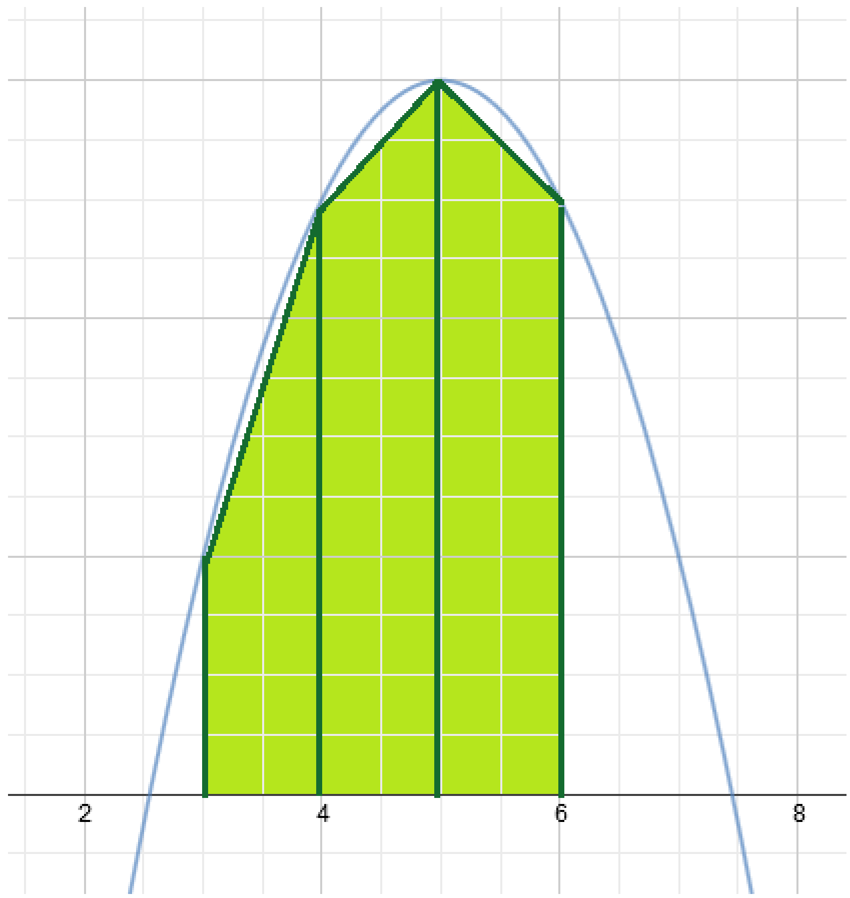
Calculus: Trapezoid Rule



Notes, Examples, and Practice Test (with Solutions)

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Calculus: Trapezoid Rule

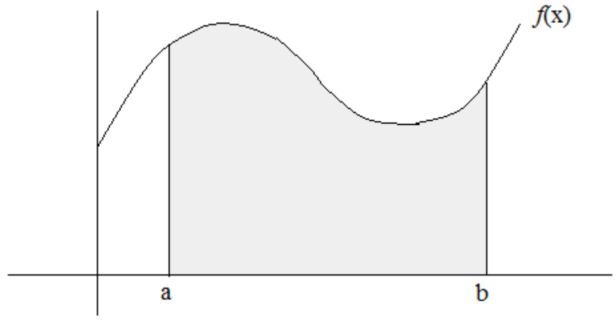
What is it? A method for estimating the *area under a curve*.
A method for approximating the value of a *definite integral*
It uses linear measures of a function to create "trapezoidal areas"

Definition:

$$\int_a^b f(x)\,dx = \frac{\triangle x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n) \right]$$

where $\triangle x$ is the measure of each sub-interval $\frac{b - a}{n}$

Explanation and General Example: Find the area *under the curve* from a to b.

