

CROSS-ENERGY OF DIRICHLET PARTIAL SUMS. PART 3: THE REFLECTED CROSS-ENERGY

PEDRO CACERES

ABSTRACT. The cross-energy $C_2(n, s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma}$ of the partial sums X_n of ζ pairs s with $\bar{s}$. We pair s with $1-\bar{s}$ and with $1-s$, the two reflections of the functional equation. At a reflected pair of zeros, $C_2(n, \rho)C_2(n, 1-\bar{\rho})$ has a finite positive limit for every $\beta = \operatorname{Re} \rho$; it is a square on the critical line, and off it, beyond $|\gamma|^{3/2}$ and above height 16, it is negative exactly between two crossings around $n = \gamma^2$, with relative depth $(2\beta-1)^2/4\beta(1-\beta)$; in two variables the window is the real part of a Wronskian. The reflected cross-energy $C_2^{\text{FR}}(N, s) = N^{-s}X_{N-1}(1-s) + N^{s-1}X_{N-1}(s)$ is entire and equals a function of $\lambda = s(1-s)$ plus $\zeta(s)$ times a carrier of the functional equation; it reproduces Hardy's function from the integers below N , its oscillating part vanishes at every zero, on the line or off it, and it is a fixed-height form of the symmetric truncation of Gonek and Montgomery. Computed from the integers of a Beurling system it detects the off-line zeros, which drift with N unless the reflected point is a zero. Weil's criterion restricted to the test functions $f * \mu_n$ built from the integers below n is still equivalent to RH; its positivity lives in the primes of the dual range $[n, n^2]$, which $\Lambda * 1 * 1 = \log * 1$ trades for the primes up to $ne^{2\alpha}$ and the divisor function; its margin at height T is of order T^{-2} to T^{-4} of the Gamma term; with the data of a Beurling system it is not a criterion. As $n \rightarrow \infty$ it becomes the positivity of the Hankel matrices of the power sums of $1/\rho(1-\rho)$, computable from ξ at $s = 1$, a criterion classical in substance. We do not prove the Riemann hypothesis.

1. INTRODUCTION

1.1. Two reflections. The completed zeta function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ satisfies $\xi(s) = \xi(1-s)$ and $\xi(\bar{s}) = \overline{\xi(s)}$. The set of nontrivial zeros is therefore invariant under two involutions of the critical strip: the holomorphic reflection $s \mapsto 1-s$, through the point $\frac{1}{2}$, and the anti-holomorphic reflection $\iota(s) = 1-\bar{s}$, about the critical line, whose fixed set is the line itself. A zero $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$ comes with its reflected partner $1-\bar{\rho} = (1-\beta) + i\gamma$ at the same height; the Riemann hypothesis (RH) is the statement that every such pair collapses.

Two classical coordinates make ι an inversion. For a prime p and $w = p^{-s}$, $p^{-(1-\bar{s})} = 1/(p\bar{w})$: ι is the inversion in the circle $|w| = p^{-1/2}$, which is the image of the critical line, so that the critical line is the fixed circle of the Euler factor of every prime. For the Möbius coordinate $w_\rho = 1 - 1/\rho$ of Li's criterion [Li], which maps the critical line onto the unit circle, $|w_{1-\bar{\rho}}| = |\rho|/|1-\rho| = |w_\rho|^{-1}$: an off-line pair has reciprocal radii. Both statements are elementary.

The cross-energy of Part 1 [CE1],

$$C_2(n, s) = |X_n(s)|^2 - |X_{n-1}(s)|^2 - n^{-2\sigma} = 2 \operatorname{Re} (X_{n-1}(s) n^{-\bar{s}}), \quad X_n(s) = \sum_{k \leq n} k^{-s}, \quad s = \sigma + it,$$

pairs s with $\bar{s}$: it is sesquilinear. This paper collects everything we know about pairing s with $\iota(s) = 1-\bar{s}$ and with $1-s$ instead. The first pairing gives the product $C_2(n, \rho)C_2(n, 1-\bar{\rho})$ at a

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reflected pair (Section 2); the second gives the *reflected cross-energy*

$$C_2^{\text{FR}}(N, s) = X_N(s)X_N(1-s) - X_{N-1}(s)X_{N-1}(1-s) - \frac{1}{N} = N^{-s}X_{N-1}(1-s) + N^{s-1}X_{N-1}(s),$$

which is entire in s (Section 3). On the critical line $1-s = \bar{s}$, and both coincide with C_2 or its square.

1.2. The one fact. Write $\lambda = s(1-s)$ and $\lambda_\rho = \rho(1-\rho)$. The reflected cross-energy splits exactly as

$$C_2^{\text{FR}}(N, s) = \mathcal{G}_N^*(\lambda) + N^{s-1}\zeta(s) + N^{-s}\zeta(1-s),$$

$$\mathcal{G}_N^*(\lambda) = \frac{1}{\lambda} - \frac{1}{N} - \frac{1}{12N^2} + \left(\frac{1}{120} - \frac{\lambda}{80}\right)N^{-4} + 1.1 \cdot 10^{-4} \theta(|s| + 7)^6 N^{-6}$$

with $|\theta| \leq 1$, for $0 < \sigma < 1$ and $N \geq 2$ (Theorem 8). The second part is $\zeta(s)$ times a carrier of the functional equation, and it vanishes at every zero of ζ , on the critical line or off it. At every zero, therefore, the reflected cross-energy of the integers below N is a fixed function of λ_ρ , and nothing else. This one fact organizes the paper: it makes C_2^{FR} a detector of zeros that works for any system of integers and tells the controls apart (Sections 3 and 4); it turns Weil's form, restricted to the test functions built from the integers below n , into a weighted sum over the zeros that is still equivalent to RH (Section 5); it fixes the precision that the primes would have to supply (Section 7); and as $n \rightarrow \infty$ it reduces the restricted criterion to finite Hankel matrices of the numbers $1/\lambda_\rho$ (Section 8).

1.3. Results and their status. We state the status of every result where it appears; here is the list.

- *The reflected pair* (Proposition 3, proved, with the explicit height $t_1 = 16$): beyond $|\gamma|^{3/2}$ and above height 16, $C_2(n, \rho)C_2(n, 1-\bar{\rho})$ is negative exactly on an interval around $n = \gamma^2$ of length about $|2\beta - 1|\gamma^2/(2\beta(1-\beta))$, by the fraction $(2\beta - 1)^2/(4\beta(1-\beta))$ of its limit $4\beta(1-\beta)/(|\rho|^2|1-\rho|^2)$, and only if $\beta \neq \frac{1}{2}$. In two variables the window is where the real part of the Wronskian of consecutive partial sums exceeds the real part of the reflected cross-energy (Proposition 5, proved). Off the zeros, positivity of the product characterizes the critical line as a set (proved without the wave, numerical with it).
- *The zero detector* (Theorem 8, proved): the decomposition above, its Hardy form on the line, the location of the zeros of the carrier, and the identification of C_2^{FR} with a fixed-height form of the symmetric truncation of Gonek and Montgomery [GoM].
- *Number systems* (Proposition 10, proved): for generalized integers with $N_B(x) = cx + c'x^b + O(x^\theta)$, $\theta < \frac{1}{2}$, the same construction detects every zero of ζ_B with real part in $(\frac{1}{2}, 1-\theta)$; the detected zero is fixed in N if and only if the reflected point is a zero, and otherwise drifts by an explicit amount. Numerically, the off-line zero of the Davenport–Heilbronn function is fixed and the real zero 0.6 of a Beurling system of the type of Broucke, Debruyne and Révész [BDR] drifts. For that system the hypothesis of the proposition is proved only with $\theta = b + \varepsilon$, which does not reach the zero 0.6; the drift at 0.6 is checked under a measured value of θ (Section 4).
- *Weil's form on the test functions of the cross-energy* (Proposition 11: proved; Proposition 12: proved): the restricted criterion is equivalent to RH, and positivity for a single n already forces every zero with $\beta > \frac{1}{2}$ onto the zero set of the weight $\mathcal{G}_n^*$; and its prime side trades the dual range of primes for the primes up to ne^{2a} , the divisor function and the edge of the integers below n . Numerically the positivity lives in the dual range, Chebyshev's identity used as an inequality falls short by a factor 10^7 , and with the data of a Beurling system the same functional is not a criterion.

- *Precision* (Remark 13, heuristic reading of numerical exponents): at height T the margin of the form is of order T^{-2} at a generic cut and T^{-4} at the filter, relative to the Gamma term.
- *The limit $n \rightarrow \infty$* (Proposition 14: proved; Proposition 15: proved, classical in substance): the positivity of the Hankel matrices of the power sums S_k of $1/\lambda_\rho$, computable from the Taylor coefficients of ξ at $s = 1$, is equivalent to RH; computed to size 43 it returns the first zeros and detects an inserted off-line pair at about twice its index. Heine's formula (Proposition 16, classical) decomposes the Hankel determinants into contributions of sets of zeros, in which an off-line quadruple appears as a negative family.

None of this proves RH. Every exact reformulation obtained here is equivalent to RH, and Section 9 lists what the reflected cross-energy does and does not do. Table 1 lists the dependencies between the results and separates what is new from what is a classical reformulation.

result	uses	status	new or classical
Prop. 1 (precursor)	Euler–Maclaurin	proved	reformulation of [C1]
Lemma 2 (crossing law)	[CE1, Prop. 3.6], Euler–Maclaurin	proved, $t_0 = 16$	law from [CE1]; explicit constants new
Prop. 3 (reflected pair)	Lemma 2	proved, $t_1 = 16$; two asymptotic formulas with absolute constants	new
Prop. 5 (two variables)	Euler–Maclaurin, Lemma 2	proved	new
Thm. 8 (zero detector)	Euler–Maclaurin, functional equation	proved	new object; related to [GoM]
Prop. 10 (number systems)	partial summation, Hurwitz, Rouché	proved under its hypothesis; the [BDR] control is built under RH	new
Prop. 11 (restricted Weil)	Thm. 8, explicit formula [W2, Bo], Bochner–Schwartz [GV]	proved	restriction of Weil's criterion
Prop. 12 (dual range)	Chebyshev's identity	proved	new in this form
Prop. 14 (limit)	Thm. 8, Prop. 11	proved	new
Prop. 15 (Hankel–Jacobi)	Hamburger [Ak], orthogonal polynomials [Ch]	proved	classical in substance [CNV, Nu, Ma, GORZ]
Prop. 16 (Heine)	Cauchy–Binet	proved	classical formula
Rem. 4, 6, 7	[CE1, Thm. 3.2], [CE5, Prop. 2.5]	proved	elementary
Rem. 13 (precision)	numerical exponents	heuristic	–
Rem. 9 (integers and primes)	explicit formula	conditional on RH	classical [Con]

TABLE 1. Dependencies and status of the results. “New” means new in this series; every exact reformulation of RH obtained here is equivalent to RH and is not a step towards a proof.

1.4. **A precursor.** The earliest reflected object in this series is the reflected normalized defect of [C1, Sect. 1]. With $Y_N(s) = N^{1-s}/(1-s)$, the Euler–Maclaurin formula gives, for fixed s with $\sigma > 0$,

$$\frac{X_N(s)}{Y_N(s)} = 1 + (1-s)\zeta(s)N^{s-1} + \frac{1-s}{2N} - \frac{s(1-s)}{12N^2} + O_s(N^{-4}),$$

so the ratio tends to 1 at every point of the strip, and only its first-order defect carries information. For $0 < a < \frac{1}{2}$ and real t put $s_\pm = \frac{1}{2} \pm a + it$, so that $s_- = 1 - \bar{s}_+$, and

$$A_N(a, t) = N \left[\frac{X_N(s_+)}{Y_N(s_+)} - \frac{X_N(s_-)}{Y_N(s_-)} \right].$$

Proposition 1 (the reflected normalized defect). *Let $0 < a < \frac{1}{2}$ and $t \in \mathbb{R}$.*

- (i) $A_N(a, t) = (1 - s_+) \zeta(s_+) N^{s_+} - (1 - s_-) \zeta(s_-) N^{s_-} - a + \frac{iat}{3N} + O_{a,t}(N^{-3})$.
- (ii) $A_N(a, t) = O(1)$ if and only if $\zeta(s_+) = 0$. This is a single condition: $\zeta(s_+) = 0$ if and only if $\zeta(s_-) = 0$, by the functional equation and $s_- = 1 - \bar{s}_+$.
- (iii) If $\zeta(s_+) = 0$, then $A_N(a, t) = -a + iat/(3N) + O(N^{-3}) \rightarrow -a$. The limit $-a$ comes from the term $\frac{1-s}{2N}$, that is, from the endpoint term $\frac{1}{2}N^{-s}$ of X_N , and does not depend on ζ ; with the centred sums $X_N - \frac{1}{2}N^{-s}$ it disappears, and the defect tends to 0 at every reflected pair of zeros, whatever a is.

Proof. (i) Substitute the expansion at $s_{\pm}$; $s_+ - s_- = 2a$ and $s_+(1 - s_+) - s_-(1 - s_-) = (s_+ - s_-)(1 - s_+ - s_-) = -4iat$. (ii) The two ζ -terms have moduli proportional to $N^{\frac{1}{2}+a}$ and $N^{\frac{1}{2}-a}$, with distinct positive exponents, so boundedness forces $(1 - s_+) \zeta(s_+) = 0$; conversely, if $\zeta(s_+) = 0$ then $\zeta(s_-) = \zeta(1 - s_+) = 0$ and (i) gives boundedness. (iii) is (i). $\square$

In [C1] the second step of the proof of (ii) was presented as a separate condition, and the “Reflected Asymptotic Rigidity Problem”, $\text{RH} \iff [A_N(a, t) = O(1) \Rightarrow A_N(a, t) \rightarrow 0 \text{ for all } 0 < a < \frac{1}{2}, t \in \mathbb{R}]$, was stated as a reformulation. By (ii) and (iii) the implication in brackets says exactly that no zero with $\text{Re } s = \frac{1}{2} + a$ exists: the problem is RH restated, as [C1] itself notes, and the limit $-a$ is fixed by the parametrization, not produced by ζ . Both reflected objects of this paper share this feature, in a weaker form: what they record at a zero is computed from its position (Proposition 3(v) and Theorem 8). What they add is the window of negativity of Section 2 and, for C_2^{FR} , holomorphy: it can be summed over the zeros by the explicit formula (Section 5).

1.5. Dependencies and notation. From Part 1 [CE1] we use the expansion of C_2 (Theorem A, [CE1, Thm. 3.2]) and the crossing law, restated in Lemma 2. The Beurling system of [BDR, Thm. 1.3] that serves as a control is constructed and studied in Part 11 [CE11]; we restate in Section 4 only what we need. Weil’s quadratic form on fixed intervals, computed from the pole, the Gamma factor and the primes, is in Part 12 [CE12], and the Davenport–Heilbronn function and the other controls are described in Part 17 [CE17]. Throughout, $\rho = \beta + i\gamma$ is a nontrivial zero of ζ , sums over ρ are taken with multiplicity and in symmetric order, $\gamma_\rho = (\rho - \frac{1}{2})/i$, γ_0 is Euler’s constant, $\rho_1 = \frac{1}{2} + i\gamma_1$, $\gamma_1 = 14.1347\dots$, and χ is the factor of the functional equation, $\zeta(s) = \chi(s)\zeta(1-s)$. We extend $X_{n-1}(s) = \zeta(s) - \zeta(s, n)$ to real $n \geq 1$ by the Hurwitz zeta function $\zeta(s, n)$. Part 11 writes D_N for X_N .

1.6. Organization. Section 2 studies the reflected pair and its two-variable form; Section 3 the reflected cross-energy as a zero detector; Section 4 number systems; Section 5 Weil’s form on the test functions of the cross-energy; Section 6 the same functional with the data of a Beurling system; Section 7 the precision required at large height; Section 8 the limit $n \rightarrow \infty$. Section 9 summarizes by status and Section 10 lists open problems.

2. THE REFLECTED PAIR

2.1. The crossing law. We recall from [CE1, Sect. 3] the facts about C_2 at a single point that we need. For $s = \sigma + it$ and real $n \geq 1$ put

$$g_s(n) = -\text{Re} [n^s \zeta(s, n)], \quad A(s) = \frac{|1-s|^2}{2(1-\sigma)}, \quad c(\sigma) = (1-\sigma) \left(\frac{1-\sigma^2}{30} - \frac{\sigma^2}{18} \right).$$

Lemma 2 (the crossing law, [CE1, Prop. 3.6], with explicit constants). *For $0 < \sigma < 1$, the Euler–Maclaurin formula gives*

$$g_s(n) = \frac{n}{2A(s)} - \frac{1}{2} - \frac{\sigma}{12n} + \frac{\text{Re}(s)_3}{720n^3} + E_s(n), \quad |E_s(n)| \leq \kappa |(s)_5| n^{-4}, \quad \kappa = \frac{\|B_5\|_\infty}{480} < 5.1 \cdot 10^{-5},$$

$(s)_k = s(s+1)\cdots(s+k-1)$. Let $0 < \sigma < 1$ and $|t| \geq 16$, and let θ denote numbers with $|\theta| \leq 1$.

(a) On $n \geq |t|^{3/2}$, $g_s < 0$ for $n \leq 0.99A$ and $g_s > 0$ for $n \geq 1.01A$; on $[0.99A, 1.01A]$, $2A \partial_n g_s(n) = 1 + 0.36\theta t^{-2}$. Hence g_s has exactly one zero $n^*(s)$ in $n \geq |t|^{3/2}$, it changes sign there from negative to positive, and $n^*(s) = A(s)(1 + 0.34\theta t^{-2})$.

(b) $n^*(s) = A(s) + \frac{\sigma}{6} + \frac{c(\sigma)}{t^2} + O(t^{-4})$, with an absolute implied constant.

If $\zeta(\rho) = 0$, then $C_2(n, \rho) = 2n^{-2\beta} g_\rho(n)$ for every real $n \geq 1$; on the critical line $n^*(\rho) = \gamma^2 + \frac{1}{3} + \frac{1}{180\gamma^2} + O(\gamma^{-4})$.

Part (b) and the last statements are proved in [CE1, Props. 3.6 and 3.8]; the last statement follows from $X_{n-1}(\rho) = -\zeta(\rho, n)$ and $\overline{n^{-\rho}} = n^{\rho-2\beta}$. Part (a) is the first half of the proof of [CE1, Prop. 3.6] with every constant made explicit, which gives the value $t_0 = 16$ for the constant t_0 of [CE1]. With $\|B_5\|_\infty = 0.02446$ and $|(s)_k| \leq \prod_{j < k} (|t| + j + 1)$, the three terms after $-\frac{1}{2}$ are bounded on $n \geq |t|^{3/2}$ by $r_1(t) = \frac{1}{12}|t|^{-3/2} + \frac{(|t|+3)^3}{720}|t|^{-9/2} + \kappa(|t|+5)^5|t|^{-6}$, and $r_1(16) < 0.0014 < 0.005$; this gives the signs outside $[0.99A, 1.01A]$, because $A \geq t^2/2$ and $|t|^{3/2} < 0.99t^2/2$. On $[0.99A, 1.01A]$ we have $n \geq 0.495t^2$, and the derivative of the error terms is bounded by $\frac{1}{12n^2} + \frac{(|t|+3)^3}{240n^4} + \frac{\|B_5\|_\infty}{120}(|t|+1)\cdots(|t|+5)(1+\frac{|t|+1}{4})n^{-5}$, where the last term uses $\partial_n(n^s R_2(s, n)) = sn^{s-1}R_2(s, n) + \frac{(s)_5}{120}\tilde{B}_5(n)n^{-5}$; multiplied by $2A \leq 2n/0.99$ this is at most $0.36t^{-2}$. Finally $g_s(n^*) = 0$ gives $|n^*/(2A) - \frac{1}{2}| \leq r(n^*)$ with the same three terms at $n = n^* \geq 0.495t^2$, that is $|n^*/A - 1| \leq 0.34t^{-2}$. On $[0.99A, 1.01A]$ each bound is a sum of terms $P(|t|)n^{-k}$ with $n \geq 0.495t^2$, where P is a product of factors $|t| + a$, $a \geq 0$, of degree at most $2k - 2$; such a term times t^2 is non-increasing in $|t|$. The bound $r_1(t)$ on $n \geq |t|^{3/2}$ is itself non-increasing, and only $r_1 < 0.005$ is needed. So it suffices to evaluate the bounds at $|t| = 16$; this is the one numerical step of the proof (script `explicit_constants.py`). It holds at every point of the strip; the zero condition enters only through the removal of the oscillating term $2\operatorname{Re}(\zeta(s)n^{-\bar{s}})$ of C_2 .

