

Cross-Energy of Dirichlet Partial Sums.

Part 14: The constant in the $C/\log X$ law for triangle-inequality upper bounds on the de Bruijn–Newman constant

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Part 14 of the series *Cross-energy of Dirichlet partial sums* (seventeen parts; the plan of the series is in Part 1 [CE1])

Abstract

The method of Polymath bounds the de Bruijn–Newman constant by $\Lambda \leq t_0 + \frac{1}{2}y_0^2$, where H_{t_0} is shown to have no zeros with $\text{Im } z \geq y_0$ to the right of a height X (RH verified for ζ up to $X/2$), and a barrier condition holds at X . Polymath observed that this gives $\Lambda = O(1/\log X)$. We study the constant C in bounds $\Lambda \leq (C + o(1))/\log X$ for the arguments in which the zero-free region is established by a triangle inequality applied to Polymath’s approximation, with or without a mollifier; C is a property of this method, not of Λ . We define the best constant C_{opt} attainable in this class and prove $4 \leq C_{\text{opt}} \leq 4w_0 = 4.9146\dots$, where $\zeta(\frac{1}{2} + w_0) = 2$, for mollifiers of length at most $N/2$ ($N \approx \sqrt{X/4\pi}$ is the length of the sum), and $C_{\text{opt}} = 4w_0$ exactly without a mollifier. The upper value comes from an explicit criterion for the non-vanishing of H_t in which each term of the sum keeps its own effective exponent; its hypotheses are explicit inequalities at the single point $x = X$, and the resulting bounds for Λ are conditional on the verification of RH and on the barrier. The lower value comes from the primes in $(N/2, N]$, whose coefficients no such mollifier can change. At the present height the bound $\Lambda \leq 0.174$ has $\log(X/4\pi) \cdot 0.174 = 4.68$. Numerically, for optimised mollifiers the boundary of the zero-free region is close to a level set of the effective exponent of the last term of the sum; we report how the level depends on the mollifier length. Finally, the triangle inequality amounts to giving every cross-energy of the partial sums its most negative value; at the present height the phases of the terminal primes are close to equidistributed. We do not prove the Riemann hypothesis, and none of these bounds says anything about it.

1 Introduction

Let $H_t(z) = \int_0^\infty e^{tu^2} \Phi(u) \cos(zu) du$, where $\Phi(u) = \sum_{n \geq 1} (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) e^{-\pi n^2 e^{4u}}$, so that $H_0(z) = \frac{1}{8}\xi(\frac{1}{2} + \frac{iz}{2})$. By the work of de Bruijn and Newman [1, 7] there is a constant Λ such that H_t has only real zeros exactly when $t \geq \Lambda$. The Riemann hypothesis (RH) is equivalent to $\Lambda \leq 0$, and Rodgers and Tao proved $\Lambda \geq 0$ [12]. Polymath [11] proved $\Lambda \leq 0.22$; with the verification of RH up to height $3 \cdot 10^{12}$ [10] their criterion gives $\Lambda \leq 0.2$, and with optimised mollifiers we obtained $\Lambda \leq 0.174$ [CE13]. Other claims announced in 2026 are compared with our results in Section 8.

Three different objects. It is important to keep apart: (A) the constant Λ , a fixed number, equal to 0 if RH holds; (B) the bound $B(X) = t_0 + \frac{1}{2}y_0^2$ that a method produces when RH has been verified up to $X/2$; and (C) the coefficient $C(X) = B(X) \log \frac{X}{4\pi}$, which has the same limiting behaviour as $B(X) \log X$; at finite heights the value depends on this normalisation, and we always use $\log \frac{X}{4\pi}$. This paper is about (C), and its main results concern the limit $X \rightarrow \infty$. A statement such as $C(X) \geq 4 - o(1)$ does *not* mean $\Lambda \geq 4/\log X$, which would contradict RH; it means that a certain kind of proof cannot produce a bound $B(X)$ with a smaller coefficient. Such limitations are fully compatible with $\Lambda = 0$.

Figure 1(a) shows the mechanism at low height: as t decreases, neighbouring real zeros of H_t collide and leave the real axis. Polymath's criterion (Theorem 1 below) controls three regions (Figure 1(b)): to the left of a height X the zeros are real because RH has been verified there; a barrier near X prevents zeros from entering that region; and to the right of X one shows that H_{t_0} has no zeros with $\text{Im } z \geq y_0$. Zeros may remain in the strip $0 < \text{Im } z < y_0$, but by de Bruijn's argument they reach the real axis by time $t_0 + \frac{1}{2}y_0^2$.

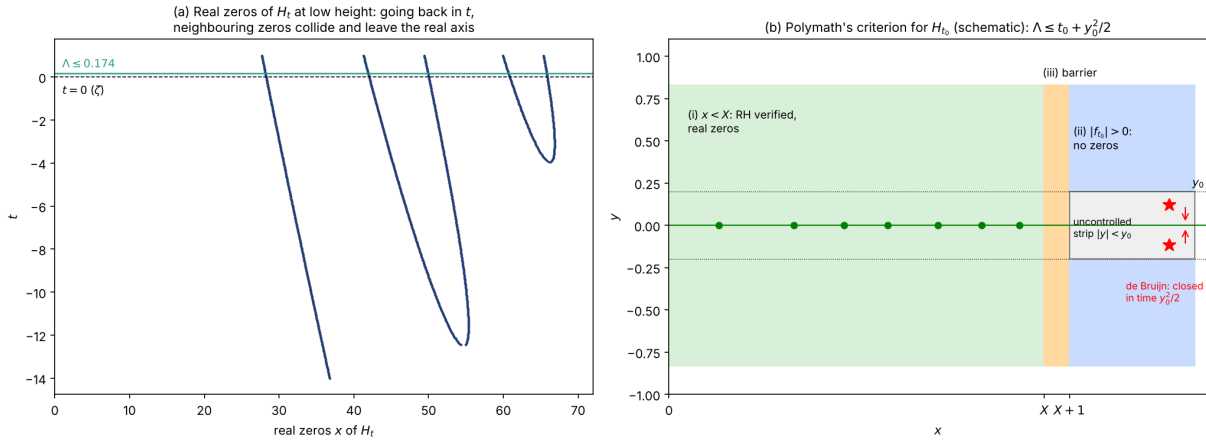


Figure 1: (a) Real zeros of H_t for $x \leq 72$, computed by direct numerical integration (double precision). Going back in t , pairs of neighbouring zeros collide and become complex. The horizontal line is the bound $\Lambda \leq 0.174$ of [CE13] (preprint). (b) Schematic of Polymath's criterion: (i) RH verified for $x < X$; (iii) barrier; (ii) no zeros of H_{t_0} with $|y| \geq y_0$ for $x \geq X + \sqrt{1 - y_0^2}$. The strip $|y| < y_0$ is not controlled; de Bruijn's argument closes it in time $\frac{1}{2}y_0^2$.

Polymath showed that the zeros of H_t are real and simple for $x \geq \exp(C/t)$, with an unspecified absolute constant C [11, Thm. 1.5], refining Ki, Kim and Lee [5]. They observed that, combined with their criterion and ignoring the barrier condition, this gives $\Lambda = O(1/\log T)$ once RH is verified up to height T , and they gave a heuristic argument that one cannot do substantially better without major new information on RH [11, §10]. Their table of conditional bounds [11, Table 1] illustrates the $1/\log T$ behaviour. The $O(1/\log X)$ law is therefore not new; the purpose of this paper is to determine the constant for a precisely defined class of arguments. Throughout, $L = \log \frac{x}{4\pi}$, or $L = \log \frac{X}{4\pi}$ when $x = X$.

The class consists of the *triangle certificates* of Definition 1: hypothesis (ii) of Polymath's criterion is established by bounding the modulus of a mollified version of Polymath's main term from below by 1 minus the sum of the absolute values of the other terms. For a class $\mathcal{D}$ of admissible mollifier lengths, $C_{\text{opt}}(\mathcal{D})$ denotes the smallest asymptotic coefficient $\liminf B(X) \log X$ that such certificates can reach (Definition 1). Let $w_0 > 0$ be defined by $\zeta(\frac{1}{2} + w_0) = 2$, so

$w_0 = 1.2286472389981 \dots$ and $4w_0 = 4.9145889559 \dots$

Theorem A. (a) Without a mollifier, $C_{\text{opt}}(\{1\}) = 4w_0$.

(b) For mollifiers of length $D \leq N/2$, where $N \approx \sqrt{X/4\pi}$ is the length of the sum, $4 \leq C_{\text{opt}}(\{D \leq N/2\}) \leq 4w_0$.

(c) The families that realise the value $4w_0$ in (a) establish hypothesis (ii) of Theorem 1 on the whole region, including Polymath's error terms. Hence $\Lambda \leq (4w_0 + o(1))/\log X$ along any sequence of heights X at which RH has been verified up to $X/2$ and the barrier condition (iii) holds for the parameters (t_0, y_0) of Corollary 7.