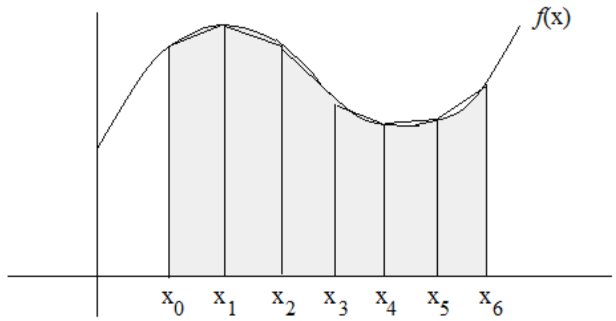


Step 1: Determine the number of partitions (n)

(note: the greater the number of partitions, the more accurate the approximation of the area)

Step 2: Divide the interval [a, b] into n sub-intervals

Since $n = 6$, there are 6 subintervals (creating 6 regions)



Step 3: Draw segments "connecting the tops" of the vertical lines

This creates 6 trapezoids!

Step 4: Add up the areas of the trapezoids

Area of Trapezoid: $\frac{1}{2}(b_1 + b_2)h$

where h = height
b_1 = upper base
b_2 = lower base

(After taking out the greatest common factors and collecting the like terms,)

The sum of the 6 trapezoids:

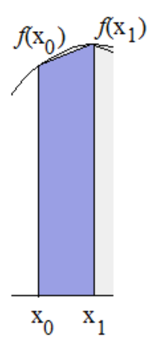
$$\frac{\triangle x}{2}\left[(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6) \right]$$

Area of first trapezoid:

$h = \frac{b - a}{n} = \triangle x$ (the distance from x_0 to x_1)

$b_1 = f(x_0)$ (the distance from the x-axis to $f(x_0)$)

$b_2 = f(x_1)$ (the distance from the x-axis to $f(x_1)$)



$\frac{1}{2}\left[(f(x_0) + f(x_1) \right] \triangle x$

Area of second trapezoid:

$h = \frac{b - a}{n} = \triangle x$ (the distance from x_1 to x_2)

$b_1 = f(x_1)$ (the distance from the x-axis to $f(x_0)$)

$b_2 = f(x_2)$ (the distance from the x-axis to $f(x_2)$)

$\frac{1}{2}\left[(f(x_1) + f(x_2) \right] \triangle x$

Calculus: Trapezoid Rule

Example: Using the trapezoid rule, where the number of sub-intervals n = 4, approximate the area under $f(x)$ in the interval [0, 2].

Then, using the definite integral $\int_0^2 f(x)\,dx$ compare your estimate with the true value.

Step 1: Divide into n sub-intervals

Each sub-interval will be 1/2 $\quad \frac{2-0}{4} = 1/2$

Step 2: Draw line segments "connecting the tops"

Find the values at each sub-interval:

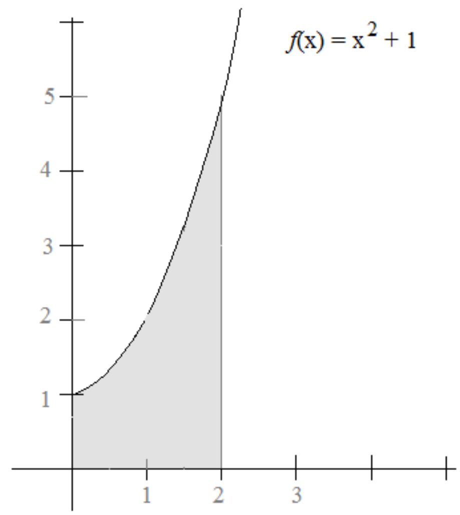
$f(0) = 1$

$f(1/2) = 5/4$

$f(1) = 2$

$f(3/2) = 13/4$

$f(2) = 5$



Step 3: Add up the trapezoids.