2.2. The product of a reflected pair. Let $\Phi_s(n) = n^s X_{n-1}(s)$; then $C_2(n, s) = 2n^{-2\sigma} \operatorname{Re} \Phi_s(n)$, and for every s and every real $n \geq 1$

$$C_2(n, s) C_2(n, 1 - \bar{s}) = 4n^{-2} \operatorname{Re} \Phi_s(n) \operatorname{Re} \Phi_{1-\bar{s}}(n) :$$

the weights $n^{-2\sigma}$ and $n^{-2(1-\sigma)}$ multiply to n^{-2} whatever σ is. For integers $n \geq 2$, from $C_2(n, s) = 2 \sum_{k < n} (nk)^{-\sigma} \cos(t \log \frac{n}{k})$ and $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$,

$$(1) \quad C_2(n, s) C_2(n, 1 - \bar{s}) = \frac{2}{n} \sum_{j, k < n} k^{-\sigma} j^{\sigma-1} \left[\cos\left(t \log \frac{j}{k}\right) + \cos\left(t \log \frac{n^2}{jk}\right) \right],$$

valid at every s ; at $\sigma = \frac{1}{2}$ it is $C_2(n, s)^2$, a square at every point of the critical line. At a pair of zeros the factor that tends to 0 like $n^{1-2\beta}$ and the factor that grows like $n^{2\beta-1}$ compensate, and the product has a finite positive limit for every β . For $\frac{1}{2} \leq \sigma < 1$ let $s' = 1 - \bar{s} = (1 - \sigma) + it$ and

$$\Pi_s(n) = 4n^{-2} g_s(n) g_{s'}(n), \quad A = A(s), \quad A' = A(s') = \frac{|s|^2}{2\sigma}.$$

Proposition 3 (reflected pairs). Let $|t| \geq t_1 = 16$, and let θ denote numbers with $|\theta| \leq 1$.

- (i) $\sigma \mapsto n^*(\sigma + it)$ is a strictly increasing C^1 function on $(0, 1)$, and $\frac{\partial n^*}{\partial \sigma} = \frac{t^2 - (1 - \sigma)^2}{2(1 - \sigma)^2} (1 + 1.04\theta t^{-2})$.
- (ii) If $\sigma = \frac{1}{2}$, then $s' = s$ and $\Pi_s = 4n^{-2} g_s^2 \geq 0$.

- (iii) If $\sigma > \frac{1}{2}$, then $n^*(s') < n^*(\frac{1}{2} + it) < n^*(s)$, and on $n \geq |t|^{3/2}$ the product Π_s is negative exactly on $I_s = (n^*(s'), n^*(s))$ and positive outside its closure. The length of I_s satisfies the explicit bounds

$$n^*(s) - n^*(s') = (1 + 1.04\theta t^{-2}) L(\sigma, t), \quad L(\sigma, t) = (2\sigma - 1) \left(\frac{t^2}{2\sigma(1 - \sigma)} - \frac{1}{2} \right) \geq (2\sigma - 1) \left(2t^2 - \frac{1}{2} \right),$$

and the asymptotic formula

$$n^*(s) - n^*(s') = (2\sigma - 1) \left(\frac{t^2}{2\sigma(1 - \sigma)} - \frac{1}{3} \right) + \frac{c(\sigma) - c(1 - \sigma)}{t^2} + O(t^{-4});$$

moreover

$$\min_{n \geq |t|^{3/2}} \Pi_s(n) = -\frac{1}{4} \left(\frac{1}{A} - \frac{1}{A'} \right)^2 (1 + O(t^{-2})) = -\frac{(2\sigma - 1)^2}{4\sigma(1 - \sigma)} \frac{1}{AA'} (1 + O(t^{-2})),$$

attained at $n = t^2 + O(1)$. In the last two formulas the implied constants are absolute.

- (iv) $\Pi_s(n) \rightarrow 1/(AA') = 4\sigma(1 - \sigma)/(|s|^2|1 - s|^2)$ as $n \rightarrow \infty$, and $1/(AA') \leq (\frac{1}{4} + t^2)^{-2}$, with equality if and only if $\sigma = \frac{1}{2}$.
- (v) Extend Π_s to $\sigma < \frac{1}{2}$ by $\Pi_s = \Pi_{s'}$. If $\zeta(\rho) = 0$, then $\Pi_\rho(n) = C_2(n, \rho) C_2(n, 1 - \bar{\rho})$ for every real $n \geq 1$. The following are equivalent: (a) every zero with $|\gamma| \geq 16$ has $\beta = \frac{1}{2}$; (b) $C_2(n, \rho) C_2(n, 1 - \bar{\rho}) \geq 0$ for every such zero and every real $n \geq |\gamma|^{3/2}$; (c) $C_2(n^*(\frac{1}{2} + i\gamma), \rho) = 0$ for every such zero.
- (vi) (Integers.) Let $\zeta(\rho) = 0$ with $|\gamma| \geq 16$ and

$$|\beta - \frac{1}{2}| > \delta_{\mathbb{Z}}(\gamma) := \frac{1}{(1 - 1.04\gamma^{-2})(4\gamma^2 - 1)} \quad \left(\text{so } \delta_{\mathbb{Z}}(\gamma) < \frac{1.006}{4\gamma^2} \right).$$

Then I_ρ (or $I_{1-\bar{\rho}}$) contains an integer $n \geq |\gamma|^{3/2}$, at which $C_2(n, \rho) C_2(n, 1 - \bar{\rho}) < 0$. Hence, if $C_2(n, \rho) C_2(n, 1 - \bar{\rho}) \geq 0$ for every integer $n \geq |\gamma|^{3/2}$, then $|\beta - \frac{1}{2}| \leq \delta_{\mathbb{Z}}(\gamma)$.

Proof. Write $F(s, n) = n^s \zeta(s, n)$, so that $g_s(n) = -\operatorname{Re} F(s, n)$ and, F being holomorphic in s , $\partial_\sigma g_s(n) = -\operatorname{Re} \partial_s F(s, n)$. The Euler–Maclaurin formula with two Bernoulli terms reads $F(s, n) = \frac{n}{s-1} + \frac{1}{2} + \frac{s}{12n} - \frac{(s)_3}{720n^3} + n^s R_2(s, n)$, where $n^s R_2(s, n)$ is holomorphic in $\operatorname{Re} s > -4$ and bounded there by $\frac{\|B_5\|_\infty}{120} \frac{|(s)_5|}{\operatorname{Re} s + 4} n^{-4}$. Since $\partial_n \zeta(z, n) = -z \zeta(z + 1, n)$ and the main terms transform in the same way, $\partial_n R_2(z, n) = -z R_2(z + 1, n)$, so $|\partial_n (n^z R_2(z, n))| \leq C|t|^6 n^{-5}$ on the discs below. Cauchy's inequality on the disc of radius $\frac{1}{2}$ about s gives

$$\partial_s F(s, n) = -\frac{n}{(s-1)^2} + \frac{1}{12n} + O(t^2 n^{-3} + |t|^5 n^{-4}).$$

- (i) With the explicit remainder, $\partial_\sigma g_s(n) = \operatorname{Re} \frac{n}{(s-1)^2} - \frac{1}{12n} + \frac{\operatorname{Re}(3s^2 + 6s + 2)}{720n^3} - \operatorname{Re} \partial_s (n^s R_2(s, n))$, and Cauchy's inequality on the circle of radius $\frac{1}{2}$ about s , on which $\operatorname{Re} z > -\frac{1}{2}$ and $|(z)_5| \leq (|t| + 2) \cdots (|t| + 6)$, gives $|\partial_s (n^s R_2(s, n))| \leq \frac{2\|B_5\|_\infty}{120 \cdot 3.5} (|t| + 2) \cdots (|t| + 6) n^{-4}$. At $n = n^*(s)$, Lemma 2(a) gives $n = A(1 + 0.34\theta t^{-2})$, and $\operatorname{Re} \frac{n}{(s-1)^2} = -\frac{n}{A} \cdot \frac{t^2 - (1 - \sigma)^2}{2(1 - \sigma)|1 - s|^2}$, whose absolute value is at least $(1 - 0.34t^{-2}) \frac{t^2 - 1}{2(t^2 + 1)} > 0.49$, since $u \mapsto \frac{t^2 - u^2}{u(t^2 + u^2)}$ decreases on $(0, 1]$. The other three terms are at most $0.35t^{-2}$ times this main term (script `explicit_constants.py`, with the same monotonicity in $|t|$ as in Lemma 2). With $2A \partial_n g_s(n^*) = 1 + 0.36\theta t^{-2} > 0$, the implicit function theorem and the uniqueness of n^* give

$$\frac{\partial n^*}{\partial \sigma} = -\frac{\partial_\sigma g_s}{\partial_n g_s} = 2A \frac{n^*}{A} \frac{t^2 - (1 - \sigma)^2}{2(1 - \sigma)|1 - s|^2} \cdot \frac{1 + 0.35\theta t^{-2}}{1 + 0.36\theta t^{-2}} = \frac{t^2 - (1 - \sigma)^2}{2(1 - \sigma)^2} (1 + 1.04\theta'' t^{-2}) > 0,$$

because $2A = |1 - s|^2/(1 - \sigma)$. (ii) is immediate. (iii) The ordering follows from (i). By Lemma 2, for $z = s, s'$ and $n \geq |t|^{3/2}$, $g_z < 0$ for $n < n^*(z)$ and $g_z > 0$ for $n > n^*(z)$; hence $g_s < 0 < g_{s'}$ on I_s , while outside its closure g_s and $g_{s'}$ have the same sign. The explicit bounds for the length are the integral of (i) over $[1 - \sigma, \sigma]$, whose integrand is positive: $\int_{1-\sigma}^{\sigma} (\frac{t^2}{2(1-u)^2} - \frac{1}{2}) du = (2\sigma - 1)(\frac{t^2}{2\sigma(1-\sigma)} - \frac{1}{2}) = L(\sigma, t)$, and $\sigma(1 - \sigma) \leq \frac{1}{4}$. The asymptotic formula is Lemma 2(b) at s and s' , with $A(s) - A(s') = (2\sigma - 1)(\frac{t^2}{2\sigma(1-\sigma)} - \frac{1}{2})$ and $\frac{\sigma}{6} - \frac{1-\sigma}{6} = \frac{2\sigma-1}{6}$. For the minimum put $x = 1/n$, $p = 1/A$, $q = 1/A'$ and $h_z = 2g_z(n)/n$, so that $\Pi_s = h_s h_{s'}$. By Lemma 2, $h_z = 1/A(z) - x + e_z(x)$ with $e_z(x) = -\frac{\text{Re } z}{6}x^2 + O(t^2x^4 + |t|^5x^5)$, and the Cauchy estimate above, applied also to $\partial_x e_z$, gives $\partial_\sigma e_z = O(x^2)$ and $\partial_\sigma \partial_x e_z = O(t^{-2})$ for $x \leq 3t^{-2}$. On I_s , where $x \leq (2 + o(1))t^{-2}$, put $y = x - e_s(x)$ and $\delta = e_{s'}(x) - e_s(x)$; then $\Pi_s = (p - y)(q + \delta - y)$, $dy/dx = 1 + O(t^{-2})$, $\delta = O((2\sigma - 1)t^{-4})$ and $d\delta/dy = O((2\sigma - 1)t^{-2})$, while

$$p - q = -\frac{2(2\sigma - 1)(t^2 - \sigma(1 - \sigma))}{|s|^2|1 - s|^2} = -2(2\sigma - 1)t^{-2}(1 + O(t^{-2})),$$

so $\delta = O(t^{-2})(p - q)$. Since $(p - y)(q' - y) \geq -\frac{1}{4}(p - q')^2$ for every q' , $\Pi_s \geq -\frac{1}{4}(|p - q| + |\delta|)^2$, and at $y = \frac{p+q}{2}$ the value is $-\frac{1}{4}(p - q)^2 + \frac{1}{2}\delta(p - q)$; this gives the minimum. It is attained at $y = \frac{p+q}{2} + O((2\sigma - 1)t^{-4})$, hence at $n = \frac{2}{p+q} + O(1) = \frac{|s|^2|1-s|^2}{t^2+\sigma(1-\sigma)} + O(1) = t^2 + O(1)$. The second form uses $pq = 4\sigma(1 - \sigma)/(|s|^2|1 - s|^2)$ and $|s|^2|1 - s|^2 = (t^2 - \sigma(1 - \sigma))^2 + t^2$, so that $(p - q)^2/(pq) = \frac{(2\sigma-1)^2}{\sigma(1-\sigma)}(1 - \frac{t^2}{|s|^2|1-s|^2})$. (iv) $\Pi_s = h_s h_{s'} \rightarrow pq$ as $x \rightarrow 0$. With $w = \sigma(1 - \sigma) \in (0, \frac{1}{4}]$, $AA' = \frac{w^2+t^2(1-2w)+t^4}{4w}$, whose derivative in w is $\frac{1}{4} - \frac{t^2+t^4}{4w^2} < 0$; its minimum on $(0, \frac{1}{4}]$ is $(t^2 + \frac{1}{4})^2$, at $\sigma = \frac{1}{2}$. (v) At a zero, $1 - \bar{\rho}$ is a zero, and $C_2(n, z) = 2n^{-2\text{Re } z}g_z(n)$ for $z = \rho, 1 - \bar{\rho}$ (Lemma 2). (a) $\Rightarrow$ (b) is (ii). If a zero with $|\gamma| \geq 16$ has $\beta \neq \frac{1}{2}$, one of $\rho, 1 - \bar{\rho}$ has real part $> \frac{1}{2}$, the product is the same for both, and by (iii) it is negative on an interval contained in $n \geq |\gamma|^{3/2}$, because both crossings lie in $[0.99A', 1.01A]$ and $A' \geq \gamma^2/2$; so (b) $\Rightarrow$ (a). If $\beta = \frac{1}{2}$, then $n^*(\frac{1}{2} + i\gamma) = n^*(\rho)$ is the crossing of $C_2(\cdot, \rho)$; if $\beta \neq \frac{1}{2}$, then $n^*(\frac{1}{2} + i\gamma) \neq n^*(\rho)$ by (i), and $n^*(\rho)$ is the only zero of $C_2(\cdot, \rho)$ in $n \geq |\gamma|^{3/2}$; so (a) $\Leftrightarrow$ (c). (vi) Let $\beta > \frac{1}{2}$ (otherwise use $1 - \bar{\rho}$). By (iii), $|I_\rho| \geq (1 - 1.04\gamma^{-2})(2\beta - 1)(2\gamma^2 - \frac{1}{2})$, which exceeds 1 exactly when $\beta - \frac{1}{2} > \delta_{\mathbb{Z}}(\gamma)$; an open interval of length greater than 1 contains an integer, and $I_\rho \subset [|\gamma|^{3/2}, \infty)$. The bound $\delta_{\mathbb{Z}}(\gamma) < 1.006/(4\gamma^2)$ holds at $\gamma = 16$ (where $4\gamma^2\delta_{\mathbb{Z}} = 1.00506$) and $4\gamma^2\delta_{\mathbb{Z}}(\gamma)$ decreases in γ . $\square$

Proposition 3 is the picture of Figure 1. On the critical line a reflected pair is a single zero, and the product is a square, which touches 0 at the crossing $n^*(\rho) = \gamma^2 + \frac{1}{3} + \dots$ and increases to $(\frac{1}{4} + \gamma^2)^{-2}$. Off the line the two crossings separate, the zero with the larger real part crossing later, and beyond $|\gamma|^{3/2}$ the product dips below zero between them and only there, by the fraction $(2\beta - 1)^2/(4\beta(1 - \beta))$ of its limit, up to a factor $1 + O(\gamma^{-2})$, at every height $|\gamma| \geq t_1$: in the variable $u = n/\gamma^2$ the product times AA' tends to the profile $(1 - \frac{1}{2(1-\beta)u})(1 - \frac{1}{2\beta u})$, whose minimum is at $u = 1$. The window always contains the crossing $n^*(\frac{1}{2} + i\gamma)$ of the line at the same height; its length grows like $2|2\beta - 1|\gamma^2$ near the line and like $|2\beta - 1|\gamma^2/(2\beta(1 - \beta))$ in general; and its relative depth does not decay with γ . An off-line pair would therefore be visible in C_2 evaluated at the pair itself, at a predictable place, $n \approx \gamma^2$, and with a predictable size, and by (iv) its limit is smaller than on the line. Below $|\gamma|/2\pi$ the partial sums are short, the two factors oscillate, and the product can also be negative there, by much more; the window is the last region of negativity. Part (v), however, is a reformulation and not a constraint, and it concerns only the zeros above height 16; the only zero below, ρ_1 , is on the line. Π_ρ is computed from the position of ρ alone, through the Hurwitz function, and its sign on I_ρ is decided by β alone. A proof of RH along these lines would

need an independent reason for $C_2(n, \rho) C_2(n, 1 - \bar{\rho}) \geq 0$ near $n = \gamma^2$, for instance an expression of the product through the primes in which its sign is visible; Section 5 is the closest we come to one.

Real versus integer n . The statements (iii) and (v) are about real n , through the Hurwitz extension, and they detect every $\beta \neq \frac{1}{2}$ above height 16. With integers, which is what the partial sums X_n provide, (vi) guarantees detection only for $|\beta - \frac{1}{2}| > \delta_{\mathbb{Z}}(\gamma) \approx 1/(4\gamma^2)$. Below that threshold the window has length about $4|\beta - \frac{1}{2}|\gamma^2 < 1 + O(\gamma^{-2})$, it contains the crossing $n^*(\frac{1}{2} + i\gamma) = \gamma^2 + \frac{1}{3} + O(\gamma^{-2})$ of the line, and it contains an integer or not according to the position of that crossing relative to the integers; so the integer test may miss such a zero. Above the height $3 \cdot 10^{12}$ to which RH is verified [PT], the threshold is $\delta_{\mathbb{Z}} < 3 \cdot 10^{-26}$.

Numerical checks. We computed Π_s with the Hurwitz function at real n for $s = \beta + i\gamma$, $\gamma \in \{\gamma_1, \gamma_2, \gamma_{10}, \gamma_{100}\}$, $\beta \in \{0.5005, 0.51, 0.55, 0.6, 0.7, 0.8, 0.9\}$; these s are not zeros, and by (v) Π_s is what $C_2(n, \rho) C_2(n, 1 - \bar{\rho})$ would be at a zero $\rho = s$. The length of I_s agrees with the formula in (iii) to a relative error below $4 \cdot 10^{-9}$; the minimum agrees with $-\frac{1}{4}(A^{-1} - A'^{-1})^2$ to a relative error within 0.3% of $1/(6\gamma^2)$ (from $8.4 \cdot 10^{-4}$ at γ_1 to $3.0 \cdot 10^{-6}$ at γ_{100} ; it comes from the terms $-\sigma/(12n)$ and $-(1 - \sigma)/(12n)$ of g_s and $g_{s'}$, which multiply $p - q$ by $1 + \frac{1}{12t^2}$); and the minimum is attained at $n^*(\frac{1}{2} + i\gamma) + \frac{7}{3}(\beta - \frac{1}{2})^2$ to within $2 \cdot 10^{-4}$. At $\sigma \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ the derivative $\partial n^*/\partial \sigma$ agrees with the main term of (i) to a relative error below $1.4 \cdot 10^{-3}$ at γ_1 and $1.1 \cdot 10^{-4}$ at γ_{10} , the difference being $\frac{1}{6} + O(t^{-2})$, the derivative of the term $\frac{\sigma}{6}$ of Lemma 2. With integers and the partial sums, for $s = 0.7 + i\gamma_1$ the product is negative exactly for $144 \leq n \leq 333$ ($I_s = (143.108, 333.251)$), with minimum $-3.9801 \cdot 10^{-6}$ at $n = 200$ against $-3.9768 \cdot 10^{-6}$ predicted, and limit $1/(AA') = 2.0983 \cdot 10^{-5}$ against $(\frac{1}{4} + \gamma_1^2)^{-2} = 2.4990 \cdot 10^{-5}$ on the line; for $\beta = 0.5005$ the window $(199.924, 200.324)$ has length $0.40 \approx 2(2\beta - 1)\gamma_1^2$, below 1, so (vi) would not guarantee an integer in it (and $\gamma_1 < 16$ is outside the range of (vi) anyway), but it happens to contain the integer 200. Status: numerical (`reflected_pair.py`, `reflected_pair_fig.py`).

Remark 4 (the limits at a reflected pair). By Theorem A [CE1, Thm. 3.2], at every point of the strip, zero or not, $C_2(n, s) = A_s n^{1-2\sigma} + 2\operatorname{Re}(\zeta(s)n^{-\bar{s}}) - n^{-2\sigma} + O(|s|n^{-2\sigma-1})$ with $A_s = 1/A(s) = 2(1 - \sigma)/|1 - s|^2$. Multiplying the expansions at s and $1 - \bar{s}$, the cross terms are $O(n^{-\sigma} + n^{\sigma-1})$, so $C_2(n, s) C_2(n, 1 - \bar{s}) \rightarrow A_s A_{1-\bar{s}} = 1/(AA')$ at every point; at a zero the wave is absent and

$$C_2(n, \rho) C_2(n, 1 - \bar{\rho}) = A_{\rho} A_{1-\bar{\rho}} - \frac{A_{\rho} + A_{1-\bar{\rho}}}{n} + O_{\rho}(n^{-2}), \quad \frac{C_2(n, \rho)}{C_2(n, 1 - \bar{\rho})} \sim \frac{A_{\rho}}{A_{1-\bar{\rho}}} n^{2-4\beta},$$

with $A_{1-\bar{\rho}} = 2\beta/|\rho|^2$; so $[\log(C_2(n, \rho)/C_2(n, 1 - \bar{\rho}))/\log n]^2 \rightarrow 4(1 - 2\beta)^2$. The same two coefficients give the off-line term of the Jensen formula of Balazard, Saias and Yor [BSY]: $\log |\rho/(1 - \rho)| = \frac{1}{2} \log(\beta A_{\rho}/((1 - \beta)A_{1-\bar{\rho}}))$, which vanishes exactly on the line. Status: proved (elementary, from [CE1, Thm. 3.2]).