Theorem A is proved in Section 5. The value $4w_0$ is the *constructed* coefficient (Corollary 7), and (a) shows that it is optimal without a mollifier (Corollary 12). We do not know where $C_{\text{opt}}(\{D \leq N/2\})$ lies in $[4, 4w_0]$. The results are, in order:

- Theorem 2, an explicit lower bound for $|f_t|$, the main term of Polymath's approximation to H_t , in which each term keeps its own effective exponent $\alpha_n = \frac{1+y}{2} + \frac{t}{4}(L - \log n)$: a finite sum is computed exactly and the rest is bounded by elementary integrals. With Lemmas 4 and 5 it reduces hypothesis (ii) to two inequalities at the single point $x = X$ (Theorem 6), whose asymptotic form gives $4w_0$ (Corollary 7).
- Proposition 9 and Corollary 10: the primes $N/2 < p \leq N$ are not divisible by any $d \leq N/2$ other than 1, so their coefficients survive every mollifier of length $D \leq N/2$; this forces $C \geq 4$. Proposition 11: without a mollifier the first terms of the sum force $C \geq 4w_0$.
- Section 6 (numerical, not certified): for optimised mollifiers the boundary of the zero-free region is close to a level set $\frac{y}{2} + \frac{t}{8}L \approx w$ of the effective exponent of the last term, and we report how w depends on the mollifier length D .
- Section 7: the triangle inequality is the same as giving every cross-energy $|F_n|^2 - |F_{n-1}|^2 - |a_n|^2$ of the partial sums its most negative value. This is the link with the rest of the series, and it is where a phase-sensitive certificate would have to gain.

The bounds for Λ obtained in this paper are conditional on the verification of RH up to $X/2$ and on the barrier condition at X ; the unconditional bounds quoted are 0.22 [11], 0.2 [10] and 0.174 [CE13] (preprint, not yet refereed). Appendix A lists every constant imported from [11] with its source and every inequality used in the proofs; Appendix B lists the computations and says which are certified.

2 Polymath's criterion and approximation

Theorem 1 ([11, Thm. 1.2]). Let $t_0, X > 0$ and $0 < y_0 \leq 1$, and assume: (i) $\zeta(\sigma + iT) \neq 0$ for $\frac{1+y_0}{2} \leq \sigma \leq 1$ and $0 \leq T \leq X/2$; (ii) $H_{t_0}(x + iy) \neq 0$ for $x \geq X + \sqrt{1 - y_0^2}$ and $y_0 \leq y \leq \sqrt{1 - 2t_0}$; (iii) $H_t(x + iy) \neq 0$ for $X \leq x \leq X + \sqrt{1 - y_0^2}$, $\sqrt{y_0^2 + 2(t_0 - t)} \leq y \leq \sqrt{1 - 2t}$ and $0 \leq t \leq t_0$. Then $\Lambda \leq t_0 + \frac{1}{2}y_0^2$.

Hypothesis (iii) is the *barrier*. It is a finite computation at a given X ; at $t = 0$ it concerns ζ itself near height $X/2$, so it cannot be replaced by an analytic argument, and in this paper it is always an assumption.

In the region $0 < t \leq \frac{1}{2}$, $0 \leq y \leq 1$, $x \geq 200$, Polymath [11, Thm. 1.3] write $H_t/B_t = f_t + O_{\leq}(e_A + e_B + e_{C,0})$, where B_t is an explicit non-vanishing factor, and

$$f_t(x + iy) = \sum_{n=1}^N \frac{b_n}{n^{s_*}} + \gamma \sum_{n=1}^N \frac{n^y b_n}{n^{s_* + \kappa}}, \quad b_n = e^{\frac{t}{4} \log^2 n}, \quad N = \left\lfloor \sqrt{\frac{x}{4\pi} + \frac{t}{16}} \right\rfloor,$$

with $s_* = \frac{1+y-ix}{2} + \frac{t}{2}\alpha(\frac{1+y-ix}{2})$, $\alpha(s) = \frac{1}{2s} + \frac{1}{s-1} + \frac{1}{2} \log \frac{s}{2\pi}$, and explicit κ, γ satisfying [11, (20)–(22)]

$$|\gamma| \leq e^{0.02y} \left(\frac{x}{4\pi} \right)^{-y/2}, \quad \operatorname{Re} s_* \geq \frac{1+y}{2} + \frac{t}{4}L - \frac{t}{x^2}, \quad |\kappa| \leq \frac{ty}{2(x-6)}. \quad (1)$$

In the middle inequality we weakened the last term of [11, (21)], $-\frac{t}{2x^2}(1 - 3y + \frac{4y(1+y)}{x^2})_+$, using that the bracket is at most 1 for $0 \leq y \leq 1$ and $x \geq 200$ (Appendix A). Polymath also give explicit bounds [11, (23), (24)] for $e_A + e_B$ and $e_{C,0}$, and if $|f_t| > e_A + e_B + e_{C,0}$, then $H_t \neq 0$ [11, Cor. 1.4]. From the formula for s_* one also has the upper bound

$$\operatorname{Re} s_* \leq \frac{1+y}{2} + \frac{t}{4}L + \frac{3t}{2x^2}, \quad (2)$$

since, with $s_0 = \frac{1+y-ix}{2}$, $\operatorname{Re} \frac{1}{s_0} \leq \frac{2}{x^2}$, $\operatorname{Re} \frac{1}{s_0-1} \leq 0$ for $y \leq 1$, and $\frac{1}{2} \log \frac{|s_0|}{2\pi} \leq \frac{L}{2} + \frac{1}{x^2}$.

Mollified triangle inequalities. Polymath [11] and the author [CE13] establish hypothesis (ii) with a mollifier $\sum_{d \leq D} \lambda_d d^{-s_*}$, $\lambda_1 = 1$, and the inequality

$$|f_t| \sum_{d \leq D} |\lambda_d| d^{-\operatorname{Re} s_*} \geq \mathcal{L} := 1 - \sum_{n \geq 2} |c_n| n^{-\sigma} - g \sum_{n \geq 1} |c'_n| n^{y-\sigma+k}, \quad (3)$$

where $c_n = \sum_{d|n, d \leq D, n/d \leq N} \lambda_d b_{n/d}$, c'_n is defined in the same way, $\sigma \leq \operatorname{Re} s_*$, $g \geq |\gamma|$ and $k \geq |\kappa|$ [CE13, Lemma 3]. Any real weights give a valid bound, and $D = 1$ (so $c_n = b_n$ for $n \leq N$) is the unmollified triangle inequality.

Definition 1. Let $0 < t_0 \leq \frac{1}{2}$, $0 < y_0 \leq \sqrt{1 - 2t_0}$, $x_X = X + \sqrt{1 - y_0^2}$ and $N = N(x_X)$. A *triangle certificate of length D at height X* is a choice of real weights $\lambda_1 = 1, \lambda_2, \dots, \lambda_D$ and of admissible σ, g, k as in (3) for which $\mathcal{L} > 0$ at $t = t_0$, $x = x_X$, $y = y_0$. Its value is $B = t_0 + \frac{1}{2}y_0^2$. For a set $\mathcal{D}$ of lengths (we use $\mathcal{D} = \{1\}$ and $\mathcal{D} = \{D \leq N/2\}$), $C_{\text{opt}}(\mathcal{D})$ is the infimum of $\liminf_{X \rightarrow \infty} B(X) \log X$ over all families $X \mapsto$ (certificate at height X with length in $\mathcal{D}$) defined for all large X .

Positivity of $\mathcal{L}$ at the corner point (x_X, y_0) is necessary for any proof of (ii) by (3); a complete proof also needs $\mathcal{L}$ to exceed the error terms on the whole region. So C_{opt} is a lower bound for the coefficient of every such proof, and an upper bound for C_{opt} is obtained by exhibiting complete proofs (Section 4). C_{opt} is a purely analytic quantity: hypotheses (i) and (iii) play no role in its definition.