$$\text{Area} = \frac{1}{2}(\text{base 1} + \text{base 2})(\text{height})$$

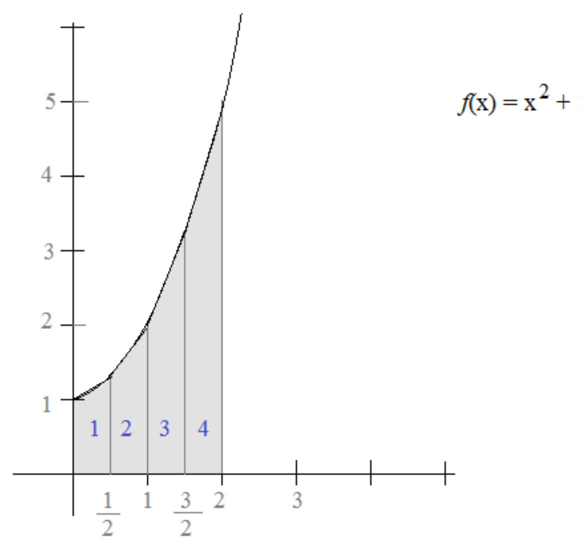
trapezoid 1: $\frac{1}{2}(1 + 5/4)(1/2) = \frac{9}{16}$

trapezoid 2: $\frac{1}{2}(5/4 + 2)(1/2) = \frac{13}{16}$

trapezoid 3: $\frac{1}{2}(2 + 13/4)(1/2) = \frac{21}{16}$

trapezoid 4: $\frac{1}{2}(13/4 + 5)(1/2) = \frac{33}{16}$

Total: $\frac{76}{16} = 4.75$



> **Observation:** If the curve is *concave up*, the trapezoid rule will *overestimate* the area.
> If the curve is *concave down*, the trapezoid rule will *underestimate* the area.
> (And, if it is a straight line, it will give the exact area.)

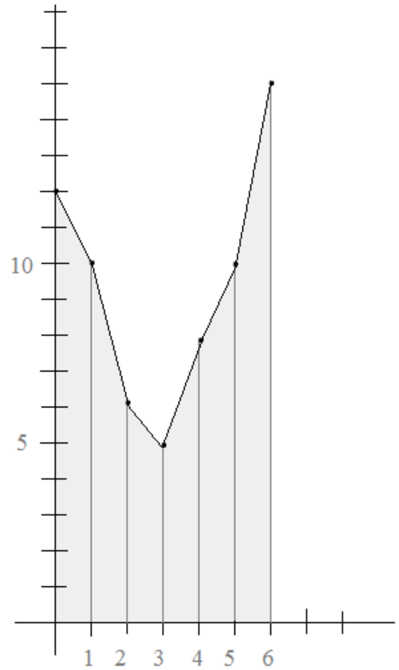
The actual area:

$$\int_0^2 f(x)\,dx = \int_0^2 x^2 + 1 \;\; dx \qquad \left.\frac{x^3}{3} + x\,\right|_0^2 = \frac{(2)^3}{3} + (2) - \left(\frac{(0)^3}{3} + (0)\right) = \frac{14}{3} \quad \boxed{= 4.6\overline{6}}$$

Since the curve is concave up in the interval, the trapezoid rule overestimated the actual area!

Example: Use the table of values to estimate $\int_0^6 f(x)\,dx$

x	0	1	2	3	4	5	6
f(x)	12	10	6	5	8	10	17



Since the partitions (intervals) are uniform, we can use the trapezoid rule formula.

$$\int_0^6 f(x)\,dx = \frac{\triangle x}{2}\left[f(x_0) + 2f(x_1) + \ldots + 2f(x_5) + f(x_6)\right]$$

$$= \frac{1}{2}\left[12 + 2(10) + 2(6) + 2(5) + 2(8) + 2(10) + 17\right]$$

$$= \frac{1}{2}\left[107\right] = \boxed{53.5}$$

Or, adding the 6 trapezoids individually:

11 + 8 + 5.5 + 6.5 + 9 + 13.5 = 53.5

Example: Use the table of values to estimate $\int_1^{11} f(x)\,dx$

x	0	1	4	8	11	16	18
f(x)	2	3	8	16	13	7	3

Since the intervals are not uniform, we need to evaluate each trapezoid separately (rather than use the formula).