2.3. Positivity on the whole line. Parts (ii) and (iii) of Proposition 3 are statements about Π_s at every point, not only at zeros: for $|t| \geq t_1$ the set of points of the strip at which $\Pi_s(n) \geq 0$ for every real $n \geq |t|^{3/2}$ is exactly the critical line. On the line the same holds for the product with the wave, $C_2(n, s) C_2(n, 1 - \bar{s}) = C_2(n, s)^2 \geq 0$ for every n and every t , zero or not. Off the line the wave changes the negative set: at the 20 points with $\sigma \in \{0.51, 0.55, 0.6, 0.7, 0.9\}$ and $t \in \{\gamma_1, 20, 30, 50\}$, none of them a zero, the product $C_2(n, s) C_2(n, 1 - \bar{s})$ is negative at between 63 and 53019 integers of $[t^{3/2}, 40t^2]$, in three cases up to the end of that range, and this larger set need not contain the window I_s (for $t \geq 20$, between 37% and all of the integers of I_s give a nonnegative product with the wave). At a zero the wave vanishes and the negative set beyond $|\gamma|^{3/2}$ is exactly I_{ρ} . Positivity of the product of a reflected pair therefore characterizes the critical line as a set, and RH is the statement that the zeros lie in that set; a proof along these lines would have to explain why the points where the wave vanishes avoid the region where the wave-free product is negative. Status:

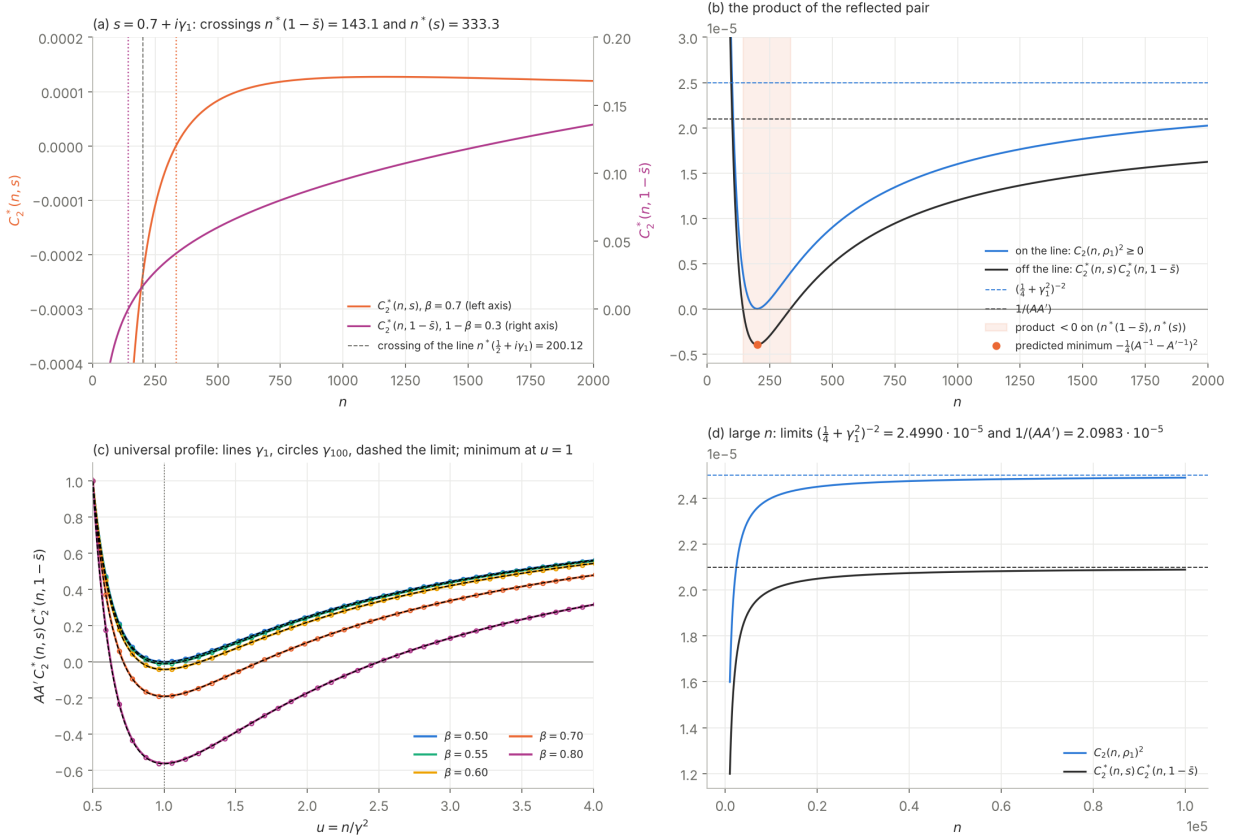


FIGURE 1. The reflected pair (Proposition 3), linear axes. The function $C_2^*(n, s) = 2 \operatorname{Re}(n^{-\bar{s}}(X_{n-1}(s) - \zeta(s))) = 2n^{-2\sigma}g_s(n)$ is $C_2(n, s)$ with the wave removed, that is, $C_2(n, \rho)$ as it would be at a zero $\rho = s$. (a) $C_2^*(n, s)$ (left axis) and $C_2^*(n, 1-\bar{s})$ (right axis) for $s = 0.7 + i\gamma_1$, with their crossings 143.1 and 333.3 (dotted) and the crossing 200.12 of the line $n^*(\frac{1}{2} + i\gamma_1) = 200.12$ (dashed). (b) The products: $C_2(n, \rho_1)^2 \geq 0$ on the line, and $C_2^*(n, s)C_2^*(n, 1-\bar{s})$, negative exactly between the two crossings (shaded), with the predicted minimum (dot) and the two limits (dashed). (c) The product times AA' against $u = n/\gamma^2$ for $\beta = 0.5, 0.55, 0.6, 0.7, 0.8$, at γ_1 (lines) and γ_{100} (circles), and the limit profile $(1 - \frac{1}{2(1-\beta)u})(1 - \frac{1}{2\beta u})$ (dashed). (d) Larger n : the products tend to $(\frac{1}{4} + \gamma_1^2)^{-2}$ and to $1/(AA')$.

the characterization by Π_s follows from Proposition 3; the statement with the wave is proved on the line and numerical off it (`reflected_pair.py`, Table 5 of its output).

2.4. Two variables: the Wronskian. The cross-energy and the reflected cross-energy are two specializations of the *two-variable cross-energy*

$\mathcal{C}_2(N; s, w) = X_N(s)X_N(w) - X_{N-1}(s)X_{N-1}(w) - N^{-s-w} = N^{-s}X_{N-1}(w) + N^{-w}X_{N-1}(s) :$
 $w = \bar{s}$ gives $C_2(N, s)$ and $w = 1-s$ gives $C_2^{\text{FR}}(N, s)$. Its antisymmetric companion is the Wronskian of consecutive partial sums,

$$\mathcal{W}(N; s, w) = X_N(s)X_{N-1}(w) - X_{N-1}(s)X_N(w) = N^{-s}X_{N-1}(w) - N^{-w}X_{N-1}(s).$$

Proposition 5 (the reflected pair in two variables). *For every s and every real $N \geq 1$,*

$$C_2(N, s)C_2(N, 1-\bar{s}) = (\operatorname{Re} C_2(N; s, 1-s))^2 - (\operatorname{Re} \mathcal{W}(N; s, 1-s))^2.$$

- (i) If $\sigma = \frac{1}{2}$, then $\mathcal{W}(N; s, 1-s)$ is purely imaginary, $\mathcal{C}_2(N; s, 1-s) = \mathcal{C}_2(N, s)$, and the product is $\mathcal{C}_2(N, s)^2 \geq 0$.
- (ii) If $\zeta(\rho) = 0$, $\lambda = \rho(1-\rho)$, $|\gamma| \geq 16$ and $N \geq |\gamma|^{3/2}$, then, with $|\theta_1|, |\theta_2| \leq 1$ and κ as in Lemma 2,

$$\begin{aligned} \operatorname{Re} \mathcal{C}_2(N; \rho, 1-\rho) &= \operatorname{Re} \frac{1}{\lambda} - \frac{1}{N} - \frac{1}{12N^2} + \theta_1 \left(\frac{6+9|\lambda|}{720N^4} + \frac{2\kappa(|\gamma|+5)^5}{N^5} \right), \\ \operatorname{Re} \mathcal{W}(N; \rho, 1-\rho) &= (2\beta-1) \left(\frac{\gamma^2 - \beta(1-\beta)}{|\rho|^2|1-\rho|^2} + \frac{1}{12N^2} + \theta_2 \left(\frac{(|\gamma|+2)^2}{240N^4} + \frac{1.2 \cdot 10^{-4}(|\gamma|+6)^5}{N^5} \right) \right). \end{aligned}$$

Consequently, if $\beta \neq \frac{1}{2}$, then $\operatorname{Re} \mathcal{C}_2(\cdot; \rho, 1-\rho)$ vanishes at some $N_0 = \gamma^2(1 + O(\gamma^{-2}))$, and

$$\mathcal{C}_2(N_0, \rho) \mathcal{C}_2(N_0, 1-\rho) = -(\operatorname{Re} \mathcal{W}(N_0; \rho, 1-\rho))^2 = -(2\beta-1)^2 \gamma^{-4} (1 + O(\gamma^{-2})) < 0.$$

Proof. Put $a = X_{N-1}(s)$, $b = X_{N-1}(1-s)$, $x = N^{-s}$, $y = N^{s-1}$. Then $\mathcal{C}_2(N, s) = 2\operatorname{Re}(a\bar{x})$, $\mathcal{C}_2(N, 1-s) = 2\operatorname{Re}(b\bar{y})$, $\mathcal{C}_2(N; s, 1-s) = xb + ya$ and $\mathcal{W}(N; s, 1-s) = xb - ya$. Since $y = N^{2\sigma-1}\bar{x}$ and $x = N^{1-2\sigma}\bar{y}$, $\operatorname{Re}(ya)\operatorname{Re}(xb) = \operatorname{Re}(a\bar{x})\operatorname{Re}(b\bar{y})$, and $4\operatorname{Re} u \operatorname{Re} v = (\operatorname{Re}(u+v))^2 - (\operatorname{Re}(u-v))^2$ with $u = xb$, $v = ya$ gives the identity (for real N with the Hurwitz extension). (i) On the line $b = \bar{a}$ and $y = \bar{x}$, so $xb = \bar{y}\bar{a}$. (ii) At a zero, $X_{N-1}(\rho) = -\zeta(\rho, N)$ and $X_{N-1}(1-\rho) = -\zeta(1-\rho, N)$, so $N\mathcal{C}_2 = \Phi_\rho + \Phi_{1-\rho}$ and $N\mathcal{W} = \Phi_{1-\rho} - \Phi_\rho$, with

$$\Phi_z(N) = -N^z \zeta(z, N) = \frac{N}{1-z} - \frac{1}{2} - \frac{z}{12N} + \frac{(z)_3}{720N^3} - r(z), \quad r(z) = N^z R_2(z, N),$$

by the Euler–Maclaurin formula. Here r is holomorphic in $\operatorname{Re} z > -4$, $|r(z)| \leq \kappa|(z)_5|N^{-4}$ for $0 < \operatorname{Re} z < 1$, and $r(\bar{z}) = \overline{r(z)}$ because N is real. Adding, $\mathcal{C}_2 = \frac{1}{\lambda} - \frac{1}{N} - \frac{1}{12N^2} + \frac{(\rho)_3 + (1-\rho)_3}{720N^4} - \frac{r(\rho) + r(1-\rho)}{N}$ with $(\rho)_3 + (1-\rho)_3 = 6 - 9\lambda$; this gives the first formula. Subtracting,

$$\mathcal{W} = \frac{1-2\rho}{\lambda} + \frac{2\rho-1}{12N^2} + \frac{(1-\rho)_3 - (\rho)_3}{720N^4} - \frac{r(1-\rho) - r(\rho)}{N}.$$

The real part of $\frac{1-2\rho}{\lambda}$ is $(2\beta-1)\frac{\gamma^2-\beta(1-\beta)}{|\rho|^2|1-\rho|^2}$, and that of $\frac{2\rho-1}{12N^2}$ is $\frac{2\beta-1}{12N^2}$. For a function h with $h(\bar{z}) = \overline{h(z)}$ we have $\operatorname{Re} h(\rho) = \operatorname{Re} h(\bar{\rho})$, and $1-\rho$ and $\bar{\rho}$ have the same imaginary part and real parts $1-\beta$ and β . Hence $|\operatorname{Re}(h(1-\rho) - h(\rho))| = |\operatorname{Re}(h(1-\rho) - h(\bar{\rho}))| \leq |2\beta-1| \max |h'|$ on the segment $[\bar{\rho}, 1-\rho]$. For $h(z) = (z)_3$, $|h'(z)| = |3z^2 + 6z + 2| \leq 3(|\gamma|+2)^2$ there. For $h = r$, Cauchy's inequality on circles of radius $\frac{1}{2}$ gives $|r'| \leq \frac{2\|B_5\|_\infty}{120 \cdot 3.5} (|\gamma|+6)^5 N^{-4} < 1.2 \cdot 10^{-4} (|\gamma|+6)^5 N^{-4}$. This gives the second formula. Finally, $\operatorname{Re} \frac{1}{\lambda} = \frac{\gamma^2 + \beta(1-\beta)}{|\lambda|^2} \in (0, \gamma^{-2}]$ because $|\lambda|^2 \geq (\gamma^2 + \beta(1-\beta))^2$; so at $N = |\gamma|^{3/2}$ the first formula gives $\operatorname{Re} \mathcal{C}_2 \leq \gamma^{-2} - |\gamma|^{-3/2} + 10^{-3} \gamma^{-2} < 0$ for $|\gamma| \geq 16$, while $\operatorname{Re} \mathcal{C}_2 \rightarrow \operatorname{Re} \frac{1}{\lambda} > 0$. At a zero $N_0 \geq |\gamma|^{3/2}$ of $\operatorname{Re} \mathcal{C}_2$ the first formula gives $\frac{1}{N_0} = \operatorname{Re} \frac{1}{\lambda} (1 + O(\gamma^{-1/2}))$, because there the error terms are $O(\gamma^{-5/2})$; hence $N_0 \asymp \gamma^2$, the error terms are then $O(\gamma^{-4})$, and a second application gives $\frac{1}{N_0} = \operatorname{Re} \frac{1}{\lambda} (1 + O(\gamma^{-2}))$, that is $N_0 = \gamma^2(1 + O(\gamma^{-2}))$, and there the error terms in the second formula are $O(\gamma^{-5})$, which gives $\operatorname{Re} \mathcal{W}(N_0) = (2\beta-1)\gamma^{-2}(1 + O(\gamma^{-2}))$. $\square$

In two variables Proposition 3 becomes transparent (Figure 2). The coordinates $u = \operatorname{Re} \mathcal{C}_2 + \operatorname{Re} \mathcal{W}$ and $v = \operatorname{Re} \mathcal{C}_2 - \operatorname{Re} \mathcal{W}$ are, up to the factors $2N^{-2\beta}$ and $2N^{-2(1-\beta)}$, the two crossing functions g_ρ and $g_{1-\bar{\rho}}$, and the product is uv . As N grows, $\operatorname{Re} \mathcal{W}$ stays essentially constant, at $\operatorname{Re} \frac{1-2\rho}{\lambda}$, while $\operatorname{Re} \mathcal{C}_2$ sweeps from $-\frac{1}{N}$ to $\operatorname{Re} \frac{1}{\lambda}$; the curve must cross the cone $|\operatorname{Re} \mathcal{W}| > |\operatorname{Re} \mathcal{C}_2|$ unless $\operatorname{Re} \mathcal{W} = 0$. The window of Proposition 3 is the time spent in the cone, its depth $\frac{1}{4}(A^{-1} - A'^{-1})^2$ is $(\operatorname{Re} \mathcal{W})^2$, and its centre $N_0 \approx |\lambda|^2 / \operatorname{Re} \lambda$ is where $\operatorname{Re} \mathcal{C}_2$ vanishes. For the zeros above height 16, RH is the statement that the Wronskian of the partial sums at ρ and $1-\rho$ has no real part.

The obstruction in two variables. Consider the holomorphic combination $\mathcal{C}_2^2 - \mathcal{W}^2$. It equals $4N^{-1}X_{N-1}(s)X_{N-1}(1-s)$, which on the line is $4N^{-1}|X_{N-1}(s)|^2 \geq 0$, and it can be summed over the zeros by the explicit formula (it is Weil's form for Dirichlet-polynomial test functions), but at

every zero, on the line or not, it equals $\frac{4}{\lambda} - \frac{2}{N\lambda} + O(N^{-2})$, with positive real part: it never sees an off-line zero. The product is

$$C_2(N, s) C_2(N, 1 - \bar{s}) = \frac{1}{2} \operatorname{Re} (\mathcal{C}_2^2 - \mathcal{W}^2) + \frac{1}{2} (|\mathcal{C}_2|^2 - |\mathcal{W}|^2),$$

and the second, sesquilinear term, which carries the detection, is not a holomorphic function of ρ and cannot be summed over the zeros. At height γ_1 the identity of Proposition 5 holds to a relative error of $1.2 \cdot 10^{-8}$ or better; $\operatorname{Re} \mathcal{W}$ varies by less than 0.1% between $N = 143$ and $N = 10^5$ ($1.99583 \cdot 10^{-3}$ to $1.99420 \cdot 10^{-3}$ for $\beta = 0.7$, against the limit $1.994202 \cdot 10^{-3}$); the product at N_0 equals $-(\operatorname{Re} \mathcal{W}(N_0))^2$ to five digits ($-3.98016 \cdot 10^{-6}$ for $\beta = 0.7$, $-8.95091 \cdot 10^{-6}$ for $\beta = 0.8$); and $\operatorname{Re}(\mathcal{C}_2^2 - \mathcal{W}^2)$ stays between 0.0199 and 0.0200 for $30 \leq N \leq 10^5$ and $\beta \in \{0.55, 0.7, 0.9\}$. Status: Proposition 5 is proved; the numbers are numerical (`reflected_pair_two_var.py`).

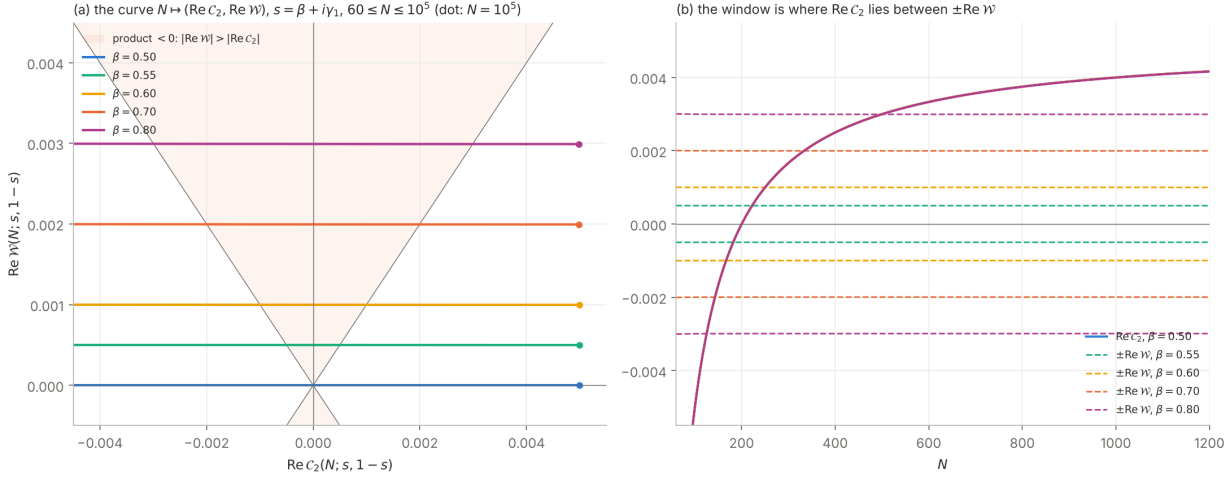


FIGURE 2. The reflected pair in two variables (Proposition 5), $s = \beta + i\gamma_1$ without the wave, linear axes. (a) The curve $N \mapsto (\operatorname{Re} \mathcal{C}_2(N; s, 1-s), \operatorname{Re} \mathcal{W}(N; s, 1-s))$ for $60 \leq N \leq 10^5$; shaded, the cone $|\operatorname{Re} \mathcal{W}| > |\operatorname{Re} \mathcal{C}_2|$ where $\mathcal{C}_2(N, s) \mathcal{C}_2(N, 1 - \bar{s}) < 0$. On the line the curve stays on the axis; off it, it runs at the height $\operatorname{Re} \frac{1-2s}{s(1-s)}$ and crosses the cone. (b) $\operatorname{Re} \mathcal{C}_2$ (one curve for all β) and $\pm \operatorname{Re} \mathcal{W}$ (dashed) against N : the window of Proposition 3 is where the curve lies between the dashed lines.

