}{2})e^{-|u|/2}$, $\int C_2 L d\mu = \frac{4(N-1)}{N} + \frac{2}{N} \sum_{j < N} \log \frac{N}{j}$. Expanding B_N^2 and using $\sum_{j < N} \log(N/j) = \log(N^{N-1}/(N-1)!)$ gives the exact form of (2.6). Stirling's formula, $\log \frac{N^{N-1}}{(N-1)!} = N - \frac{1}{2} \log N - \frac{1}{2} \log 2\pi + O(N^{-1})$, and $H_{N-1} = \log N + \gamma_0 + O(N^{-1})$ give the asymptotic form; the constants and the terms $\pm 2 \log N/N$ cancel. Summing over $N \leq M$ and dividing by $\log M$ gives the limit. $\square$

Status: proved.

Numerically, (2.6) agrees with direct quadrature: 0.51891 against 0.51884 at $N = 5$ and 0.050561 against 0.050547 at $N = 50$, the difference coming from truncating at $|t| \leq 4000$ (`anc/c2_jensen/c2_part4_numerics.py`, option `finite_identity`). Proposition 2.12 recovers $\mathcal{M}(1)$ from finite C_2 data by Stirling's formula, without the Mellin transform. It gives $\lim_M \int \Omega_M d\mu = \int \lim_M \Omega_M d\mu$; we do not have a uniform-integrability argument that would give the same interchange for $\log \Omega_M$.

2.5. Why the criterion is sharp, and why it is hard. The filters. Against the filters of [CE2, Definition 2.22], the pointwise criteria of Remark 2.3 fail (F1) and (F3). Corollary 2.8 passes (F1) and (F2); for (F3), its left side is computable from C_2 on the line, but through Theorem 2.7 it is the statement that no zero has $\beta > \frac{1}{2}$, so it is not a new mechanism. For (F1), Theorem 2.7 holds for Z_Q normalized to residue 1 with its own zeros, and its real zero alone contributes $\log(\beta_0/(1-\beta_0)) = 1.4720$, so $\mathcal{J}_Q \geq 1.47$; for the Davenport–Heilbronn function the off-line zeros contribute $1.527 \cdot 10^{-4}$ below height 400 (Corollary 2.8). A proof of $\mathcal{J} = 0$ must therefore use a property of ζ that both controls lack: the Euler product together with the functional equation. For (F2), the pole is the base point of the harmonic measure, not a positive contribution to be dominated. For (F3), $\mathcal{J}$ is computed from C_2 on the critical line, with no reference to the zeros, as are the criteria of Weil and Li.

Insensitivity. An off-line zero contributes

$$\log \left| \frac{\rho}{1-\rho} \right| = \frac{1}{2} \log \left(1 + \frac{2\beta-1}{(1-\beta)^2 + \gamma^2} \right) \approx \frac{2\beta-1}{2\gamma^2}.$$

A zero at height 10^{12} with $\beta - \frac{1}{2} = 10^{-3}$ changes $\mathcal{J}$ by about 10^{-27} . No computation can decide $\mathcal{J} = 0$; a proof must establish the exact vanishing.

Corollary 2.13 (the Jensen gap). Unconditionally,

$$0 \leq \sum_{\beta > 1/2} \log \left| \frac{\rho}{1-\rho} \right| \leq \frac{1}{2} \log (\log 2\pi - \gamma_0) = 0.115818 \dots,$$

and more generally $\mathcal{J} \leq \frac{1}{2k} \log \mathcal{M}(k)$ for every $0 < k \leq 1$.

Proof. Corollary 2.8, Proposition 2.11 and Jensen's inequality for the concave logarithm. $\square$

Status: proved.

Proving RH in this form amounts to controlling $\mathcal{M}(k)$ analytically as $k \rightarrow 0$ with an error $o(k)$. Asymptotic moment results as $T \rightarrow \infty$ do not help, because μ is a fixed measure concentrated at low height. Mollified versions of the bound, the barrier $\frac{1}{2} \log(1+c_0) = 0.0386$ for Dirichlet-polynomial mollifiers and the Euler–Caceres factor, of which the Poisson identity for partial Euler products at $s = 1$ in [C4, Remark 8.5] is a precursor, are the subject of [CE11].

Proposition 2.14 (moment obstruction). Let $X \geq 0$ be a random variable with $\mathbb{E}X = m$ and $\text{Var } X = v > 0$.

- (i) Over all such X , $\sup \mathbb{E} \log X = \log m$, and the supremum is not attained.
- (ii) For every fixed $k > 0$, $\frac{1}{k} \log \mathbb{E} X^k > \mathbb{E} \log X$ unless X is almost surely constant.

Proof. (i) The upper bound is Jensen's inequality, strict because $v > 0$. For the lower bound let $X = m - \sqrt{vp/(1-p)}$ with probability $1-p$ and $X = m + \sqrt{v(1-p)/p}$ with probability p ; it has

mean m and variance v , and $\mathbb{E} \log X \rightarrow \log m$ as $p \rightarrow 0$, because $p \log(1/\sqrt{p}) \rightarrow 0$. (ii) is the strict form of Jensen's inequality. $\square$

Status: proved.

Applied to $X = \Omega/2$ under μ , the proposition says that knowing the mean $\mathcal{M}(1)$ and the variance does not improve the bound $\mathcal{J} \leq \frac{1}{2} \log \mathcal{M}(1)$ at all. The variance is what correlations such as $\int B_N^2 B_K^2 d\mu$ would provide; and no bound through a fixed positive moment can reach 0, because $|\zeta(\frac{1}{2} + it)|$ is not constant. An argument that controls the C_2 wave only through its moments or correlations cannot prove RH. What is needed is information on the lower tail of Ω , the small values of $|\zeta|$ near the zeros, encoded in $\mathcal{M}(k)$ as $k \rightarrow 0^+$.

Remark 2.15 (finite M). One might try to bound $\int \log(\Omega_M/2) d\mu$ by some $\varepsilon_M \rightarrow 0$. Restricted to $|t| \leq 150$, where the reference value computed from ζ is -0.00006 :

M	10^2	10^3	10^4	$4 \cdot 10^4$
$\int \log(\Omega_M/2) d\mu$	+0.0442	+0.0320	+0.0259	+0.0233
$\int (\Omega_M/2) d\mu$	1.492	1.411	1.370	1.355

The finite quantity is positive and decays roughly like $1/\log M$, so ε_M must be at least of that order, and showing $\varepsilon_M \rightarrow 0$ is equivalent to the limiting statement. There is also a structural reason why truncation does not help: Ω_M is a sum of squares of real parts, not $|F|^2$ for an analytic F , so it admits no Jensen formula, and the natural analytic truncations, the partial sums X_N , have zeros to the right of the critical line, for large N even to the right of $\operatorname{Re} s = 1$ [Mo83]. Status: numerical (`anc/c2_jensen/c2_part4_numerics.py`, option `omegaM`).

2.6. The Szegő form and the Nyman–Beurling connection. Proposition 2.16 (Szegő form). Let H^∞ denote the bounded analytic functions on $\operatorname{Re} s > \frac{1}{2}$. Then

$$e^{2\mathcal{J}} = \inf \left\{ \int \frac{\Omega(t)}{2} |F(\tfrac{1}{2} + it)|^2 d\mu(t) : F \in H^\infty, F(1) = 1 \right\}.$$

Hence RH holds if and only if there are F with $F(1) = 1$ for which $\int (\Omega/2)|F|^2 d\mu$ is arbitrarily close to 1.

Proof. The map $z = (s-1)/s = 1-1/s$, Li's variable, sends $\operatorname{Re} s > \frac{1}{2}$ onto the unit disc and $s = 1$ to $z = 0$, and carries μ to normalized arc length. The weight $w = \Omega/2 = |\zeta(\frac{1}{2} + it)|^2$ is integrable, and $\log w$ is integrable. Szegő's theorem [Ga] gives $\inf_{F(0)=1} \int |F|^2 w dm = \exp \int \log w dm$ over analytic polynomials, and the infimum over H^∞ is the same: polynomials belong to H^∞ , and for $F \in H^\infty$ with $F(0) = 1$, $\int |F|^2 w dm \geq \exp \int \log(|F|^2 w) dm \geq \exp \int \log w dm$ because $\int \log |F| dm \geq \log |F(0)| = 0$. Finally $\int \log w d\mu = 2\mathcal{J}$. $\square$

Status: proved (classical in substance).

The Jensen gap of Corollary 2.13 is the choice $F \equiv 1$. To prove $\mathcal{J} \leq 0$ one must construct functions F that flatten the C_2 wave energy in $L^2(\mu)$, and Proposition 2.14 explains why no information about Ω alone, without such an F , can suffice.

The common sawtooth. The C-transform has the sawtooth $\{x\}$ as its kernel: for every N , by partial summation,

$$X_N(s) - Y_N(s) = 1 - \frac{1}{1-s} - s \int_1^N \{x\} x^{-s-1} dx,$$

and letting $N \rightarrow \infty$ gives, for $0 < \sigma < 1$, the formula $\zeta(s)/s = -\int_0^\infty \{x\} x^{-s-1} dx$ used in Proposition 2.11. The same sawtooth, as $\{\theta/x\}$, is the basic function of the Nyman–Beurling criterion. This is why the energy of C_2 leads to that criterion: for a Dirichlet polynomial $A(s) = \sum_{n \leq N} a_n n^{-s}$ the

quantity

$$D^2(A) = \int |1 - \zeta(\tfrac{1}{2} + it)A(\tfrac{1}{2} + it)|^2 d\mu(t)$$

is the Nyman–Beurling distance of the coefficients $c_k = -a_k$, and its Gram data are exact arithmetic ([CE5, Proposition 2.14]); its minimum over A of length N is d_N^2 . Báez-Duarte proved that RH holds if and only if $d_N \rightarrow 0$ [BD]; Burnol proved that $\liminf_N d_N^2 \log N \geq \sum_{\beta=1/2} m_\rho^2 / |\rho|^2$ [Bu02], which is C_B under RH with simple zeros; and Báez-Duarte, Balazard, Landreau and Saias conjectured $d_N^2 \log N \rightarrow C_B$ [BBLS] (Part 5 [CE5]). Both problems approximate $1/\zeta$ by functions analytic in $\operatorname{Re} s > \frac{1}{2}$, measured against the harmonic measure of the pole: the Nyman–Beurling problem asks for Dirichlet polynomials A with ζA close to 1 in $L^2(\mu)$, and Proposition 2.16 asks for functions F with $F(1) = 1$ for which $|\zeta F|^2$ has μ -mean close to 1, the value of its geometric mean under RH.

Bettin–Conrey–Farmer. Assume RH and $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-2} \ll T^{3/2-\delta}$ for some $\delta > 0$. Then the polynomial $V_N(s) = \sum_{n \leq N} \mu(n)(1 - \frac{\log n}{\log N})n^{-s}$ satisfies $D^2(V_N) \sim C_B / \log N$, so it attains Burnol’s bound [BCF].

Remark 2.17 (the hypothesis of [BCF] and the residual). (i) The sum $\sum_{\gamma \leq T} |\zeta'(\rho)|^{-2}$ is, up to a factor $1+o(1)$, the total normalized monotonicity window of C_2 at the zeros up to height T [C3, §3]: near a simple zero on the line, the range of n in which $C_2(n, \rho + i\varepsilon)$ remains monotone, as at the zero itself, is of size $1/(4|\zeta'(\rho)|^2|\rho|^2\varepsilon^2)$ up to a factor in $[1, e^{2\pi/\gamma}]$. The hypothesis of [BCF] is therefore a bound on the total size of the C_2 windows. (ii) For $A_{N,a}(s) = \sum_{n \leq N} \mu(n)(1 - a \frac{\log n}{\log N})n^{-s}$, the Dirichlet coefficients of $1 - \zeta A_{N,a}$ (for $\operatorname{Re} s > 1$) are 0 at $k = 1$ and $-a\Lambda(k)/\log N$ for $2 \leq k \leq N$: the Möbius mollifier generates exactly the weights of the arithmetic cross-energy C_2^Λ of Part 5 [CE5] inside its range. Status: (i) restates the window law of [C3, §3]; (ii) is a classical computation.