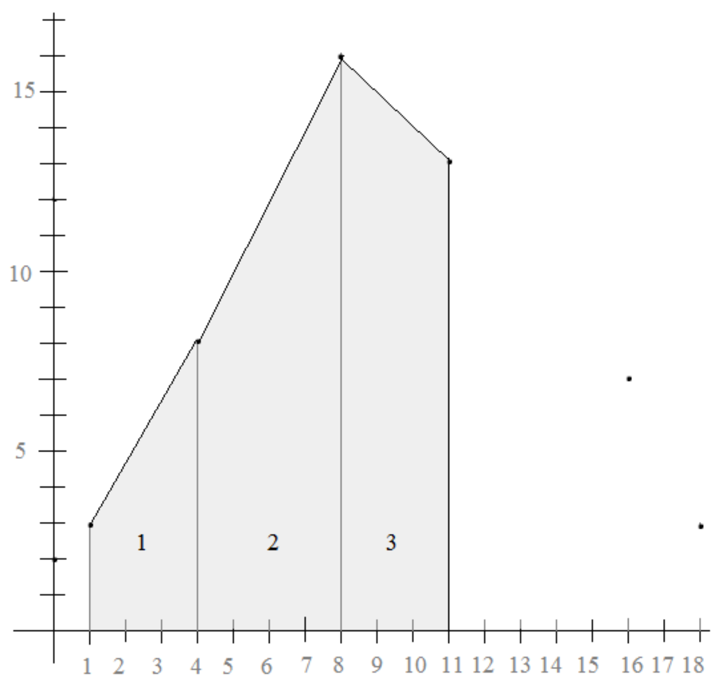
Also, since the integral is evaluating the interval from 1 to 11, we'll only use part of the table.

Trapezoid 1: $\frac{1}{2}(3 + 8)(3) = 33/2$

Trapezoid 2: $\frac{1}{2}(8 + 16)(4) = 48$

Trapezoid 3: $\frac{1}{2}(16 + 13)(3) = 87/2$

Total Area: $\boxed{108}$



Example: Use the trapezoid rule to approximate the area under the function $f(x) = -2x + 8$ for the interval [1, 4]. Use six sub-intervals. $(n = 6)$
Then, use the definite integral to determine the exact area.

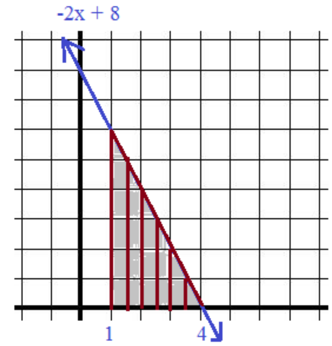
six sub-intervals

$f(1) = 6$
$f(1.5) = 5$
$f(2) = 4$
$f(2.5) = 3$
$f(3) = 2$
$f(3.5) = 1$
$f(4) = 0$

The total interval length is 3 units (from 1 to 4)
After dividing by 6, each sub-interval = .5

$$\frac{1}{2}(.5)\,[\,6 + 2(5) + 2(4) + 2(3) + 2(2) + 2(1) + 0\,]$$

$$= \frac{1}{2}(.5)\,[\,36\,] = \boxed{9}$$



Since the function is linear, the definite integral should equal the trapezoid approximation!

$$\int_1^4 -2x + 8 \; dx = -x^2 + 8x \Big|_1^4 = -(4)^2 + 8(4) - [\,-(1)^2 + 8(1)\,] = -16 + 32 - 7 = 9 \; \checkmark$$

Example: Use the trapezoid rule to approximate the area *between* the function $g(x) = -(\frac{1}{2}x)^2 + 5$, $x = 4$, the x-axis, and the y-axis.

a) Use 4 sub-intervals
b) Use 8 sub-intervals
c) Compare with the definite integral