Remark 6 (the holomorphic and the modulus sides). Since $\overline{X_N(1-\bar{s})} = X_N(1-s)$, Theorem A [CE1, Thm. 3.2] gives, at every point of the strip, zero or not,

$$\lim_{N \rightarrow \infty} \frac{X_N(s) \overline{X_N(1-\bar{s})}}{N} = \lim_{N \rightarrow \infty} \frac{X_N(s) X_N(1-s)}{N} = \frac{1}{s(1-s)} = \frac{1}{\lambda},$$

the terms with $\zeta(s)$ and $\zeta(1-s)$ decaying like $N^{\sigma-1}$ and $N^{-\sigma}$; on the line the quotient is $|X_N|^2/N$ (at $s = \frac{1}{2} + 20i$, $N = 10^6$, it is 0.002389 against $1/\lambda = 0.002498$). Summed over the zeros the holomorphic side gives $\sum_{\rho} 1/\lambda_{\rho} = 2 + \gamma_0 - \log 4\pi$, the constant seen by the primes, while the limits of $\mathcal{C}_2$ give $\sum_{\rho} |\rho|^{-2}$ under RH. The difference is the non-holomorphic gap of [CE5, Prop. 2.5], $\sum_{\rho} |\rho|^{-2} = 2 + \gamma_0 - \log 4\pi + \mathcal{Q}$ with $\mathcal{Q} = 2 \sum_{\rho} (\beta - \frac{1}{2})^2 / (|\rho|^2 |1-\rho|^2) = \frac{1}{2} \sum_{\rho} (|w_{\rho}| - |w_{\rho}|^{-1})^2 \geq 0$, w_{ρ} as in Section 1.1, which vanishes if and only if RH holds: an off-line pair has reciprocal radii and contributes a positive amount. The explicit formula evaluates only holomorphic functions of ρ and gives $\sum 1/\lambda_{\rho}$ without information on β ; $\mathcal{Q}$ is reached only through a positivity statement, which is the obstruction of the preceding paragraph in its simplest form. Status: the limit is proved; the identity for $\mathcal{Q}$ is [CE5, Prop. 2.5]; the number is numerical (`functional_reflection_check.py`).

Remark 7 (derivatives at a reflected pair). Differentiating $\xi(s) = \xi(1-s)$ and $\xi(\bar{s}) = \overline{\xi(s)}$ gives $\xi^{(k)}(s) = (-1)^k \xi^{(k)}(1-s)$, hence $\xi'(1-\bar{\rho}) = -\overline{\xi'(\rho)}$ and $\xi''(1-\bar{\rho}) = \overline{\xi''(\rho)}$; at a simple zero $\xi''/\xi'(1-\bar{\rho}) = -\overline{\xi''/\xi'(\rho)}$, with $\xi''/\xi'(\rho) = 2h'/h(\rho) + \zeta''/\zeta'(\rho)$, $h(s) = \xi(s)/\zeta(s)$. These identities hold equally for hypothetical off-line zeros. Status: proved (elementary).

3. THE REFLECTED CROSS-ENERGY AND THE ZERO DETECTOR

The specialization $w = 1-s$ of the two-variable cross-energy is the reflected cross-energy

$$C_2^{\text{FR}}(N, s) = C_2(N; s, 1-s) = N^{-s} X_{N-1}(1-s) + N^{s-1} X_{N-1}(s).$$

It is entire in s , satisfies $C_2^{\text{FR}}(N, s) = C_2^{\text{FR}}(N, 1-s)$, and equals $C_2(N, s)$ on the critical line. With $s = \frac{1}{2} + \delta + it$,

$$C_2^{\text{FR}}(N, s) = \frac{2}{N} \sum_{k < N} \sqrt{\frac{N}{k}} \left[\cosh\left(\delta \log \frac{N}{k}\right) \cos\left(t \log \frac{N}{k}\right) + i \sinh\left(\delta \log \frac{N}{k}\right) \sin\left(t \log \frac{N}{k}\right) \right],$$

which is real on the line. Put

$$\mathcal{G}_N^*(s) = -\frac{1}{N} (N^s \zeta(s, N) + N^{1-s} \zeta(1-s, N)), \quad \mathcal{G}_N(\lambda) = \frac{1}{\lambda} - \frac{1}{N} - \frac{1}{12N^2} + \left(\frac{1}{120} - \frac{\lambda}{80}\right) N^{-4}.$$

$\mathcal{G}_N^*$ is symmetric under $s \leftrightarrow 1-s$ and analytic for $s \neq 0, 1$, hence a function of $\lambda = s(1-s)$, and we write $\mathcal{G}_N^*(\lambda)$.

Theorem 8 (the reflected cross-energy as a zero detector). *For every $s \neq 0, 1$ and real $N \geq 1$, exactly,*

$$C_2^{\text{FR}}(N, s) = \mathcal{G}_N^*(\lambda) + V_N(s), \quad V_N(s) = N^{s-1} \zeta(s) + N^{-s} \zeta(1-s) = \zeta(s) \mathcal{K}_N(s),$$

with the carrier $\mathcal{K}_N(s) = N^{s-1} + \chi(1-s)N^{-s}$ of the functional equation $\zeta(1-s) = \chi(1-s)\zeta(s)$; and for $0 < \sigma < 1$, $N \geq 2$,

$$|\mathcal{G}_N^*(\lambda) - \mathcal{G}_N(\lambda)| \leq 1.1 \cdot 10^{-4} (|s| + 7)^6 N^{-6}.$$

In particular, at every zero of ζ , on the critical line or off it,

$$C_2^{\text{FR}}(N, \rho) = \mathcal{G}_N^*(\lambda_\rho).$$

(i) On the critical line, with $\zeta(\frac{1}{2} + it) = e^{-i\vartheta(t)} Z(t)$ (Hardy's function),

$$N^{1/2} (C_2^{\text{FR}}(N, \frac{1}{2} + it) - \mathcal{G}_N(\frac{1}{4} + t^2)) = 2Z(t) \cos(t \log N - \vartheta(t)) + 1.1 \cdot 10^{-4} \theta (|t| + 8)^6 N^{-11/2}, \quad |\theta| \leq 1.$$

(ii) V_N vanishes at every zero of ζ , on the line or off it, and at the zeros of $\mathcal{K}_N$; for $|t| \geq 2$ and $N^2 > \frac{|t|}{2\pi} e^{1/|t|}$ the zeros of $\mathcal{K}_N$ with ordinate t and $0 < \sigma < 1$ lie on the critical line. At every zero, $C_2^{\text{FR}}(N, \rho) = \mathcal{G}_N^*(\lambda_\rho)$.

(iii) $N^{1-s} C_2^{\text{FR}}(N, s) = X_{N-1}(s) + N^{1-2s} X_{N-1}(1-s)$.

Proof. The exact identity is $X_{N-1}(z) = \zeta(z) - \zeta(z, N)$ at $z = s$ and $z = 1-s$. For the expansion, the Euler–Maclaurin formula to order m reads

$$N^z \zeta(z, N) = \frac{N}{z-1} + \frac{1}{2} + \sum_{j=1}^m \frac{B_{2j}}{(2j)!} (z)_{2j-1} N^{1-2j} + r_m(z),$$

$$|r_m(z)| \leq \frac{\|B_{2m+1}\|_\infty}{(2m+1)!} \frac{|(z)_{2m+1}|}{\text{Re } z + 2m} N^{-2m},$$

for $\text{Re } z > -2m$. Adding it at $z = s$ and $z = 1-s$ and dividing by $-N$, the constant terms give $\frac{1}{s} + \frac{1}{1-s} = \frac{1}{\lambda}$, and $s(s+1)(s+2) + (1-s)(2-s)(3-s) = 6 - 9\lambda$ gives the term in N^{-4} . Note that $|(z)_k| \leq (|s| + k)^k$ for $z \in \{s, 1-s\}$. If $|s| + 7 > N$, take $m = 2$: then $|\mathcal{G}_N^* - \mathcal{G}_N| \leq \frac{|r_2(s)| + |r_2(1-s)|}{N} \leq \frac{\|B_5\|_\infty}{240} (|s| + 5)^5 N^{-5} \leq 1.02 \cdot 10^{-4} (|s| + 7)^6 N^{-6}$. If $|s| + 7 \leq N$, take $m = 3$: the term $j = 3$

contributes at most $\frac{2(|s|+5)^5}{30240 N^6}$ and the remainder at most $\frac{2\|B_7\|_\infty(|s|+7)^7}{5040 \cdot 6 N^7}$, with $\|B_7\|_\infty < 0.027$; both are below $10^{-5}(|s|+7)^6 N^{-6}$. At a zero, $V_N(\rho) = 0$ because $\zeta(\rho) = \zeta(1-\rho) = 0$. (i) On the line $\chi(\frac{1}{2} - it) = e^{2i\vartheta(t)}$, so $\zeta(s)\mathcal{K}_N(s) = N^{-1/2}Z(t)(e^{i(t \log N - \vartheta)} + e^{-i(t \log N - \vartheta)})$, and $|s|+7 \leq |t|+8$. (ii) A zero of $\mathcal{K}_N$ needs $|\chi(1-s)|N^{1-2\sigma} = 1$. This holds at $\sigma = \frac{1}{2}$, and for $0 < \sigma < 1$ and $t \geq 2$ (the case $t \leq -2$ follows by conjugation) $\partial_\sigma \log |\chi(1-s)| = \operatorname{Re} \frac{\Gamma'}{\Gamma}(s) - \log 2\pi - \frac{\pi}{2} \operatorname{Re} \tan \frac{\pi s}{2} \leq \log \frac{t}{2\pi} + \frac{1}{t}$, because $\operatorname{Re} \tan \frac{\pi s}{2} = \frac{\sin \pi \sigma}{\cos \pi \sigma + \cosh \pi t} > 0$ and, by Stirling's formula, $\operatorname{Re} \frac{\Gamma'}{\Gamma}(s) \leq \log |s| + O(|s|^{-2}) \leq \log t + \frac{1}{t}$ (we also checked this last inequality numerically on $0 < \sigma < 1$, $2 \leq t \leq 10^4$); so $\sigma \mapsto |\chi(1-s)|N^{1-2\sigma}$ is strictly decreasing when $2 \log N > \log \frac{t}{2\pi} + \frac{1}{t}$. At a zero $\zeta(\rho) = \zeta(1-\rho) = 0$. (iii) is the definition. $\square$

Two properties. Theorem 8 contains two different facts, and only the second can distinguish the zeros on the line from those off it.

Property A (the wave vanishes at every zero). $V_N(\rho) = 0$ at every zero, on the line or off it, so $C_2^{\text{FR}}(N, \rho) = \mathcal{G}_N^*(\lambda_\rho)$ is a fixed function of λ_ρ . This holds for every zero and therefore does not by itself say anything about β ; it is what makes C_2^{FR} summable over the zeros (Section 5).

Property B (the spectral variable is real exactly on the line). For $0 < \beta < 1$ and $\gamma \neq 0$, $\lambda_\rho = \beta(1-\beta) + \gamma^2 + i\gamma(1-2\beta)$ is real if and only if $\beta = \frac{1}{2}$. Since $\mathcal{G}_N^*(\bar{\lambda}) = \overline{\mathcal{G}_N^*(\lambda)}$, the weight $\mathcal{G}_N^*(\lambda_\rho)$ is real at a zero on the line, and at a zero off the line its imaginary part tends to $\operatorname{Im} \frac{1}{\lambda_\rho} \neq 0$. The restricted Weil criterion and its limit (Propositions 11 and 14) use Property A together with the fact that the weights do not vanish for large n ; the Hankel–Jacobi criterion (Proposition 15) uses Property B, since its conclusion is that every λ_ρ is real.

The Gonek–Montgomery link. By (iii), C_2^{FR} is the symmetric truncation $X_N(s) + \chi(s)X_N(1-s)$ of Gonek and Montgomery [GoM], almost all of whose zeros lie on the critical line, with $\chi(s)$ replaced by N^{1-2s} , the factor of the functional equation at the fixed height $2\pi N^2$: since $|\chi(\sigma + it)| \sim (t/2\pi)^{\frac{1}{2}-\sigma}$, the moduli of $\chi(s)$ and N^{1-2s} agree at $t = 2\pi N^2$. The symmetric truncation is the approximate functional equation with equal lengths, and it is the functional equation, not the Euler product, that keeps its zeros near the line; the reflected cross-energy freezes its height and moves the dependence on s into the carrier $\mathcal{K}_N$, whose zeros (ii) places on the line as soon as N^2 exceeds $t/2\pi$. Status: Theorem 8, with (i)–(iii), is proved.

Numerics. The decomposition holds to $2.4 \cdot 10^{-11}$ for $N = 200$ and to $3 \cdot 10^{-15}$ for $N = 1000$ at five points of the strip, and (i) to $4.7 \cdot 10^{-10}$ for $10 \leq t \leq 40$, $N = 200$. The zeros of V_N , $N = 200$, in $0.02 < \sigma < 0.98$, $10 < t < 40$, are 50, all on the line: the six zeros of ζ and 44 zeros of the carrier, at which $|\cos(t \log N - \vartheta(t))| < 10^{-13}$. For the Davenport–Heilbronn function f [DH], [T, Sect. 10.25], [CE17], which satisfies a functional equation of the type of ζ but has no Euler product, put $V_N^f(s) = N^{s-1}f(s) + N^{-s}f(1-s)$: its 17 zeros with $80 < t < 90$ include exactly two off the line, $0.808517 + 85.699348i$ and $0.191483 + 85.699348i$ (Figure 3). The reflected cross-energy sees every zero with its real part, and the integers below N alone reproduce Hardy's function on the line; since $V_N = \zeta\mathcal{K}_N$ and the zeros of $\mathcal{K}_N$ lie on the line, the statement that V_N has no zeros off the line is RH. It detects the zeros; it does not place them. Status: numerical (`c2fr_zero_map.py`).

What the smooth part sees off the line. At a zero off the line, $\operatorname{Im} C_2^{\text{FR}}(N, \rho) = \operatorname{Im} \mathcal{G}_N^*(\lambda_\rho) \rightarrow \operatorname{Im} \frac{1}{\lambda_\rho} = \frac{2\delta\gamma}{|\lambda_\rho|^2}$, $\delta = \beta - \frac{1}{2}$, which vanishes if and only if $\delta = 0$. This characterizes the line, not the zeros: $\mathcal{G}_N^*$ is computed from the position of ρ alone, and its imaginary part is essentially constant in N ; for $s = 0.7 + i\gamma_1$, $\operatorname{Im} \mathcal{G}_N^*(\lambda) = 1.41234 \cdot 10^{-4}$ at every N from 10^2 to 10^6 , the value of $\operatorname{Im} \frac{1}{\lambda}$. What the reflection adds, compared with Proposition 3, is holomorphy: unlike C_2 , C_2^{FR} can be summed over all the zeros by the explicit formula (Section 5). Status: proved; the number is numerical (`functional_reflection_check.py`).

Remark 9 (the integers and the primes). The reflected cross-energy sees a zero through the integers below N with the functional equation as its carrier. The arithmetic cross-energy of [CE5, Sect. 2],

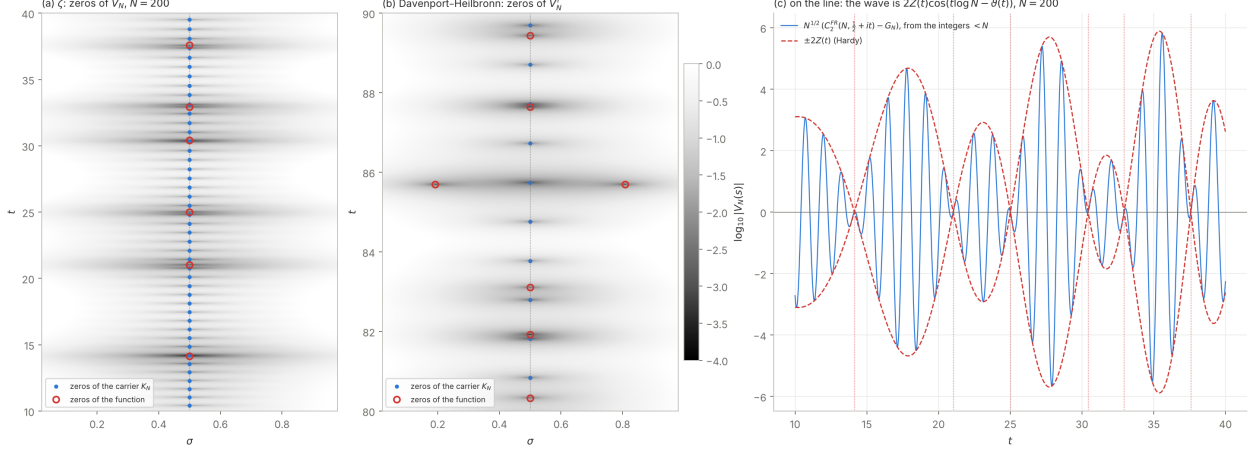


FIGURE 3. The reflected cross-energy as a zero detector (Theorem 8), $N = 200$. (a) $\log_{10} |V_N(s)|$ for ζ , $10 < t < 40$: the zeros of ζ (circles) and of the carrier $\mathcal{K}_N$ (dots), all on $\sigma = \frac{1}{2}$. (b) The same for the Davenport–Heilbronn function, $80 < t < 90$: its zeros $0.8085 + 85.699i$ and $0.1915 + 85.699i$ appear off the line. (c) $N^{1/2}(C_2^{\text{FR}}(N, \frac{1}{2} + it) - \mathcal{G}_N)$ computed from the integers below N , and $\pm 2Z(t)$; linear axes.

built from $\mathcal{P}_N(s) = \sum_{n \leq N} \Lambda(n)n^{-s}$, sees it through the primes, without the functional equation: under RH, by the explicit formula, at a zero ρ_0 of multiplicity m , $\mathcal{P}_x(\rho_0) - \frac{x^{1-\rho_0}}{1-\rho_0} = -m \log x + x^{-\rho_0}(\psi(x) - x) + O_{\rho_0}(1)$, and the middle term has bounded mean square, so the resonance $-m \log x$ of Conrad [Con] dominates in logarithmic mean. The two are joined by Chebyshev’s identity $\log = \Lambda * 1$, which is the subject of [CE11] at a single zero and of Proposition 12 below after summation over the zeros. The helper `lambda_c2.py`, which supplies the prime powers to the scripts of Section 5, also prints the resonance at ρ_1 from the primes up to 10^8 . Status: conditional on RH, classical [Con]; see [CE1, Prop. 2.9].

4. NUMBER SYSTEMS: DETECTION AND DRIFT

The definition of C_2^{FR} uses only the integers below N , so it applies to any system of generalized integers $1 = b_1 \leq b_2 \leq \dots$ with counting function N_B and zeta function $\zeta_B(s) = \sum_j b_j^{-s}$:

$$C_2^{\text{FR},B}(N, s) = N^{-s} \sum_{b_j < N} b_j^{s-1} + N^{s-1} \sum_{b_j < N} b_j^{-s}.$$

Proposition 10 (the reflected cross-energy of a number system). *Let $N_B(x) = cx + c'x^b + E(x)$ with $c > 0$, $0 < b < 1$ and $E(x) = O(x^\theta)$, $0 \leq \theta < \frac{1}{2}$. Uniformly on compact subsets of $\theta < \sigma < 1 - \theta$ not containing b and $1 - b$,*

$$C_2^{\text{FR},B}(N, s) = \frac{c}{\lambda} + c' b N^{b-1} \left(\frac{1}{b-s} + \frac{1}{b-1+s} \right) + N^{s-1} \zeta_B(s) + N^{-s} \zeta_B(1-s) + O(N^{\theta-1}).$$

Let $U_N(s)$ be N^{1-s} times $C_2^{\text{FR},B}(N, s)$ minus its first two terms.

- (i) *If $\zeta_B(\rho) = 0$ with multiplicity m and $\frac{1}{2} < \text{Re } \rho < 1 - \theta$, then for N large U_N has exactly m zeros in a fixed disc around ρ , all at distance $O((N^{1-2\text{Re } \rho} + N^{\theta-\text{Re } \rho})^{1/m})$ from ρ ; and every limit of zeros of U_N in $\frac{1}{2} < \sigma < 1 - \theta$, $s \neq 1 - b$, is a zero of ζ_B (if $\frac{1}{2} < 1 - b < 1 - \theta$, U_N has a pole at $1 - b$, from the subtracted term c' , and the disc around ρ is taken not to contain it).*

- (ii) ρ is a zero of $\zeta_B(s) + N^{1-2s}\zeta_B(1-s)$ for every N if and only if $\zeta_B(1-\rho) = 0$. If ρ is simple and $\zeta_B(1-\rho) \neq 0$, the zero of U_N near ρ is $\rho - N^{1-2\rho}\zeta_B(1-\rho)/\zeta'_B(\rho) + O(N^{2-4\operatorname{Re}\rho} \log N + N^{\theta-\operatorname{Re}\rho})$.