For the explicit polynomials $A_{N,a}$, with a^* the best a for each N , computed on $|t| \leq 2000$ with step 0.01 and a tail estimate, and checked against the exact Gram evaluation to $5 \cdot 10^{-5}$ at $N = 100$:

N	$D^2(a=0)$	$D^2(a=\frac{1}{2})$	$D^2(a=1)$	a^*	$\log N \cdot D^2(a=\frac{1}{2})$	$\log N \cdot D^2(a=1)$
100	0.1473	0.1178	0.1784	0.41	0.543	0.822
200	0.2621	0.0702	0.1347	0.62	0.372	0.713
500	0.0758	0.0373	0.0990	0.44	0.232	0.615
1000	0.0541	0.0403	0.0813	0.38	0.278	0.561
2000	0.0665	0.0385	0.0674	0.50	0.292	0.512
5000	0.0418	0.0260	0.0543	0.43	0.222	0.463

TABLE 6. The distance $D^2(A_{N,a}) = \int |1 - \zeta A_{N,a}|^2 d\mu$ for the mollifiers $A_{N,a}(s) = \sum_{n \leq N} \mu(n)(1 - a \log n / \log N)n^{-s}$.

In this range the half-taper $a = \frac{1}{2}$ is about 35 to 60% better than $a = 1$, and a^* fluctuates between 0.38 and 0.62 (Table 6). Asymptotically no choice can beat Burnol’s bound $C_B / \log N$, and under the hypotheses of [BCF] $a = 1$ attains it; the column for $a = 1$ is still an order of magnitude above C_B at $N = 5000$, which shows how slowly the asymptotic regime is reached. We make no asymptotic claim for $a = \frac{1}{2}$. Status: numerical (`anc/c2_jensen/c2_part4_numerics.py`, options `gram, mollifiers`); the optimal coefficients for $N \leq 1200$ are in Part 5 [CE5].

What C_2 does not add. The C_2 energies define positive measures $d\nu_M = \frac{1}{2}\Omega_M d\mu$ with $\nu_M \rightarrow |\zeta|^2 d\mu$ (Lemma 2.5, Proposition 2.12). One might use them as finite approximations of the Nyman–Beurling weight, writing $D^2(A) = \lim_M \int |1/\zeta - A|^2 d\nu_M$. This fails for two reasons. First, the finite measures do not vanish at the zeros: at the first zero $\Omega_M(\gamma_1) = 0.243, 0.182$ and 0.146 for $M = 10^3, 10^4, 10^5$, decaying only like $1/\log M$, while $|1/\zeta|^2 \asymp (t - \gamma)^{-2}$ near a simple zero, so $\int |1/\zeta - A|^2 d\nu_M = +\infty$ for every finite M . Second, on Dirichlet polynomials the action of $|\zeta|^2 d\mu$ is already exact arithmetic ([CE5, Proposition 2.14]), whereas ν_M approximates

it only with an error of order $1/\log M$. The contribution of C_2 to the Nyman–Beurling problem is therefore explanatory, not quantitative: it explains why the sawtooth, the harmonic measure of the pole, the Gram matrix and the window sums all appear, but it supplies no new bound for d_N . A regularization removes the divergence, but not the dependence on ζ . Fix a Dirichlet polynomial A , put $W_M = \Omega_M/2$, and assume that the zeros are simple and $\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{-1} \ll T^{2-\delta}$ for some $\delta > 0$, so that $\sum_\rho |\rho|^{-2} |\zeta'(\rho)|^{-1} < \infty$. Then for every $\eta(M) \rightarrow 0$ with $\eta(M) \log^2 M \rightarrow \infty$,

$$D^2(A) = \lim_{M \rightarrow \infty} \int \frac{|1 - \zeta A|^2}{|\zeta|^2 + \eta(M)} W_M d\mu.$$

Indeed the difference of the integrands is $|1 - \zeta A|^2 (W_M - |\zeta|^2 - \eta) / (|\zeta|^2 + \eta)$; the part with η tends to 0 by dominated convergence, and the rest is at most $(C/\log M) \int |1 - \zeta A|^2 (|\zeta|^2 + \eta)^{-1} d\mu$ in the relevant range, where near a simple zero $|\zeta|^2 \sim |\zeta'(\rho)|^2 (t - \gamma)^2$ gives a local integral $\pi / (|\zeta'(\rho)| \sqrt{\eta})$, so the total is $O((\log M)^{-1} \eta^{-1/2} \sum_\rho |\rho|^{-2} |\zeta'(\rho)|^{-1}) \rightarrow 0$; for a zero of multiplicity m the condition becomes $\eta^{1-1/(2m)} \log M \rightarrow \infty$. The zeros enter only through the weak negative moment $\sum_\rho |\rho|^{-2} |\zeta'(\rho)|^{-1}$, of the same family as J_{-1} and the monotonicity windows of Part 2 [CE2]. The formula still contains ζ , so it is not a ζ -free approximation of D^2 , and it has no bearing on whether $d_N \rightarrow 0$. Status: proved in outline (the uniform bound $|W_M - |\zeta|^2| \leq C/\log M$ in the relevant range is assumed from Lemma 2.5 and not made explicit).

2.7. A weaker target: averaged prime–integer balance. Let $R_x(s) = \sum_{n \leq x} (\Lambda(n) - 1)n^{-s}$. The poles of $-\zeta'/\zeta$ and ζ at $s = 1$ cancel in $\sum_n (\Lambda(n) - 1)n^{-s} = -\frac{\zeta'}{\zeta}(s) - \zeta(s)$, and by Landau’s theorem on Dirichlet series with an oscillating summatory function $\psi(x) - [x] = \Omega_\pm(x^{\Theta-\varepsilon})$, the partial sums $R_x(s)$ converge for $\sigma > \Theta$ and diverge for $\sigma < \Theta$ [In, Ch. V]. So RH is equivalent to the convergence, or the boundedness, of $R_x(s)$ for every $\sigma > \frac{1}{2}$: the prime and integer trajectories stay balanced. Averaging over t gives a hierarchy of weaker statements.

Proposition 2.18 (averaged balance). For $\frac{1}{2} < \sigma < 1$ and $T \geq 2$,

$$\frac{1}{T} \int_T^{2T} |R_x(\sigma + it)|^2 dt = \sum_{n \leq x} (\Lambda(n) - 1)^2 n^{-2\sigma} + O\left(\frac{x^{2-2\sigma} \log x}{T}\right).$$

The diagonal sum is bounded as $x \rightarrow \infty$, so the mean is bounded unconditionally for $x \leq (T/\log T)^{1/(2-2\sigma)}$.

Proof. The mean value theorem of Montgomery and Vaughan [MV74] gives $\int_T^{2T} |\sum_n a_n n^{-it}|^2 dt = \sum_n |a_n|^2 (T + O(n))$; take $a_n = (\Lambda(n) - 1)n^{-\sigma}$ and use $\sum_{n \leq x} (\Lambda(n) - 1)^2 n^{1-2\sigma} \ll x^{2-2\sigma} \log x$. The diagonal converges because $\sigma > \frac{1}{2}$. If $x^{2-2\sigma} \leq T/\log T$ then $\log x \leq \log T/(2-2\sigma)$ and the error is $O(1)$. $\square$

Status: proved.

For $\sigma = 0.6$ and $t \in [200, 400]$ the mean agrees with the diagonal sum in the range of Proposition 2.18, and far beyond it, up to $x \approx 3000T$, it stays bounded, as RH predicts (`anc/c2-jensen/c2-part4_numerics.py`, option `averaged`):

x	39	630	10^4	$1.6 \cdot 10^5$	$6.3 \cdot 10^5$
mean of $ R_x ^2$	2.63	6.37	8.88	9.98	10.50
diagonal sum	2.64	6.50	10.40	13.57	14.85

Remark 2.19 (a heuristic dictionary). For $x \gg T$ an off-line zero with $\beta > \sigma$ and $\gamma \in [T, 2T]$ contributes roughly $\frac{\pi}{\beta - \sigma} x^{2(\beta - \sigma)} / T$ to the mean. Boundedness of the mean for $x \leq T^a$ therefore corresponds to zero-density bounds of the shape $N(\sigma + \delta, T) \ll T^{1-2a\delta}$; at $a = 1/(2-2\sigma)$ and $\sigma \rightarrow \frac{1}{2}$ this is the shape of the density hypothesis. Making the correspondence rigorous requires a

zero-detection lemma that controls the interference of nearby zeros, which is where the difficulty of the density hypothesis lies. This route leads to zero-density theorems, not to RH. Status: heuristic.

The remaining statement. By Corollary 2.8 and $\int L dt = 2\pi$, RH is equivalent to

$$\int_{-\infty}^{\infty} \log \Omega(t) L(t) dt \leq 2\pi \log 2,$$

the reverse inequality being Theorem 2.7; equivalently $\int |\zeta(\frac{1}{2} + it)|^{2k} L(t) dt \leq 2\pi(1 + o(k))$ as $k \rightarrow 0^+$. By the locality principle (Proposition 2.1, Remark 2.3) no argument that works one point s at a time can reach it; by Proposition 2.14 no argument that uses only the moments or correlations of the C_2 wave can reach it either; and by Proposition 2.16 it is the problem of constructing functions F with $F(1) = 1$ that flatten the C_2 wave energy to 1 in $L^2(\mu)$, of which the Nyman–Beurling problem, $d_N \rightarrow 0$, is the version with Dirichlet polynomials. Its difficulty is precise: a logarithmic mean of $|\zeta|$ against the harmonic measure of the pole must be shown to vanish exactly, while the first moment of the same measure is the explicit constant $\log 2\pi - \gamma_0$. We do not prove it.

3. THE ENERGY SLOPE AND WEIL’S FUNCTIONAL

The theorems on the mean energy of the prime signal in Part 5 [CE5] concern averages over $u = \log x$. This section formulates an inequality at each fixed height t , the positivity of the *energy slope*, describes its local structure, determines which parts of it can be proved, and identifies it with Weil’s functional evaluated at a Poisson kernel. The material first appeared in Part VIII of the earlier series [C8] and in version 3 of this paper; the version given here supersedes both.

3.1. The energy slope and the criterion of Lagarias. Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and define the *energy slope*

$$S(\sigma, t) = \partial_{\sigma} \log |\xi(\sigma + it)| = G(\sigma, t) + \operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it), \quad (3.1)$$

where

$$G(\sigma, t) = \operatorname{Re} \left[\frac{1}{s} + \frac{1}{s-1} - \frac{1}{2} \log \pi + \frac{1}{2} \psi\left(\frac{s}{2}\right) \right]$$

and $\psi = \Gamma'/\Gamma$. We call G the *gamma pressure*; by Stirling’s formula $G(\sigma, t) = \frac{1}{2} \log \frac{|t|}{2\pi} + O(|t|^{-1})$ uniformly for σ in compact sets. The term $\log |\zeta|$ is half the logarithm of the C_2 energy: for $\sigma > 1$,

$$|\zeta(s)|^2 = \zeta(2\sigma) + \sum_{n \geq 2} C_2(n, s),$$

by the energy identity $|X_n|^2 - |X_{n-1}|^2 = n^{-2\sigma} + C_2(n, s)$ summed over n , and for $0 < \sigma < 1$ the modulus $|\zeta(s)|$ is the radius of the spiral described by the partial sums around the centre of the continuous component. The slope can therefore be evaluated from a finite C_2 sum: with

$$E_N(s) = \left| X_N(s) + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + \frac{s}{12}N^{-s-1} \right|^2,$$

the quantity $\frac{1}{2}\partial_{\sigma} \log E_N + G$ agrees with S to six digits at $N = 200$ at the points tested ($\sigma = 0.55, 0.7, 0.9$; $t = 27.7, 100$).