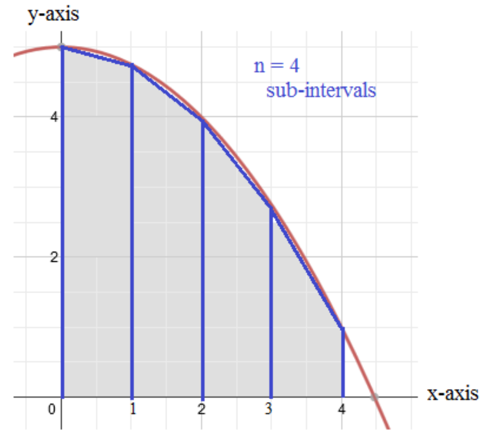
a) Using 4 sub-intervals, we evaluate:

interval is [0, 4] -- each sub-interval is 1

(the height height of each trapezoid)

$g(0) = 5$
$g(1) = 19/4$
$g(2) = 4$
$g(3) = 11/4$
$g(4) = 1$

(the bases of the trapezoids)



(sum of 4 trapezoids)

$$\frac{1}{2}(5 + 19/4)(1) + \frac{1}{2}(19/4 + 4)(1) + \frac{1}{2}(4 + 11/4)(1) + \frac{1}{2}(11/4 + 1)(1) =$$

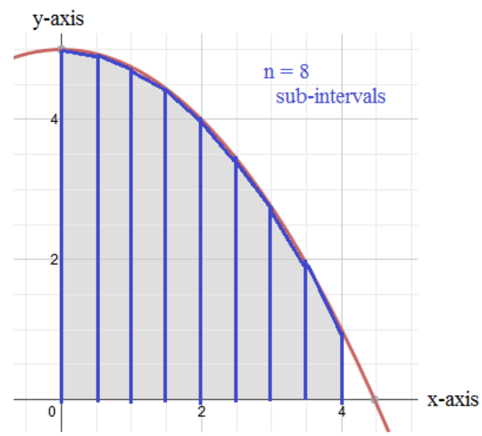
$$\frac{39}{8} + \frac{35}{8} + \frac{27}{8} + \frac{15}{8} = \boxed{14.5}$$

b) Using 8 sub-intervals:

interval is [0, 4] -- each sub-interval is .5

$g(0) = 5$
$g(1) = 19/4$
$g(2) = 4$
$g(3) = 11/4$
$g(4) = 1$

$g(.5) = 79/16$
$g(1.5) = 71/16$
$g(2.5) = 55/16$
$g(3.5) = 31/16$



(height) (all the bases of the trapezoids)

$$\frac{1}{2}(.5)\,[\,5 + 2(79/16) + 2(19/4) + 2(71/16) + 2(4) + 2(55/16) + 2(11/4) + 2(31/16) + 1\,]$$

$$\frac{1}{2}(.5)\,[\,14 + 236/8 + 30/2\,] = \frac{1}{2}(.5)\,[\,58.5\,] = \boxed{14.625}$$

c) Evaluate

$$\int_0^4 -(\tfrac{1}{2}x)^2 + 5 = \int_0^4 -\frac{x^2}{4} + 5 = -\frac{x^3}{12} + 5x \Big|_0^4$$

$$= \frac{-64}{12} + 20 = \boxed{14.6\overline{6}}$$

NOTE: Since the curve is concave down, the trapezoid rule underestimates the true area. Also, the more sub-intervals that are used, the more accurate the approximation!)

Math Parents -- and, embarrassed children -- on Halloween

LanceAF #57 (10-31-12)
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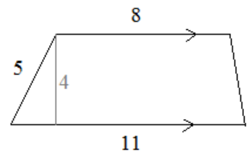
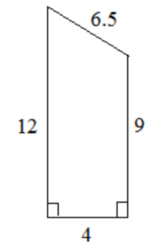
Practice Test

Calculus: Trapezoid Rule Quiz

I. Area of Trapezoid

Find the area of each trapezoid





II. Trapezoid Sums

1) Using the trapezoid rule, where the number of sub-intervals n = 4, approximate the area under $f(x)$ in the interval $[0, 4]$.