Proof. For $\sigma > \theta$, partial summation gives $\sum_{b_j < N} b_j^{-s} = \zeta_B(s) + \frac{cN^{1-s}}{1-s} + \frac{c'bN^{b-s}}{b-s} + N^{-s}E(N^-) - s \int_N^\infty E(x)x^{-s-1}dx$, and the last two terms are $O(N^{\theta-\sigma})$ uniformly on compact sets. Multiply by N^{s-1} , do the same at $1-s$ (where $1-\sigma > \theta$) multiplied by N^{-s} , and add: $\frac{c}{s} + \frac{c'}{1-s} = \frac{c}{\lambda}$. Hence $U_N(s) = \zeta_B(s) + N^{1-2s}\zeta_B(1-s) + O(N^{\theta-\sigma})$, and on compact subsets of $\frac{1}{2} < \sigma < 1-\theta$ the last two terms tend to 0 uniformly. (i) follows from Hurwitz's theorem, and the distance from Rouché's theorem with $|\zeta_B(s)| \gg |s-\rho|^m$ near ρ . In (ii) the first statement is immediate; for the second, take one Newton step for $g(s) = \zeta_B(s) + N^{1-2s}\zeta_B(1-s)$ from ρ , with $g(\rho) = N^{1-2\rho}\zeta_B(1-\rho)$ and $g'(\rho) = \zeta'_B(\rho) + O(N^{1-2\operatorname{Re}\rho} \log N)$, and add the remainder of U_N . $\square$

The detector needs neither the functional equation nor the Euler product. What the functional equation adds is (ii): for ζ and for the Davenport–Heilbronn function every zero has $f(1-\rho) = 0$, so the zeros of the wave off the carrier are the zeros of the function at every N ; for a system without a functional equation the detected zero drifts, by $-N^{1-2\rho}\zeta_B(1-\rho)/\zeta'_B(\rho)$.

The Beurling control. The systems of [BDR, Thm. 1.3], which are constructed under RH, modify the rational primes [BDR, Sect. 5]. For $\frac{1}{2} < \alpha < \frac{2}{3}$ and $\frac{2\alpha}{\alpha+2} \leq b < \frac{1}{2}$ the numbers $p^{1/b}$, p prime, are added to the primes, and a set $\mathcal{S}$ of rational primes is removed, with $\pi_{\mathcal{S}}(x) = F(x) + O(1)$, $F(x) = \sum_{p \leq x} p^{\alpha-1} \sim \operatorname{Li}(x^\alpha)$ for the deterministic set of [CE11] ([BDR] choose it at random), so that

$$\zeta_B(s) = \frac{\zeta(s)\zeta(s/b)}{\zeta_{\mathcal{S}}(s)}, \quad \zeta_{\mathcal{S}}(s) = \zeta(s+1-\alpha)e^{H(s)},$$

where $H(s) = \sum_p (-\mathbf{1}_{\mathcal{S}}(p) \log(1-p^{-s}) + \log(1-p^{-(s+1-\alpha)}))$ is analytic in $\operatorname{Re} s > \frac{\alpha}{2}$ [CE11, Sect. 13.5]; this is an identity of meromorphic functions there, whatever the zeros of ζ . Such a system has $\Lambda_B \geq 0$ and Chebyshev's identity $\log = \Lambda_B * 1$; under RH, and for the random sets of [BDR], its integers satisfy $N_B(x) = cx + O(x^{b+\varepsilon})$ for every $\varepsilon > 0$ [BDR, Thm. 1.3], the pole at b contributing a term $c'x^b$. It also has every zero of ζ in $\operatorname{Re} s > \frac{\alpha}{2}$ (under RH; unconditionally ρ_1), and a simple real zero at α . We use the system of [CE11] with $\alpha = 0.6$, $b = 0.47$, built up to $3 \cdot 10^7$: $c = 0.69529746$, $c' = 0.23799$, and from H , $\zeta_B(1-\alpha) = \zeta_B(0.4) = -1.6569$ and $\zeta'_B(\alpha) = -11.573$. Chebyshev's identity at the zero ρ_1 that it shares with ζ cannot separate the system from ζ by the size of the integer error alone; this is shown in [CE11], where the construction is given in full.

What is proved for this system. The hypothesis of Proposition 10 is an expansion of N_B with an exponent $\theta < \frac{1}{2}$. For the systems of [BDR], under RH, $N_B(x) = cx + O(x^{b+\varepsilon})$, so the proposition applies with $c' = 0$ and $\theta = b + \varepsilon = 0.47 + \varepsilon$, and it detects the zeros of ζ_B with real part in $(\frac{1}{2}, 0.53 - \varepsilon)$; the expansion with the term $c'x^b$ that [BDR, Sect. 5] obtain under RH has a remainder exponent above 0.4 for these parameters. The real zero $\alpha = 0.6$ lies outside these ranges: to reach it one needs the expansion with some $\theta < 0.4$, which is not known, even under RH. Moreover we do not prove for our deterministic set the integer bound that [BDR] prove for random sets. Below we measure $|E(N)| \leq 0.53N^{0.3}$. So, for this system, the drift observed at α illustrates Proposition 10 under a measured hypothesis; it is not a theorem. (Unconditionally, the logic of the controls is not affected: if RH fails, ζ itself is a system with the Euler product, integers $N(x) = x + O(1)$ and an off-line zero [CE11, Table 11].)

Numerics. For this system we measured $|E(N)| \leq 0.53N^{0.3}$ at 40 cuts $N \in [10^4, 3 \cdot 10^7]$ (we do not prove a bound), so (i) applies to its real zero α with $\theta = 0.3$. Computed from the integers of the system alone, U_N has two real zeros in $[0.56, 0.6]$ from $N \approx 1.1 \cdot 10^6$ on, the lower of which, near 0.57, leaves $[0.56, 0.95]$ near $N \approx 8 \cdot 10^6$; the upper one is the detected zero. It is 0.5817 at $N = 1.12 \cdot 10^6$, 0.5918 at $1.07 \cdot 10^7$ and 0.5938 at $3 \cdot 10^7$, against 0.5919 and 0.5939 for the zero of

$\zeta_B(\sigma) + N^{1-2\sigma}\zeta_B(1-\sigma)$, which appears near α at $N \approx 1.4 \cdot 10^6$ (and 0.5944, 0.5954 from the first-order term of (ii)); the difference is the integer remainder: for $N \geq 10^6$, $U_N(\alpha) - N^{1-2\alpha}\zeta_B(1-\alpha)$ follows the boundary term $2E(N)N^{-\alpha}$ of the proof to within $4 \cdot 10^{-3}$. With the ordinary integers the same wave is negative on $[0.56, 0.95]$ for every cut. For the Davenport–Heilbronn function the zero 0.8085171825 + 85.6993484854*i* of $f(s) + N^{1-2s}f(1-s)$ is the same for $N = 200, 10^3, 10^4, 10^5$ to $3 \cdot 10^{-15}$, and the zero computed from the partial sums of its coefficients converges to it like $N^{-\sigma}$, the size of the partial-sum remainder ($1.8 \cdot 10^{-2}, 4.0 \cdot 10^{-3}, 6.1 \cdot 10^{-4}, 9.5 \cdot 10^{-5}$), without drift (Figure 4). Status: numerical (`c2fr.beurling.py`, `dh_wave_fixed.py`).

The two controls thus fail on opposite sides of (ii). The Davenport–Heilbronn function has the functional equation, and its off-line zero is a fixed zero of the wave. The system of [BDR] has the Euler product, $\Lambda_B \geq 0$ and integers better than $\sqrt{x}$, and the reflected cross-energy of its integers sees its off-line zero, drifting. For ζ the wave is $\zeta \mathcal{K}_N$ (Theorem 8), and the statement that it has no zero off the line, at any height and for any N , is RH.

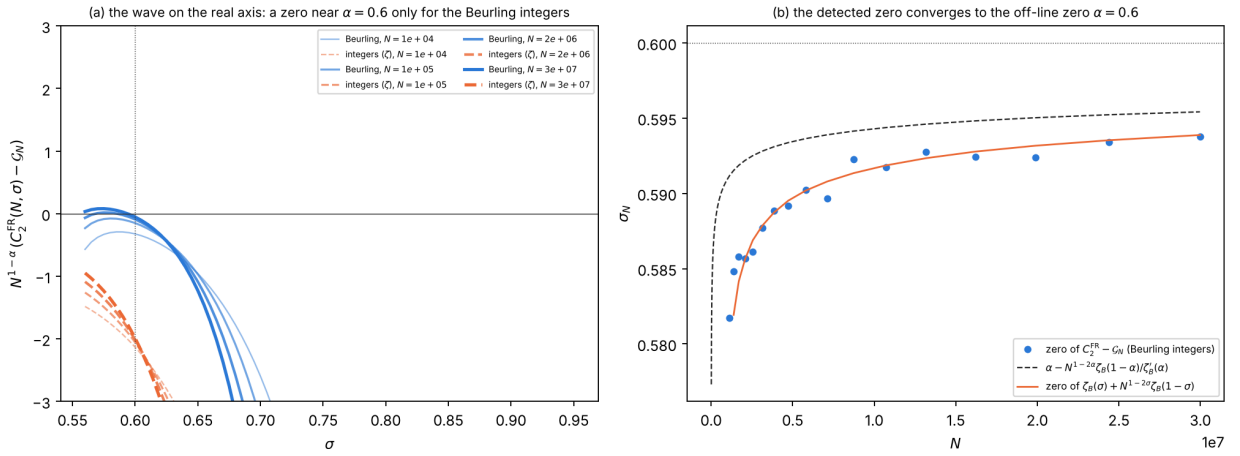


FIGURE 4. The reflected cross-energy of the Beurling system ($\alpha = 0.6$, $b = 0.47$; Proposition 10). (a) $N^{1-\alpha}(C_2^{FR,B}(N, \sigma)$ minus its smooth part) on the real axis, from the integers of the system (solid) and from the ordinary integers (dashed), for $N = 10^4, 10^5, 2 \cdot 10^6, 3 \cdot 10^7$: only the system has a zero, near $\alpha = 0.6$. (b) That zero against N (dots), the zero of $\zeta_B(\sigma) + N^{1-2\sigma}\zeta_B(1-\sigma)$ (solid) and the first-order term of (ii) (dashed); linear axes.

5. WEIL'S FORM ON THE TEST FUNCTIONS OF THE CROSS-ENERGY

5.1. Weil's form. For real $f \in C_c^\infty(\mathbb{R})$ let $\hat{f}(z) = \int f(u)e^{izu}du$, $G = f * \tilde{f}$ with $\tilde{f}(u) = f(-u)$, and $\Phi(z) = \hat{f}(z)\hat{f}(-z)$, the entire function equal to $|\hat{f}(r)|^2$ for real r . Weil's quadratic form [W2] is

$$Q(f) = \Phi\left(\frac{i}{2}\right) + \Phi\left(-\frac{i}{2}\right) + \mathcal{A}_\infty(G) - 2 \sum_{m \geq 2} \Lambda(m)m^{-1/2}G(\log m),$$

$$\mathcal{A}_\infty(G) = \frac{1}{2\pi} \int \Phi(r) \left(\operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{ir}{2} \right) - \log \pi \right) dr,$$

pole + Gamma – primes, and by the explicit formula $Q(f) = \sum_\rho \Phi(\gamma_\rho)$. Weil's criterion states that RH holds if and only if $Q(f) \geq 0$ for every real $f \in C_c^\infty(\mathbb{R})$ [W2], [Bo]. Its numerical behaviour on fixed intervals, with near-null directions, is studied in [CE12].

5.2. The reflected pair summed over the zeros. Let

$$\mu_n = \frac{1}{n} \sum_{k < n} \sqrt{\frac{n}{k}} (\delta_{\log(n/k)} + \delta_{-\log(n/k)}),$$

the integers below n seen from n . Then $\hat{\mu}_n(z) = C_2^{\text{FR}}(n, \frac{1}{2} + iz)$, an even entire function, and $f * \mu_n \in C_c^\infty(\mathbb{R})$, with $\widehat{f * \mu_n} = \hat{f} \hat{\mu}_n$. Since $\frac{1}{2} + i\gamma_\rho = \rho$,

$$(2) \quad Q(f * \mu_n) = \sum_{\rho} \Phi(\gamma_\rho) C_2^{\text{FR}}(n, \rho)^2 = \sum_{\rho} \Phi(\gamma_\rho) \mathcal{G}_n^*(\lambda_\rho)^2,$$

Weil's form with the weight $\mathcal{G}_n^*(\lambda_\rho)^2$ at each zero, a function of λ_ρ alone (Theorem 8(ii)). Under RH, $Q(f * \mu_n) = \sum_{\gamma} \Phi(\gamma) C_2(n, \frac{1}{2} + i\gamma)^2 \geq 0$: the positivity of the reflected pair, summed over the zeros, is Weil's positivity for the test functions $f * \mu_n$, and its prime side is the explicit formula. The identity also holds for weights $w(r) = \Phi(r)$ that are even, entire, nonnegative on $\mathbb{R}$ and of Gaussian decay in horizontal strips, such as the Gaussian pair $w(r) = e^{-(r-T)^2/2D^2} + e^{-(r+T)^2/2D^2}$ that we use below; then f is not compactly supported, and in the computations its kernel is cut where it is negligible. Table 2 compares the two sides for five such windows; the zero side is the small difference of the Gamma term and the prime term, and the prime term is positive in all five cases, so $\Lambda \geq 0$ does not by itself help the positivity: it is the inequality between the Gamma term and the prime term, as in Weil's form. Status: the identity is the explicit formula; the comparison is numerical (`c2_weil_pair.py`).

n	T	D	zeros (first 1000)	pole+ Γ -primes	Gamma term	prime term
5	60	3	2.287350114	2.287350114	4.809402	2.522052
10	60	3	0.3922842607	0.3922842607	3.297325	2.905040
20	100	4	0.03339746178	0.03339746178	2.719853	2.686455
50	100	4	0.003491452142	0.003491452142	1.091944	1.088452
100	200	5	0.001376729390	0.001376729390	1.561078	1.559701

TABLE 2. Identity (2) for the Gaussian pair window at $\pm T$, width D : the sum over the first 1000 zeros and the explicit formula, with its Gamma and prime terms; the pole terms are below 10^{-85} .

At an off-line zero λ is not real, and the four zeros $\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}$ contribute $4 \operatorname{Re} [w(\gamma_\rho) \mathcal{G}_n^*(\lambda)^2]$, which for $w(\gamma_\rho)$ real is $4w(\gamma_\rho)((\operatorname{Re} \mathcal{G}_n^*)^2 - (\operatorname{Im} \mathcal{G}_n^*)^2)$. The energy of deviation $(\operatorname{Im} \mathcal{G}_n^*)^2 \approx (\operatorname{Im} \frac{1}{\lambda})^2$ enters with a minus sign, but the contribution is negative only where $|\operatorname{Re} \mathcal{G}_n^*(\lambda)| < |\operatorname{Im} \mathcal{G}_n^*(\lambda)|$, an interval of length about $4|\delta\gamma|$ around $n = |\lambda|^2 / \operatorname{Re} \lambda \approx \gamma^2$, where $\operatorname{Re} \mathcal{G}_n^*$ changes sign. For $s = 0.7 + i\gamma_1$ it is (194.74, 206.07), with minimum $-(\operatorname{Im} \frac{1}{\lambda})^2 = -1.995 \cdot 10^{-8}$, inside the window (143.1, 333.3) of Proposition 3, whose product dips to $-3.98 \cdot 10^{-6}$. Away from $n \approx \gamma^2$ the term $(\operatorname{Re} \mathcal{G}_n^*)^2$, of order γ^{-4} , dominates $(\operatorname{Im} \mathcal{G}_n^*)^2$, of order $\delta^2 \gamma^{-6}$, and an off-line zero is invisible in the sum. Status: proved; the numbers are numerical (`functional_reflection_check.py`).

Proposition 11 (Weil's criterion on the test functions of the cross-energy). (a) *If RH holds, then $Q(f * \mu_n) \geq 0$ for every real $f \in C_c^\infty(\mathbb{R})$ and every $n \geq 2$.*
 (b) *If, for some $n \geq 2$, $Q(f * \mu_n) \geq 0$ for every real $f \in C_c^\infty(\mathbb{R})$, then $C_2^{\text{FR}}(n, \rho) = \mathcal{G}_n^*(\lambda_\rho) = 0$ at every zero with $\beta > \frac{1}{2}$.*
 (c) *Consequently, for any n_0 , RH holds if and only if $Q(f * \mu_n) \geq 0$ for every real $f \in C_c^\infty(\mathbb{R})$ and every $n \geq n_0$.*

Proof. Write $\gamma_\rho = (\rho - \frac{1}{2})/i = \gamma - i(\beta - \frac{1}{2})$ and $W_n(z) = \hat{\mu}_n(z)^2$, an even entire function, real on $\mathbb{R}$, with $W_n(\gamma_\rho) = C_2^{\text{FR}}(n, \rho)^2$. For $0 \leq \operatorname{Re} s \leq 1$ and $k < N$, $|N^{-s} k^{s-1}| \leq k^{-1}$ and $|N^{s-1} k^{-s}| \leq k^{-1}$, so $|\hat{\mu}_n(z)| \leq 2H_{n-1}$ for $|\operatorname{Im} z| \leq \frac{1}{2}$.

(a) Under RH every γ_ρ is real, $\Phi(\gamma_\rho) = |\hat{f}(\gamma_\rho)|^2 \geq 0$ and $W_n(\gamma_\rho) = C_2(n, \rho)^2 \geq 0$, so every term of (2) is nonnegative; the series converges absolutely by Step 1 below.

(b) *Step 1: a distribution.* For complex $G \in C_c^\infty(\mathbb{R})$ with support in $[-a, a]$, two integrations by parts give $(1 + |z|^2)|\hat{G}(z)| \leq e^{a|\operatorname{Im} z|}(\|G\|_1 + \|G''\|_1)$. Since $|\operatorname{Im} \gamma_\rho| < \frac{1}{2}$ and $\sum_\rho (1 + |\gamma_\rho|^2)^{-1} < \infty$, the series

$$T_n(G) = \sum_\rho W_n(\gamma_\rho) \hat{G}(\gamma_\rho)$$

converges absolutely, and $|T_n(G)| \leq C_n e^{a/2}(\|G\|_1 + \|G''\|_1)$, so T_n is a distribution. For real f , the function $f * \tilde{f}$ has Fourier transform Φ , and (2) reads $Q(f * \mu_n) = T_n(f * \tilde{f})$.

Step 2: positive definiteness. The multiset $\{\gamma_\rho\}$ is invariant under $z \mapsto -z$ ($\rho \mapsto 1 - \rho$) and under $z \mapsto -\bar{z}$ ($\rho \mapsto \bar{\rho}$), and W_n is even with $W_n(\bar{z}) = \overline{W_n(z)}$. Hence $T_n(\check{G}) = \overline{T_n(G)}$ for $\check{G}(x) = G(-x)$, and $T_n(\bar{G}) = \overline{T_n(G)}$. For complex $\varphi = u + iv \in C_c^\infty(\mathbb{R})$ put $\varphi^*(x) = \overline{\varphi(-x)}$; then $\varphi * \varphi^* = u * \tilde{u} + v * \tilde{v} + i(v * \tilde{u} - u * \tilde{v})$ and $(v * \tilde{u})^\vee = u * \tilde{v}$, so $T_n(\varphi * \varphi^*) = Q(u * \mu_n) + Q(v * \mu_n) \geq 0$. Thus T_n is a positive-definite distribution. By the Bochner–Schwartz theorem [GV, Ch. II, §3] it is the Fourier transform of a positive tempered measure; in particular it is tempered, so there are C, k with $|T_n(\psi)| \leq C \sum_{j,l \leq k} \sup_x |x^j \psi^{(l)}(x)|$ for all $\psi \in C_c^\infty(\mathbb{R})$.

Step 3: translates. Fix $g \in C_c^\infty(\mathbb{R})$ and put $F(c) = T_n(g(\cdot - c))$. Then $F(c) = \sum_\rho a_\rho e^{i\gamma_\rho c}$ with $a_\rho = W_n(\gamma_\rho) \hat{g}(\gamma_\rho)$ and $\sum |a_\rho| < \infty$, and Step 2 gives $|F(c)| \leq C'(1 + |c|)^k$.

