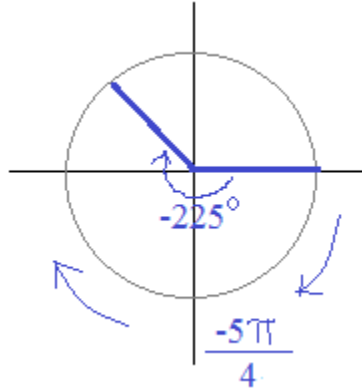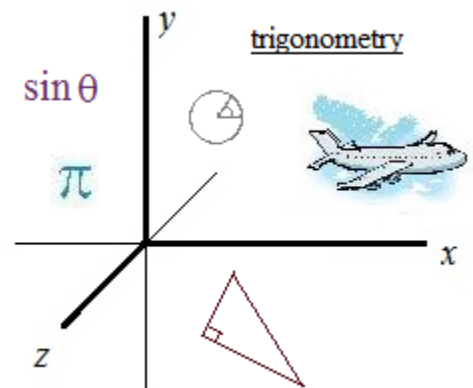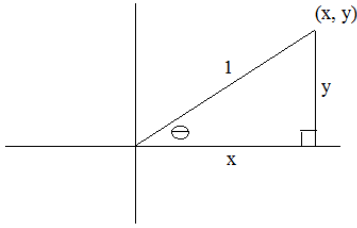# Unit Circle and Trig Measures

Examples and Practice Exercises (with solutions)

The Unit Circle

What is it?   A circle with a radius of one unit...

It's also a visual representation of special angles that give exact trig values...
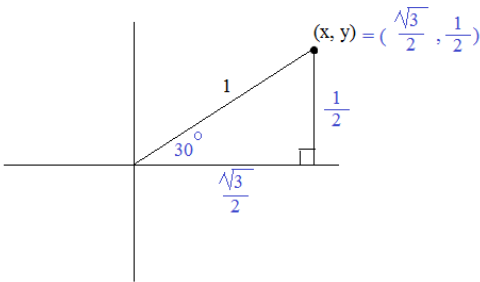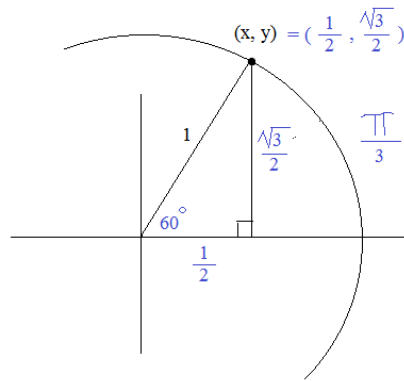
Here is a triangle with hypotenuse length 1...

$$\sin\theta = \frac{y}{1} = y \qquad \csc\theta = \frac{1}{y}$$

$$\cos\theta = \frac{x}{1} = x \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

We can determine points on a circle with radius 1...

Degrees

$(x, y) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$

Radians

$(x, y) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$
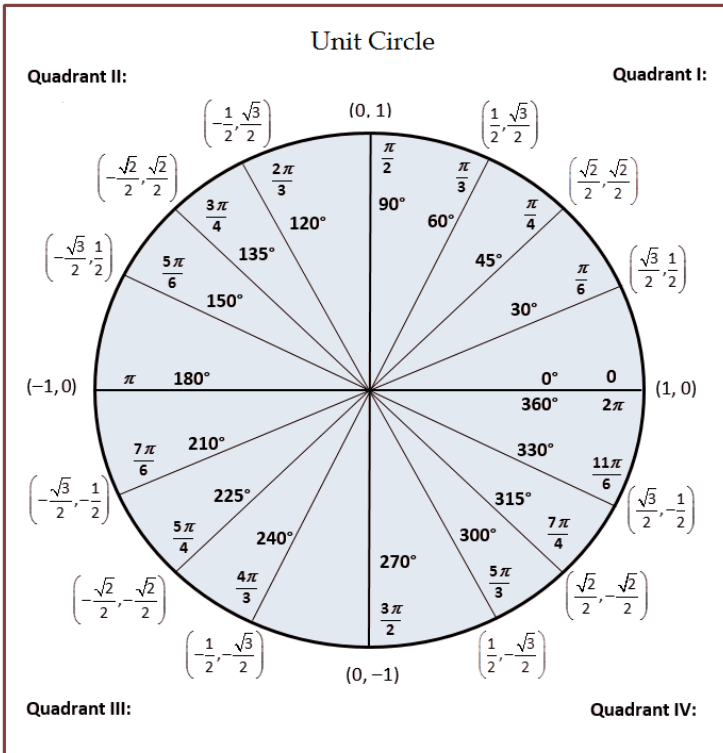
Using the angles related to 45-45-90 and 30-60-90 right triangles, using the hypotenuse of 1, and applying it to a coordinate plane, we create a "unit circle centered on the origin"....