3 An explicit criterion

For $0 < t \leq \frac{1}{2}$, $x \geq 200$ and $0 \leq y \leq 1 - \frac{4}{x}$ put

$$M = \frac{1}{2} \log \left(\frac{x}{4\pi} + \frac{t}{16} \right), \quad A = \frac{1+y}{2} + \frac{t}{4}L - \frac{t}{x^2}, \quad k = \frac{ty}{2(x-6)},$$

$$\alpha_n = A - \frac{t}{4} \log n, \quad \beta_n = \alpha_n - y - k,$$

so that $N \leq e^M$ and $\frac{L}{2} \leq M \leq \frac{L}{2} + \frac{t}{32}e^{-L}$, and let $\sigma_A = A - \frac{t}{4}M$ and $\sigma_B = \sigma_A - y - k$ be the smallest values of α_n and β_n for $n \leq N$. Thus α_n decreases from about $\frac{1+y}{2} + \frac{t}{4}L$ for small n to $\sigma_A \approx \frac{1+y}{2} + \frac{t}{8}L$ for $n \approx N$. For an integer $1 \leq n_0 \leq N$ let

$$S_A = \sum_{n=2}^{n_0} n^{-\alpha_n} + \int_{\log n_0}^M e^{(1-A)v + \frac{t}{4}v^2} dv, \quad S_B = \sum_{n=2}^{n_0} n^{-\beta_n} + \int_{\log n_0}^M e^{(1-A+y+k)v + \frac{t}{4}v^2} dv.$$

Theorem 2. *In this range,*

$$|f_t(x + iy)| \geq G(t, x, y) := 1 - S_A - e^{0.02y} e^{-yL/2} (1 + S_B).$$

Consequently $H_t(x + iy) \neq 0$ whenever $G(t, x, y) > e_A + e_B + e_{C,0}$. Moreover $G \leq \mathcal{L}$ for the certificate of length $D = 1$ with $\sigma = A$, $g = e^{0.02y} e^{-yL/2}$ and k as above.

Proof. For $n \leq N$, $b_n n^{-\text{Re } s_*} = \exp(-\text{Re } s_* \log n + \frac{t}{4} \log^2 n) \leq n^{-\alpha_n}$ by (1). Hence the first sum in f_t has modulus at least $1 - \sum_{2 \leq n \leq N} n^{-\alpha_n}$. The function $u \mapsto u^{-\alpha(u)} = \exp(-Av + \frac{t}{4}v^2)$, $v = \log u$, is decreasing on $[1, N]$, because its logarithmic derivative in v is $-A + \frac{t}{2}v \leq -(A - \frac{t}{2}M) < 0$. So by the integral test $\sum_{n_0 < n \leq N} n^{-\alpha_n} \leq \int_{\log n_0}^N u^{-\alpha(u)} du \leq \int_{\log n_0}^M e^{(1-A)v + \frac{t}{4}v^2} dv$, and the first sum has modulus at least $1 - S_A$. In the second sum, $|n^{-s_* - \kappa}| \leq n^{-\text{Re } s_* + |\kappa|}$, so its terms are bounded by $n^{y+k} b_n n^{-\text{Re } s_*} \leq n^{-\beta_n}$. The same argument applies with A replaced by $A - y - k$, since $A - y - k - \frac{t}{2}M = \frac{1-y}{2} - k - \frac{t}{x^2} - \frac{t}{4}(2M - L) > 0$ for $y \leq 1 - \frac{4}{x}$. So the second sum is at most $1 + S_B$. Together with $|\gamma| \leq e^{0.02y} e^{-yL/2}$ this gives $|f_t| \geq G$. The second statement is [11, Cor. 1.4]. For the last one, with $D = 1$ and $\sigma = A$ the sums in (3) are $\sum_{2 \leq n \leq N} n^{-\alpha_n}$ and $\sum_{n \leq N} n^{-\beta_n}$, which were just bounded by S_A and $1 + S_B$. $\square$

Remark 3. Bounding every term by the exponent of the last one, $\alpha_n \geq \sigma_A$, gives the simpler criterion $|f_t| \geq 2 - \zeta(\sigma_A) - |\gamma|(\dots)$, whose boundary is $\zeta(\sigma_A) = 2$. That loses a factor of about 2 in the constant of Corollary 7. The point of Theorem 2 is that the small values of n , which carry most of the sum, have effective exponent close to $\frac{1}{2} + \frac{t}{4}L$, twice as far from $\frac{1}{2}$ as the last term.

To apply Theorem 2 on a whole region we need monotonicity and a majorant for the error terms. Write $G(x, y)$, $\sigma_A(x, y)$ and so on when t is fixed. In Lemma 4 the integer n_0 is fixed and does not change with x ; we regard all quantities as functions of $L = \log \frac{x}{4\pi}$, so $x = 4\pi e^L$ and $\frac{dx}{dL} = x$.

Lemma 4. *Let $0 < t \leq \frac{1}{2}$, $x_1 \geq 200$ and let $n_0 \leq N(x_1)$ be fixed.*

- (a) *For fixed x , $G(x, y)$ is increasing in $y \in [0, 1 - \frac{4}{x}]$.*
- (b) *Fix $y \in (0, 1 - \frac{4}{x_1}]$. Assume $\sigma_A(x_1, y) > 1$ and $\sigma_B(x_1, y) \geq 1 + \frac{2}{L_1} \log \frac{1}{y}$, where $L_1 = \log \frac{x_1}{4\pi}$. Then for all $x \geq x_1$,*

$$G(x, y) \geq G(x_1, y) - R_A, \quad R_A = \frac{e^{-(\sigma_A(x_1, y) - 1)L_1/2}}{\sigma_A(x_1, y) - 1}.$$

Proof. (a) We have $\partial_y A = \partial_y \alpha_n = \frac{1}{2}$, so every term of S_A decreases with y : $\partial_y n^{-\alpha_n} = -\frac{1}{2} \log n \cdot n^{-\alpha_n}$, and the integrand $e^{(1-A)v + \frac{t}{4}v^2}$ has y -derivative $-\frac{v}{2} e^{(1-A)v + \frac{t}{4}v^2}$; the limits of integration do not depend on y . Let $T = e^{0.02y} e^{-yL/2} (1 + S_B)$ and $\theta = \frac{1}{2} + \frac{t}{2(x-6)} = -\partial_y(\beta_n) = \partial_y(1 - A + y + k)$. Then $\partial_y n^{-\beta_n} = \theta \log n \cdot n^{-\beta_n} \leq \theta M n^{-\beta_n}$ for $n \leq n_0 \leq N \leq e^M$, and the y -derivative of the

integrand of S_B is θv times the integrand, with $v \leq M$. Hence $\partial_y(1 + S_B) \leq \theta M S_B \leq \theta M(1 + S_B)$ and

$$\partial_y \log T \leq 0.02 - \frac{L}{2} + \theta M \leq 0.02 - \frac{L}{2} + \left(\frac{1}{2} + \frac{1}{776}\right) \left(\frac{L}{2} + 10^{-3}\right) \leq 0.021 + \frac{L}{1552} - \frac{L}{4} < 0,$$

since $L \geq \log \frac{200}{4\pi} > 2.76$. So $G = 1 - S_A - T$ increases with y .

(b) As functions of L : $\frac{dA}{dL} = \frac{t}{4} + \frac{2t}{x^2} > 0$; $M'(L) = \frac{1}{2} \frac{e^L}{e^L + t/16} \in (0, \frac{1}{2}]$; $\frac{d\sigma_A}{dL} = \frac{t}{4} + \frac{2t}{x^2} - \frac{t}{4} M' \geq \frac{t}{8} > 0$; $\frac{dk}{dL} = -\frac{tyx}{2(x-6)^2} < 0$, so σ_B is increasing; and $\frac{2}{L} \log \frac{1}{y}$ is decreasing. Hence for $L \geq L_1$, $\sigma_A(L) \geq \sigma_A(L_1) > 1$ and $\sigma_B(L) - 1 \geq \sigma_B(L_1) - 1 \geq \frac{2}{L_1} \log \frac{1}{y} \geq \frac{2}{L} \log \frac{1}{y}$: the hypotheses persist. Also $n_0 \leq N(x_1) \leq N(x)$.

The finite part of S_A decreases, since each α_n increases with L . For $I_A(L) = \int_{\log n_0}^{M(L)} e^{\phi(v,L)} dv$, $\phi = (1 - A)v + \frac{t}{4}v^2$, the total derivative is

$$\frac{dI_A}{dL} = M'(L) e^{\phi(M,L)} + \int_{\log n_0}^M \partial_L \phi e^{\phi} dv \leq \frac{1}{2} e^{\phi(M,L)},$$

because $\partial_L \phi = -\frac{dA}{dL}v \leq 0$. Here $\phi(M, L) = M(1 - A + \frac{t}{4}M) = M(1 - \sigma_A) \leq \frac{L}{2}(1 - \sigma_A(L_1))$, using $M \geq \frac{L}{2}$ and $\sigma_A(L) \geq \sigma_A(L_1) > 1$. Integrating from L_1 to ∞ gives $S_A(L) \leq S_A(L_1) + R_A$.