Step 4: Laplace transform. Since $|e^{i\gamma_\rho c}| = e^{(\beta - \frac{1}{2})c} \leq e^{c/2}$ for $c \geq 0$, Fubini's theorem gives, for $\operatorname{Re} w > \frac{1}{2}$,

$$\int_0^\infty F(c) e^{-wc} dc = \sum_\rho \frac{a_\rho}{w - i\gamma_\rho}, \quad i\gamma_\rho = \beta - \frac{1}{2} + i\gamma.$$

The left side is holomorphic in $\operatorname{Re} w > 0$, by Step 3. The right side converges absolutely and locally uniformly on the complement of the discrete set $\{i\gamma_\rho\}$, so it is meromorphic in $\mathbb{C}$, with a simple pole of residue $m_\rho a_\rho$ at each point $i\gamma_\rho$ (m_ρ the multiplicity). By uniqueness of analytic continuation it has no pole in $\operatorname{Re} w > 0$, so $W_n(\gamma_\rho) \hat{g}(\gamma_\rho) = 0$ for every zero with $\beta > \frac{1}{2}$. Taking $g(x) = h(x) e^{-i\gamma_\rho x}$ with $h \in C_c^\infty$, $\int h = 1$, gives $\hat{g}(\gamma_\rho) = 1$, hence $W_n(\gamma_\rho) = C_2^{\text{FR}}(n, \rho)^2 = 0$.

(c) The direct implication is (a). Conversely, if a zero is off the line, then it or $1 - \bar{\rho}$ has $\beta > \frac{1}{2}$; call it ρ . By Theorem 8, $C_2^{\text{FR}}(n, \rho) = \mathcal{G}_n^*(\lambda_\rho) \rightarrow 1/\lambda_\rho \neq 0$, so $C_2^{\text{FR}}(n, \rho) \neq 0$ for all $n \geq n_1(\rho)$, and (b) with $n \geq \max(n_0, n_1)$ gives an f with $Q(f * \mu_n) < 0$. $\square$

The restriction loses nothing, and (b) says more: positivity of the form for a single n already forces every zero with $\beta > \frac{1}{2}$ onto the zero set of $\lambda \mapsto \mathcal{G}_n^*(\lambda)$. This uses Property A, which makes the weight a function of λ_ρ , and the fact that this function does not vanish at λ_ρ for large n .

5.3. The filter and the arithmetic of the prime side. Two features are specific to these test functions. First, a filter: the weight of a zero on the line is $\mathcal{G}_n^*(\lambda)^2$, and $\mathcal{G}_n(\lambda)$ vanishes at $\lambda = n - \frac{1}{12} + O(n^{-1})$, that is, at $n \approx \gamma^2 + \frac{1}{3}$, the crossing of Lemma 2; so near $n \approx \gamma^2$ the zeros on the line at height $\sqrt{n}$ are switched off, while an off-line zero keeps a weight at least $(\operatorname{Im} \frac{1}{\lambda})^2 \approx 4\delta^2 \gamma^{-6}$. Its neighbours on the line at distance $2\pi/\log \gamma$ keep weights of the same order γ^{-6} , so the filter separates displacements comparable to the spacing. Second, the prime side has an explicit arithmetic. The autocorrelation is

$$\mu_n * \mu_n = \frac{1}{n} \sum_{j, k < n} (jk)^{-1/2} (\delta_{\log(k/j)} + \delta_{\log(j/k)} + \delta_{\log(n^2/jk)} + \delta_{-\log(n^2/jk)}),$$

so, with $G = f * \tilde{f}$, the prime term of $Q(f * \mu_n)$ is

$$(3) \quad \frac{2}{n} \sum_m \frac{\Lambda(m)}{\sqrt{m}} \sum_{j,k < n} \frac{G(\log \frac{mj}{k}) + G(\log \frac{mk}{j}) + G(\log \frac{mjk}{n^2}) + G(\log \frac{mn^2}{jk})}{\sqrt{jk}}.$$

each prime power is compared with the ratios of the integers below n . The exact coincidences $mj = k$ (and $mk = j$) contribute $\frac{4G(0)}{n} \sum_{k < n} \frac{1}{k} \sum_{m|k} \Lambda(m) = \frac{4G(0)}{n} \sum_{k < n} \frac{\log k}{k}$ in closed form: Chebyshev's identity. Everything else is in the near coincidences, $|\log(mj/k)|$ below the width of G , the distribution of the primes at fine scale relative to the ratios of integers, which is information of the strength of pair correlation; this is why Chebyshev's identity with $\Lambda \geq 0$ falls short at the window (Section 5.5). Status: proved.

5.4. The margin at the window. We evaluated (2) at the window, with the Gaussian pair centred at a zero, $T = \gamma_j$, and n the integer nearest to $\gamma_j^2 + \frac{1}{3}$, the filter; the prime term is split into its smooth part $2 \int_0^\infty e^{u/2} g_n(u) du$ and its fluctuating part $P_{\text{fluct}} = 2 \int_0^\infty e^{-u/2} g_n(u) d(\psi(e^u) - e^u)$, where $g_n(x) = \frac{1}{2\pi} \int w(r) C_2^{\text{FR}}(n, \frac{1}{2} + ir)^2 e^{-irx} dr$ (Table 3).

j	n	D	Gamma term	P_{fluct}	$Z = \sum_\rho w \mathcal{G}_n^*(\lambda_\rho)^2$	Z/Gamma	δ^*	accuracy for $\delta = 0.1$
1	200	2.0	0.0194955	0.0196098	$4.0 \cdot 10^{-8}$	$2.1 \cdot 10^{-6}$	0.142	$1.0 \cdot 10^{-6}$
1	200	1.5	0.0103824	0.0104428	$4.2 \cdot 10^{-10}$	$4.0 \cdot 10^{-8}$	0.015	$1.9 \cdot 10^{-6}$
2	442	2.0	0.0145448	0.0145772	$1.6 \cdot 10^{-7}$	$1.1 \cdot 10^{-5}$	–	$1.3 \cdot 10^{-7}$
2	442	1.3	0.0068538	0.0068681	$8.0 \cdot 10^{-9}$	$1.2 \cdot 10^{-6}$	0.207	$2.7 \cdot 10^{-7}$
3	626	1.5	0.0108848	0.0108963	$2.6 \cdot 10^{-8}$	$2.4 \cdot 10^{-6}$	–	$6.0 \cdot 10^{-8}$
4	926	2.0	0.0118394	0.0118485	$3.7 \cdot 10^{-8}$	$3.1 \cdot 10^{-6}$	–	$1.7 \cdot 10^{-8}$
5	1085	2.0	0.0115452	0.0115511	$2.9 \cdot 10^{-8}$	$2.5 \cdot 10^{-6}$	–	$1.1 \cdot 10^{-8}$

TABLE 3. The explicit formula at the window $n \approx \gamma_j^2 + \frac{1}{3}$, $w(r) = e^{-(r-T)^2/2D^2} + e^{-(r+T)^2/2D^2}$, $T = \gamma_j$. Z is computed from the zeros and agrees with pole + Gamma – primes to about 10^{-12} in absolute terms. δ^* is the smallest displacement $\delta = \beta - \frac{1}{2}$ for which replacing ρ_j by the quadruple $\frac{1}{2} \pm \delta \pm i\gamma_j$ makes the zero side negative (–: none below $\frac{1}{2}$); the last column is the relative accuracy of the prime term that detecting $\delta = 0.1$ requires.

Four features stand out. (a) The zero side is a fraction $4 \cdot 10^{-8}$ to 10^{-5} of the Gamma term: the primes cancel the Gamma term to five to eight digits, and almost all of the prime term is its fluctuating part (the smooth part is below 1% of it and of the opposite sign). Read through these test functions, the error term of the prime number theorem reproduces the Gamma factor to that accuracy, and positivity sits in the last digits. (b) The zero ρ_j itself contributes almost nothing, a fraction between $2 \cdot 10^{-10}$ and $5 \cdot 10^{-2}$ of Z , because of the filter; the margin is carried by the neighbouring zeros. (c) An off-line quadruple at height γ_j would contribute about $-16\delta^2\gamma_j^{-6}$, and it makes the zero side negative only if the weight is narrower than the spacing of the neighbours: at γ_1 with $D = 1.5$ every $\delta \geq 0.015$ is detected, at γ_2 with $D = 1.3$ every $\delta \geq 0.21$, and at $\gamma_3, \gamma_4, \gamma_5$ with the widths of the table none below $\frac{1}{2}$; narrower weights need primes up to $n^2 e^{7.5/D}$. (d) Detecting $\delta = 0.1$ requires the prime term to a relative accuracy that falls from 10^{-6} at γ_1 to 10^{-8} at γ_5 , about like γ^{-6} for a fixed width. No upper bound for $|\psi(x) - x|$ can give this, because P_{fluct} is not small: it is the Gamma term, to five to eight digits. Positivity at the window is an exact cancellation between the primes and the Gamma factor, which the explicit formula supplies through the zeros; a proof would have to supply it from the functional equation and the Euler product directly. Status: numerical (`c2_window_margin.py`).

5.5. Chebyshev’s identity as an inequality. The convolution $\log = \Lambda * 1$ is exact, and together with the Euler–Maclaurin expansion of the integers it is equivalent to the explicit formula; the question is whether it gives the one-sided bound $P \leq \text{pole} + \text{Gamma}$ that positivity requires, where $P = \sum_m \Lambda(m)G_n(m)$, $G_n(m) = 2m^{-1/2}g_n(\log m)$. Using only $\Lambda \geq 0$ gives $P \leq P^+ = \sum_m \Lambda(m)G_n(m)^+$. Using $\Lambda \geq 0$ and $\sum_{d|N} \Lambda(d) \leq \log N$ for every $N \leq M$, Chebyshev’s method, gives the value B of a linear program in the unknowns $\lambda(2), \dots, \lambda(M)$, and imposing equality for $N \leq M_0$ fixes $\lambda = \Lambda$ on $[2, M_0]$. A bound proves positivity for the test function only if it exceeds P by less than the margin Z (Table 4).

n	T	D	M	Z	$P^+ - P$	$B - P, M_0 = 1$	$M_0 = \sqrt{M}$	$M_0 = M/10$
5	60	3	1364	2.29	2.27	2.77	0.096	$2.3 \cdot 10^{-7}$
10	60	3	5459	0.392	4.78	5.56	0.68	$3.5 \cdot 10^{-7}$
20	100	4	8034	0.0334	3.74	6.06	2.92	$6.7 \cdot 10^{-3}$

TABLE 4. Chebyshev’s identity used as an inequality, for the weights of Table 2; M covers the support of $g_n(\log m)$, and Z is the margin of the explicit formula.

The bound P^+ alone proves positivity only for the shortest test functions: for $(T, D) = (60, 3)$ for $n \in \{2, 4, 5\}$, for $(100, 4)$ for $n \leq 5$ and $n = 7$, for $(200, 5)$ for $n \leq 7$ and $n = 10$, and it fails for all larger n that we tried ($n \leq 12$), as Weil’s form is positive without the primes only for supports that admit almost none. The linear program does better only when Λ is fixed on almost the whole effective support of g_n ($M_0 = M/10$; beyond it $|G_n(m)|$ is below $2 \cdot 10^{-4}$). At the window the gap is enormous: for $\gamma_1, \gamma_2, \gamma_3$ with $D = 2$, $P^+ = 1.75, 2.22, 3.05$ against $P = 0.0195, 0.0145, 0.0170$, so the prime sum itself cancels to about 99%, and $P^+ - P$ exceeds the margin by a factor between $1.4 \cdot 10^7$ and $4.3 \cdot 10^7$. The convolution identity certifies positivity at the window only when the primes are known exactly on the support of the test function, that is, by computation. This is the familiar limitation of sieve-type arguments, which use $\Lambda \geq 0$ and divisor sums and cannot see the sign pattern that carries the cancellation. Evaluated at a single zero instead of integrated against a weight, the identity is a rewriting of the prime error in every Beurling system whose integers are better than $\sqrt{x}$ [CE11]. Status: numerical (`c2_convolution_lp.py`).

5.6. Where the positivity lives. At the window the prime powers below n meet the integers below n through exact divisibility, where Chebyshev’s identity fixes Λ , and the prime powers in $[n, M]$, $M = e^{2 \log n + 7.5/D}$, meet them only through the near coincidences $mjk \approx n^2$: the dual, or hyperbola, range. We split the prime term accordingly and replaced Λ on $[n, M]$, keeping it exact below n , by the smooth density 1, by three Cramér random sets (each $m \geq n$ taken with probability $1/\log m$, weight $\log m$) and by the Beurling perturbation of Section 4 (the primes of $\mathcal{S}$ removed, the numbers $p^{1/b}$ added), all with $\Lambda \geq 0$ (Table 5). The primes below n alone give $P(m < n)$ within 2.4% of pole + Gamma at the first zero and within 0.06 to 0.23% at the next four; the dual range contributes between $-4.6 \cdot 10^{-4}$ and $3.3 \cdot 10^{-5}$ and restores positivity with a margin of $3 \cdot 10^{-8}$ to $1.6 \cdot 10^{-7}$, a fraction $2 \cdot 10^{-6}$ to $1.1 \cdot 10^{-5}$ of pole + Gamma, which the sum over the zeros reproduces to four digits. No substitute reproduces it: the smooth density gives $-1.4 \cdot 10^{-4}$ to $+6.4 \cdot 10^{-5}$ (negative in four cases), the Cramér sets errors of order 10^{-2} , the Beurling perturbation $-2 \cdot 10^{-3}$ to $+1.3 \cdot 10^{-3}$, and each substitute is negative for at least one of the five test functions. So the restriction of Weil’s criterion to the test functions of C_2^{FR} places the whole decision in the fluctuation of the primes in the dual range $[n, n^2 e^{O(1/D)}]$. Neither $\Lambda \geq 0$, nor the density of the primes, nor Chebyshev’s identity, nor a random or Beurling model of the same density does this; only the true primes do, and their fluctuation in that range is the explicit formula of the zeros. Status: numerical (`window_prime_regions.py`).

j	n	pole+ Γ	$P(m < n)$	$P(m \geq n)$	margin	smooth	Cramér (3 seeds)	Beurling
1	200	0.0194955	0.0199538	$-4.58 \cdot 10^{-4}$	$4.0 \cdot 10^{-8}$	$-1.4 \cdot 10^{-4}$	$-5.2, -5.4, +0.8 (10^{-2})$	$-2.0 \cdot 10^{-3}$
2	442	0.0145448	0.0145113	$+3.33 \cdot 10^{-5}$	$1.6 \cdot 10^{-7}$	$+6.4 \cdot 10^{-5}$	$-1.2, +1.2, +0.6 (10^{-2})$	$+1.3 \cdot 10^{-3}$
3	626	0.0170472	0.0170731	$-2.60 \cdot 10^{-5}$	$1.3 \cdot 10^{-7}$	$-1.9 \cdot 10^{-5}$	$+2.6, +2.3, -1.0 (10^{-2})$	$-5.1 \cdot 10^{-5}$
4	926	0.0118394	0.0118470	$-7.6 \cdot 10^{-6}$	$3.7 \cdot 10^{-8}$	$-7.1 \cdot 10^{-8}$	$-2.0, -3.0, -0.9 (10^{-2})$	$+3.3 \cdot 10^{-4}$
5	1085	0.0115452	0.0115608	$-1.56 \cdot 10^{-5}$	$2.9 \cdot 10^{-8}$	$-8.9 \cdot 10^{-6}$	$-0.6, -2.9, -2.5 (10^{-2})$	$+6.4 \cdot 10^{-4}$

TABLE 5. Weil's form on the test functions $f * \mu_n$ at the window of the first five zeros ($D = 2$): the prime term split at $m = n$, the margin pole + Gamma - P with the true primes, and the margin when Λ on $[n, M]$ is replaced by the smooth density, by Cramér random sets or by the Beurling perturbation. With the true primes the margin is positive and equals the zero side; each substitute is negative for at least one test function.

5.7. The dual range traded for small primes. The dual range is not independent data.

Proposition 12 (the dual range traded for small primes). *Let $f \in C_c^\infty(\mathbb{R})$ be real with support in $[-a, a]$, $G = f * \tilde{f}$, $n \geq 2$, and $\mathcal{T}_n(d) = \sum_{mjk=d, \max(j,k) \geq n} \Lambda(m)$. The prime term of $Q(f * \mu_n)$ equals*

$$\frac{2}{n} \sum_m \frac{\Lambda(m)}{\sqrt{m}} \sum_{j,k < n} \frac{2G(\log \frac{mj}{k}) + G(\log \frac{mn^2}{jk})}{\sqrt{jk}} + \frac{2}{n} \sum_d \frac{G(\log \frac{d}{n^2})}{\sqrt{d}} \left(\frac{1}{2} \tau(d) \log d - \mathcal{T}_n(d) \right),$$

and every value $\Lambda(m)$ in it has $m \leq ne^{2a}$, whereas the prime term (3) computed directly uses Λ up to $n^2 e^{2a}$.

Proof. The families $G(\log \frac{mj}{k})$ and $G(\log \frac{mk}{j})$ in (3) are equal by the symmetry $j \leftrightarrow k$. In the family $G(\log \frac{mj}{k})$ the weights combine, $m^{-1/2}(jk)^{-1/2} = d^{-1/2}$ with $d = mjk$, so it is $\frac{2}{n} \sum_d d^{-1/2} G(\log \frac{d}{n^2}) \sum_{mjk=d, j,k < n} \Lambda(m)$. Without the truncation $j, k < n$, Chebyshev's identity gives $\sum_{mjk=d} \Lambda(m) = (\log * 1)(d) = \frac{1}{2} \tau(d) \log d$, and the truncation removes $\mathcal{T}_n(d)$; if $\max(j, k) \geq n$ and $d \leq n^2 e^{2a}$, then $m = d/(jk) \leq ne^{2a}$. In the other families $G \neq 0$ forces $m \leq e^{2a} k/j < ne^{2a}$ or $m \leq e^{2a} jk/n^2 < e^{2a}$. $\square$

So the identity $\Lambda * 1 * 1 = \log * 1$ replaces the primes of the dual range by the divisor function near n^2 , an integer quantity, and by the small primes $m \leq ne^{2a}$ weighted by the lattice points (j, k) that cross the edge $\max(j, k) = n$; it is the analogue, for Weil's form, of the approximate functional equation, and it halves the length. At the window of γ_1 ($n = 200$, $D = 2$, Gaussian kernel cut at $|x| \leq 3.5$, so $e^{2a} \approx 33$) the four families give the prime term 0.0194954637, the value of the explicit formula; the divisor part is $-5.3 \cdot 10^{-9}$, equal to its smooth main term up to $1.8 \cdot 10^{-11}$, because the kernel oscillates with the ordinate of a zero; and the dual family, 0.0036971, is minus the edge term up to the divisor part, computed from the prime powers up to $6623 = 33n$ instead of $1.3 \cdot 10^6 = 33n^2$. At γ_2 ($n = 442$) the same holds with the prime powers up to 14637 and a divisor part $-1.5 \cdot 10^{-9}$. The positivity of $Q(f * \mu_n)$ is therefore an inequality for the primes up to ne^{2a} with explicit weights built from the integers below n and their edge, with a required margin of $2 \cdot 10^{-6}$ to 10^{-5} of pole + Gamma at the window. The edge of the integers is the sawtooth, whose Poisson summation is the functional equation, so the inequality couples the Euler product and the functional equation, as the controls require [CE11], [CE17]; for each n it is a finite computation, and for all n it is RH. Status: Proposition 12 is proved; the numbers are numerical (`dual_range_hyperbola.py`).

6. CONTROLS: WEIL'S FORM WITH THE DATA OF A BEURLING SYSTEM

Proposition 11 is a criterion for ζ : the functional equation turns the form into a sum over the zeros. To see what the same functional says about a system without a functional equation, we evaluated pole + Gamma – primes on $f * \mu_n$ at 83 values $2 \leq n \leq 1000$ with the primes Λ_B of the system of Section 4 and with those of two controls that separate its two ingredients: $\zeta(s)/\zeta_S(s)$, the primes of $\mathcal{S}$ removed, which has the real zero 0.6; and $\zeta(s)\zeta(s/b)$, the numbers $p^{1/b}$ added, which has no zero with real part $> \frac{1}{2}$ but has the zeros $b\rho$ of $\zeta(s/b)$ on $\operatorname{Re} s = 0.235$. We used two windows: the Gaussian pair at $\pm\gamma_1$ of Table 5 (f even), and $\Phi(r) = r^2 e^{-r^2/8}$ (f odd), which is negative at $r = \pm i\delta$, the direction in which Weil's form sees a real zero $\frac{1}{2} + \delta$ (Table 6, Figure 5).

primes (and integers in μ_n)	window at $\pm\gamma_1$ (f even)	$\Phi(r) = r^2 e^{-r^2/8}$ (f odd)
ζ	> 0 for every n ; equals the sum over the zeros to $5 \cdot 10^{-9}$	sum over the zeros 10^{-8} to 10^{-13} ; computed to 10^{-10}
the system (integers of ζ)	first < 0 at $n = 8$; < 0 at 4 of 83	< 0 from $n = 2$, at 83 of 83; -0.761 at $n = 1000$
the system (its integers)	first < 0 at $n = 7$; < 0 at 69 of 83	< 0 from $n = 2$, at 82 of 83
ζ/ζ_S (zero at 0.6)	never < 0	< 0 at every n ; -1.244 at $n = 1000$
$\zeta(s)\zeta(s/b)$ (no zero with $\operatorname{Re} s > \frac{1}{2}$)	first < 0 at $n = 3$; < 0 at 70 of 83	> 0 from $n = 3$; $+0.483$ at $n = 1000$

TABLE 6. Weil's functional of ζ , pole + Gamma – primes, on the test functions $f * \mu_n$ at 83 values $2 \leq n \leq 1000$, with the primes of ζ , of the Beurling system ($\alpha = 0.6$, $b = 0.47$) and of two controls; $D = 2$.