Unit Circle

Example: What is $\sin(150°)$? What is $\cos(\frac{5\pi}{6})$?

"TRIANGLES"

$$\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \sin(150) = \frac{1}{2}$$

$$\text{cosine} = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \cos(\frac{5\pi}{6}) = \frac{-\sqrt{3}}{2}$$

Now, look at the unit circle....

The point that corresponds to 150 degrees and $\frac{5\pi}{6}$ radians is
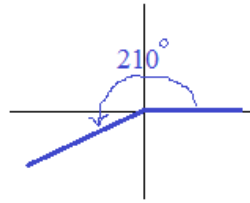
$$(\frac{-\sqrt{3}}{2}, \frac{1}{2})$$

since $\sin = y$, $\sin(150) = \frac{1}{2}$

$\cos = x$, $\cos(\frac{5\pi}{6}) = \frac{-\sqrt{3}}{2}$

"UNIT CIRCLE"
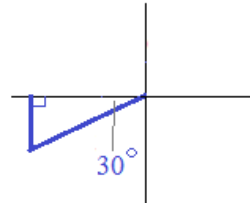
Unit Circle: Finding Trig Values
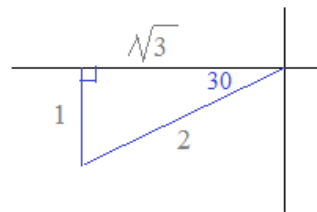
Example: find $\sin 210^\circ$

Step 1: Draw angle in standard position
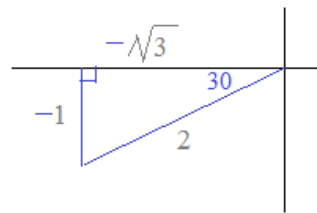
210°

Step 2: Find reference angle

30°

Step 3: Identify the triangle and label

$\sqrt{3}$

30

1

2

30-60-90 right triangle

Step 4: distinguish 'negatives' and 'positives'

$-\sqrt{3}$

30

-1

2

Quadrant III

Step 5: Find Trigonometry Value

$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{-1}{2}$$

$$\boxed{\text{Sin}(210) = \frac{-1}{2}}$$

Note: --- If the angle is given in radians, convert to degrees and begin.
 --- Step 4 is important! Don't forget the negatives.
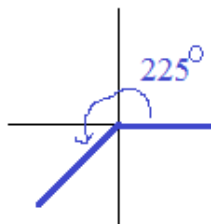 Also, the hypotenuse is always positive.

Unit Circle: Finding Trig Values

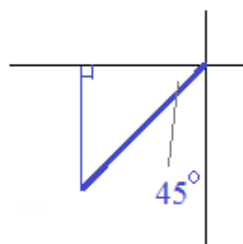*Example:* $\text{Tan } \dfrac{5\pi}{4}$

(Convert to Degrees)

$$\dfrac{5\pi \text{ radians}}{4} \cdot \dfrac{180^\circ}{\pi \text{ radians}} = 225^\circ$$

Step 1: Draw Angle in standard position

225°

Step 2: Find Reference Angle

45°

Step 3: Identify Triangle and label

-1

45

-1

$\sqrt{2}$
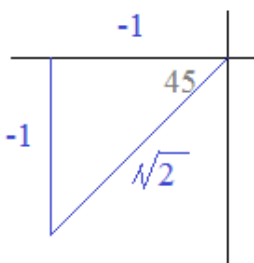
45-45-90 right triangle

Step 4: Negatives and Positives

Quadrant III

Step 5: Find Trigonometry Value

$$\text{tangent} = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{-1}{-1} = 1$$

Note: $180^\circ = \pi$ radians

$\text{Tan } \dfrac{5\pi}{4} = 1$

What is a quadrantal angle?