For T : the finite part of S_B decreases, since each β_n increases with L . For $I_B(L) = \int_{\log n_0}^{M(L)} e^{\psi(v,L)} dv$ with $\psi = (1 - A + y + k)v + \frac{t}{4}v^2$ we have $\partial_L \psi = (\frac{dk}{dL} - \frac{dA}{dL})v \leq 0$; so, as before,

$$\frac{dI_B}{dL} \leq \frac{1}{2} e^{\psi(M,L)} = \frac{1}{2} e^{M(1-\sigma_B)} \leq \frac{1}{2} e^{-(\sigma_B-1)L/2}.$$

Since $1 + S_B \geq 1$,

$$\frac{d}{dL} \left(e^{-yL/2} (1 + S_B) \right) \leq e^{-yL/2} \left(-\frac{y}{2} (1 + S_B) + \frac{1}{2} e^{-(\sigma_B-1)L/2} \right) \leq \frac{1}{2} e^{-yL/2} (e^{-(\sigma_B-1)L/2} - y) \leq 0,$$

the last step by $\sigma_B - 1 \geq \frac{2}{L} \log \frac{1}{y}$. So $T(x) \leq T(x_1)$, and $G(x, y) \geq G(x_1, y) - R_A$. $\square$

Lemma 5. For $0 < t \leq \frac{1}{2}$, $x \geq 200$, $0 < y_0 \leq y \leq 1$ and $\sigma_A(x, y_0) > 1$, we have $e_A + e_B + e_{C,0} \leq E(t, x, y_0)$, where

$$E = \left(\frac{x}{4\pi} \right)^{-\frac{1+y_0}{4}} \exp \left(-\frac{t}{16} L^2 + \frac{4.96}{N - 0.125} + \frac{3|L + \frac{i\pi}{2}| + 10.44}{x - 12} \right) + (1 + 1.03 e^{\frac{tM}{2(x-6)}}) \zeta(\sigma_A(x, y_0)) (e^\delta - 1),$$

with $\delta = (\frac{t^2}{16} L^2 + 0.626)/(x - 6.66)$. Moreover E is non-increasing in x on the set where $\sigma_A(x, y_0) > 1$.

Proof. The first term is Polymath's bound [11, (24)] for $e_{C,0}$, with $(x/4\pi)^{-(1+y)/4} \leq (x/4\pi)^{-(1+y_0)/4}$ and $1.24(3^y + 3^{-y}) \leq 1.24 \cdot \frac{10}{3} < 4.96$. For [11, (23)], $e_A + e_B \leq \sum_{n \leq N} (1 + |\gamma| N^{|\kappa|} n^y) b_n n^{-\text{Re } s_*} (e^{\delta_n} - 1)$, with $\delta_n = (\frac{t^2}{16} \log^2 \frac{x}{4\pi n^2} + 0.626)/(x - 6.66) \leq \delta$ because $|\log \frac{x}{4\pi n^2}| \leq L$ for $n \leq N$. Since $n \leq N$, $|\gamma| n^y \leq e^{0.02(4\pi N^2/x)^{y/2}} \leq 1.03$, and $N^{|\kappa|} \leq e^{tM/(2(x-6))}$. Finally $\sum b_n n^{-\text{Re } s_*} \leq \sum_n n^{-\sigma_A(x,y)} \leq \zeta(\sigma_A(x, y_0))$.

For the monotonicity, every factor of E is positive, and we check that its logarithmic derivative in x is ≤ 0 (for $x \geq 200$, $L \geq 2.76$):

factor	$\frac{d}{dx} \log(\text{factor})$	sign
$(x/4\pi)^{-(1+y_0)/4}$	$-\frac{1+y_0}{4x}$	< 0
$e^{-tL^2/16}$	$-\frac{tL}{8x}$	< 0
$e^{4.96/(N-0.125)}$	N is a non-decreasing step function of x	non-increasing
$\frac{e^h}{\frac{3 L+i\pi/2 +10.44}{x-12}}$	$= h' = \frac{3L/(x L+i\pi/2)}{x-12} - \frac{h}{x-12}; \frac{3L(x-12)}{x L+i\pi/2 } < 3 < 3 L + \frac{i\pi}{2} + 10.44$	< 0
$\frac{M}{x-6}$ (in $1 + 1.03e^{tM/2(x-6)}$)	$+\frac{M'(x)}{M} - \frac{1}{x-6}$, with $M'(x) \leq \frac{1}{2x}$ and $M \geq 1.38$	< 0
$\zeta(\sigma_A(x, y_0))$	$\frac{\zeta'}{\zeta}(\sigma_A) \cdot \frac{d\sigma_A}{dx}, \frac{d\sigma_A}{dx} \geq \frac{t}{8x} > 0, \zeta' < 0$ on $(1, \infty)$	< 0
$e^\delta - 1$	increasing in δ ; $\frac{\delta'}{\delta} = \frac{t^2 L/(8x)}{\frac{t^2}{16} L^2 + 0.626} - \frac{1}{x-6.66}$, negative because $\frac{t^2}{8} L \cdot \frac{x-6.66}{x} < \frac{t^2}{16} L^2 + 0.626$ for $L \geq 2$	< 0

A product and a sum of positive non-increasing functions is non-increasing. $\square$

4 Explicit conditional upper bounds for Λ

Theorem 6. *Let $0 < t_0 \leq \frac{1}{2}$, $X \geq \max(200, 4/t_0)$, $0 < y_0 \leq \sqrt{1-2t_0}$ and $1 \leq n_0 \leq N(X)$. Assume hypotheses (i) and (iii) of Theorem 1, and, at the single point $x = X$, with $L = \log \frac{X}{4\pi}$:*

$$(H1) \quad \sigma_A(X, y_0) > 1, \quad \sigma_B(X, y_0) \geq 1 + \frac{2}{L} \log \frac{1}{y_0}; \quad (H2) \quad G(X, y_0) - R_A > E(t_0, X, y_0).$$

Then $\Lambda \leq t_0 + \frac{1}{2}y_0^2$.

Proof. Let $x \geq X$ and $y_0 \leq y \leq \sqrt{1-2t_0}$. Since $\sqrt{1-2t_0} \leq 1-t_0 \leq 1-\frac{4}{x}$, Theorem 2 and Lemma 4 apply (with $x_1 = X$ and n_0 fixed), and give $G(x, y) \geq G(x, y_0) \geq G(X, y_0) - R_A$. By Lemma 5 and the persistence of $\sigma_A > 1$ shown in Lemma 4(b), $e_A + e_B + e_{C,0} \leq E(t_0, x, y_0) \leq E(t_0, X, y_0)$. Hence (H2) and Theorem 2 give hypothesis (ii) of Theorem 1, and the conclusion follows. $\square$

The hypotheses (H1) and (H2) involve a finite sum, two one-dimensional integrals, ζ at one point and elementary functions. Table 1 lists admissible parameters, with $n_0 = 10^4$. These are not new bounds for Λ : each row is a consequence of Theorem 6 only if RH has been verified up to $X/2$ and the barrier has been certified at that X , which has been done at none of these heights except, for RH, the first two. At the present height the bound is weaker than the mollified ones; its interest is that it is explicit for every X , and its asymptotic behaviour.

Corollary 7. *For every $\varepsilon > 0$ there is X_ε such that, for every $X \geq X_\varepsilon$, the parameters t_0, y_0, n_0 chosen in the proof satisfy (H1) and (H2) and $(t_0 + \frac{1}{2}y_0^2) \log X \leq 4w_0 + \varepsilon$. Consequently $\Lambda \leq (4w_0 + \varepsilon)/\log X$ for every such X at which (i) and (iii) of Theorem 1 hold.*

Proof. The limits are taken in the order: $\eta > 0$ fixed; then $n_0 = n_0(\eta)$; then $L \rightarrow \infty$; finally $\eta \rightarrow 0$.

Parameters. Let $0 < \eta < \frac{1}{10}$, $w = \frac{w_0}{2} + \eta$, $y_0 = \frac{4 \log L}{L}$ and $t_0 = \frac{4}{L}(2w - y_0)$. Then $A = \frac{1}{2} + 2w - \frac{y_0}{2} + O(x^{-2})$ and $\frac{t_0}{4}M = w - \frac{y_0}{2} + O(x^{-1})$, so

$$\sigma_A = \frac{1}{2} + w + O(x^{-1}), \quad \sigma_B = \frac{1}{2} + w - y_0 + O(x^{-1}).$$

Since $\frac{1}{2} + \frac{w_0}{2} = 1.114\dots$, there is $L_1(\eta)$ with $\sigma_A, \sigma_B \geq 1.1$ for $L \geq L_1$. Let $c(\eta) = 2 - \zeta(\frac{1}{2} + w_0 + 2\eta)$, which is positive because ζ is decreasing on $(1, \infty)$.