With ζ the form is the sum over the zeros and is positive. With the even window the system is first negative at $n = 7$ with its own integers in μ_n and at $n = 8$ with those of ζ , but this is not its off-line zero: the control with the zero at 0.6 is never negative, and the control without it is negative from $n = 3$ on, through the zeros $b\rho_k$ at heights $0.47\gamma_k$ (14.30, 15.48) inside the window. With the odd window the sign follows the zero at 0.6: the two systems that have it are negative at every $n \geq 2$, the control without it is positive from $n = 3$ on, and the system is the sum of the two effects ($-1.244 + 0.483 = -0.761$ against -0.7614 at $n = 1000$). The smooth density of the removed primes, $2 \int_0^\infty g(x) e^{(\alpha - \frac{1}{2})x} dx$, gives the sign, the trend and half the size; the rest comes from the small primes of $\mathcal{S}$ (3, 7, 13, ...). The violations are of order 1, against at most 10^{-8} for ζ : they measure the distance of Λ_B from Λ from the prime 3 on, not a fine effect of the zero. Without a functional equation the form is not a sum over the zeros of the system, and the controls show that it can be negative without a zero to the right of the line and positive with one. Proposition 11 is a criterion for ζ only; what separates ζ from the system cleanly is the drift of Proposition 10(ii). Status: numerical (`weil_bdr_scan.py`).

7. THE PRECISION THE PRIMES MUST SUPPLY

At the window of a zero γ_j the terms of Weil's form on $f * \mu_n$ have very different sizes. The pole and Gamma terms see $C_2^{\text{FR}}(n, \frac{1}{2} + ir)^2$ on the whole window, where by Theorem 8(i) the wave $2Z(r) \cos(r \log n - \vartheta(r))/\sqrt{n}$ dominates the smooth part, so they are of size n^{-1} times a local mean value of Z^2 . The zeros see only $\mathcal{G}_n^*(\lambda_\rho)^2$, because the wave vanishes there. The ratio

$$\mathcal{R}_j(n) = \frac{Q(f * \mu_n)}{\text{pole} + \Gamma} = \frac{\text{pole} + \Gamma - \text{primes}}{\text{pole} + \Gamma}$$

is therefore the relative precision with which the prime term must cancel the Gamma term for the positivity of the form to be read from the primes. For a cut $n \asymp \gamma_j^2$ away from the filter,

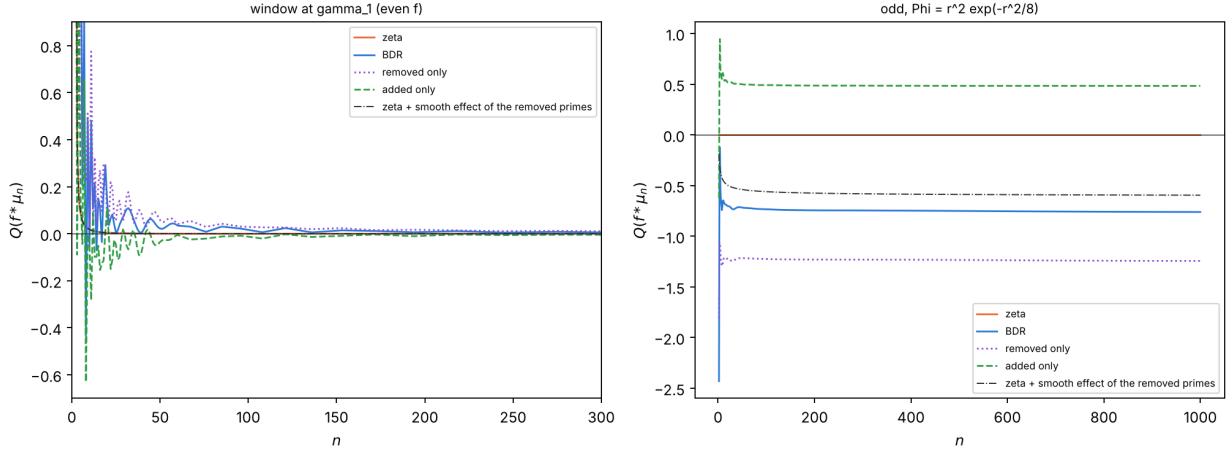


FIGURE 5. Weil's functional of ζ on $f * \mu_n$, $2 \leq n \leq 1000$, with the primes of ζ , of the Beurling system, of ζ/ζ_S (removed only, zero at 0.6) and of $\zeta(s)\zeta(s/b)$ (added only, no zero with real part $> \frac{1}{2}$); all with the integers of ζ in μ_n ; the dash-dotted line is ζ plus the smooth effect of the removed primes. Left: the window at $\pm\gamma_1$ (f even), $n \leq 300$. Right: $\Phi(r) = r^2 e^{-r^2/8}$ (f odd). Linear axes.

$\mathcal{G}_n(\lambda_\rho) \asymp \gamma_j^{-2}$ for the zeros in the window and $\mathcal{R}_j \asymp \gamma_j^{-2}$ up to logarithms. At the filter $n \approx \gamma_j^2 + \frac{1}{3}$, $\mathcal{G}_n$ vanishes at γ_j , its value at a zero at distance Δ is about $2\Delta\gamma_j^{-3}$, and $\mathcal{R}_j \asymp \gamma_j^{-4}$ up to logarithms. These predictions are heuristic.

We computed $\mathcal{R}_j$ for 52 windows $\gamma_1 \leq \gamma_j \leq \gamma_{960}$ ($D = 2$), with the Hardy form of Theorem 8(i) for the Gamma term (it agrees with the partial sums to $4.1 \cdot 10^{-10}$) and the zeros for Q ; at γ_2 it reproduces Table 5 (0.0145448 and $1.6088 \cdot 10^{-7}$). A least-squares fit for $\gamma_j > 100$ gives $\mathcal{R}_j \propto \gamma_j^{-4.03}$ at the filter and $\propto \gamma_j^{-2.01}$ at the cut $n \approx \gamma_j^2/2$; Figure 6 shows the scaled ratios, which stay between 0.08 and 5, and between 0.02 and 0.26. Status: numerical (`margin_vs_height.py`).

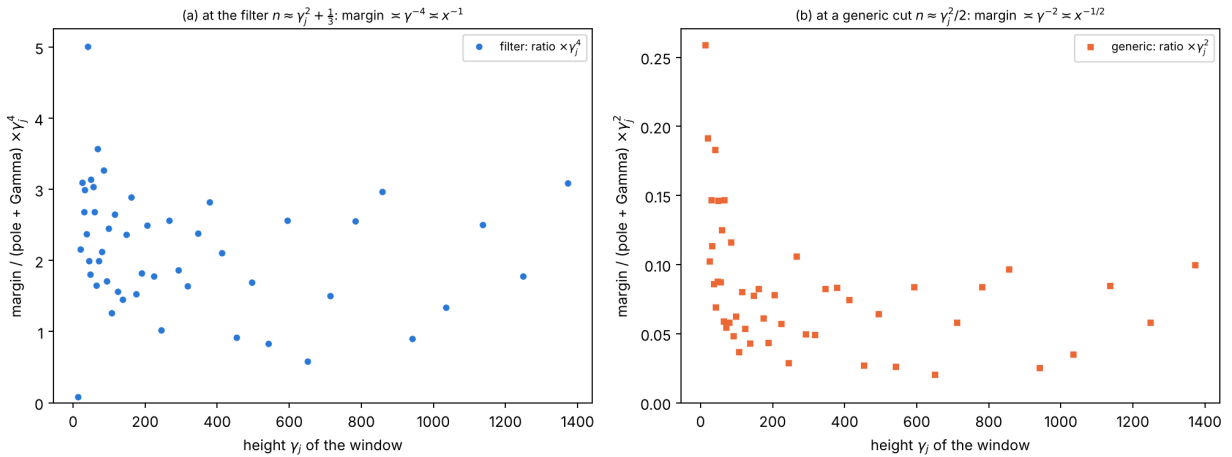


FIGURE 6. The relative margin $\mathcal{R}_j(n)$ of Weil's form on $f * \mu_n$ at the window of γ_j ($D = 2$), for 52 zeros $\gamma_1 \leq \gamma_j \leq \gamma_{960}$, scaled by the predicted powers. (a) At the filter $n \approx \gamma_j^2 + \frac{1}{3}$, $\gamma_j^4 \mathcal{R}_j$. (b) At the cut $n \approx \gamma_j^2/2$, $\gamma_j^2 \mathcal{R}_j$. Both stay bounded; linear axes.

Remark 13 (the precision required). The prime powers that enter go up to $x = n^2 e^{7.5/D} \asymp \gamma^4$, so the measured exponents say that the weighted prime sum must be known to relative accuracy about $x^{-1/2}$ at a generic cut and x^{-1} at the filter. A bound for $\psi(x) - x$ of the strength of RH gives the first by partial summation only up to logarithmic factors and does not give the second, and no unconditional estimate gives a power saving: the best known error term in the prime number theorem, of Vinogradov–Korobov type, is $x \exp(-c(\log x)^{3/5}(\log \log x)^{-1/5})$. In this sense the positivity of Weil’s form on the test functions of the cross-energy cannot be obtained at large height from estimates for the primes; it has to come from an exact identity. This quantifies, for the reflected cross-energy, the near-null directions of Weil’s form on fixed intervals [CE12] and the rigidity of positivity at large height noted in [CE12]. Status: heuristic; it is a reading of the measured exponents, not a theorem.

8. THE LIMIT $n \rightarrow \infty$: A HANKEL–JACOBI FORM OF THE CRITERION

For $0 < \sigma < 1$, $C_2^{\text{FR}}(n, s) \rightarrow 1/\lambda$ as $n \rightarrow \infty$, since $n^{s-1}\zeta(s)$ and $n^{-s}\zeta(1-s)$ tend to 0; at a zero the convergence is through $\mathcal{G}_n^*(\lambda_\rho)$ alone, with no oscillating term. Put

$$t_\rho = \frac{1}{\lambda_\rho} = \frac{1}{\rho(1-\rho)} = \lim_{n \rightarrow \infty} C_2^{\text{FR}}(n, \rho),$$

the kernel of the cross-energy at the zero ($t_\rho = \lim_n C_2(n, \rho) = 1/(\frac{1}{4} + \gamma^2)$ on the line [CE1]). Under RH, $t_\rho \in (0, 1/\lambda_1]$, $\lambda_1 = \frac{1}{4} + \gamma_1^2 = 200.0026 \dots$; an off-line zero has $\text{Im } t_\rho = -\gamma(1-2\beta)/|\lambda_\rho|^2 \neq 0$.

Proposition 14 (the limit of the restricted criterion). *For every real $f \in C_c^\infty(\mathbb{R})$,*

$$\lim_{n \rightarrow \infty} Q(f * \mu_n) = \sum_{\rho} \Phi(\gamma_\rho) t_\rho^2,$$

the series converging absolutely, and RH holds if and only if this limit is ≥ 0 for every real $f \in C_c^\infty(\mathbb{R})$.

Proof. By (2), $Q(f * \mu_n) - \sum_{\rho} \Phi(\gamma_\rho) t_\rho^2 = \sum_{\rho} \Phi(\gamma_\rho) (C_2^{\text{FR}}(n, \rho)^2 - t_\rho^2)$, and we bound the right side termwise. Since f has compact support, $|\Phi(z)| \leq C_f(1+|z|)^{-4}$ for $|\text{Im } z| \leq \frac{1}{2}$. Also $|C_2^{\text{FR}}(n, \rho)| \leq 2H_{n-1} \leq 2 + 2 \log n$ (proof of Proposition 11), and $|t_\rho| = |\lambda_\rho|^{-1} \leq \gamma^{-2} < 1$, because $|\lambda_\rho|^2 = (\gamma^2 + \beta(1-\beta))^2 + \gamma^2(1-2\beta)^2 \geq \gamma^4$. Put $Y = n^{1/4}$.

Zeros with $|\gamma| > Y$. Each term is at most $5C_f(1 + \log n)^2(1 + |\gamma|)^{-4}$. Since there are $O(\log T)$ zeros with $T < |\gamma| \leq T + 1$, their sum is $O((\log n)^2 Y^{-3} \log Y) = o(1)$.

Zeros with $|\gamma| \leq Y$. By Theorem 8, $C_2^{\text{FR}}(n, \rho) = \mathcal{G}_n^*(\lambda_\rho)$ and

$$\left| \mathcal{G}_n^*(\lambda_\rho) - \frac{1}{\lambda_\rho} \right| \leq \frac{1}{n} + \frac{1}{12n^2} + \left(\frac{1}{120} + \frac{|\lambda_\rho|}{80} \right) n^{-4} + 1.1 \cdot 10^{-4} (|\rho| + 7)^6 n^{-6} \leq \frac{2}{n}$$

for $n \geq n_1$, uniformly in these zeros, because $|\lambda_\rho| = |\rho||1-\rho| \leq (Y+1)^2$ and $(|\rho| + 7)^6 \leq (Y+8)^6 = O(n^{3/2})$. Hence $|C_2^{\text{FR}}(n, \rho)^2 - t_\rho^2| \leq \frac{2}{n}(2 + \frac{2}{n}) \leq \frac{5}{n}$, and the sum of these terms is at most $\frac{5}{n} \sum_{\rho} |\Phi(\gamma_\rho)| = O(n^{-1})$.

So the difference tends to 0, and $\sum_{\rho} |\Phi(\gamma_\rho) t_\rho^2| < \infty$. For the equivalence, if RH holds then $t_\rho = 1/(\frac{1}{4} + \gamma^2) > 0$ and every term of the limit is $|\hat{f}(\gamma)|^2 t_\rho^2 \geq 0$. Conversely, the limit equals $T_\infty(f * \hat{f})$ with $T_\infty(G) = \sum_{\rho} t_\rho^2 \hat{G}(\gamma_\rho)$. The weights $w(z) = (\frac{1}{4} + z^2)^{-2}$, evaluated at $z = \gamma_\rho$ (note $\lambda_\rho = \frac{1}{4} + \gamma_\rho^2$), are bounded, even and satisfy $w(\bar{z}) = \overline{w(z)}$. So Steps 1–4 of the proof of Proposition 11 apply verbatim and give $t_\rho^2 \hat{g}(\gamma_\rho) = 0$ for every zero with $\beta > \frac{1}{2}$ and every g , which is impossible because $t_\rho \neq 0$. $\square$

Both the limit and the equivalence are proved. The test functions of the limit are $f * k$ with $\hat{k}(r) = 1/(\frac{1}{4} + r^2)$, $k(x) = e^{-|x|/2}$, the kernel of the cross-energy; they are not compactly supported,

and the prime side of the limit is a regularized one (the pole term at $r = \pm \frac{i}{2}$ diverges like $\log^2 n$ and is compensated by the primes). The limit is a sum over the zeros of a function of t_ρ alone, and taking polynomials in t_ρ instead of Φ gives a criterion with finite matrices whose entries do not involve the zeros.

The function $\Xi(\lambda) = \xi(s)$, $\lambda = s(1-s)$, is well defined because $\xi(s) = \xi(1-s)$; it is entire of order $\frac{1}{2}$ in λ , real for real λ , with zeros λ_ρ , one for each pair $\{\rho, 1-\rho\}$, and $\sum |\lambda_\rho|^{-1} < \infty$. So $\Xi(\lambda) = \xi(1) \prod (1 - \lambda/\lambda_\rho)$ over the pairs, and for $|\lambda| < \lambda_1$

$$\log \frac{\Xi(\lambda)}{\xi(1)} = - \sum_{k \geq 1} \frac{S_k}{k} \lambda^k, \quad S_k = \sum_{\{\rho, 1-\rho\}} t_\rho^k, \quad S_1 = \sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma_0}{2} - \frac{1}{2} \log 4\pi.$$

The S_k are therefore determined by the Taylor coefficients of ξ at $s = 1$, that is, by γ_0 , $\log \pi$, the Stieltjes constants and the values of the digamma function and its derivatives at $\frac{1}{2}$, without the zeros.

Proposition 15 (the Hankel–Jacobi criterion). *Let $s_k = S_{k+2}$ ($k \geq 0$), $H_m = (s_{i+j})_{0 \leq i, j < m}$ and $D_m = \det H_m$, $D_0 = 1$. The s_k are real.*

- (i) *If RH holds, then $D_m > 0$ for every $m \geq 1$.*
- (ii) *If H_m is positive semidefinite for every $m \geq 1$, then RH holds.*
- (iii) *Hence RH holds if and only if $D_m > 0$ for every $m \geq 1$. Equivalently, since $s_0 = D_1 = S_2 = 3.7100636 \dots \cdot 10^{-5} > 0$: RH holds if and only if the recurrence coefficients*

$$\beta_m = \frac{D_{m+1} D_{m-1}}{D_m^2} \quad (m \geq 1)$$

of the monic orthogonal polynomials of (s_k) are all defined and positive; that is, if and only if the Jacobi matrix, with diagonal entries α_m and off-diagonal entries $\sqrt{\beta_m}$, is a real symmetric matrix.

- (iv) *Under RH, the eigenvalues of the $m \times m$ section of the Jacobi matrix are the nodes of the m -point Gauss quadrature for $\nu = \sum t_\rho^2 \delta_{t_\rho}$ (one atom for each pair, with multiplicity), and for each fixed k the k -th largest of them tends to $1/(\frac{1}{4} + \gamma_k^2)$ as $m \rightarrow \infty$, where $0 < \gamma_1 < \gamma_2 < \dots$ are the distinct positive ordinates.*

Proof. Reality and size. The multiset $\{\lambda_\rho\}$ over the pairs is closed under conjugation, because $\lambda_\rho = \overline{\lambda_\rho}$, so every S_k is real. Let $M = \max_\rho |t_\rho| = 1/\min |\lambda_\rho|$ and $S^{\text{abs}} = \sum |t_\rho| < \infty$ (over the pairs; $\sum |\lambda_\rho|^{-1} < \infty$ since Ξ has order $\frac{1}{2}$). Then $|s_k| \leq M^{k+1} S^{\text{abs}}$.

(i) Under RH every t_ρ lies in $(0, 1/\lambda_1]$, and $s_k = \int x^k d\nu$ for the positive measure ν , which has infinitely many atoms. For a nonzero real vector c , $c^T H_m c = \int |\sum_j c_j x^j|^2 d\nu > 0$, since a nonzero polynomial of degree $< m$ cannot vanish at infinitely many atoms. So H_m is positive definite and $D_m > 0$.

(ii) By Hamburger's theorem [Ak, Ch. 2, §1], a real sequence whose Hankel matrices are all positive semidefinite is the moment sequence of a positive measure σ on $\mathbb{R}$. Its support lies in $[-M, M]$: for $M' > M$, $\sigma(\{|x| > M'\}) M'^{2k} \leq s_{2k} \leq M^{2k+1} S^{\text{abs}}$ for every k . In particular σ is determined by its moments. For $|w| < 1/M$,

$$\int \frac{d\sigma(x)}{1-wx} = \sum_{k \geq 0} s_k w^k = \sum_{\rho} \frac{t_\rho^2}{1-wt_\rho} = \sum_{\rho} \frac{1}{\lambda_\rho(\lambda_\rho - w)},$$

the double series converging absolutely. The left side is holomorphic on the connected set $\Omega = \mathbb{C} \setminus ((-\infty, -1/M] \cup [1/M, \infty))$. The right side converges absolutely and locally uniformly off the discrete set $\{\lambda_\rho\}$, so it is meromorphic in $\mathbb{C}$, with a simple pole of residue $-m_\rho/\lambda_\rho \neq 0$ at each λ_ρ ; distinct pairs have distinct λ_ρ , since λ determines $\{s, 1-s\}$. The two sides agree on

$|w| < 1/M$, hence on Ω minus the poles, so the right side has no pole in Ω : every λ_ρ is real. Since $\lambda_\rho = \beta(1 - \beta) + \gamma^2 + i\gamma(1 - 2\beta)$ and ζ has no zeros on $(0, 1)$, $\beta = \frac{1}{2}$.

(iii) The first equivalence follows from (i), (ii) and Sylvester's criterion. If $D_m \neq 0$ for all m , the monic orthogonal polynomials of the moment functional exist and satisfy $p_{m+1} = (x - \alpha_m)p_m - \beta_m p_{m-1}$ with real α_m and $\beta_m = D_{m+1}D_{m-1}/D_m^2$ [Ch, Ch. I, §4]. From $D_{m+1} = \beta_m D_m^2/D_{m-1}$ and $D_1 = s_0 > 0$, all D_m are positive if and only if all β_m are positive. The value of S_2 is the one computed from ξ below.