Theorem 3.1 (Lagarias [La]; Sondow and Dumitrescu [SD]). RH holds if and only if $S(\sigma, t) > 0$ for all $\sigma > \frac{1}{2}$ and all real t , that is, if and only if $|\xi(\sigma + it)|$ is strictly increasing in σ on $\sigma > \frac{1}{2}$ for every fixed t . Status: classical [La, SD].

The proof rests on the Hadamard product, which gives

$$S(\sigma, t) = \sum_{\rho} \operatorname{Re} \frac{1}{s - \rho}, \quad (3.2)$$

with the zeros summed in symmetric order (that is, grouping ρ with $1 - \bar{\rho}$ and summing over $|\gamma| \leq T$, $T \rightarrow \infty$) [La]. The next result makes the contribution of each zero explicit.

Theorem 3.2 (disc lemma). Let $\sigma > \frac{1}{2}$. A zero on the critical line contributes $(\sigma - \frac{1}{2})/((\sigma - \frac{1}{2})^2 + (t - \gamma)^2) > 0$ to $S(\sigma, t)$. A pair of zeros $\rho = \beta + i\gamma$ and $\iota(\rho) = 1 - \beta + i\gamma$ with $\beta > \frac{1}{2}$ contributes a negative amount if and only if

$$\left(\sigma - \frac{1}{2}\right)^2 + (t - \gamma)^2 < \left(\beta - \frac{1}{2}\right)^2,$$

that is, if and only if s lies in the open disc D_ρ with centre $\frac{1}{2} + i\gamma$ passing through ρ and $\iota(\rho)$; on the boundary circle the contribution is 0.

Proof. The first assertion is $\operatorname{Re} 1/(\delta + i(t - \gamma))$ with $\delta = \sigma - \frac{1}{2}$. For the pair put $a = \beta - \sigma$, $b = \sigma - 1 + \beta$ and $d = t - \gamma$. The pair contributes

$$-\frac{a}{a^2 + d^2} + \frac{b}{b^2 + d^2} = \frac{(a - b)(ab - d^2)}{(a^2 + d^2)(b^2 + d^2)}.$$

Here $a - b = 1 - 2\sigma < 0$ and $ab = (\beta - \frac{1}{2})^2 - (\sigma - \frac{1}{2})^2$, so the contribution is negative exactly when $ab > d^2$ and vanishes when $ab = d^2$. $\square$ Status: proved.

Corollary 3.3. If $\sigma > \frac{1}{2}$ and $s = \sigma + it$ lies in no disc D_ρ , then $S(\sigma, t) > 0$. In particular, the inequality $S > 0$ can fail only within distance $\beta - \frac{1}{2} < \frac{1}{2}$ of the ordinate of an off-line zero, and only at abscissae $\sigma < \beta$.

Proof. By Theorem 3.2 every term of (3.2), grouped in pairs, is nonnegative, and the terms of the zeros on the critical line, of which there are infinitely many [T, Ch. X], are strictly positive. $\square$ Status: proved.

The criterion is therefore local: each off-line pair can make S negative only inside its own disc. Near an off-line zero, $S \rightarrow -\infty$ as $s \rightarrow \rho$ from the left, so no positive contribution from the other zeros can compensate there, and a proof of $S > 0$ inside a disc D_ρ is equivalent to the absence of the zero ρ .

Proposition 3.4 (concavity in the square of the distance to the line). For real t and $v > 0$ put $\ell_t(v) = \log |\xi(\frac{1}{2} + \sqrt{v} + it)| \in [-\infty, \infty)$. Then $\ell'_t(v) = S(\sigma, t)/(2(\sigma - \frac{1}{2}))$ with $\sigma = \frac{1}{2} + \sqrt{v}$ wherever ℓ_t is finite, and RH holds if and only if ℓ_t is concave on $(0, \infty)$ for every real t . Under RH, for every t that is not an ordinate of a zero,

$$\lim_{v \rightarrow 0^+} \ell'_t(v) = \frac{1}{2} \sum_{\rho} \frac{1}{(t - \gamma)^2}, \quad \ell'_t(v) \downarrow 0 \quad (v \rightarrow \infty).$$

Proof. The formula for ℓ'_t is the chain rule, $dv/d\sigma = 2(\sigma - \frac{1}{2})$. By the functional equation $|\xi(\frac{1}{2} + a + it)| = |\xi(\frac{1}{2} - a + it)|$, so ℓ_t is a function of $v = a^2$. Write $\xi(\frac{1}{2} + w) = \xi(\frac{1}{2}) \prod_z (1 + w^2/z^2)$ over the zeros $\rho = \frac{1}{2} + iz$ with $\operatorname{Re} z > 0$, the product converging absolutely since $\sum |z|^{-2} < \infty$. For $w = a + it$ and real $z = \gamma$, $|\gamma^2 + w^2|^2 = (v + (t - \gamma)^2)(v + (t + \gamma)^2)$, so under RH

$$2\ell_t(v) = \text{const} + \sum_{\gamma > 0} \left[\log(v + (t - \gamma)^2) + \log(v + (t + \gamma)^2) - 4 \log \gamma \right],$$

a convergent sum of concave functions of v , finite for $v > 0$; its derivative at $v = 0^+$ is $\frac{1}{2} \sum_{\rho} (t - \gamma)^{-2}$, and every term of ℓ'_t decreases to 0. Conversely, let $\rho = \frac{1}{2} + b + i\gamma$ be a zero with $0 < b < \frac{1}{2}$, so $z = \gamma - ib$. At height $t = \gamma$ the four zeros $\pm z, \pm \bar{z}$ contribute to $|\xi(\frac{1}{2} + w)|^2$ the factor $|(z^2 + w^2)(\bar{z}^2 + w^2)|^2/|z|^8$, and $z^2 + w^2 = (z + iw)(z - iw)$ with $z + iw = i(a - b)$, $\bar{z} + iw = i(a + b)$; the factor contains $(a^2 - b^2)^2 = (v - b^2)^2$. Hence $\ell_\gamma(b^2) = -\infty$, while ℓ_γ is finite at all nearby $v \neq b^2$ (the zeros are isolated). The set where a concave function is finite is an interval, so ℓ_γ is not concave. $\square$ Status: proved.

The proposition combines the three structures used in the series. The functional equation makes $v = (\sigma - \frac{1}{2})^2$ the natural variable; ℓ_t is, up to the gamma factor, half the logarithm of the C_2

energy; the Euler product makes $\ell'_t > 0$ for $\sigma > 1$; and the derivative at the line is half of $\sum_\gamma (t - \gamma)^{-2} = -\frac{d^2}{dt^2} \log |\xi(\frac{1}{2} + it)|$, which under RH, for simple zeros, is the mean energy of the Euler helix [CE8]. Under RH the normalized slope $S/(2(\sigma - \frac{1}{2}))$ starts at this value on the line and decreases to 0 without changing sign. Numerically, on $10 \leq t \leq 100$ (step 0.05) and $0 < v \leq 0.36$, the largest value of ℓ''_t computed by second differences is -0.0017 . For the completed Davenport–Heilbronn function $(5/\pi)^{s/2} \Gamma(\frac{s+1}{2}) f(s)$ the same quantity reaches $1.9 \cdot 10^5$, and it is positive exactly for $84.98 \leq t \leq 86.40$, around the height 85.699 of its off-line pair (`anc/concavity/v.py`). One off-line pair breaks the concavity only near its own height. The criterion is a direct consequence of the Hadamard product and is presumably known in equivalent forms (the Laguerre–Pólya property in the variable y^2 [CV]).

The valley of log $|\xi|$. Theorem 3.1 says that under RH, and only then, the critical line is the floor of the valley of the potential $\log |\xi|$: the depth $D(\sigma, t) = \log |\xi(\sigma + it)| - \log |\xi(\frac{1}{2} + it)| = \int_{1/2}^\sigma S(u, t) du$ is positive for $\sigma > \frac{1}{2}$. An off-line pair turns the horizontal profile at its height into a W, with two wells at $\frac{1}{2} \pm (\beta - \frac{1}{2})$ and a crest on the line. The local form is the curvature at the line, $\kappa(t) = \partial_\sigma^2 \log |\xi|(\frac{1}{2} + it) = 2 \lim_{v \rightarrow 0^+} \ell'_t(v)$; since $\log |\xi|$ is harmonic off the zeros, $\kappa = -(\log |\Xi|)''$ with $\Xi(t) = \xi(\frac{1}{2} + it)$, so $\kappa > 0$ is the first Laguerre inequality (see “The margin near the critical line” below), and under RH $\kappa(t) = \sum_\gamma (t - \gamma)^{-2}$, while an off-line pair at height γ contributes $-2/(\beta - \frac{1}{2})^2$ to $\kappa(\gamma)$. Numerically, for ζ on $10 \leq t \leq 45$ and $\frac{1}{2} < \sigma \leq 1$ the depth is positive at every grid point, and the valley is shallow: at $\sigma = 0.75$ its median is 0.035 and its minimum 0.003. On $10 \leq t \leq 100$, $\kappa > 0$, with minimum 0.106 at the end $t = 10$ of the range (in the gaps above 14.13 the minimum is 0.229, at $t \approx 17.5$). For the completed Davenport–Heilbronn function, $\kappa < 0$ exactly on $85.41 \leq t \leq 85.98$, with $\kappa(85.699) = -20.16$ against $-2/(\beta - \frac{1}{2})^2 = -21.01$ for its zero $0.808517 + 85.699348i$, the other zeros adding about $+0.85$; around that zero the depth is negative on an island and tends to $-\infty$ at the zero itself. Status: the identities are classical; the numbers are numerical (`anc/valley/valley.py`, `anc/valley/margin.py`).

3.2. What is proved. Proposition 3.5 (unconditional positivity). (a) For $\sigma > 1$ and real t , $S(\sigma, t) \geq G(\sigma, t) + \frac{\zeta'}{\zeta}(\sigma)$, so $S(\sigma, t) > 0$ whenever $G(\sigma, t) > -\frac{\zeta'}{\zeta}(\sigma)$; moreover $|\zeta(s)| \geq \zeta(2\sigma)/\zeta(\sigma)$, so that $\sum_{n \geq 2} C_2(n, s) \geq (\zeta(2\sigma)/\zeta(\sigma))^2 - \zeta(2\sigma)$. (b) $S(\sigma, t) > 0$ for all $\sigma > \frac{1}{2}$ and $0 < |t| \leq 3 \cdot 10^{12} - \frac{1}{2}$.

Proof. (a) By the Euler product, $\operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it) = -\sum_n \Lambda(n) n^{-\sigma} \cos(t \log n) \geq \frac{\zeta'}{\zeta}(\sigma)$, and $|\zeta(s)|^{-1} = |\prod_p (1 - p^{-s})| \leq \prod_p (1 + p^{-\sigma}) = \zeta(\sigma)/\zeta(2\sigma)$; the bound for the C_2 sum follows from the energy identity. (b) All zeros with $0 < \gamma \leq 3 \cdot 10^{12}$ lie on the critical line [PT], so no disc D_ρ meets the range, and Corollary 3.3 applies. $\square$ Status: proved.