X	t_0	y_0	$t_0 + \frac{1}{2}y_0^2$	$L(t_0 + \frac{1}{2}y_0^2)$	σ_A	G	R_A	E
$6.12 \cdot 10^{10}$	0.2709	0.2259	0.2964	6.61	1.368	0.0745	0.045	$2 \cdot 10^{-7}$
$5.998 \cdot 10^{12}$	0.2167	0.1999	0.2367	6.36	1.328	0.0675	0.037	$2 \cdot 10^{-8}$
10^{20}	0.1242	0.1430	0.1344	5.85	1.247	0.0487	0.019	$2 \cdot 10^{-12}$
10^{50}	0.0450	0.0683	0.0473	5.33	1.168	0.0252	$5 \cdot 10^{-4}$	$< 10^{-28}$
10^{100}	0.0221	0.0376	0.0228	5.19	1.148	0.0295	$3 \cdot 10^{-7}$	$< 10^{-56}$

Table 1: Parameters satisfying the analytic conditions (H1), (H2) of Theorem 6 ($n_0 = 10^4$), conditional on zero verification up to $X/2$ and barrier certification at X ; they are chosen with a margin rather than at the optimum. The margin $G - R_A - E$ is at least 0.024, large compared with the error of the double-precision evaluation of the sums and integrals.

Choice of n_0 . Since $(1 - A)v + \frac{t}{4}v^2 = v(1 - A + \frac{t}{4}v) \leq v(1 - \sigma_A)$ for $v \leq M$, the integral in S_A is at most $\int_{\log n_0}^{\infty} e^{-(\sigma_A - 1)v} dv = \frac{n_0^{1 - \sigma_A}}{\sigma_A - 1} \leq 10 n_0^{-0.1}$ for $L \geq L_1$. Fix $n_0 = n_0(\eta)$ with $10 n_0^{-0.1} \leq c(\eta)/8$.

$L \rightarrow \infty$ with η, n_0 fixed. For each fixed $n \leq n_0$, $\alpha_n = A - \frac{t_0}{4} \log n \rightarrow \frac{1}{2} + 2w = \frac{1}{2} + w_0 + 2\eta$, so $\sum_{n=2}^{n_0} n^{-\alpha_n} \rightarrow \sum_{n=2}^{n_0} n^{-1/2 - w_0 - 2\eta} < \zeta(\frac{1}{2} + w_0 + 2\eta) - 1 = 1 - c(\eta)$. Hence $S_A \leq 1 - \frac{3}{4}c(\eta)$ for $L \geq L_2(\eta)$. The quantity $1 + S_B$ is at most $n_0 + 10 n_0^{-0.1}$ (each $n^{-\beta_n} \leq 1$ because $\beta_n \geq \sigma_B > 0$), while $e^{0.02y_0} e^{-y_0 L/2} \leq 2L^{-2}$; so the last term of G tends to 0. Also $R_A \leq 10e^{-0.05L} \rightarrow 0$, and $E \rightarrow 0$ because its first term is $O(e^{-L/4})$ and its second is at most $3\zeta(1.1)(e^\delta - 1)$ with $\delta = O(L^2 e^{-L})$. Hence $G(X, y_0) - R_A - E \geq c(\eta)/2 > 0$ for $L \geq L_3(\eta)$, which is (H2). (H1) holds because $\sigma_A, \sigma_B \geq 1.1$ and $\frac{2}{L} \log \frac{1}{y_0} = O(\frac{\log L}{L})$. The side conditions $t_0 \leq \frac{1}{2}$, $y_0 \leq \sqrt{1 - 2t_0}$, $X \geq 4/t_0$ and $n_0 \leq N(X)$ hold for large L .

Value. $t_0 + \frac{1}{2}y_0^2 = \frac{8w}{L} - \frac{4y_0}{L} + \frac{y_0^2}{2} = \frac{4w_0 + 8\eta}{L} + O(\frac{\log^2 L}{L^2})$, so, with $\log X = L + \log 4\pi$, $(t_0 + \frac{1}{2}y_0^2) \log X = (4w_0 + 8\eta)(1 + \frac{\log 4\pi}{L}) + O(\frac{\log^2 L}{L})$. Given ε , take $\eta = \varepsilon/16$ and X_ε such that $L \geq \max(L_1, L_2, L_3)$ and the last two terms are below $\varepsilon/2$. $\square$

The optimum over (t_0, y_0) in Theorem 6, computed in Figure 3, gives $L\Lambda = 6.67$ at $X = 10^{10}$, 6.31 at $6 \cdot 10^{12}$ and 5.13 at 10^{100} , decreasing slowly towards $4w_0 = 4.915$.

5 Lower bounds for mollified triangle certificates

This section gives lower bounds for C_{opt} in the restricted class of Definition 1; they are limitations of a method of proof, not lower bounds for Λ . We first record the explicit prime count that is used.

Lemma 8. *For every real $N \geq 2 \cdot 10^5$, $\pi(N) - \pi(N/2) \geq \frac{N}{3 \log N}$.*

Proof. Rosser and Schoenfeld [13, (3.5), (3.6)] proved $\pi(x) > \frac{x}{\log x}$ for $x \geq 17$ and $\pi(x) < 1.25506 \frac{x}{\log x}$ for $x > 1$. Hence

$$\pi(N) - \pi(N/2) - \frac{N}{3 \log N} > \frac{N}{\log N} \left(\frac{2}{3} - 0.62753 \frac{\log N}{\log N - \log 2} \right) =: \frac{N}{\log N} h(N).$$

The function h is increasing, and $h(N) \geq 0$ exactly when $\log N \geq \frac{(2/3) \log 2}{2/3 - 0.62753}$, that is $N \geq 134228$. At $N = 2 \cdot 10^5$, $h = 0.0013556 \dots > 0$ (checked in ball arithmetic; a sieve also confirms the inequality at every integer and half-integer $N \in [2 \cdot 10^5, 2 \cdot 10^7]$, with ratio at least 1.52). $\square$

Proposition 9. Let $0 < t \leq \frac{1}{2}$, $0 \leq y \leq 1$, $x \geq 200$ with $N \geq 2 \cdot 10^5$, and $D \leq N/2$. For any real weights λ_d with $\lambda_1 = 1$ and any admissible σ, g, k in (3),

$$\mathcal{L} \leq 1 - \frac{N^{1-\sigma'}}{3 \log N}, \quad \sigma' = \sigma_A + \frac{t}{4} \log 2 + \frac{t}{N}.$$

In particular, $\mathcal{L} > 0$ implies $\sigma_A > 1 - \frac{t}{4} \log 2 - \frac{t}{N} - \frac{\log(3 \log N)}{\log N}$.

Proof. If p is a prime with $D < p \leq N$, its only divisor $d \leq D$ is $d = 1$, so $c_p = \lambda_1 b_p = b_p$: the coefficient cannot be cancelled by the other weights. Dropping the non-negative g -term and all terms with n not a prime in $(N/2, N]$, and using $\sigma \leq \operatorname{Re} s_*$, we get $\mathcal{L} \leq 1 - \sum_{N/2 < p \leq N} b_p p^{-\operatorname{Re} s_*}$. For such p , $b_p p^{-\operatorname{Re} s_*} = p^{-e_p}$ with $e_p = \operatorname{Re} s_* - \frac{t}{4} \log p$. By (2) and $\log p \geq \log N - \log 2 \geq M - \log 2 - \frac{2}{N}$ (since $N \geq e^M - 1$),

$$e_p \leq A + \frac{t}{x^2} + \frac{3t}{2x^2} - \frac{t}{4} \left(M - \log 2 - \frac{2}{N} \right) = \sigma_A + \frac{t}{4} \log 2 + \frac{t}{2N} + \frac{5t}{2x^2} \leq \sigma',$$

because $\frac{5}{2x^2} \leq \frac{1}{2N}$. If $e_p \geq 0$, then $p^{-e_p} \geq N^{-e_p} \geq N^{-\sigma'}$; if $e_p < 0$, then $p^{-e_p} > 1 \geq N^{-\sigma'}$ (as $\sigma' > 0$). By Lemma 8 there are at least $\frac{N}{3 \log N}$ such primes, which gives the bound. The last statement follows by taking logarithms in $\frac{N^{1-\sigma'}}{3 \log N} < 1$. $\square$

Corollary 10. Suppose that, for every large X , a certificate of length $D \leq N/2$ in the sense of Definition 1 is given. Then $\liminf_{X \rightarrow \infty} (t_0 + \frac{1}{2} y_0^2) \log X \geq 4$. Hence $C_{\text{opt}}(\{D \leq N/2\}) \geq 4$.

Proof. If $(t_0 + \frac{1}{2} y_0^2) \log X$ is unbounded along a subsequence there is nothing to prove there, so we may assume it bounded; then $t_0 \rightarrow 0$ and $y_0 \rightarrow 0$. Proposition 9 at $x = x_X$ gives, since $\sigma_A = \frac{1+y_0}{2} + \frac{t_0}{8} L + O(x^{-1})$, that $\frac{1+y_0}{2} + \frac{t_0}{8} L \geq 1 - \delta$ with $\delta = \frac{\log(3 \log N)}{\log N} + \frac{t_0}{4} \log 2 + O(x^{-1/2}) \rightarrow 0$. So $t_0 \geq \frac{4}{L}(1 - y_0 - 2\delta)$ and

$$t_0 + \frac{1}{2} y_0^2 \geq \min_y \left(\frac{4}{L}(1 - y) + \frac{1}{2} y^2 \right) - \frac{8\delta}{L} = \frac{4}{L} - \frac{8}{L^2} - \frac{8\delta}{L},$$

the minimum being at $y = 4/L$. Since $\log X/L \rightarrow 1$ and $\delta \rightarrow 0$, the claim follows. $\square$

Without a mollifier the obstruction comes instead from the first terms of the sum, and it is stronger.