(iv) The zeros of p_m are the eigenvalues of the $m \times m$ section of the Jacobi matrix and the nodes of the m -point Gauss quadrature [Ch, Ch. I, §6]. Under RH, ν is a positive measure on $[0, t_{\rho_1}]$ whose support is $\{1/(\frac{1}{4} + \gamma_k^2) : k \geq 1\} \cup \{0\}$, with 0 as the only accumulation point. If $x_{m,1} > x_{m,2} > \dots$ are the zeros of p_m , then $x_{m,k}$ increases to a limit ξ_k as $m \rightarrow \infty$. Every point of the support above $\lim_k \xi_k$ is one of the ξ_k , and conversely [Ch, Ch. II, §4]. The limit $\lim_k \xi_k$ is the largest limit point of the support [Ch, Ch. II, §4], which here is 0; so $\xi_k = 1/(\frac{1}{4} + \gamma_k^2)$. $\square$

Status: proved; classical in substance (see below). The computations that follow check the positivity of β_m , a necessary condition for RH, numerically (without interval arithmetic) for $m \leq 43$; this is neither a proof for all m nor a rigorous verification of RH up to any height.

size m	$\gamma = (1/t - \frac{1}{4})^{1/2}$ for the five largest eigenvalues of the Jacobi matrix, from ξ alone
5	14.1347256812, 21.0460880624, 25.8351028389, 36.8147548709, 73.779087465
10	14.1347251417, 21.0220396389, 25.0108582694, 30.430358594, 33.0247999988
20	14.1347251417, 21.0220396388, 25.0108575801, 30.4248761259, 32.9350615877
40	14.1347251417, 21.0220396388, 25.0108575801, 30.4248761259, 32.9350615877
two consecutive zeros on the line replaced by an off-line pair at their mean height, real parts $\frac{1}{2} \pm \delta$:	
zeros 3, 4 (27.72)	first $\beta_m < 0$ at $m = 8$ ($\delta = 0.1$) and $m = 9$ ($\delta = 0.01$)
zeros 10, 11 (51.37)	$m = 20$ ($\delta = 0.1$), $m = 23$ ($\delta = 0.01$)
zeros 20, 21 (78.24)	$m = 38$ ($\delta = 0.1$), $m = 41$ ($\delta = 0.01$)
zeros 30, 31 (102.52)	no $\beta_m < 0$ for $m \leq 43$ ($\delta = 0.1$ and 0.001)

TABLE 7. Proposition 15 computed from the Taylor coefficients of ξ at $s = 1$ (200 and 300 digits). Top: the zeros recovered as the spectrum of the Jacobi matrix. Bottom: the first non-positive Hankel matrix when an off-line pair is inserted.

Numerics. We computed the Taylor coefficients of Ξ at $\lambda = 0$ by Cauchy's formula on $|\lambda| = 100$ with 400 points and 200 and 300 digits (mpmath), and the S_k , $k \leq 90$, by Newton's identities. S_1 agrees with $1 + \frac{\gamma_0}{2} - \frac{1}{2} \log 4\pi = 0.0230957089661210338143 \dots$ to 30 digits; S_2, S_3, S_5, S_{10} agree with the sums over the first 1000 zeros to the size of the omitted tail ($2.9 \cdot 10^{-6}$, $2.2 \cdot 10^{-10}$, $1.9 \cdot 10^{-14}$ and $3.9 \cdot 10^{-14}$ in relative terms). By Chebyshev's algorithm for the moments of ν (scaled by 200), $\beta_m > 0$ for every $m \leq 43$, the two precisions agreeing to $4 \cdot 10^{-27}$; the recurrence coefficients decrease, $\alpha_0 = 0.7745$, $\alpha_{20} = 0.0332$, $\alpha_{43} = 0.0137$, $\beta_1 = 0.1115$, $\beta_{20} = 3.4 \cdot 10^{-4}$, $\beta_{43} = 5.1 \cdot 10^{-5}$, and for $3 \leq m \leq 43$ the ratio $4\beta_m/\alpha_m^2$ stays between 0.84 and 1.48 (it is 2.10 and 1.60 at $m = 1, 2$), so that the Jacobi matrix looks locally like that of an interval $[0, 2\alpha_m]$. Its spectrum returns the zeros: the 20×20 section gives the first five ordinates to ten digits, and the 40×40 section the first eight (Table 7). Status: numerical (`hanke1-jacobi.py`).

The bottom of Table 7 shows what the criterion sees. We replaced two consecutive zeros on the line by an off-line pair at their mean height, the way a pair would leave the line, and recomputed the Jacobi coefficients from the modified S_k . The first negative β_m appears at $m \approx 2j$ for the zeros $j, j+1$, with $\delta = 0.01$ detected almost as early as $\delta = 0.1$, and the pair at the zeros 30, 31 is not seen with $m \leq 43$. Elsewhere $4\beta_m/\alpha_m^2$ stays near 1: the regularity reflects the density of the zeros, not their position, and an off-line pair breaks it only at its own index. So, numerically, an

off-line pair inserted at index j is first seen at $m \approx 2j$ for the displacements tested. No theorem turns positivity for $m \leq M$ into the absence of off-line zeros below a height: by Proposition 16 the negative family of a quadruple carries the factor $(\operatorname{Im} t_\rho)^2 \approx (2\beta - 1)^2 \gamma^{-6}$ and can be outweighed by the positive families. Finite sections therefore play the role of Li's coefficients [Li], not of Turing's method [Tu], which is a rigorous verification; no finite section implies RH. Status: numerical (`hankel_offline_test.py`).

The determinants. One can study the determinants D_m and the ratios β_m directly, and ask whether the particular structure of the S_k gives information on them that does not reduce to a classical form of RH. The basic tool is the following exact formula.

Proposition 16 (Heine's formula for the Hankel determinants). *For every $m \geq 1$, with the sum over the m -element sets J of pairs $\{\rho, 1 - \rho\}$ (with multiplicity) and $t_j = t_{\rho_j}$,*

$$D_m = \sum_{|J|=m} \prod_{j \in J} t_j^2 \prod_{\substack{i < j \\ i, j \in J}} (t_i - t_j)^2,$$

the series converging absolutely.

- (i) *Under RH every term is nonnegative, and $D_m \geq \prod_{j \in J} t_j^2 \prod_{i < j} (t_i - t_j)^2$ for every single J .*
- (ii) *Suppose that there is an off-line zero, and let $t = t_\rho$ and $\bar{t} = t_{\bar{\rho}}$, $\operatorname{Im} t \neq 0$. Each set J that contains both pairs and otherwise only pairs on the line contributes*

$$-4(\operatorname{Im} t)^2 |t|^4 \prod_{j \in J'} t_j^2 |t - t_j|^4 \prod_{\substack{i < j \\ i, j \in J'}} (t_i - t_j)^2 < 0, \quad J' = J \setminus \{\rho, \bar{\rho}\},$$

and each set that contains exactly one of them is paired with its conjugate set, and the two contribute twice a real part, of either sign.

Proof. For a finite set P of pairs, Cauchy–Binet applied to $H_m^{(P)} = \sum_{\rho \in P} t_\rho^2 v_\rho v_\rho^T$, with $v_\rho = (1, t_\rho, \dots, t_\rho^{m-1})$, gives $\det H_m^{(P)} = \sum_{J \subset P, |J|=m} \prod_J t_j^2 \det(V_J)^2$, where V_J is the Vandermonde matrix of the t_j , $j \in J$. As P increases to all pairs, $H_m^{(P)} \rightarrow H_m$ entrywise, and the right side converges absolutely because $\sum_J \prod_J |t_j|^2 |\det V_J|^2 \leq m! (\sum_\rho |t_\rho|^2 \|v_\rho\|^2)^m < \infty$. (i) Under RH all t_j are real. (ii) $t^2 \bar{t}^2 = |t|^4$, $(t - \bar{t})^2 = -4(\operatorname{Im} t)^2$, and $(t - t_j)^2 (\bar{t} - t_j)^2 = |t - t_j|^4$ for real t_j . $\square$

So $D_m > 0$ is a competition between nonnegative families and the negative families created by off-line quadruples. Each negative family carries the factor $(\operatorname{Im} t_\rho)^2 = \gamma^2 (2\beta - 1)^2 / |\lambda_\rho|^4 \approx (2\beta - 1)^2 \gamma^{-6}$. In particular a lower bound for D_m , or for β_m , that does not assume RH must control these families, and this is RH itself; the formula does not provide one. What it does provide is a decomposition of the determinant into contributions of sets of zeros, which explains two features of the numerics. First, D_m is not dominated by the first m zeros: the set $J = \{1, \dots, m\}$ carries 67%, 25% and 4% of D_1 , D_2 , D_3 , and the rest is spread over many sets (script `hankel_heine_check.py`, which also checks the formula against the D_m computed from ξ). Second, an off-line pair at index j first appears in the sets J that contain it, and these carry a non-negligible share of D_m only once m is comparable to j , which agrees with the detection index $m \approx 2j$ of Table 7. Status: Proposition 16 is proved (it is classical, the formula of Heine for Hankel determinants of a discrete measure); the shares are numerical.

What is classical. The criterion is classical in substance: it is Hermite's theorem on the Newton sums of a polynomial with real roots, extended by the Hamburger moment problem [Ak], applied to Ξ in the variable λ , and it belongs to the circle of determinantal and Turán-type conditions on the Taylor coefficients of Ξ [CNV, Nu, Ma, GORZ]. What the reflected cross-energy adds is the variable: $t_\rho = 1/\lambda_\rho$ is its value at the zeros and the kernel of the cross-energy, and the Hankel form $\sum t_\rho^2 p(t_\rho)^2$ is the limit, along polynomial windows, of Weil's form on its test functions; and the

computation of the Jacobi matrix from ξ alone. The sum-of-squares identity sought for $Q(f * \mu_n)$ exists in this limit, $\sum t_\rho^2 p(t_\rho)^2 = \|Rp\|^2$ with R the Cholesky factor of the Hankel matrix, and exists for every size if and only if RH holds; proving that requires a reason for $\beta_m > 0$ for every m in the Taylor coefficients of ξ , which neither we nor, as far as we know, anyone has.

9. WHAT THE REFLECTED CROSS-ENERGY CONTRIBUTES

Proved. (i) The window of negativity of the reflected pair product, with its length, depth, location and limit, and its two-variable form through the Wronskian (Propositions 3 and 5); positivity of the wave-free product characterizes the critical line as a set. (ii) The exact decomposition $C_2^{\text{FR}} = \mathcal{G}_N^*(\lambda) + \zeta(s)\mathcal{K}_N(s)$, the expansion of $\mathcal{G}_N^*$, the Hardy form on the line, the location of the carrier zeros, and the identification with a fixed-height form of the symmetric truncation of Gonek and Montgomery (Theorem 8): the integers below N reproduce $Z(t)$, and the reflected cross-energy minus an explicit function of $s(1-s)$ vanishes at every zero of ζ , on the line or off it. (iii) For any system of integers with $N_B(x) = cx + c'x^b + O(x^\theta)$, $\theta < \frac{1}{2}$, the same construction detects every zero of ζ_B with real part in $(\frac{1}{2}, 1 - \theta)$, and the detected zero is fixed in N if and only if the reflected point is a zero, which is what a functional equation guarantees (Proposition 10). (iv) The identity (2), the restricted Weil criterion and its limit, with complete proofs of their equivalence with RH (Propositions 11 and 14); the Hankel–Jacobi criterion (Proposition 15, classical in substance) and Heine’s formula for its determinants (Proposition 16, classical). (v) The identity $\Lambda * 1 * 1 = \log * 1$ trades the dual range of primes for the primes up to ne^{2a} , the divisor function and the edge of the integers (Proposition 12). (vi) The precursor of Proposition 1, the limits of Remark 4, the gap identity of Remark 6.

Numerical. The checks of the window (Figure 1) and of the two-variable picture (Figure 2); the negative sets with the wave off the line; the zero maps of ζ and of the Davenport–Heilbronn function (Figure 3); the fixed zero of the Davenport–Heilbronn function and the drifting zero of the Beurling system, against exact predictions (Figure 4); the explicit formula on the windows, the margin at the window and the Chebyshev linear program (Tables 2–4); the location of the positivity in the dual range (Table 5) and its trade for small primes; Weil’s functional with the data of the Beurling system and of two controls (Table 6); the exponents -4 and -2 of the margin at large height (Figure 6); the Hankel–Jacobi computation from ξ alone to $m = 43$, with the zeros as its spectrum and the detection index $m \approx 2j$ of an off-line pair (Table 7).

Heuristic. The predicted margins at large height and their reading as a precision of RH strength or beyond (Remark 13).

What it does not do. It does not prove RH. Every exact reformulation obtained here (the sign of the reflected pair product, the real part of the Wronskian, the absence of zeros of the wave off the line, the restricted Weil criterion, the edge inequality, the Hankel–Jacobi positivity) is equivalent to RH or, for Proposition 3(v), to RH above height 16; the objects of Section 2 and the smooth part of C_2^{FR} are computed from the position of a zero, so they characterize the line, not the zeros; the precision that the primes would have to supply is that of RH or beyond; finite Hankel sections only verify RH up to a height; and Weil’s functional of ζ is not a criterion for systems without a functional equation. What the reflected cross-energy contributes is a single object, built from the integers below N , that evaluates a function of λ_ρ at the zeros, contains the functional equation as its carrier, meets the Euler product through $\log = \Lambda * 1$, and separates the two controls, the Davenport–Heilbronn function by a fixed off-line zero and the Beurling system by a drifting one. A proof along this route needs an exact positivity identity for $Q(f * \mu_n)$, equivalently a reason for $\beta_m > 0$ for all m , and that is where the problem stands.

10. OPEN PROBLEMS

1. The edge inequality. By (2), Proposition 11 and Proposition 12, RH is equivalent to $Q(f * \mu_n) \geq 0$ for all real $f \in C_c^\infty(\mathbb{R})$ and all n , an inequality for the primes up to ne^{2a} weighted by the integers below n and their edge $\max(j, k) = n$, with the divisor function near n^2 . Prove it for one family $n \rightarrow \infty$ with f at the window of a fixed zero, or for all n with one fixed f . By Table 4, any such proof must use more than $\Lambda \geq 0$ and Chebyshev's divisor-sum inequalities.

2. A criterion for number systems. For the Beurling system of Section 4 the reflected cross-energy of its integers sees its off-line zero (Proposition 10), but Weil's functional of ζ evaluated with its data is not a criterion for it: a control without a zero to the right of the line makes it negative, and one with the zero at 0.6 leaves it positive for the window at γ_1 (Table 6). Find a form of the restricted criterion that is a criterion for number systems without a functional equation, or show that there is none. Is the drift of Proposition 10(ii) the only invariant that separates them?

3. A reason for $\beta_m > 0$. Find a structural reason, from the Taylor coefficients of ξ at $s = 1$, for the positivity of the recurrence coefficients of the moment sequence (S_{k+2}) (Proposition 15), even for m beyond the range where it has been checked; or explain the near-constancy $4\beta_m/\alpha_m^2 \approx 1$ from the density of the zeros alone. By Proposition 16, a lower bound for β_m that does not assume RH must control the negative families of off-line quadruples; is there a structure in the S_k , beyond their being power sums, that does this?

4. The product through the primes. Express $C_2(n, \rho) C_2(n, 1 - \bar{\rho})$ near $n = \gamma^2$, or the real part of the Wronskian of Proposition 5, through the primes in a form in which its sign is visible. The obstruction of Section 2.4 says that the sesquilinear part of the product cannot be summed over the zeros; is there a positive combination of reflected pairs that can?

5. The range of the window. The constant $t_1 = 16$ of Proposition 3 is explicit. Decide whether the range $n \geq |\gamma|^{3/2}$ can be lowered to the threshold $|\gamma|/2\pi$ of the approximate functional equation, beyond which the sign change is unique for large $|\gamma|$ [CE2, Cor. 2.13], and give explicit constants for the asymptotic formulas of Proposition 3(iii).

DATA AND CODE

The ancillary Python scripts reproduce all tables and figures. They need NumPy, SciPy and Matplotlib, and the two Hankel scripts of Section 8 also need mpmath; the file `anc/README.txt` lists each script, what it reproduces and its output file. The scripts in `anc/reflected_pair/` are self-contained in that folder; among those at the top level of `anc/`, `c2fr_beurling.py` and `weil.bdr_scan.py` import `beurling_real_zero.py`, which builds the Beurling system of [CE11], and several read `zeros.1000.npy`, the ordinates of the first 1000 zeros. Double precision throughout except where stated, without interval arithmetic: the numbers illustrate the propositions, and the proofs do not use them, with two exceptions. Lemma 2(a) and Proposition 3(i) use the evaluation of closed-form bounds at $|t| = 16$ (`explicit_constants.py`, in 60-digit arithmetic with large margins). Proposition 15(iii) uses $S_2 > 0$, which also follows without computation: the pairs with $|\gamma| \leq 3 \cdot 10^{12}$ are on the line [PT] and give positive t_ρ^2 , at least $(\frac{1}{4} + \gamma_1^2)^{-2} > 2.4 \cdot 10^{-5}$, while $\sum_{|\gamma| > 3 \cdot 10^{12}} |t_\rho|^2 \leq \sum \gamma^{-4} < 10^{-20}$.

Use of AI tools. The proofs, computations and drafting of this paper were carried out with the assistance of an AI system (Claude, Anthropic), under the author's direction; an independent AI review agent refereed the proofs. The author checked the results and takes responsibility for them.

REFERENCES

- [Ak] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver & Boyd, Edinburgh, 1965.
- [BDR] F. Broucke, G. Debruyne, Sz. Révész, Some examples of well-behaved Beurling number systems, *Trans. Amer. Math. Soc.* 378 (2025), 477–501, doi:10.1090/tran/9274, arXiv:2309.01567.

- [Bo] E. Bombieri, Remarks on Weil’s quadratic functional in the theory of prime numbers, I, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 11 (2000), 183–233.
- [BSY] M. Balazard, E. Saias, M. Yor, Notes sur la fonction ζ de Riemann, 2, *Adv. Math.* 143 (1999), 284–287.
- [C1] P. Caceres, Cross-energy of Dirichlet partial sums and the zeros of the Riemann zeta function: critical-line detection, a Fresnel transition law, and structural limits, preprint, Zenodo, 2026, doi:10.5281/zenodo.23066932.
- [Ch] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [CNV] G. Csordas, T. S. Norfolk, R. S. Varga, The Riemann hypothesis and the Turán inequalities, *Trans. Amer. Math. Soc.* 296 (1986), 521–541.
- [Con] K. Conrad, Partial Euler products on the critical line, *Canad. J. Math.* 57 (2005), 267–297.
- [DH] H. Davenport, H. Heilbronn, On the zeros of certain Dirichlet series, I, II, *J. London Math. Soc.* 11 (1936), 181–185, 307–312.
- [GV] I. M. Gel’fand, N. Ya. Vilenkin, *Generalized Functions*, Vol. 4: *Applications of Harmonic Analysis*, Academic Press, New York, 1964.
- [GoM] S. M. Gonek, H. L. Montgomery, Zeros of a family of approximations of the Riemann zeta-function, *Int. Math. Res. Not. IMRN* 2013, no. 20, 4712–4733, doi:10.1093/imrn/rns182.
- [GORZ] M. Griffin, K. Ono, L. Rolin, D. Zagier, Jensen polynomials for the Riemann zeta function and other sequences, *Proc. Natl. Acad. Sci. USA* 116 (2019), 11103–11110.
- [Li] X.-J. Li, The positivity of a sequence of numbers and the Riemann hypothesis, *J. Number Theory* 65 (1997), 325–333.
- [Ma] Yu. Matiyasevich, Hidden life of Riemann’s zeta function 2. Electrons and trains, arXiv:0709.0028, 2007.
- [Nu] J. Nuttall, Determinantal approach to a proof of the Riemann hypothesis, arXiv:1111.1128, 2011.
- [PT] D. Platt, T. Trudgian, The Riemann hypothesis is true up to $3 \cdot 10^{12}$, *Bull. London Math. Soc.* 53 (2021), 792–797.
- [T] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, 1986.
- [Tu] A. M. Turing, Some calculations of the Riemann zeta-function, *Proc. London Math. Soc. (3)* 3 (1953), 99–117.
- [W2] A. Weil, Sur les “formules explicites” de la théorie des nombres premiers, *Comm. Sém. Math. Univ. Lund*, tome supplémentaire (1952), 252–265.
- [CE1] P. Caceres, Cross-energy of Dirichlet partial sums. Part 1: The C-transform and the cross-energy, preprint, 2026.
- [CE2] P. Caceres, Cross-energy of Dirichlet partial sums. Part 2: The transition at the critical line and Hardy’s function, preprint, 2026.
- [CE5] P. Caceres, Cross-energy of Dirichlet partial sums. Part 5: The energy of the primes, preprint, 2026.
- [CE11] P. Caceres, Cross-energy of Dirichlet partial sums. Part 11: The Euler–Caceres bound, preprint, 2026.
- [CE12] P. Caceres, Cross-energy of Dirichlet partial sums. Part 12: Weil positivity at the windows of the zeros, preprint, 2026.
- [CE17] P. Caceres, Cross-energy of Dirichlet partial sums. Part 17: Controls for the Riemann hypothesis: Davenport–Heilbronn, Beurling systems and function fields, preprint, 2026.

UNIVERSIDAD EUROPEA DE VALENCIA, VALENCIA, SPAIN

Email address: pedrojesus.caceres@universidadeuropea.es