An angle in standard position whose terminal side lies on the x or y axis.
These angles include:

$$0, \ \pm 90, \ \pm 180, \pm 270, \ \pm 360...$$

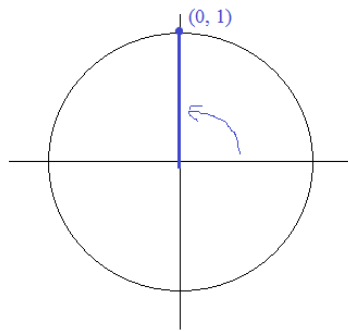$$\pm \frac{-\pi}{2} \quad \pm -\pi \quad \pm \frac{3\pi}{2} \quad \pm 2\pi$$

The trig values of a quadrantal angle will be 0, -1, 1, or undefined

---

Using the unit circle or utilizing a "fictional triangle" and 'Soh Cah Toa'

Here are 2 approaches to finding the trig values of a quadrantal angle...

Example: Find $\sin(90^\circ)$

Using the Unit Circle

(0, 1)

We know that $\sin = \dfrac{y}{r}$

so, $\sin(90) = \dfrac{1}{1} = 1$

Creating an (imaginary) Triangle

1

0

1

1

0
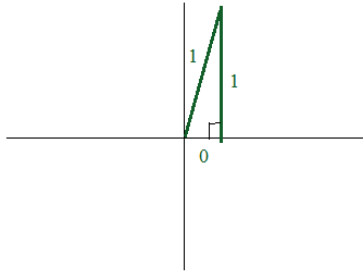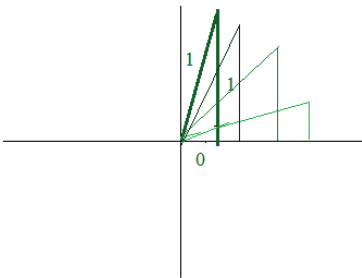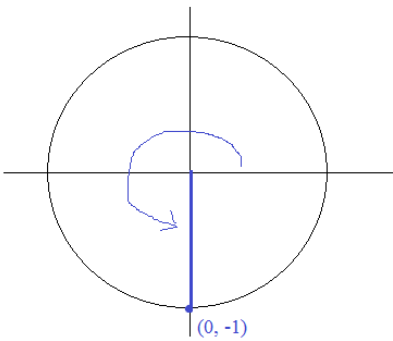
This triangle has a reference angle of 90..

It's opposite side is 1,
   adjacent side is 0,
and, hypotenuse is 1

$$\sin = \frac{opposite}{hypotenuse} = \frac{1}{1} = 1$$

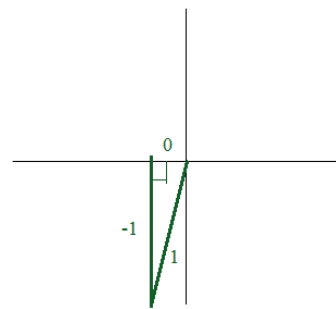*Example:* Find the value of $\cot\left(\dfrac{3\pi}{2}\right)$

Unit Circle

(0, -1)

We know that $\cot = \dfrac{x}{y}$

Therefore, $\cot\left(\dfrac{3\pi}{2}\right) = \dfrac{0}{-1} = 0$

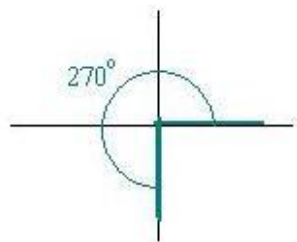### Creative Triangle

0

-1

1

('reference angle' is 90 degrees)

Using an imaginary triangle, we know that
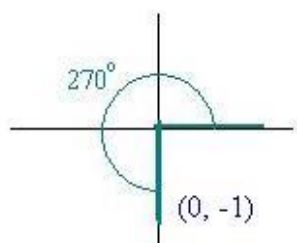
$$\cot = \frac{adjacent}{opposite} = \frac{0}{-1} = 0$$

Example: Sin 270°

Step 1: Draw angle in standard position
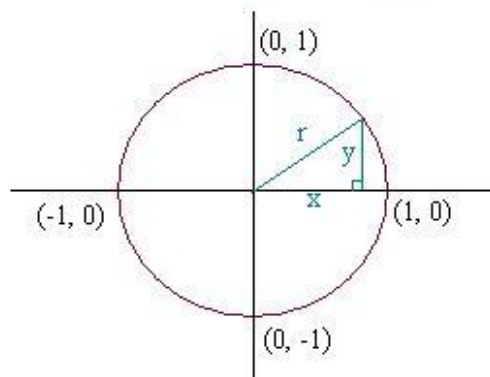
270°

Step 2: "Label the point"
(Reminder: it is a UNIT circle; the radius is 1)

270°

(0, -1)

Step 3: Apply the trig function

$$\text{Sine} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} = \frac{-1}{1} = -1$$