Proposition 11. Let $D = 1$ in (3), and let $0 < t \leq \frac{1}{2}$, $0 \leq y \leq 1$, $x \geq 200$. If $\mathcal{L} > 0$, then for every $2 \leq n_0 \leq N$,

$$\sum_{n=2}^{n_0} n^{-\alpha_n^+} < 1, \quad \alpha_n^+ = \frac{1+y}{2} + \frac{t}{4}(L - \log n) + \frac{3t}{2x^2}.$$

Proof. With $D = 1$, $c_n = b_n$ for $n \leq N$. Dropping the g -term and the terms $n > n_0$, and using $\sigma \leq \operatorname{Re} s_*$ and (2), $\mathcal{L} \leq 1 - \sum_{n=2}^{n_0} b_n n^{-\operatorname{Re} s_*} \leq 1 - \sum_{n=2}^{n_0} n^{-\alpha_n^+}$. $\square$

Corollary 12. Suppose that, for every large X , a certificate of length $D = 1$ is given. Then $\liminf_{X \rightarrow \infty} (t_0 + \frac{1}{2} y_0^2) \log X \geq 4w_0$. Hence $C_{\text{opt}}(\{1\}) \geq 4w_0$.

Proof. As before we may assume that $B \log X$ is bounded along the sequence considered, so $t_0 L$ and $y_0^2 L$ are bounded and $y_0 \rightarrow 0$, $t_0 \rightarrow 0$. Let $L = \log \frac{xX}{4\pi}$ and $a_X = \frac{y_0}{2} + \frac{t_0}{4} L$, which is bounded; pass to a subsequence with $a_X \rightarrow a$. For fixed n , $\alpha_n^+ = \frac{1}{2} + a_X - \frac{t_0}{4} \log n + O(x^{-2}) \rightarrow \frac{1}{2} + a$. Proposition 11 gives $\sum_{n=2}^{n_0} n^{-1/2-a} \leq 1$ for every n_0 . Hence $\frac{1}{2} + a > 1$ and $\zeta(\frac{1}{2} + a) \leq 2$, so $a \geq w_0$. Now $t_0 L = 4a_X - 2y_0$, so

$$BL = t_0 L + \frac{1}{2} y_0^2 L = 4a_X - 2y_0 + \frac{L}{2} y_0^2 \geq 4a_X - \frac{2}{L},$$

the minimum over y_0 being at $y_0 = 2/L$. Since $\log X/L \rightarrow 1$, $\liminf B \log X \geq 4a \geq 4w_0$ along the subsequence, and the claim follows. $\square$

Proof of Theorem A. (a) Corollary 12 gives $C_{\text{opt}}(\{1\}) \geq 4w_0$. For the reverse inequality, the parameters of Corollary 7 satisfy (H1) and (H2), so by Lemma 4 $G(x_X, y_0) \geq G(X, y_0) - R_A > E \geq 0$, and by Theorem 2 $\mathcal{L} \geq G > 0$ for the certificate of length 1 with $\sigma = A$; its value satisfies $B \log X \leq 4w_0 + \varepsilon$ for $X \geq X_\varepsilon$. (b) The lower bound is Corollary 10, and the upper bound follows from (a) since $1 \leq N/2$. (c) is Theorem 6 and Corollary 7. $\square$

Remark 13 (Scope of the floor). (1) The floor concerns certificates of the form (3): absolute values of the coefficients, a mollifier of length $D \leq N/2$, and positivity near X . The proof uses only primes $p \in (D, N]$, so the same argument, with the prime number theorem in the interval $((1-\epsilon)N, N]$ in place of Lemma 8, gives the floor 4 also for $D \leq (1-\epsilon)N$ with $\epsilon > 0$ fixed. For $D = N - o(N)$ we claim nothing, and for $D \geq N$ the weights can be chosen to cancel every c_n with $2 \leq n \leq N$, so the argument says nothing. (2) It does not apply to methods that use the phases of the terms, that is, cancellation between the n^{-s} ; see Section 7. (3) The two obstructions are different: without a mollifier the small n force $\frac{1}{2} + 2w \geq \frac{1}{2} + w_0$ (Corollary 12); a mollifier removes them, and then the terminal primes force $\frac{1}{2} + w \geq 1$ (Corollary 10), where $\frac{1}{2} + w \approx \sigma_A$ is the effective exponent of the last term.

6 Optimised mollifiers: numerical observations

For given (t, x) and mollifier length D , let $y_{\min}(t, x; D)$ be the smallest y for which the optimised value of $\mathcal{L}$ in (3) (iteratively reweighted least squares over the weights, as in [CE13]) is positive. We computed it in double precision and by bisection (resolution $8 \cdot 10^{-4}$): with $D = 30$ for $t \in \{0.10, 0.13, 0.154, 0.18, 0.22\}$ and $x \in \{6 \cdot 10^{12}, 2 \cdot 10^{13}, 6 \cdot 10^{13}\}$, and for $t = 0.154$ at nine heights between $6 \cdot 10^{12}$ and $3 \cdot 10^{14}$ (Figure 2(b)); and, at $t = 0.154$, for several mollifier lengths (Table 2). These computations are not certified, and the optimiser is not guaranteed to find the global optimum, so $y_{\min}$ is an upper estimate of the true threshold of the method. The data and scripts are among the ancillary files.

Observation 14. For $D = 30$, whenever $y_{\min} \gtrsim 0.1$, the data satisfy

$$w := \frac{y_{\min}}{2} + \frac{t}{8} L \in [0.614, 0.631], \quad \text{mean } 0.624,$$

for all the values of t and x computed. Equivalently, the boundary of the zero-free region is where the effective exponent of the last term, $\sigma_A \approx \frac{1}{2} + w$, reaches ≈ 1.124 . For smaller y the γ -term takes over and w increases. The quantity w drifts slowly with the height, by at most about 0.01 per factor 10 in x , in a direction that depends on t .

This is a numerical regularity, not a law: it was measured with one mollifier length, on a limited range of heights and of t , and we do not claim that w has a limit, nor that a limit would be independent of D or of the optimisation. Two different limits are relevant: $\lim_{X \rightarrow \infty} w_D(X)$ for fixed D , and the limit along $D = D(X) \rightarrow \infty$; we expect only the second to approach the floor $w = \frac{1}{2}$ of Corollary 10, but we do not know whether $w_D(X)$ can approach $\frac{1}{2}$ for fixed D , and the two limits need not coincide. Table 2 shows the dependence on D at fixed t and x .

	$x = 6 \cdot 10^{12}$ ($L = 26.89$)		$x = 6 \cdot 10^{13}$ ($L = 29.19$)	
D	$y_{\min}$	w_D	$y_{\min}$	w_D
10	0.2336	0.6345	0.1438	0.6339
30	0.2133	0.6243	0.1273	0.6257
60	0.2047	0.6200	0.1211	0.6225
120	0.1984	0.6169	—	—
240	0.1945	0.6149	—	—

Table 2: Dependence on the mollifier length at $t = 0.154$: $y_{\min}(t, x; D)$ and $w_D = \frac{y_{\min}}{2} + \frac{t}{8}L$ (numerical, double precision, not certified; script `ymin.D.py`).

At fixed height, w_D decreases monotonically with D : at $x = 6 \cdot 10^{12}$ by 0.010, 0.004, 0.003 and 0.002 from one length to the next. With $D = 240$ it is 0.6149, already at the level $\frac{w_0}{2} = 0.6143$, and it shows no sign of stopping there; a crude extrapolation of the last differences (ratio about 0.65 per doubling of D) would give a limit near 0.61 at this height. So the closeness of the $D = 30$ level to $\frac{w_0}{2}$ appears to be a coincidence of that length, and Observation 14 describes $D = 30$ only. Even so, at these heights w_D stays far above the floor $\frac{1}{2}$; whether it can approach $\frac{1}{2}$ as D and X grow is open problem (A) of Section 8. Between the two heights the change of w_D is at most 0.003, with no consistent sign.

The parameters of Polymath’s conditional table [11, Table 1], obtained with an Euler mollifier, give w decreasing from 0.716 at $X = 2 \cdot 10^{12}$ to 0.617 at $X = 9 \cdot 10^{21}$ (Figure 2(c)). The level 0.624 observed with $D = 30$ is close to $\frac{w_0}{2} = 0.614$, but Table 2 shows that longer mollifiers go below it.