Since $-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} - \gamma_0 + O(\sigma-1)$, part (a) gives positivity for $G(\sigma, t) > \frac{1}{\sigma-1} - \gamma_0 + O(\sigma-1)$. We checked numerically that $-\frac{\zeta'}{\zeta}(\sigma) < \frac{1}{\sigma-1}$ on $1 < \sigma \leq 12$ (the difference is at most -0.09), so $S > 0$ for $\sigma \geq 1 + 1/G(\sigma, t)$ in that range. For $\sigma = 1.2$ and $0.5 < t < 200$ the minimum of $|\zeta|$ is 0.424, against the bound $\zeta(2.4)/\zeta(1.2) = 0.247$.

Near $\sigma = 1$ the energy slope gives a zero-free region directly; this is the classical argument of de la Vallée Poussin written for S .

Proposition 3.6 (energy-slope form of the zero-free region). Let $P(\theta) = \sum_{k=0}^K a_k \cos k\theta \geq 0$ with $a_k \geq 0$ and $a_1 > a_0 > 0$. For $\sigma > 1$ and real t ,

$$\sum_{k=0}^K a_k [G(\sigma, kt) - S(\sigma, kt)] = \sum_{n \geq 2} \frac{\Lambda(n)}{n^\sigma} P(t \log n) \geq 0. \quad (3.3)$$

Consequently, for every $\varepsilon > 0$ there is $t_0(P, \varepsilon)$ such that $\zeta(\beta + i\gamma) \neq 0$ for

$$\beta > 1 - \frac{c(P) - \varepsilon}{\log |\gamma|}, \quad |\gamma| \geq t_0, \quad c(P) = \frac{2(\sqrt{a_1} - \sqrt{a_0})^2}{\sum_{k \geq 1} a_k},$$

and $S(\sigma, t) > 0$ for $\sigma \geq 1 - (c(P) - \varepsilon)/\log(|t| + 1)$ and $|t| \geq t_0 + 1$.

Proof. By (3.1), $G - S = -\operatorname{Re} \frac{\zeta'}{\zeta}$, and for $\sigma > 1$, $-\operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + ikt) = \sum_n \Lambda(n) n^{-\sigma} \cos(kt \log n)$, which gives (3.3). For $\sigma > 1$ every term of (3.2) is positive, so $S(\sigma, kt) \geq 0$ for all k , and if $\rho = \beta + it$ is a zero then $S(\sigma, t) \geq \frac{1}{\sigma - \beta}$. The term $k = 0$ is at most $a_0 G(\sigma, 0) = \frac{a_0}{\sigma - 1} + O(1)$, the pole. For $k \geq 1$, $G(\sigma, kt) = \frac{1}{2} \log |t| + O_K(1)$ by Stirling's formula. Hence, with $\Sigma = \sum_{k \geq 1} a_k$,

$$\frac{a_1}{\sigma - \beta} \leq \frac{a_0}{\sigma - 1} + \frac{\Sigma}{2} \log |t| + O_P(1).$$

Put $\sigma = 1 + \frac{\eta}{\log |t|}$. Then $(1 - \beta) \log |t| \geq \frac{a_1 \eta}{a_0 + \eta \Sigma / 2} - \eta + o(1)$, and the maximum over $\eta > 0$ of the right side is $2(\sqrt{a_1} - \sqrt{a_0})^2 / \Sigma$, attained at $\eta = 2(\sqrt{a_0 a_1} - a_0) / \Sigma$. For the statement on S : if $s = \sigma + it$ lay in a disc D_ρ , then $\beta > \sigma$ and $|\gamma - t| < \frac{1}{2}$, so $t_0 \leq |\gamma| < |t| + 1$ and $\beta > \sigma \geq 1 - (c - \varepsilon) / \log(|t| + 1) > 1 - (c - \varepsilon) / \log |\gamma|$, which is excluded; Corollary 3.3 applies. $\square$ Status: proved; the optimal constants for each K quoted below are numerical.

The classical choice $P = 3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2$ gives $c = 0.02872 = 1/34.8$. Optimizing over nonnegative cosine polynomials of degree K (sequential quadratic programming with random starts) gives $1/c = 26.6, 18.5, 17.45, 17.27$ and 17.25 for $K = 2, 3, 4, 8, 16$. These are the constants of the elementary method; refinements of the same inequality give $1/c = 9.65$ [Ste], 5.70 [Ka], 5.573 [MoT] and, in a 2026 preprint, 4.896 [BTY]. With the Vinogradov–Korobov region [T, Ch. VI], Corollary 3.3 gives $S(\sigma, t) > 0$ for $\sigma \geq 1 - c(\log t)^{-2/3}(\log \log t)^{-1/3}$ and large t . Equation (3.3) is the argument in the language of this section: a nonnegative combination of energy slopes at the heights $0, t, 2t, \dots$ is bounded by the gamma pressures and the pole, and the pole term $a_0 > 0$, which cannot be avoided, limits the method to regions of width $c/\log t$ ([CE16, Propositions 2.8 and 2.9]).

3.3. The exceptional set. In the remaining region only a statement in measure is available.

Theorem 3.7 (exceptional set). Let $\frac{1}{2} < \sigma < 1$ and $T \geq 2$, and let $N(\sigma, T)$ be the number of zeros with $\beta > \sigma$ and $0 < \gamma \leq T$. Then

$$\operatorname{meas}\{t \in [0, T] : S(\sigma, t) \leq 0\} \leq \sum_{\substack{\beta > \sigma \\ 0 < \gamma < T + 1/2}} 2\sqrt{(\beta - \tfrac{1}{2})^2 - (\sigma - \tfrac{1}{2})^2} \leq 2\sqrt{\sigma(1 - \sigma)} N(\sigma, T + 1).$$

Proof. By Corollary 3.3 the set $\{t : S(\sigma, t) \leq 0\}$ is contained in the union of the sections of the discs D_ρ at abscissa σ . The section of D_ρ is an interval of length $2\sqrt{(\beta - \frac{1}{2})^2 - (\sigma - \frac{1}{2})^2}$ if $\beta > \sigma$ and is empty otherwise; each pair $\rho, \iota(\rho)$ has exactly one member with real part $> \sigma$. Since $\beta < 1$, the length is less than $2\sqrt{\frac{1}{4} - (\sigma - \frac{1}{2})^2} = 2\sqrt{\sigma(1 - \sigma)}$. Only discs with $|\gamma - t| < \frac{1}{2}$ for some $t \in [0, T]$ meet $[0, T]$, and ζ has no zeros with $|\gamma| < 14$. $\square$ Status: proved.

The sharper constant $2\sqrt{\sigma(1 - \sigma)}$ was already proved in Part VIII of the earlier series, whose abstract states the weaker bound $N(\sigma, T + 1)$. Combined with zero-density estimates, Theorem 3.7 shows that the proportion of $t \in [0, T]$ at which the energy slope fails is $O(T^{-\theta(\sigma) + \varepsilon})$ for some $\theta(\sigma) > 0$. Selberg's bound $N(\sigma, T) \ll T^{1 - (\sigma - 1/2)/4} \log T$ [Sel] gives this for every $\sigma > \frac{1}{2}$. The bounds of Huxley, $N(\sigma, T) \ll T^{12(1 - \sigma)/5 + \varepsilon}$ [Hu], and of Guth and Maynard, $N(\sigma, T) \ll T^{30(1 - \sigma)/13 + \varepsilon}$ [GM], give the rates $\theta(\sigma) = 1 - \frac{12}{5}(1 - \sigma)$ and $\theta(\sigma) = 1 - \frac{30}{13}(1 - \sigma)$ where these are positive. By contrast, functions of Davenport–Heilbronn type have $\gg T$ zeros in strips $\sigma_1 < \sigma < \sigma_2$ inside $\frac{1}{2} < \sigma < 1$

[KK], so their exceptional sets have positive proportion. The Euler product makes the failures of the pointwise inequality rare, and RH is the statement that they do not occur.

Proposition 3.8 (the integrated slope counts zeros). Let $\sigma < 2$ and $T_1 < T_2$, and suppose that ξ has no zeros on the boundary of $R = [\sigma, 2] \times [T_1, T_2]$. Then

$$\int_{T_1}^{T_2} S(\sigma, t) dt = \arg_h \xi(\sigma + iT_2) - \arg_h \xi(\sigma + iT_1) - 2\pi N(\sigma; T_1, T_2),$$

where $\arg_h$ is continued along the horizontal sides of R from $\sigma = 2$, and $N(\sigma; T_1, T_2)$ counts the zeros with $\beta > \sigma$ and $T_1 < \gamma < T_2$.

Proof. Away from the zeros $\log \xi$ is holomorphic, and the Cauchy–Riemann equations give $\partial_\sigma \log |\xi| = \partial_t \arg \xi$. Hence $\int_{T_1}^{T_2} S(\sigma, t) dt$ is the variation of $\arg \xi$ along the left side of R , traversed upward. The argument principle on ∂R , traversed counterclockwise, gives the stated identity; the variation along the right side is part of $\arg_h$. $\square$ Status: proved (the argument principle); the checks below are numerical (`anc/zerocount/`).

Each zero to the right of σ lowers the integrated slope by exactly 2π , and the boundary terms are $O(\log T_2)$ [T, §9.4]. Integrating in σ gives Littlewood’s lemma, used in Proposition 3.9; in the form $2\pi \sum_{\beta > \sigma} \min(\beta - \sigma, d)$ it measures how far the zeros lie to the right of σ . The identity holds for every function holomorphic on R ; we checked it for ζ and for f by numerical integration of $\operatorname{Re}(f'/f)$ and continuation of the argument from $\sigma = 3$: for ζ , $N = 4.6 \cdot 10^{-14}$ at $\sigma = 0.6$, $10 \leq t \leq 30$, and $N = 1.0$ at $\sigma = 0.4$, $13 \leq t \leq 15$ (the zero ρ_1); for f , $N = 1.0$ at $\sigma = 0.75$ and $N = -3 \cdot 10^{-10}$ at $\sigma = 0.85$ on $85 \leq t \leq 86.5$, around $\rho_f = 0.8085 + 85.6993i$. On that rectangle the Littlewood sum with $\sigma = 0.75$, $d = 0.1$ is 0.058517098, against $\beta_f - 0.75 = 0.05851718$.

3.4. A density estimate from the square of the cross-energy. The energy $|X_N(s)|^2 = \sum_{n \leq N} n^{-2\sigma} + \sum_{2 \leq n \leq N} C_2(n, s)$ is a sum of squares plus cross terms, and its mean over t keeps only the diagonal: each $C_2(n, s) = 2n^{-\sigma} \sum_{j < n} j^{-\sigma} \cos(t \log \frac{n}{j})$ has mean 0 over long ranges of t . This quadratic positivity has no pole term, and it is the source of the density theorems used after Theorem 3.7. Its simplest consequence is the following.

Proposition 3.9 (Bohr–Landau, in energy form). Let $\frac{1}{2} < \sigma < 1$ and let $N(\sigma; T, 2T)$ be the number of zeros with $\beta > \sigma$ and $T < \gamma \leq 2T$. Then

$$N(\sigma; T, 2T) \leq \frac{\log \zeta(\sigma + \frac{1}{2})}{2\pi(\sigma - \frac{1}{2})} T + o(T),$$

so that the proportion of zeros with $\beta > \sigma$ among those with $T < \gamma \leq 2T$ is $O_\sigma(1/\log T)$.