Note: ---If the angle is given in radians, convert to degrees and begin.
    ---Observe the points and trig functions on the unit circle:

Sin(270) = -1

(0, 1)

r = 1

$$\sin = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$$

$$\cos = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

(-1, 0)

(1, 0)

$$\tan = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

(0, -1)

Finding Trig Values: Coterminal and "Negative" Angles

"Coterminal" Example: Find the exact trig value of $\sin(480^\circ)$

  Step 1: Find Coterminal angle (between 0 and 360 degrees)

$$480^\circ - 360^\circ = 120^\circ$$

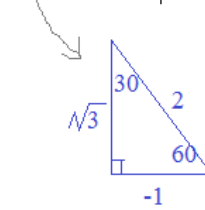Since 480 and 120 are coterminal angles, their trig values are equal.

  Step 2: Draw the angle in standard position
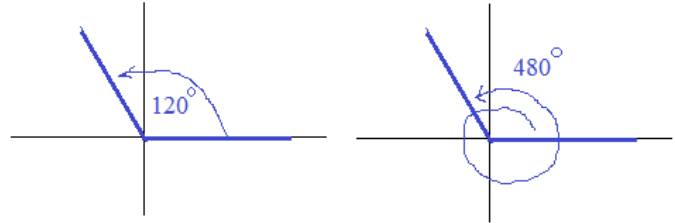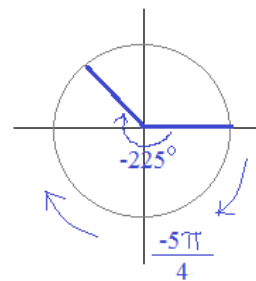

  Step 3: Find Reference Angle

The reference angle is 60°
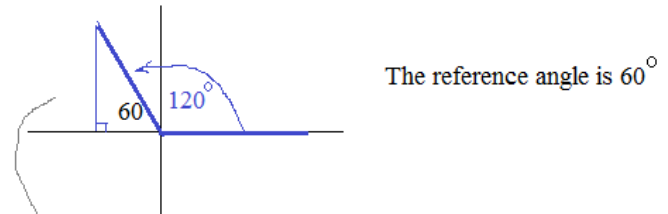
  Step 4: "Identify the triangle and label"

  Step 5: Find the trigonometry value

30-60-90 triangle...

$$\text{Sine} = \frac{\text{opposite side}}{\text{hypotenuse}} = \boxed{\frac{\sqrt{3}}{2}}$$

"Negative Angle" Example: Evaluate $\tan(-5\pi/4)$

  Step 1: Draw the angle in standard position

Since the angle is negative, move *clock-wise* along the unit circle!

  Step 2: Find the reference angle

  Step 3: Identify the triangle and label
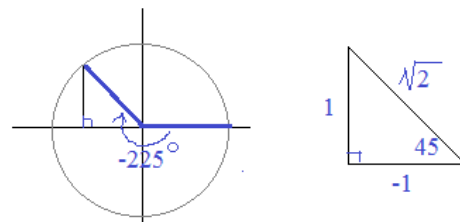
  Step 4: find the trig value