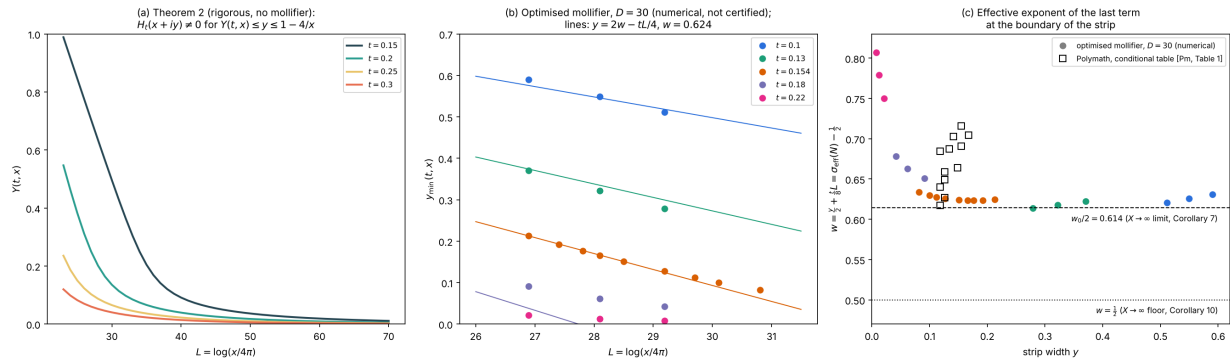


Figure 2: Width of the uncontrolled strip. (a) The zero-free bound $Y(t, x)$ of Theorem 2 and Lemma 5 ($n_0 = 10^4$; smallest y with $G > E$). (b) $y_{\min}(t, x)$ for the optimised mollifier with $D = 30$ (numerical, not certified), with the lines $y = 2w - \frac{t}{4}L$, $w = 0.624$. (c) The quantity $w = \frac{y}{2} + \frac{t}{8}L$ at the boundary, for the mollifier data (dots) and for the parameters of Polymath’s conditional table (squares), with the asymptotic values $w_0/2$ (Corollary 7) and $\frac{1}{2}$ (floor, Corollary 10).

Remark 15 (A heuristic envelope). If the zero-free region at height X contained $\{y \geq 2w - \frac{t}{4}L\}$ for all relevant t , the bound $t + \frac{1}{2}y^2$ with $t = \frac{4}{L}(2w - y)$ would be minimised at $y = 4/L$, with value $\frac{8w}{L} - \frac{8}{L^2}$, a parabola in $1/L$. With $w = 0.624$ this gives 0.1746 at $X = 6 \cdot 10^{12}$. The minima of $t + \frac{1}{2}y_{\min}^2$ computed directly from the $D = 30$ data are 0.1761, 0.1676 and 0.1605 at $X = 6 \cdot 10^{12}$, $2 \cdot 10^{13}$ and $6 \cdot 10^{13}$, and the certified bound with $D = 240$ is 0.174 [CE13]. This is only a heuristic. The law fails for $y \lesssim 0.1$, close to the optimal $y = 4/L \approx 0.15$, and asymptotically y_0 must be at least of order $\frac{\log L}{L}$ to make the γ -term negligible, as in the proof of Corollary 7.

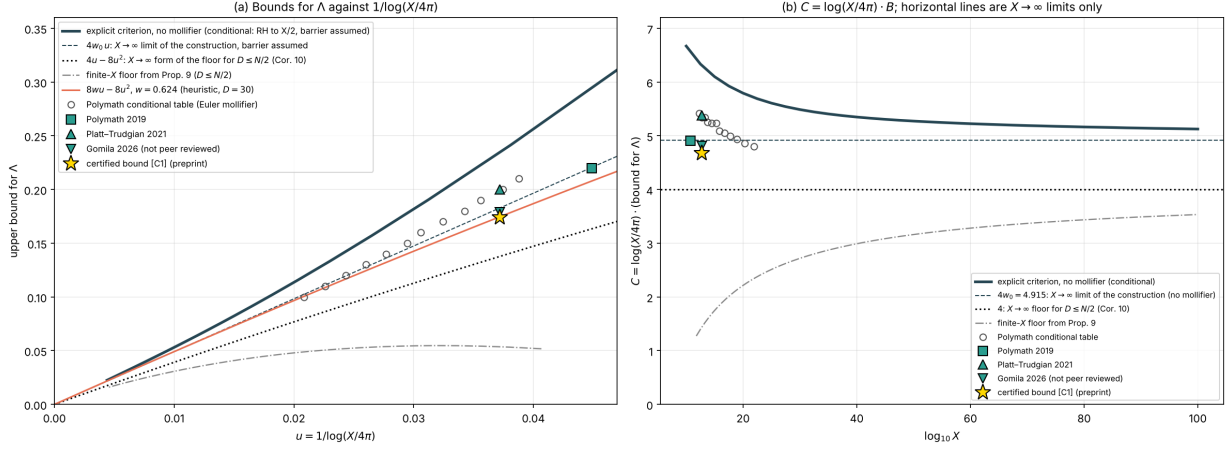


Figure 3: (a) Bounds for Λ against $u = 1/\log(X/4\pi)$, where $X/2$ is the height of RH verification. Solid dark line: the optimum of Theorem 6 over (t_0, y_0) , a constructive bound conditional on zero verification and on the barrier, for $10^{10} \leq X \leq 10^{100}$. Dashed: its limit $4w_0u$, the constructive asymptotic upper coefficient (barrier assumed). Dotted: $4u - 8u^2$, the $X \rightarrow \infty$ form of the floor of Corollary 10 for mollified triangle certificates with $D \leq N/2$; dash-dotted: the weaker finite-height floor given by Proposition 9. Both are limitations of this class of proofs, not lower bounds for Λ . Orange: the heuristic of Remark 15. Circles: Polymath’s conditional table (Euler mollifier, barrier verified, zero-free region checked at one point). Markers: published or announced bounds [11, 10, 4], [CE13]. (b) The same data as $C = \log(X/4\pi) \cdot B$ against $\log_{10} X$. The two horizontal lines are the asymptotic ($X \rightarrow \infty$) values $4w_0$ (constructed, no mollifier) and 4 (floor for $D \leq N/2$); neither is a bound at finite X , and neither bounds Λ .

7 What the triangle inequality discards: phases and cross-energy

Write the first sum of f_t (or of its mollified version) as $F_N(s) = \sum_{n \leq N} a_n$, $a_n = c_n n^{-s}$ with $c_1 = 1$, and let F_n be its partial sums. Expanding $|F_N|^2 = |F_{n-1} + a_n|^2$,

$$|F_N|^2 = 1 + \sum_{n=2}^N (|a_n|^2 + \mathcal{C}(n)), \quad \mathcal{C}(n) := |F_n|^2 - |F_{n-1}|^2 - |a_n|^2 = 2 \operatorname{Re}(a_n \overline{F_{n-1}}). \quad (4)$$

For $c_n \equiv 1$, $\mathcal{C}(n)$ is the cross-energy $C_2(n, s)$ of the partial sums of ζ studied in this series [CE1]. Write $\mathcal{C}(n) = 2|a_n||F_{n-1}|\cos\theta_n$. The step $|F_n| \geq |F_{n-1}| - |a_n|$ of the triangle inequality is an equality exactly when $\cos\theta_n = -1$ and $|F_{n-1}| \geq |a_n|$; when $1 - \sum_{n \geq 2} |a_n| > 0$, this bound is the minimum of $|F_N|$ over all choices of the phases. So the bound $|F_N| \geq 1 - \sum_{n \geq 2} |a_n|$, and with it

every certificate (3), is what (4) gives when every cross-energy takes its most negative value. Proposition 9 exploits exactly this: the terminal primes contribute $\sum_{N/2 < p \leq N} |a_p| \geq N^{1-\sigma'}/(3 \log N)$ in this worst case, whatever their phases.

Numerically the phases are far from the worst case (script `phases.py`; double precision, not certified). At $X = 6 \cdot 10^{12}$, $t = 0.154$, $y = 0.2$ ($N = 690988$, 26262 terminal primes), over 400 points $x \in [X, X + 400]$: the unmollified triangle value $1 - \sum_{n \geq 2} |a_n|$ is -1.19 , while $|F_N| \geq 0.566$ (median 0.95); for the terminal primes, $\sum |a_p| = 0.0095$ while $|\sum a_p|$ has median $4 \cdot 10^{-5}$ and maximum $9 \cdot 10^{-5}$, the size $(\sum |a_p|^2)^{1/2} = 6 \cdot 10^{-5}$ of a sum with random phases; and the mean of $\cos \theta_p$ over the terminal primes lies in $[-0.008, 0.007]$. These are values at finitely many points, not a certificate.