2) Use the table of values to estimate $\int_0^6 f(x)\,dx$

x	0	1	2	3	4	5	6
$f(x)$	.2	8	13	16	17	14	9

3) Use the table of values to estimate $\int_4^{18} g(x)\,dx$

x	0	1	4	8	11	16	18	20
$g(x)$	2	3	8	16	13	7	3	0

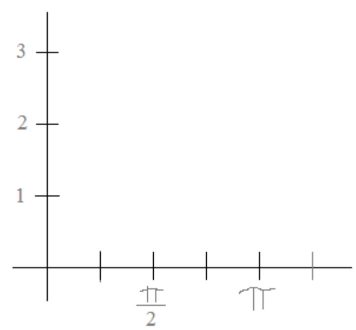
Calculus: Trapezoid Rule Quiz

III. Area under a curve

Approximate the following using the Trapezoid Rule. Then, compare with the evaluated definite integral.
(optional: Graph the functions, showing the area under the curves and the trapezoids)

1) $\int_0^4 \sqrt{x} + 2 \, dx$ (Use n = 4 sub-intervals)

2) $\int_0^{\pi} \cos x + 2 \, dx$ (Use n = 4 sub-intervals)

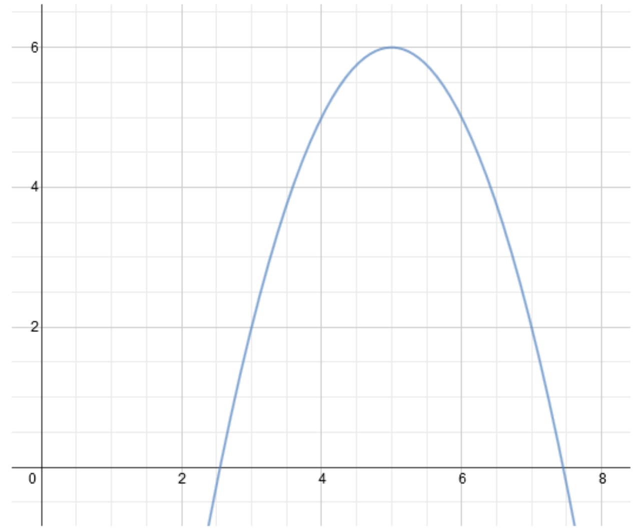


3) Using the trapezoidal rule, estimate the area between $y = x^3$ and the x-axis on the interval [-2, 2]
(Use n = 4 sub-intervals)

IV: Miscellaneous

Calculus: Trapezoid Rule Quiz

1) Approximate the area enclosed by lines x = 3 and x = 6, the function $f(x) = -x^2 + 10x - 19$, and the x-axis.



2) Determine whether the trapezoid rule would overestimate, underestimate, or equal the area under the function $h(x)$ for the given intervals:

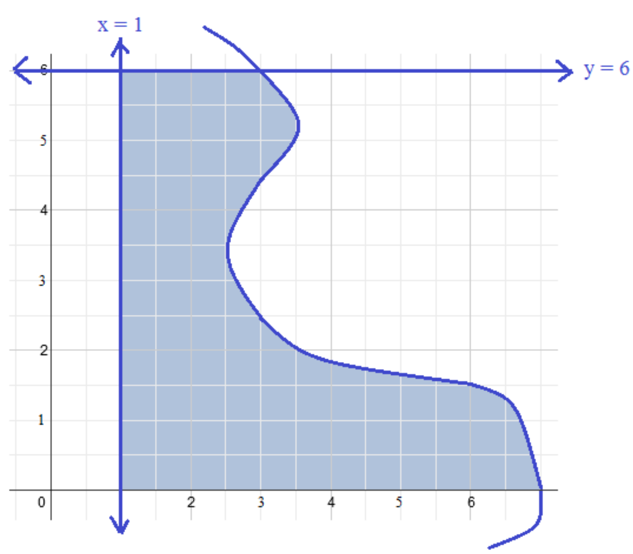
A) $y = \sin x$ $[0, \pi]$

B) $f(x) = 3x^2 + 7$ $[1, 4]$

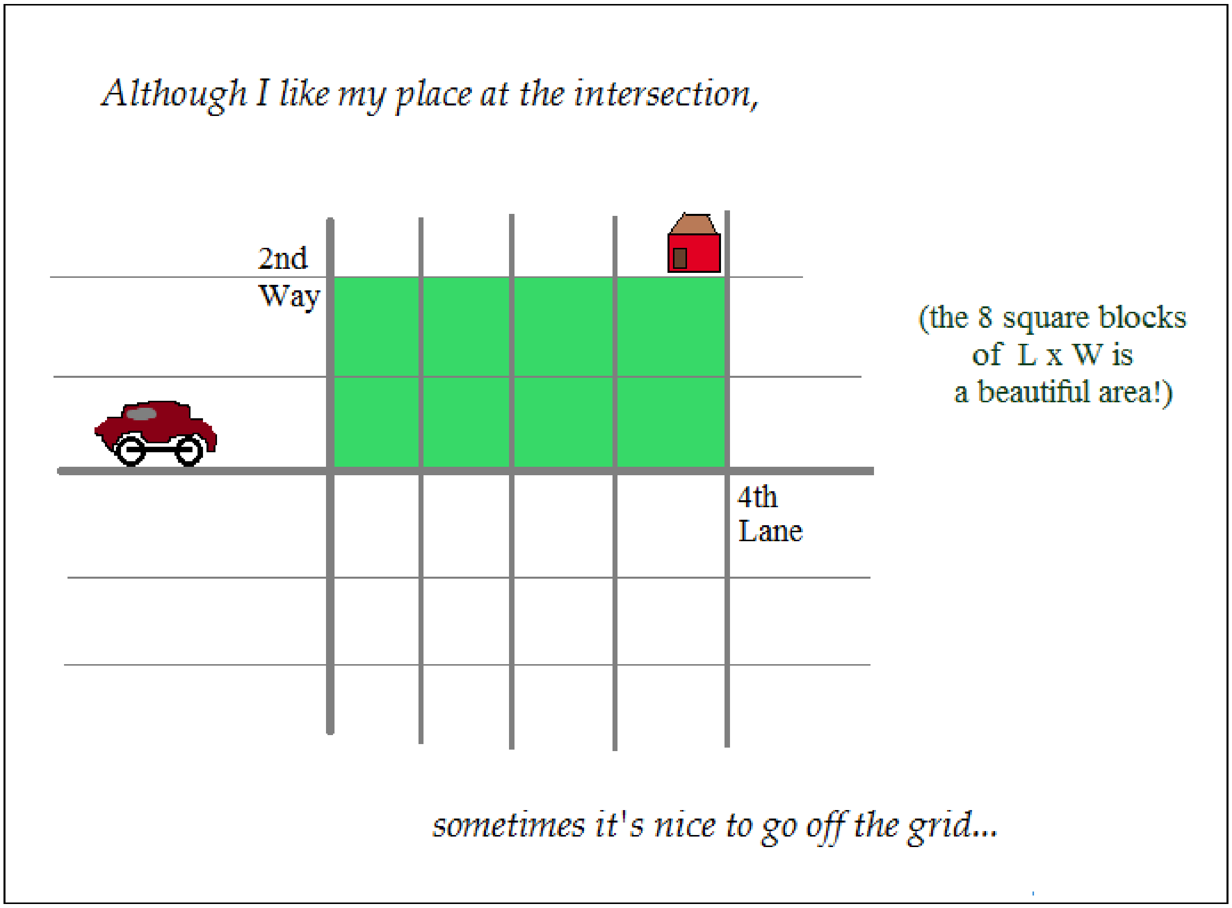
C) $y = 2|x + 5| + 7$ $[-4, 0]$

D) $g(x) = (x - 3)^3$ $[5, 10]$

3) Using the trapezoid rule, approximate the shaded area:



Area Code



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Solutions

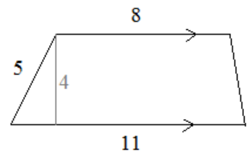
Calculus: Trapezoid Rule Quiz

SOLUTIONS

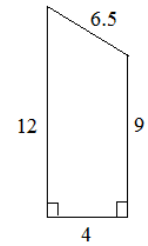
I. Area of Trapezoid

Find the area of each trapezoid

Area = $\frac{1}{2}$ (base$_1$ + base$_2$)(height)