$$\text{Tan} = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{1}{-1} = \boxed{-1}$$
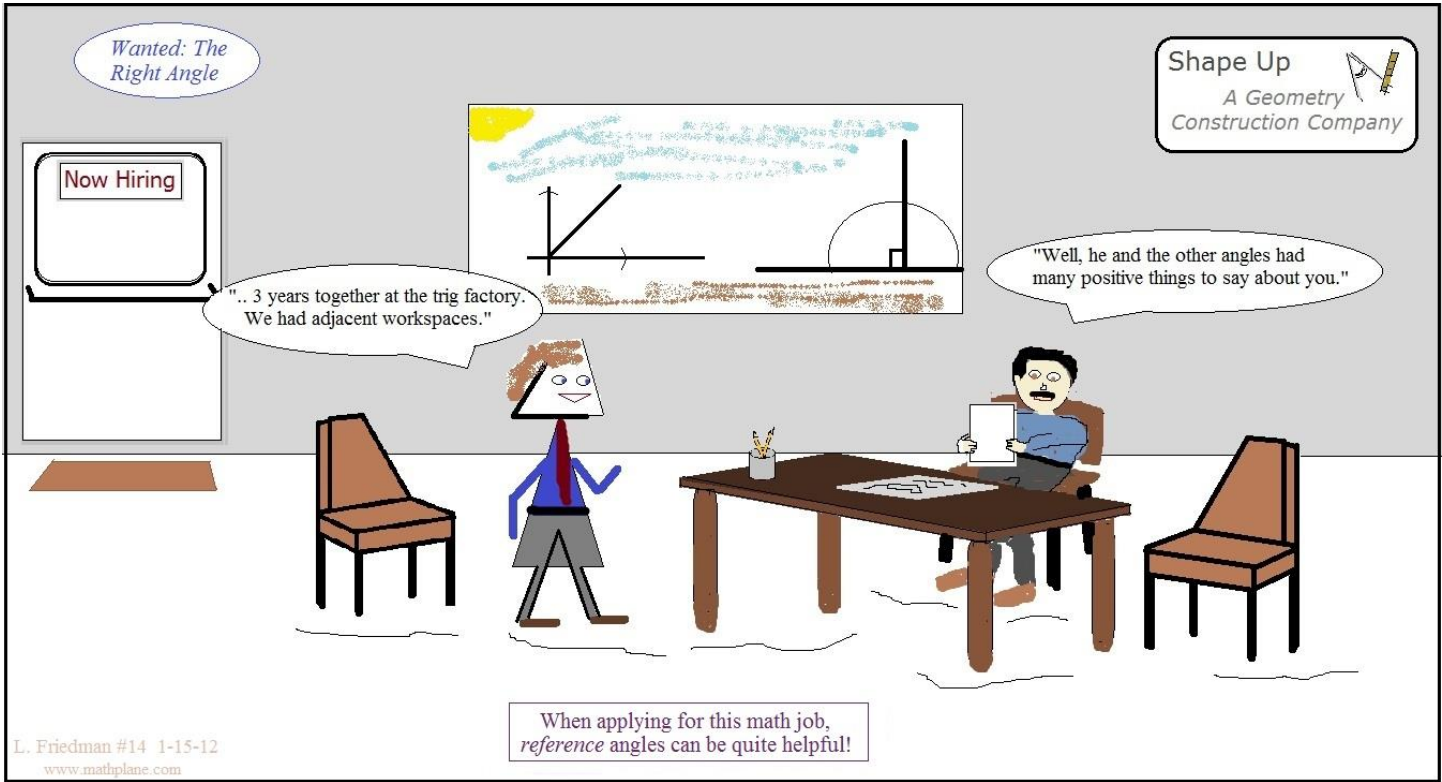
Practice -→

Evaluate the following (NO tables! NO calculators!)

1) sin 30

2) tan 210

3) csc 120

4) sec 0

5) sin 225

6) cot 315

7) cos -45

8) sin 270

9) cot 180

10) $\cos \dfrac{\pi}{3}$

11) $\sin \dfrac{3\pi}{4}$

12) $\tan \dfrac{7\pi}{4}$

13) $\csc \dfrac{5\pi}{6}$

14) $\cos 3\pi$

15) $\cos \dfrac{-\pi}{3}$

Trig Values and Unit Circle Worksheet: Coterminal and Negative Angles

Evaluate the following (without using a calculator or table)

1) $\sin(420^\circ)$

2) $\cos(\frac{9\pi}{4})$

3) $-\tan(135^\circ)$

4) $\tan(-135^\circ)$

5) $\sec(\frac{11\pi}{3})$

6) $\csc(\frac{-11\pi}{3})$

7) $\cos(7\pi)$

8) $-\cot(\frac{14\pi}{3})$

9) $\csc(900^\circ)$

10) $-\sin(450^\circ)$

11) $\csc(-450^\circ)$

12) $\cot(\frac{-29\pi}{6})$

1) $\cos(0^{\circ})$

2) $\tan(\pi)$

3) $\sin(90^{\circ})$

4) $\sec\left(-\dfrac{\pi}{2}\right)$

5) $\cot(180^{\circ})$

6) $\sin\left(\dfrac{3\pi}{2}\right)$

7) $\csc(2\pi)$

Evaluate the following:

1) sin 30

$\dfrac{1}{2}$

2) tan 210

$\dfrac{-1}{-\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

3) csc 120

$\dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

4) sec 0

$\sec = \dfrac{r}{x}$

$\dfrac{1}{1} = 1$

5) sin 225

$\dfrac{-1}{\sqrt{2}} = \dfrac{-\sqrt{2}}{2}$

6) cot 315

$\dfrac{adj}{opp} = -1$

7) cos -45

$\dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{2}}{2}$

8) sin 270

$\sin = \dfrac{y}{r}$

$\dfrac{-1}{1} = -1$

9) cot 180

$\cot = \dfrac{x}{y} = \dfrac{-1}{0}$

Undefined

10) $\cos\dfrac{\pi}{3}$ $= 60°$

$\dfrac{1}{2}$
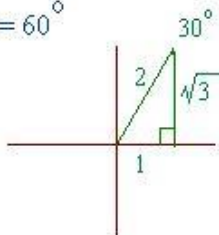
11) $\sin\dfrac{3\pi}{4}$ $= 135°$