Proof. Let $\sigma_0 = \frac{1}{2}(\sigma + \frac{1}{2})$. Littlewood’s lemma applied to ζ on $[\sigma_0, 2] \times [T, 2T]$ gives

$$2\pi \int_{\sigma_0}^2 N(u; T, 2T) du = \int_T^{2T} \log |\zeta(\sigma_0 + it)| dt - \int_T^{2T} \log |\zeta(2 + it)| dt + O(\log T),$$

the error coming from $\arg \zeta$ on the horizontal sides [T, §9.4]. The second integral is $O(1)$, since $\log \zeta(2 + it) = \sum \Lambda(n) n^{-2-it} / \log n$. By the concavity of the logarithm,

$$\int_T^{2T} \log |\zeta(\sigma_0 + it)| dt \leq \frac{T}{2} \log \left(\frac{1}{T} \int_T^{2T} |\zeta(\sigma_0 + it)|^2 dt \right) = \frac{T}{2} \log \zeta(2\sigma_0) + o(T),$$

by the mean value $\frac{1}{T} \int_T^{2T} |\zeta(\sigma_0 + it)|^2 dt \rightarrow \zeta(2\sigma_0)$ for $\sigma_0 > \frac{1}{2}$ [T, Thm 7.2], which is the vanishing of the mean of the cross terms C_2 . Since $N(u; T, 2T)$ is nonincreasing in u , the left side is at least $2\pi(\sigma - \sigma_0)N(\sigma; T, 2T)$, and $2\sigma_0 = \sigma + \frac{1}{2}$, $\sigma - \sigma_0 = \frac{1}{2}(\sigma - \frac{1}{2})$. The proportion follows from $N(2T) - N(T) \sim \frac{T}{2\pi} \log T$. $\square$ Status: classical (Bohr–Landau) [T, Ch. IX], proved; the computations below are numerical (`anc/density/`).

The zeros off the line have density zero among all zeros because the mean energy at $\sigma_0 > \frac{1}{2}$ is finite, while at $\sigma = \frac{1}{2}$, where $\zeta(2\sigma)$ has its pole, it grows like $\log T$. Numerically, on $1000 \leq t \leq 2000$

the mean of $\sum_{2 \leq n \leq 200} C_2(n, \sigma + it)$ is 0.0012, 0.0005, -0.0004 for $\sigma = 0.6, 0.75, 0.9$, against $\sum_{n \leq 200} n^{-2\sigma} = 3.86, 2.47, 1.86$; and $\frac{1}{T} \int_T^{2T} |\zeta|^2 dt = 4.10, 2.52, 1.87$ approaches $\zeta(2\sigma) = 5.59, 2.61, 1.88$ slowly, the deficit at $\sigma = 0.6$ being the tail $\sum_{n > t/2\pi} n^{-2\sigma}$. At $T = 1000$ the bound of Proposition 3.9 is 3755, 971 and 451 for these σ , against 868 zeros in $[T, 2T]$; it becomes nontrivial only for large T . The stronger density estimates use higher moments and mollifiers, but the mechanism is the same quadratic positivity, which can show that zeros off the line are rare and cannot show that there are none: a single off-line zero changes the mean energy over $[T, 2T]$ by an amount that the average does not see.

Remark 3.10 (mollified energy and the Jensen gap). The bound of Proposition 3.9 is $\frac{T}{2} \log \bar{E}$ divided by $2\pi(\sigma - \sigma_0)$, where $\bar{E}$ is the mean energy, and it improves when the energy is lowered by a mollifier $M_y(s) = \sum_{d \leq y} \mu(d) d^{-s} (1 - \frac{\log d}{\log y})$, as in Selberg's method: since the zeros of ζM_y include those of ζ , the same proof gives

$$N(\sigma; T, 2T) \leq \frac{T}{4\pi(\sigma - \sigma_0)} \log \left(\frac{1}{T} \int_T^{2T} |\zeta M_y(\sigma_0 + it)|^2 dt \right) + o(T).$$

On $1000 \leq t \leq 2000$ (868 zeros) the mean of $|\zeta M_y|^2$ at $\sigma_0 = 0.625$ is 3.72, 1.80, 1.34, 1.19 for $y = 1, 10, 100, 1000$, and the bound for $\sigma = 0.75$, without the $o(T)$ term, is 836, 373, 188, 112; at $\sigma_0 = 0.55$ the means are 5.12, 2.33, 1.66, 1.42, and the bound for $\sigma = 0.6$ is about 1350, 810, 560 for $y = 10, 100, 1000$. What limits the method is the gap in Jensen's inequality: the mean of $\log |\zeta M_y(\sigma_0 + it)|$ is 0 to within 10^{-3} (there are no zeros to the right of σ_0 in this range), and the logarithm of the mean of $|\zeta M_y|^2$ is close to twice the variance of $\log |\zeta M_y|$ (variances 0.70, 0.34, 0.18, 0.11; $e^{2\text{var}} = 4.05, 1.99, 1.43, 1.24$). To exclude every zero in $[T, 2T]$ the variance would have to be of order $1/T$. Heuristically, $\log |\zeta M_y|$ behaves like $\sum_{p > y} p^{-\sigma_0} \cos(t \log p + \dots)$, of variance $\asymp y^{1-2\sigma_0} / \log y$, which needs $y \gtrsim T^{1/(2\sigma_0-1)} > T$, beyond the length that the mean value theorems control; with $y = T$ the heuristic gives $N(\sigma, T) \ll T^{2-2\sigma+\varepsilon}$, the shape of the density hypothesis. Quadratic means of the cross-energy, however mollified, show that off-line zeros are rare but not that they are absent. The Jensen moment and its Euler-product mollifier are developed in [CE11]. Status: the inequality is proved (Littlewood's lemma and Jensen's inequality); the variance law and the resulting exponent are heuristic; the computations are numerical (`anc/density/`).

3.5. Numerical results and the margin near the line. For ζ we computed $S(\sigma, t)$ by central differences of $\log |\xi|$ on $5 \leq t \leq 600$ (step 0.01) and checked it against the zero sum (3.4526 against 3.4520 at $\sigma = 0.7, t = 14$). For $t \geq 20$ the minimum over t is attained at $t \approx 27.7$, in the gap between the zeros 25.011 and 30.425, for every σ tested (Table 8, Figure 2).

σ	0.51	0.55	0.6	0.7	0.8	0.9	1.0
$\min_t S(\sigma, t)$	0.0039	0.0195	0.0389	0.0776	0.1159	0.1534	0.1901
$\max_t (-\text{Re } \frac{\zeta'}{\zeta})/G$	0.995	0.974	0.948	0.895	0.844	0.793	0.744

TABLE 8. The energy slope of ζ on $20 \leq t \leq 600$: its minimum over t , and the fraction of the gamma pressure used by the Euler-product term.

The margin is linear near the line, $\min_t S \approx 0.39(\sigma - \frac{1}{2})$, because $S(\frac{1}{2}, t) = 0$ by the functional equation, and the Euler-product term uses almost all of the gamma pressure near $\sigma = \frac{1}{2}$. On the larger range $2 \leq t \leq 120$ we find $S(\sigma, t)/(\sigma - \frac{1}{2}) \geq 0.047$ for $\frac{1}{2} < \sigma \leq 1$, the minimum 0.0471 being at the lowest height $t = 2$, close to the limit $\sum_{\gamma} \gamma^{-2} = 0.0461$ of the quotient at $s = \frac{1}{2}$ (all ordinates of both signs) (`anc/valley/margin.py`). For the Davenport–Heilbronn function f [DH], [CE17], we used the completed function $(5/\pi)^{s/2} \Gamma(\frac{s+1}{2}) f(s)$ and its eight off-line zeros with $\gamma < 400$; Theorems 3.1, 3.2 and 3.7 apply to it with the obvious changes, since their proofs use only the Hadamard product, the symmetry of the zeros and the existence of zeros on the line. The

minimum of its slope over $10 \leq t \leq 400$ is negative for every $\sigma \leq 0.85$, with values between -30 and -273 , and positive for $\sigma \geq 0.9$, beyond its largest $\beta = 0.8695$. Near $0.8085 + 85.699i$, 99.6% of the grid points with negative slope lie in the disc of Theorem 3.2, and 91% of the disc is negative. The failure set at abscissa σ consists of exactly one interval for each off-line zero with $\beta > \sigma$ (Table 9); the bound of Theorem 3.7 is attained up to a factor of about 0.9. A Beurling system has no completed function and no functional equation, so S is not defined for it; an ad hoc version with the Gamma factor of ζ borrowed, in which the negativity is again located at the off-line zero and reaches the line, is described in [CE17, Sect. 6.4].

σ	0.55	0.65	0.75	0.85
measure of $\{t \leq 400 : S < 0\}$	3.220	2.295	1.315	0.205
bound of Theorem 3.7 (sum of chords)	3.568	2.589	1.498	0.237
number of intervals = zeros with $\beta > \sigma$	8	6	4	1

TABLE 9. The failure set of the energy slope for the Davenport–Heilbronn function.

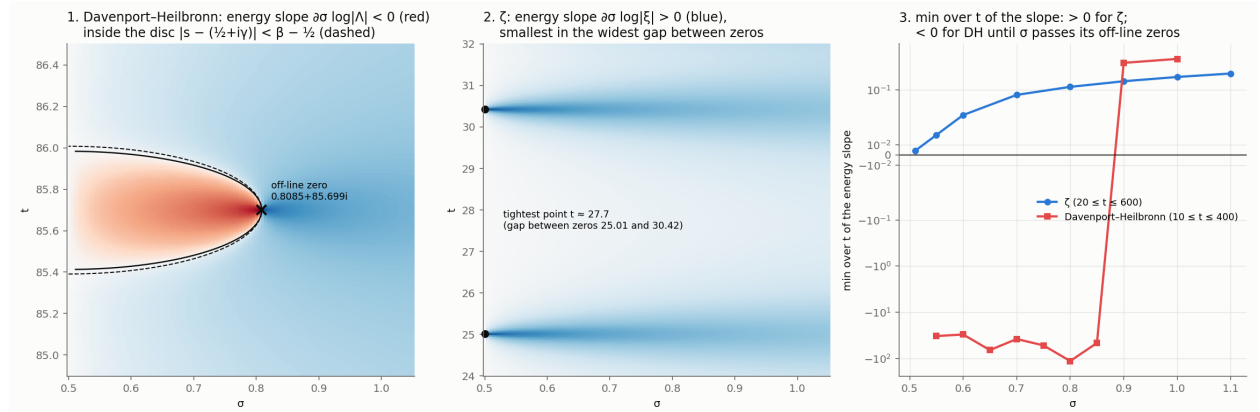


FIGURE 2. The energy slope $S(\sigma, t)$. Left: Davenport–Heilbronn function near its zero $0.8085 + 85.699i$; $S < 0$ (red) inside the disc of Theorem 3.2 (dashed), zero level solid. Centre: ζ near $t = 27.7$, where the minimum over t is attained; $S > 0$ throughout. Right: $\min_t S(\sigma, t)$ for ζ ($20 \leq t \leq 600$) and for the Davenport–Heilbronn function ($10 \leq t \leq 400$), symmetric logarithmic scale.