If the phases of the terminal block behaved like independent random phases uniformly in x , its size would be of order $(\sum_p |a_p|^2)^{1/2} \approx N^{1/2-\sigma_A}$, and the condition $\sigma_A > 1$ behind the floor would weaken to $\sigma_A > \frac{1}{2}$. But a certificate must hold for *every* $x \geq X$, and controlling $\sup_x |\sum_{N/2 < p \leq N} a_p|$ is a large-values problem for Dirichlet polynomials [6], [CE7], for which the mean-square size is not enough. A phase-sensitive certificate would therefore have to bound an energy such as $\mathcal{E}_N(s) = |\sum_{2 \leq n \leq N} c_n n^{-s}|^2$, which keeps the interactions $e^{it \log(n/m)}$, uniformly on the region. We do not claim that such a bound exists or that it would go below 4.

8 Discussion

What the constant measures. The bounds have the form $\Lambda \approx C/\log X$ with $C = 8w$, where $\frac{1}{2} + w$ is the effective exponent of the last term of the sum at the boundary of the zero-free region. Without a mollifier, the small terms force $w \geq \frac{w_0}{2}$ asymptotically, and this is attained: $C_{\text{opt}}(\{1\}) = 4w_0 = 4.915$. With any mollifier of length at most $N/2$, used near X , the terminal primes force $w \geq \frac{1}{2} - o(1)$, that is $C_{\text{opt}} \geq 4$. These are statements about $X \rightarrow \infty$. At a finite height, Proposition 9 forces much less: minimising $t + \frac{1}{2}y^2$ under its constraint gives $C \geq 1.44$ at $X = 6 \cdot 10^{12}$, 2.22 at 10^{20} , 3.16 at 10^{50} and 3.53 at 10^{100} (`floor_finite.py`, Figure 3); and the constructed coefficient approaches $4w_0$ from above. So finite-height values of C cannot be compared directly with the bracket $[4, 4w_0]$. At the present height the certified bound of [CE13] (preprint) has $C = 0.17399 \cdot \log(X/4\pi) = 4.68$. Polymath’s table has $C = 5.42$ at $X = 2 \cdot 10^{12}$ and 4.80 at $9 \cdot 10^{21}$, and Platt–Trudgian’s bound has $C = 5.38$.

What is new. (1) The law $\Lambda = O(1/\log X)$ is due to Polymath [11, §10]. (2) The explicit coefficient $4w_0$, and its optimality without a mollifier, come from keeping the individual exponents of the terms (Theorem 2, Corollaries 7 and 12); we are not aware of an earlier explicit value. (3) The floor 4 for mollified triangle certificates, from the terminal primes that no mollifier of length $\leq N/2$ can touch, explains why optimising the weights cannot improve such bounds indefinitely; it is a structural limitation of the method, not a statement about Λ .

Recent claims. Several upper bounds announced in 2026, none of them refereed at the time of writing, can be read on the same scale, all at the height of [10]. Gomila’s $\Lambda \leq 0.1787854$ [4], obtained with Euler mollifiers, has $C = 4.81$. Gordon’s $\Lambda < 0.158$ [2], which uses new analytic estimates, has $C = 4.25$. Planat’s certified comparison $\Lambda < 0.148612$ [9], conditional on three analytic premises, has $C = 4.00$. Hakimoglu–Brown’s $\Lambda \leq 0.13963$ [3] has $C = 3.75$. None of these finite-height values conflicts with the floor, which at this height only forces $C \geq 1.44$, and the floor concerns mollified triangle certificates in any case. The last bound assumes a zero-free half-plane $\text{Re } s > \frac{7}{8}$ claimed in an unrefereed manuscript [8], an input outside the class considered here.

No information on RH. All the curves in Figure 3 reach $\Lambda = 0$ only as $X \rightarrow \infty$. Since $X/2$ is the height up to which RH has been verified, this limit presupposes RH, and the conclusion $\Lambda \leq 0$ is equivalent to RH by the theorem of de Bruijn and Newman. The $C/\log X$ law is consistent with Polymath’s heuristic that bounds better than $O(1/\log X)$ would require major new information on RH [11, §10]. It does not decide RH. The barrier (iii) must be verified at each X , and the observations of Section 6 are numerical.

Open problems. (A) Determine $C_{\text{opt}}(\{D \leq N/2\}) \in [4, 4w_0]$; in particular, is it equal to 4? This requires a sequence of mollifiers whose certificates reach the floor asymptotically, not only better numerical parameters. (B) Go below 4 with a phase-sensitive certificate (Section 7): a uniform lower bound for $|F_N|$ that uses the cross-energies $\mathcal{C}(n)$, or the argument principle on the uncontrolled strip, rather than their worst case. (C) A rigorous mollified version of Theorem 2, with an explicit Euler-type mollifier and an explicit constant below 4.915.

A Audit of the imported constants

Each item imported from [11] (arXiv:1904.12438v2) was compared with the source.

item	[11]	use here
criterion	Thm. 1.2: (i) $\frac{1+y_0}{2} \leq \sigma \leq 1$, $0 \leq T \leq X/2$; (ii), (iii) as in Theorem 1; $0 < y_0 \leq 1$	exact
range	(5): $0 < t \leq \frac{1}{2}$, $0 \leq y \leq 1$, $x \geq 200$	exact; Theorem 2 uses $y \leq 1 - \frac{4}{x}$
f_t, b_n, N, s_*, α	(14), (15), (19), (17), (9)	exact
$ \gamma $	(20): $\leq e^{0.02y} (x/4\pi)^{-y/2}$	exact
$\text{Re } s_*$	(21): $\geq \frac{1+y}{2} + \frac{t}{4}L - \frac{t}{2x^2}(1 - 3y + \frac{4y(1+y)}{x^2}) + \frac{ty}{2(x-6)}$	weakened to $-\frac{t}{x^2}$: the bracket is ≤ 1 for $0 \leq y \leq 1$, $x \geq 200$
$ \kappa $	(22): $\leq \frac{ty}{2(x-6)}$	exact
$\text{Re } s_*$ (upper)	not in [11]	(2), proved in Section 2
$e_A + e_B$	(23): $\sum_{n \leq N} (1 + \gamma N^{ \kappa }n^y) \frac{b_n}{n^{\text{Re } s_*}} (e^{\delta_n} - 1)$, $\delta_n = \frac{t^2}{16} \log^2 \frac{x}{4\pi n^2} + 0.626$	majorised in Lemma 5: $\delta_n \leq \delta$, $ \gamma n^y \leq e^{0.02}(1 + \frac{\pi t}{4x})^{1/2} \leq 1.03$, $N^{ \kappa } \leq e^{tM/2(x-6)}$
$e_{C,0}$	(24): $(\frac{x}{4\pi})^{-\frac{1+y}{4}} \exp(-\frac{t}{16}L^2 + \frac{1.24(3^y + 3^{-y})}{N-0.125} + \frac{3 L+i\pi/2 +10.44}{x-12})$	$1.24(3^y + 3^{-y}) \leq 4.14 < 4.96$; $y \rightarrow y_0$
non-vanishing	Cor. 1.4 / (25): $ f_t > e_A + e_B + e_{C,0} \Rightarrow H_t \neq 0$	exact
mollified bound	[CE13, Lemma 3]	(3)
prime count	[13, (3.5), (3.6)]	Lemma 8

The inequalities derived from these in the proofs are: $b_n n^{-\text{Re } s_*} \leq n^{-\alpha_n}$ and the integral test (Theorem 2); the y - and L -derivatives in Lemma 4; the table of logarithmic derivatives in Lemma 5; $e_p \leq \sigma'$ in Proposition 9. Each item was compared with arXiv:1904.12438v2 in two independent passes; the formulas match and the simplifications are valid.

B Computations

All scripts are among the ancillary files (`anc/README.md`). Elementary: the finite-height floor of Section 8 (`floor/floor_finite.py`). Certified: the prime count of Lemma 8

(`floor/prime_floor_check.py`: ball arithmetic for $h(2 \cdot 10^5)$, exact sieve on $[2 \cdot 10^5, 2 \cdot 10^7]$). The value w_0 is computed by root finding on $\zeta(s) = 2$ and enclosed in ball arithmetic (`floor/w0.py`). Double precision, not certified: Table 1 and the curves of Theorem 6 (`thm3.py`, `curve3.py`, `ythm3.py`; the margins are at least 0.024), the mollifier thresholds of Section 6 (`ymin.g.py`, `ymin.D.py`, `optmoll3.py`), the phases of Section 7 (`floor/phases.py`), and Figure 1(a). None of the theorems depends on an uncertified computation; Table 1 illustrates Theorem 6 at finite heights.

Use of AI tools. The computations and the drafting of this paper were carried out with the assistance of an AI system (Claude, Anthropic), under the author’s direction. An independent AI review of the proofs led to the per-term form of Theorem 2, and a second one audited the constants imported from [11]. The author checked the results and takes responsibility for them.

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