$\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
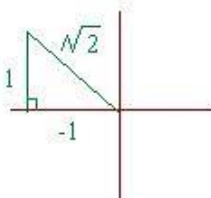
12) $\tan\dfrac{7\pi}{4}$ $= 315°$

$\dfrac{-1}{1} = -1$
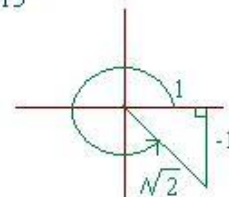
13) $\csc\dfrac{5\pi}{6}$ $= 150°$
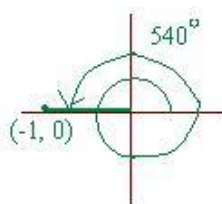
$\csc = \dfrac{hyp}{opp}$

$\dfrac{2}{1} = 2$
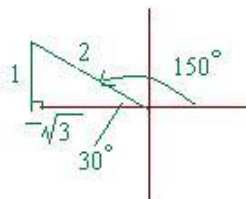
14) $\cos 3\pi$ $= 540°$

$\cos = \dfrac{x}{r}$

$\dfrac{-1}{1} = -1$

15) $\cos\dfrac{-\pi}{3}$ $= -60°$
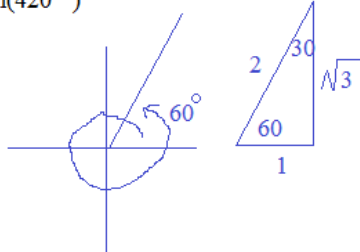
$\dfrac{1}{2}$

(same solution as question 10!!)

SOLUTIONS

Evaluate the following (without using a calculator or table)

1) $\sin(420^{\circ})$

$\boxed{\dfrac{\sqrt{3}}{2}}$
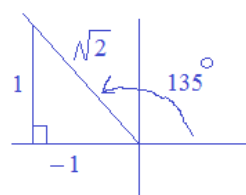
2) $\cos(\dfrac{9\pi}{4})$ coterminal angle

$\dfrac{9\pi}{4} - \dfrac{8\pi}{4} = \dfrac{\pi}{4}$

$\cos\dfrac{\pi}{4} = \boxed{\dfrac{\sqrt{2}}{2}}$
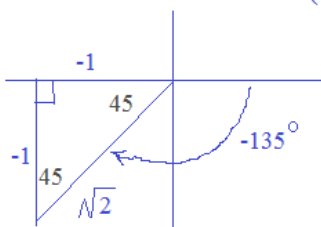
3) $-\tan(135^{\circ})$

$\tan(135) = \dfrac{1}{-1} = -1$
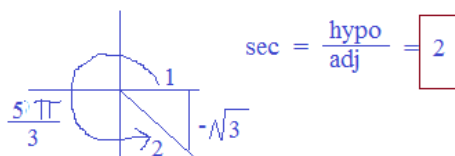
so, $\boxed{-\tan(135) = 1}$

4) $\tan(-135^{\circ})$

$\tan(-135) = \dfrac{-1}{-1}$

$\boxed{= 1}$

5) $\sec(\dfrac{11\pi}{3})$ coterminal angles

$\dfrac{11\pi}{3} - \dfrac{6\pi}{3} = \dfrac{5\pi}{3}$

$\sec = \dfrac{\text{hypo}}{\text{adj}} = \boxed{2}$
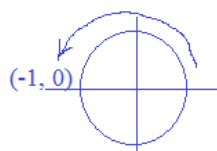
6) $\csc(\dfrac{-11\pi}{3})$ coterminal angle

$\dfrac{-11\pi}{3} + \dfrac{12\pi}{3} = \dfrac{\pi}{3}$

$\csc(60^{\circ})$ or $\csc(\dfrac{\pi}{3})$

$\boxed{\dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}}$

7) $\cos(7\pi)$

$7\pi - 2\pi - 2\pi - 2\pi$

$\cos(7\pi) = \cos(\pi)$
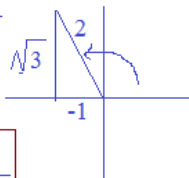
$\cos = \dfrac{x}{1}$