An Euler-product twin. The Davenport–Heilbronn function is $f = c_1 L(s, \chi) + c_2 L(s, \bar{\chi})$ with χ a complex character modulo 5, so $L(s, \chi)$ has the same conductor, the same gamma factor and a functional equation of the same type, and differs from f by having an Euler product. The proofs of Theorems 3.1, 3.2 and 3.7 apply to $S_\chi = \partial_\sigma \log |(5/\pi)^{(s+1)/2} \Gamma(\frac{s+1}{2}) L(s, \chi)|$, whose zeros are symmetric under $\rho \mapsto 1 - \bar{\rho}$. On $10 \leq |t| \leq 400$ we find $\min_t S_\chi = 0.0068, 0.034, 0.068, 0.135$ at $\sigma = 0.51, 0.55, 0.6, 0.7$, about $0.68(\sigma - \frac{1}{2})$, and no point with $S_\chi < 0$; on the same range the slope of f is negative on sets of measure 6.7, 6.5, 5.6 and 3.8. The Euler-product term of $L(s, \chi)$ uses a fraction $\max_t (-\operatorname{Re} \frac{L'}{L})/G = 0.994$ of the gamma pressure at $\sigma = 0.51$ and 0.968 at $\sigma = 0.55$, as for ζ (0.995 and 0.974). The absence of a pole does not create slack: the tightness near the line is set by the density of the zeros (anc/twin/).

The margin near the critical line and random matrices. Write $\Xi(t) = \xi(\frac{1}{2} + it)$, which is real. Expanding in $\sigma - \frac{1}{2}$ gives $S(\sigma, t) = (\sigma - \frac{1}{2}) L_1(t) + O((\sigma - \frac{1}{2})^2)$ with $L_1 = (\Xi'^2 - \Xi\Xi'')/\Xi^2 = -(\Xi'/\Xi)'$, and under RH $L_1(t) = \sum_\gamma (t - \gamma)^{-2}$. The inequality $\Xi'^2 - \Xi\Xi'' \geq 0$ is the first Laguerre inequality; it is implied by RH, and the full family of generalized Laguerre inequalities is equivalent to RH [CV]. The minimum of L_1 over a gap between consecutive zeros measures how close S comes

to 0 there. We unfolded the 341 zeros with $0 < \gamma < 600$ by the Riemann–von Mangoldt formula, computed $M_j = \min_u \sum_{|k-j| \leq 20} (u - x_k)^{-2}$ over each gap (x_j, x_{j+1}) of the unfolded zeros x_k , and compared the 300 values with 6360 values from 40 bulk spectra of 400×400 GUE matrices and with a Poisson sequence (Figure 3). The empirical quantiles at 1%, 5%, 25% and 50% are 3.86, 4.53, 7.11, 10.68 for ζ and 3.39, 4.28, 6.88, 10.75 for GUE; the Kolmogorov–Smirnov test gives $p = 0.35$ against GUE and $p = 3 \cdot 10^{-34}$ against Poisson. The margin is determined by the gap: M_j has correlation 0.9998 with Δ_j^{-2} , and an isolated gap of length Δ gives $8/\Delta^2$. The tightest points of the inequality are the largest gaps, whose distribution is the large-gap distribution of random matrix theory [Mo, Od]. Status: the expansion is proved; the statistics are numerical (**anc/slope/**).

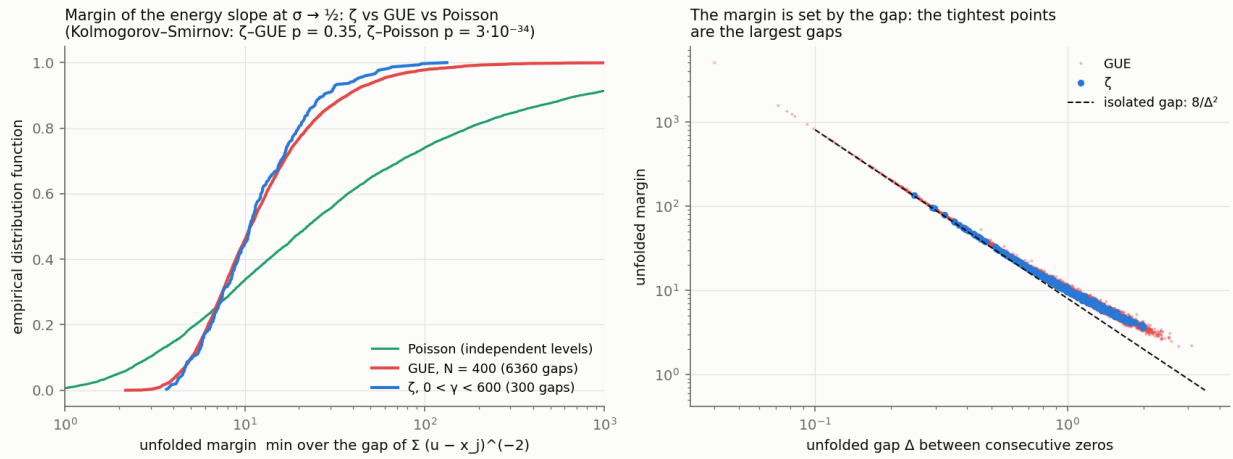


FIGURE 3. Left: empirical distribution of the unfolded margin M_j for ζ (300 gaps), GUE (6360 gaps) and a Poisson sequence. Right: M_j against the unfolded gap Δ_j ; the dashed line is $8/\Delta^2$.

Remark 3.11 (statistical mechanics). For real $\beta > 1$, $\zeta(\beta) = \sum_n e^{-\beta \log n}$ is the partition function of a system with energy levels $\log n$ [Ju]; Bost and Connes realize it as a quantum statistical system with a phase transition at $\beta = 1$ [BC]. In this language $-\frac{\zeta'}{\zeta}(\beta)$ is the mean energy, Proposition 3.5(a) compares the mean energy with the gamma pressure, and the zeros of ζ are the zeros of the continued partition function at complex temperature. The Lee–Yang theorem places the zeros of ferromagnetic Ising partition functions on a circle by means of a positivity property that is preserved under products and limits [LY]. The analogous property for ζ would have to hold in $\frac{1}{2} < \sigma < 1$, beyond the transition at $\beta = 1$, where the partition sum diverges; this is the pole barrier of Part 16 [CE16] in another form. The heat-flow deformation H_λ of Ξ studied by de Bruijn and Newman [Ne] has only real zeros for $\lambda \geq \Lambda$, and RH is equivalent to $\Lambda \leq 0$. Rodgers and Tao proved $\Lambda \geq 0$ [RT]; the upper bound $\Lambda \leq 0.22$ of the Polymath project [Pm] was improved to $\Lambda \leq 0.2$ by Platt and Trudgian [PT], and to $\Lambda \leq 0.174$ with optimised mollifiers in [CE13], so that $0 \leq \Lambda \leq 0.174$. Conversely, a zero $\frac{1}{2} + \delta + i\gamma$ off the line would force $\Lambda \geq \delta^2 / (70 \log \gamma \log \log \gamma)$ unconditionally, and for a single pair the interaction term of its dynamics under the flow is, at $\tau = 0$, the energy slope S of the other zeros evaluated at the zero itself [CE15]. Under the backward flow, zeros leave the real line first at pairs of unusually close zeros (Lehmer pairs). These are the points where the margin M_j is largest, not smallest, so the heat-flow picture and the energy-slope margin measure different aspects of the zero distribution. Status: remark (the cited results are classical).

3.6. The energy slope as Weil’s functional. Let $h(r) = \int g(u) e^{iru} du$ with g smooth and compactly supported. For a zero ρ put $\gamma_\rho = -i(\rho - \frac{1}{2})$, so that γ_ρ is real exactly when ρ lies on the

critical line. The Riemann–Weil explicit formula [We2], [IK, Thm 5.12] reads

$$\begin{aligned} \sum_{\rho} h(\gamma_{\rho}) &= h\left(\frac{i}{2}\right) + h\left(-\frac{i}{2}\right) - g(0) \log \pi + \frac{1}{2\pi} \int h(r) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) dr \\ &\quad - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} (g(\log n) + g(-\log n)). \end{aligned} \quad (3.4)$$

For the Davenport–Heilbronn function the completed function is entire and symmetric under $s \mapsto 1 - s$; its explicit formula has no pole terms, the constant is $+g(0) \log(5/\pi)$, the gamma term is $\operatorname{Re} \psi(\frac{3}{4} + \frac{ir}{2})$, and $\Lambda(n)$ is replaced by the coefficients $b(n)$ of $-f'/f$, which change sign and are supported mostly on composite n [CE17]. With $h(r) = e^{-(r/a)^2}$, $2 \leq a \leq 5$, the two sides of each formula agree to 10^{-14} for ζ (682 zeros) and to 10^{-16} for f .

Take $g = \varphi * \tilde{\varphi}$, $\tilde{\varphi}(u) = \varphi(-u)$, so that $h = |\hat{\varphi}|^2$ on the real line, and write $W(\varphi)$ for the right side of (3.4). Under RH every γ_{ρ} is real and $W(\varphi) = \sum_{\rho} |\hat{\varphi}(\gamma_{\rho})|^2 \geq 0$.

Theorem 3.12 (Weil). RH holds if and only if $W(\varphi) \geq 0$ for every $\varphi \in C_c^{\infty}(\mathbb{R})$. Status: classical [We2, Bo].

What the critical line determines. On the line $\zeta(\frac{1}{2} + it) = e^{-i\theta(t)} Z(t)$ with Z real, so $\frac{\zeta'}{\zeta}(\frac{1}{2} + it) = -\theta'(t) - iZ'(t)/Z(t)$. The real part of $F = 1 - \zeta - \zeta'/\zeta$ is therefore explicit on the line, $\operatorname{Re} F(\frac{1}{2} + it) = 1 - \operatorname{Re} \zeta(\frac{1}{2} + it) - \frac{1}{2} \log \pi + \frac{1}{2} \operatorname{Re} \psi(\frac{1}{4} + \frac{it}{2})$, which is the equality $\operatorname{Re} F = \Phi$ of [CE1, (A.25)], and the zeros on the line sit in the imaginary part, as poles $-i/(t - \gamma)$ of $-iZ'/Z$. Equivalently, $\operatorname{Re} \xi'/\xi$ vanishes on the line except for the masses $\pi\delta(t - \gamma)$ at the ordinates of the zeros on it, and the function in $\sigma > \frac{1}{2}$ that these boundary values determine is

$$\Phi_{\text{line}}(s) = \sum_{\gamma > 0, \zeta(\frac{1}{2} + i\gamma) = 0} \frac{2(s - \frac{1}{2})}{(s - \frac{1}{2})^2 + \gamma^2}.$$

By (3.2), RH holds if and only if $\xi'/\xi = \Phi_{\text{line}}$ in $\sigma > \frac{1}{2}$: the four zeros $\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}$ of an off-line pair contribute terms whose real parts cancel on the line, so they are invisible in the boundary data, and they are the whole residual $\xi'/\xi - \Phi_{\text{line}}$. This is the explicit formula once more, and it is the phase form of the problem: RH is the statement that Z has only real zeros. Numerically, for ζ with the 1517 zeros below 2000 and a tail given by the density $\frac{1}{2\pi} \log \frac{u}{2\pi}$, the residual is below $2.1 \cdot 10^{-5}$ on the grid $10 \leq t \leq 100$, $0.52 \leq \sigma \leq 0.96$. For the Davenport–Heilbronn function, with its 94 zeros on the line below 160 (sign changes of the completed function on the line), the residual is large exactly at its two off-line zeros $0.8085 + 85.699i$ and $0.6508 + 114.163i$ below that height, and $94 + 2 \cdot 2 = 98$ agrees with the main term $\frac{T}{2\pi} \log \frac{5T}{2\pi e} = 98.0$ of the zero count at $T = 160$. Status: the equivalence follows from (3.2); the numbers are numerical (`anc/valley/schwarz.py`).