$(-1, 0)$

$\cos(7\pi) = \boxed{-1}$

8) $-\cot(\dfrac{14\pi}{3})$

$\dfrac{14\pi}{3} - \dfrac{12\pi}{3} = \dfrac{2\pi}{3}$

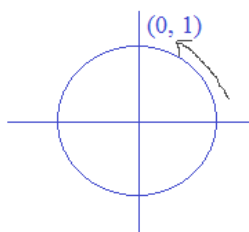$\cot(\dfrac{2\pi}{3}) = \dfrac{-1}{\sqrt{3}}$

so, $-\cot(\dfrac{2\pi}{3}) = \boxed{\dfrac{1}{\sqrt{3}}}$

9) $\csc(900^{\circ})$

$900 - 360 = 540...$
$540 - 360 = 180$

$\csc(180)$ is $\boxed{\text{undefined}}$

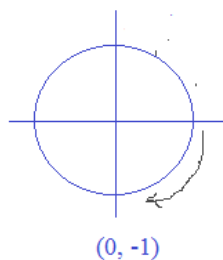10) $-\sin(450^{\circ})$

coterminal angles
$-\sin(90^{\circ})$

$(0, 1)$

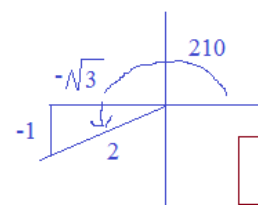$\sin = \dfrac{y}{1}$

since $\sin(90) = 1$,

then $-\sin(90) = \boxed{-1}$

11) $\csc(-450^{\circ})$

$\csc(-90) = \boxed{-1}$

$(0, -1)$

$\csc = \dfrac{1}{y}$

12) $\cot(\dfrac{-29\pi}{6})$

$\dfrac{-29\pi}{6} + \dfrac{36\pi}{6} = \dfrac{7\pi}{6}$

210

$\boxed{\sqrt{3}}$

mathplane.com

1) $\cos(0^{\circ})$

$\cos = x$

1

mathplane.com

SOLUTIONS

reference angle is 0

$\cos = \dfrac{\text{adjacent}}{\text{hypotenuse}} = 1$

2) $\tan(\pi)$

(-1, 0)

$\tan = \dfrac{y}{x}$

0

reference angle is 0

$\tan = \dfrac{\text{opposite}}{\text{adjacent}} = 0$

3) $\sin(90^{\circ})$

(0, 1)

$\sin = y$

1

reference angle is 90

$\sin = \dfrac{\text{opposite}}{\text{hypotenuse}} = 1$
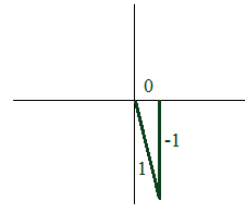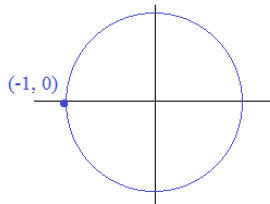
4) $\sec(-\dfrac{\pi}{2})$

$\sec = \dfrac{1}{x}$

undefined

(negative angle goes clockwise)

(0, -1)

reference angle is 90

$\sec = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \text{undefined}$

5) $\cot(180^{\circ})$

(-1, 0)

$\cot = \dfrac{x}{y}$

undefined

reference angle is 0

$\cot = \dfrac{\text{adjacent}}{\text{opposite}} = \text{undefined}$

6) $\sin(\dfrac{3\pi}{2})$

$\sin = y$

-1

(0, -1)

reference angle is 90

$\sin = \dfrac{\text{opposite}}{\text{hypotenuse}} = -1$

7) $\csc(2\pi)$

$\csc = \dfrac{1}{y}$

(1, 0)

undefined

reference angle is 0
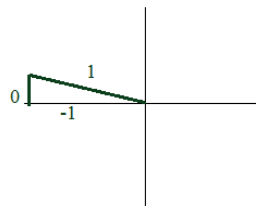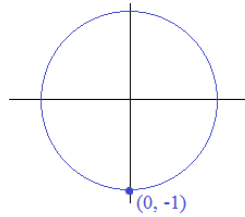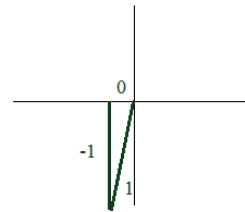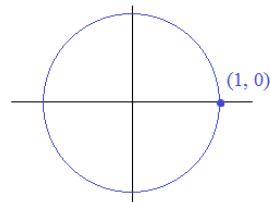
$\csc = \dfrac{\text{hypotenuse}}{\text{opposite}} = \text{undefined}$
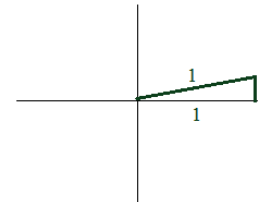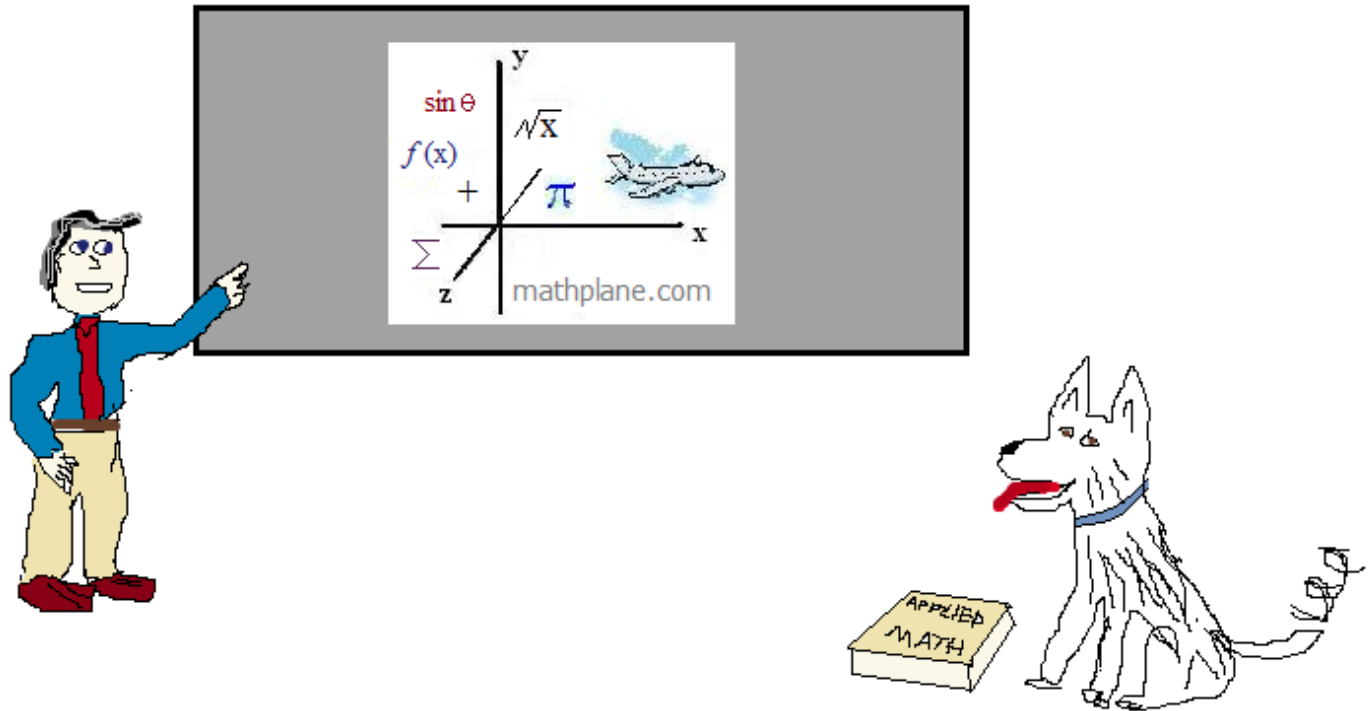
Hope this introduction helps.

If you have questions, suggestions, or requests, let us know.

Good luck!

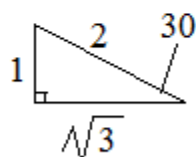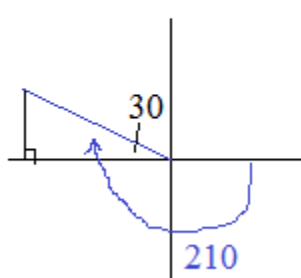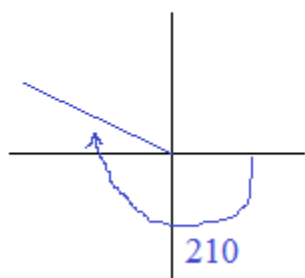Also, more trig and math resources at Mathplane *Express* for mobile at Mathplane.ORG

One more question….

What is the exact value of sin(-210)?

Answer on the next page…

What is the exact value of $\sin(-210^{\circ})$?

$$\frac{1}{2}$$

Note: the 'direction' of the angle is clock-wise...