If φ is supported in an interval of length L , only the prime powers $n \leq e^L$ enter, and by Parseval the prime part of W is the cross-energy of φ with its translates by $\log p^k$, weighted by $\Lambda(p^k)p^{-k/2}$:

$$W(\varphi) = 2 \operatorname{Re} \left[\hat{\varphi}\left(\frac{i}{2}\right) \overline{\hat{\varphi}\left(-\frac{i}{2}\right)} \right] + \frac{1}{2\pi} \int |\hat{\varphi}(r)|^2 \left(\operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) - \log \pi - 2 \operatorname{Re} \sum_{n \leq e^L} \frac{\Lambda(n)}{n^{1/2+ir}} \right) dr.$$

Theorem 3.13 (the Poisson kernel and its exact truncation). Let $\sigma > \frac{1}{2}$, $\delta = \sigma - \frac{1}{2}$ and $h_{\delta,t}(r) = \delta/(\delta^2 + (r - t)^2)$.

- (a) $S(\sigma, t) = \sum_{\rho} h_{\delta,t}(\gamma_{\rho})$, with the zeros summed in symmetric order. The function $h_{\delta,t}$ is the Fourier transform of the positive-definite function $g_{\delta,t}(u) = \frac{1}{2} e^{-\delta|u|} e^{-itu}$. For an off-line pair, $h_{\delta,t}(\gamma_{\rho}) + h_{\delta,t}(\bar{\gamma}_{\rho}) = 2 \operatorname{Re} h_{\delta,t}(\gamma_{\rho})$, which is negative exactly in the disc D_{ρ} of Theorem 3.2.

- (b) For $L > 0$ let $\varphi_L(u) = \sqrt{2\delta} e^{-\delta u} e^{-itu} \mathbf{1}_{[0,L]}(u)$ and $g_L = \varphi_L * \tilde{\varphi}_L$. Then $g_L(u) = e^{-itu} (e^{-\delta|u|} - e^{-\delta(2L-|u|)})$ for $|u| \leq L$ and $g_L = 0$ otherwise, and its Fourier transform is the entire function

$$h_L(r) = \frac{2\delta (1 - e^{-\delta L} e^{i(r-t)L}) (1 - e^{-\delta L} e^{-i(r-t)L})}{\delta^2 + (r-t)^2}.$$

Let $W_L(\sigma, t) = \sum_{\rho} h_L(\gamma_{\rho}) = W(\varphi_L)$. Then:

1. $W_L(\sigma, t)$ is given exactly by (3.4) with the prime powers $n \leq e^L$:

$$\begin{aligned} W_L &= h_L\left(\frac{i}{2}\right) + h_L\left(-\frac{i}{2}\right) - g_L(0) \log \pi + \frac{1}{2\pi} \int h_L(r) \operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) dr \\ &\quad - 2 \sum_{n \leq e^L} \frac{\Lambda(n)}{\sqrt{n}} \operatorname{Re} g_L(\log n). \end{aligned}$$

2. If all zeros lie on the critical line, then $h_L(\gamma) = 2h_{\delta,t}(\gamma) |1 - e^{-\delta L} e^{i(\gamma-t)L}|^2$ for every zero, and therefore

$$(1 - e^{-\delta L})^2 \leq \frac{W_L(\sigma, t)}{2S(\sigma, t)} \leq (1 + e^{-\delta L})^2. \quad (3.5)$$

3. A zero with $\beta > \sigma$ contributes to $W_L(\sigma, t)$ a term of size $e^{(\beta-\sigma)L}$ times the kernel factor $2\delta/|\delta^2 + (\gamma_{\rho} - t)^2|$, at every t , with a phase that rotates with L .

Proof. (a) By (3.2) and $s - \rho = \delta + i(t - \gamma_{\rho})$. The maps $\rho \mapsto 1 - \bar{\rho}$ and $\rho \mapsto 1 - \rho$ act on γ_{ρ} as $\gamma \mapsto \bar{\gamma}$ and $\gamma \mapsto -\gamma$, so the multiset $\{\gamma_{\rho}\}$ is invariant under conjugation. Hence

$$\sum_{\rho} \operatorname{Re} \frac{1}{\delta + i(t - \gamma_{\rho})} = \frac{1}{2} \sum_{\rho} \left[\frac{1}{\delta + i(t - \gamma_{\rho})} + \frac{1}{\delta - i(t - \gamma_{\rho})} \right] = \sum_{\rho} h_{\delta,t}(\gamma_{\rho}).$$

The Fourier transform of $g_{\delta,t}$ is elementary, and the last assertion is Theorem 3.2 written in the variable γ_{ρ} .

- (b) The formula for g_L follows by direct integration. On the real line $\hat{\varphi}_L(r) = \sqrt{2\delta} (1 - e^{-\delta L} e^{i(r-t)L}) / (\delta - i(r-t))$, so $h_L = |\hat{\varphi}_L|^2$ there, which gives the displayed expression; the numerator vanishes at both zeros $r = t \pm i\delta$ of the denominator, so h_L is entire, and $h_L(r) = O_L(|r|^{-2})$ in every horizontal strip. The function g_L is continuous, of bounded variation and supported in $[-L, L]$; applying (3.4) to $g_L * j_{\epsilon}$ with a smooth approximate identity j_{ϵ} and letting $\epsilon \rightarrow 0$, all terms converge (the zero sums by dominated convergence, since $|\hat{j}_{\epsilon}| \leq e^{\epsilon/2}$ in the strip $|\operatorname{Im} r| \leq \frac{1}{2}$), and $g_L(\log n) + g_L(-\log n) = 2 \operatorname{Re} g_L(\log n)$; this is (1). For real γ the factor $|1 - e^{-\delta L} e^{i(\gamma-t)L}|^2$ lies between $(1 - e^{-\delta L})^2$ and $(1 + e^{-\delta L})^2$, which with (a) gives (2). For $\gamma_{\rho} = \gamma - i(\beta - \frac{1}{2})$ the factor $e^{-\delta L} e^{i(\gamma_{\rho}-t)L}$ has modulus $e^{(\beta-\sigma)L}$, which gives (3). $\square$ Status: proved.

Part (a) identifies the pointwise inequality $S > 0$ of Theorem 3.1 with Weil positivity along the family $h_{\delta,t}$, and the disc of Theorem 3.2 with the region where $h_{\delta,t}$ is negative at complex arguments. The test function $g_{\delta,t}$ is not admissible in (3.4): $h_{\delta,t}$ has poles at $r = t \pm i\delta$, inside the strip $|\operatorname{Im} r| < \frac{1}{2}$, and the prime sum $\sum \Lambda(n) n^{-1/2} g_{\delta,t}(\log n)$ converges only for $\delta > \frac{1}{2}$, that is, for $\sigma > 1$. This is the pole barrier again (Part 16 [CE16]). Part (b) is the admissible version, and it contains no remainder term: W_L is an exact finite expression in the primes up to e^L . Under RH it is positive for every L and determines S to relative precision about $2e^{-\delta L}$, so recovering the sign of S at $\sigma = \frac{1}{2} + \delta$ with a fixed margin requires the primes up to $e^{C/\delta}$.

Corollary 3.14 (prime-resolution scale). Assume RH. Let $0 < \varepsilon < 1$, $\sigma = \frac{1}{2} + \delta$ and $P \geq 2$. If $\delta \log P \geq \log(3/\varepsilon)$, then the finite quantity $W_{\log P}(\sigma, t)$, which involves only the prime powers

$n \leq P$, satisfies, for every t ,

$$\left| \frac{W_{\log P}(\sigma, t)}{2S(\sigma, t)} - 1 \right| \leq \varepsilon.$$

Equivalently, the primes up to P determine S to relative precision ε at every distance $\delta \geq \delta_{\text{res}}(P, \varepsilon) = \log(3/\varepsilon)/\log P$ from the critical line.

Proof. By (3.5), $|W_L/2S - 1| \leq \max\{(1+a)^2 - 1, 1 - (1-a)^2\} = 2a + a^2 \leq 3a$ with $a = e^{-\delta L} \leq 1$. Take $L = \log P$, so that $3a \leq \varepsilon$. $\square$ Status: conditional on RH.

For example, primes up to 10^8 give $\delta_{\text{res}} = 0.18$ at $\varepsilon = 0.1$ and 0.31 at $\varepsilon = 0.01$; resolving $\delta = 0.01$ to 1% requires $P \geq 10^{248}$. The hypothesis cannot be removed by a better estimate. An unconditional bound for $W_L - 2S$ at abscissa σ must control the terms $e^{(\beta-\sigma)L}$ of Theorem 3.13(b3), that is, it requires knowing that no zero lies to the right of σ in the relevant range of heights. Such knowledge is available only inside the classical zero-free region and below the height $3 \cdot 10^{12}$ [PT], where $S > 0$ already follows from Corollary 3.3. A finite certificate of the form “ $W_L > B_L \Rightarrow S > 0$ ” therefore certifies nothing beyond what is known.

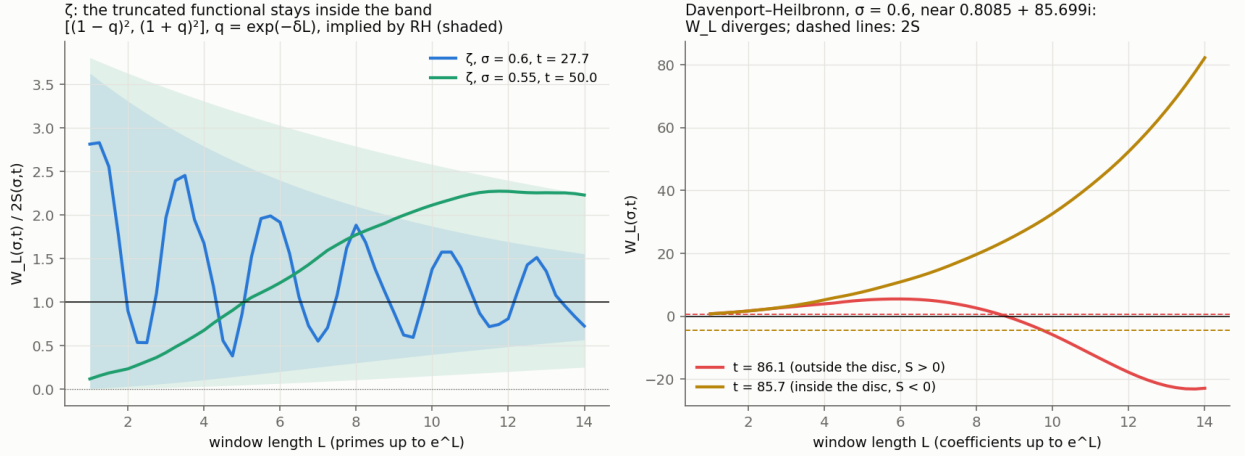


FIGURE 4. The truncated Poisson functional $W_L(\sigma, t)$ of Theorem 3.13(b), computed from the primes (or coefficients) up to e^L . Left: ζ ; the ratio $W_L/2S$ and the band (3.5) implied by RH. Right: Davenport–Heilbronn function at $\sigma = 0.6$, inside ($t = 85.7$) and outside ($t = 86.1$) the disc of its zero $0.8085 + 85.699i$; dashed lines show $2S$.

The truncated functional in practice. We computed W_L from the primes and, independently, from the zeros; the two agree to within $3 \cdot 10^{-3}$, both for ζ and for the Davenport–Heilbronn function with $b(n)$ in place of $\Lambda(n)$, the difference being the contribution of the zeros beyond the computed range. For ζ at $(\sigma, t) = (0.6, 27.7)$ and $(0.55, 50)$, $W_L/2S$ stays inside the band (3.5) for $1 \leq L \leq 14$ (Figure 4, left). For the Davenport–Heilbronn function at $\sigma = 0.6$ (Figure 4, right) the behaviour is the opposite of the disc picture. At $t = 85.7$, inside the disc, where $S < 0$, W_L is positive and grows exponentially, reaching 82 at $L = 14$. At $t = 86.1$, outside the disc, where $S > 0$, W_L becomes negative at $L = 9$ and reaches -23 at $L = 13.75$. The truncated functional detects the off-line zero from the coefficients, but not locally: by Theorem 3.13(b3) every zero with $\beta > \sigma$ affects $W_L(\sigma, t)$ at every t once L is large. The locality of Theorem 3.2 holds only in the limit $L \rightarrow \infty$, and that limit is what the Euler product does not control in $\frac{1}{2} < \sigma < 1$.

The detection length can be measured on f , whose zeros are known. For each off-line zero ρ of f with $\gamma < 400$ we computed, at $\sigma = 0.55$, the smallest L (step 0.25) at which the contribution of the pair $\rho, 1 - \bar{\rho}$ to $W_L(\sigma, t)$ exceeds in modulus the contribution of all zeros on the critical line, for some t with $8 \leq |t - \gamma| \leq 12$, that is, outside the disc (Table 10). A least-squares fit

gives $L_{\det} \approx 4.05/(\beta - \sigma) + 4.5$, with correlation 0.993 against $1/(\beta - \sigma)$; the product $L_{\det}(\beta - \sigma)$ stays between 4.0 and 5.5, its variation coming from the logarithm of the kernel factor in Theorem 3.13(b3). The zero at $\beta - \sigma = 0.024$ requires $L \approx 167$, that is, the coefficients up to $e^{167} \approx 10^{72}$. Status: numerical (`anc/weil/`).

β	γ	$\beta - \sigma$	$L_{\det}$	$L_{\det}(\beta - \sigma)$
0.8695	240.40	0.320	16.5	5.3
0.8195	320.88	0.270	16.0	4.3
0.8085	85.70	0.259	19.3	5.0
0.7682	331.05	0.218	18.5	4.0
0.7243	176.70	0.174	22.8	4.0
0.6508	114.16	0.101	49.3	5.0
0.6285	366.64	0.079	69.5	5.5
0.5744	166.48	0.024	167.0	4.1

TABLE 10. Detection length of the off-line zeros of the Davenport–Heilbronn function by the truncated Poisson functional at $\sigma = 0.55$, outside their discs.

3.7. The margin over a finite field. The smallest value of Weil’s form on test functions supported in a window of length L is a natural measure of the margin of the positivity in Theorem 3.12. Over a finite field it is explicit. Let E be an elliptic curve over $\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = p + 1 - a$ and $a = 2\sqrt{p} \cos \phi$. Its zeta function depends on s only through p^{-s} , and its zeros, all on the line by Hasse’s theorem, are $\frac{1}{2} + i\gamma$ with $\gamma \in \pm\phi/\log p + \Omega\mathbb{Z}$, $\Omega = 2\pi/\log p$.

Proposition 3.15 (finite fields). Let $f \in C_c^1(\mathbb{R})$ be supported in an interval of length $L < \log p$ and $F(r) = \int f(u)e^{iru} du$. Then

$$\sum_{\gamma} |F(\gamma)|^2 = 2 \log p \|f\|_2^2,$$

the sum running over the zeros of the zeta function of E , with multiplicity. Hence the smallest value of Weil’s form of E on such windows, normalized by $\|f\|_2^2$, is $2 \log p$, independent of $L < \log p$ and of a .

Proof. Let $g = f * \tilde{f}$, which is C^1 , supported in $(-L, L)$, and satisfies $\hat{g} = |F|^2 = O(r^{-2})$. By Poisson summation, for every θ ,

$$\sum_{k \in \mathbb{Z}} \hat{g}(\theta + k\Omega) = \frac{2\pi}{\Omega} \sum_{n \in \mathbb{Z}} g\left(\frac{2\pi n}{\Omega}\right) e^{-2\pi i n \theta / \Omega} = \log p \sum_n g(n \log p) e^{-in\theta \log p}.$$

Since $L < \log p$, only $n = 0$ contributes, and the sum is $\log p g(0) = \log p \|f\|_2^2$. The zeros form two such progressions, $\theta = \pm\phi/\log p$. $\square$ Status: proved.

The terms $n \neq 0$, which enter when $L \geq \log p$, are the closed points of E at $u = n \log p$; they play the role of the primes. Numerically the smallest value is $9.23 = 2 \log 101$ for $p = 101$ up to $L = 4$, and $2.20 = 2 \log 3$, $1.39 = 2 \log 2$ for $p = 3, 2$ up to $L \approx \log p$. For ζ , by contrast, the margin collapses superexponentially in L , and the collapse is a property of the density of its zeros (which have no ordinates below 14.13 and density $\frac{1}{2\pi} \log \frac{t}{2\pi}$), not of the primes.

What is treated in Part 12. The numerical study of Weil’s form on windows is developed in [CE12], and we summarize it here. The smallest eigenvalue $\lambda_{\min}(L)$ of the form on a window of length L involves only $n \leq e^L$; in a fixed finite sine basis it fits $10^{-6.3L^2}$, but this law holds only in that basis, and since enlarging the basis can only lower the smallest eigenvalue, it is an upper bound: high-precision computations in bases large enough to converge (up to 950 digits) show that the margin stays positive and follows $\log_{10} \lambda_{\min} \approx -(5.42e^L - 12.6)$ on $2 \leq L \leq 5$, about $e^{-12.5X}$ in the length $X = e^L$ of the prime sum, with $\lambda_{\min} \approx 10^{-793}$ at $L = 5$ [CE12]. The size of this margin

is a counting effect of the density of the zeros: points with the same density and no arithmetic content give the same values, while for curves over $\mathbb{F}_p$ it is the constant $2 \log p$ of Proposition 3.15. For the Davenport–Heilbronn function the form becomes negative at a threshold $3.42 < L^* \leq 3.43$ (determined in converged bases with 60 digits; the earlier value 3.6 is where the negative eigenvalue first exceeds double precision), that is, once the coefficients up to $n \approx e^{3.43} \approx 31$ are included; its second off-line pair $0.6508 + 114.163i$ alone would make it negative at $3.72 < L \leq 3.725$ [CE12]. At large height, a two-window family detects an isolated zero $\frac{1}{2} + \delta + i\gamma$ with the prime powers up to $(C \log \gamma)^{1/\delta}$, and positivity of that family holds for ζ at all 2400 heights sampled up to $3 \cdot 10^6$, while it fails at 1.5% of them for a random multiplicative model. The minimiser of the form reproduces the zeros of ζ to many digits on short windows (Connes [Cn26]), but does so equally well for f up to and slightly beyond L^* ; and positivity is maintained prime power by prime power with no reserve, the form becoming negative whenever the most recent prime power is removed. Dirichlet polynomials as test functions show the same collapse for ζ and for f , separated only by the sign of the minimum. All these results are in [CE12].

4. DISCUSSION

The two criteria of this part are global, and they show where the difficulty lies.

1. *Pointwise, the obstruction is local.* By Theorems 3.2 and 3.7 a failure of the energy slope at abscissa σ occurs only inside the discs of off-line zeros. The density theorems, which use the Euler product through mean values of Dirichlet polynomials, make the failure set sparse. For functions of Davenport–Heilbronn type it has positive proportion [KK]. A proof along these lines would need a version of those mean-value arguments valid inside a single disc. By Theorem 3.13 this is Weil positivity along the Poisson family. The truncated functionals W_L are exact and computable for each L , but the limit $L \rightarrow \infty$ at fixed $\sigma < 1$ is equivalent to controlling all zeros with $\beta > \sigma$, and by Corollary 3.14 the primes up to P resolve only the distances $\delta \geq \log(3/\varepsilon)/\log P$ from the line.
2. *Jensen’s formula identifies what is missing.* The C_2 –Jensen formula (Theorem 2.7) turns the trichotomy into the criterion of Balazard, Saias and Yor: a nonnegative sum over the zeros to the right of the line, which vanishes exactly under RH (Corollary 2.8). The criterion is an equality between a boundary integral and a constant. The boundary integral is an average of $\log |\zeta|$, and its mollified versions meet the gap in Jensen’s inequality (Corollary 2.13). The size of that gap is the subject of [CE11].

4.1. Open problems.

- Can the exceptional set of Theorem 3.7 be bounded inside one disc, using only the information in $\sum_{n \leq e^L} \Lambda(n)n^{-s}$ for L of size $C/(\sigma - \frac{1}{2})$?

We do not prove the Riemann hypothesis.

APPENDIX A. NUMERICAL METHODS AND DATA

All computations use double precision in Python (NumPy, SciPy), except where a number of digits is stated; those use mpmath or Python’s `decimal` arithmetic. The zeros of ζ with $0 < \gamma < 600$ (341 ordinates) and the first 1000 ordinates were computed once and are reused throughout; where a statement depends on the precision of an ordinate ([CE2, Remark 2.12]), the ordinates are refined to 10^{-12} . ζ is evaluated by Euler–Maclaurin summation with Bernoulli corrections, and the Davenport–Heilbronn function f through Hurwitz zeta functions; its eight off-line zeros with $\gamma < 400$ are those listed in [CE17]. $\Lambda(n)$ is computed by sieve up to 10^8 (up to 10^9 for ψ). Every value in the tables was obtained by the script cited in its Status line, and every script prints the numbers quoted in the text.

- *Section 2.* The C_2 -Jensen formula, the moments and the Davenport–Heilbronn data (`c2-jensen/`, `jensen/`).
- *Section 3.* $S(\sigma, t)$ by central differences with step 10^{-5} applied to $\log |\xi|$ and to the completed f (`slope/`, `concavity/`, `zerocount/`, `density/`); the twin $L(s, \chi) \bmod 5$ (`twin/`); W_L with prime powers up to e^{14} and the archimedean integral over $|r| \leq 3000$ with an analytic tail (`weil/`); the valley of $\log |\xi|$, the margin $S/(\sigma - \frac{1}{2})$ and the reconstruction of ξ'/ξ from the zeros on the line, with the 1517 ordinates below 2000 (`valley/`). The Beurling margin quoted in Section 3 is from the author’s working notes.

How this work was produced. The cross-energy C_2 and the research program are the author’s, beginning with his preprints of 2017 and the series [C1]–[C8]. The author used Claude (Anthropic), an AI assistant, as a research and writing aid: to derive and check proofs, to write and run the Python computations, to test claims against the Davenport–Heilbronn control and against counterexamples, to search the literature, and to draft and edit the text and figures. In preparing this series, the material of the consolidated paper (version 3), of the earlier series [C1]–[C8] and of the author’s manuscript on the asymptotic balance of partial sums was inventoried result by result, regrouped into the seventeen parts of the series (Table 1 of [CE1]), and checked again; the corrections found in the process are stated where they apply and listed in Part 16 [CE16]. Several claims of the author’s earlier preprints were found to be incorrect and are not used. The author directed the work, checked the results and is responsible for the content.

Data availability. The scripts are included in the ancillary directory `anc/` of the source package, one subdirectory per topic as listed above. The directory `anc/lean/` contains a draft Lean 4/mathlib formalization of the pair identity and the strict part of Theorem 3.2 (disc lemma), and of the finite core of [CE16, Proposition 2.8] (a nonnegative trigonometric polynomial has positive mean, and the weighted prime sums are nonnegative with constant part $a_0 \sum w_p$); the prime number theorem asymptotic of [CE16, Proposition 2.8] is not formalized. The files are a draft: they have not been compiled or checked by Lean, and no result of this paper depends on them (`anc/lean/README.md`). The earlier preprints [C1]–[C8] and the versions 1–4 of the consolidated paper are deposited on Zenodo under the DOIs listed in the references.

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