Area = $\frac{1}{2}(8 + 11)(4)$ = 38



Area = $\frac{1}{2}(9 + 12)(4)$ = 42

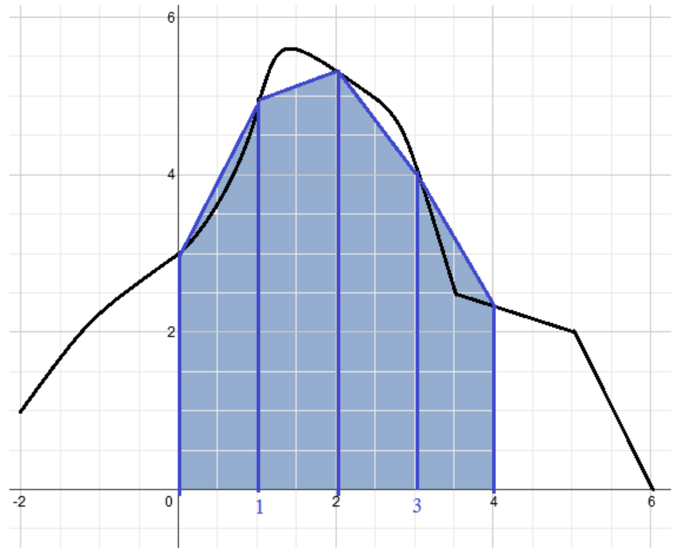
II. Trapezoid Sums

1) Using the trapezoid rule, where the number of sub-intervals n = 4, approximate the area under $f(x)$ in the interval $[0, 4]$.

The interval span is 4... So, each of the 4 sub-intervals will each have a span of 1. — height of each trapezoid

$f(0) = 3$
$f(1) = 5$
$f(2) = 5.25$
$f(3) = 4$
$f(4) = 2.33$

bases of the trapezoids



Trapezoid 1 $\frac{1}{2}(3 + 5)(1) = 4$

Trapezoid 2 $\frac{1}{2}(5 + 5.25)(1) = 5.125$

Trapezoid 3 $\frac{1}{2}(5.25 + 4)(1) = 4.625$

Trapezoid 4 $\frac{1}{2}(4 + 2.33)(1) = 3.165$

Total (approximate) area under the function: 16.915

2) Use the table of values to estimate $\int_0^6 f(x)\,dx$

x	0	1	2	3	4	5	6
$f(x)$	2	8	13	16	17	14	9

n = 6 sub-intervals (bases of the trapezoids)
sub-interval lengths are 1 (height of each trapezoid)

approximate value of above integral:

$\frac{1}{2}(1)[2 + 2(8) + 2(13) + 2(16) + 2(17) + 2(14) + 9]$

$\frac{1}{2}(1)[147]$ = 73.5

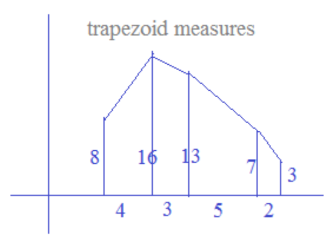
using 6 sub-intervals

3) Use the table of values to estimate $\int_4^{18} g(x)\,dx$

x	0	1	4	8	11	16	18	20
$g(x)$	2	3	8	16	13	7	3	0

Since the integral measures the function on the interval [4, 18], we'll only use part of the table!
Also, note the sub-intervals will have different lengths..

trapezoid measures



Area of trapezoids:

trapezoid 1: 48 $4 < x < 8$
trapezoid 2: 43.5 $8 < x < 11$
trapezoid 3: 50 $11 < x < 16$
trapezoid 4: 10 $16 < x < 18$

total estimate: 151.5

Calculus: Trapezoid Rule Quiz

III. Area under a curve

Approximate the following using the Trapezoid Rule. Then, compare with the evaluated definite integral.
(optional: Graph the functions, showing the area under the curves and the trapezoids)

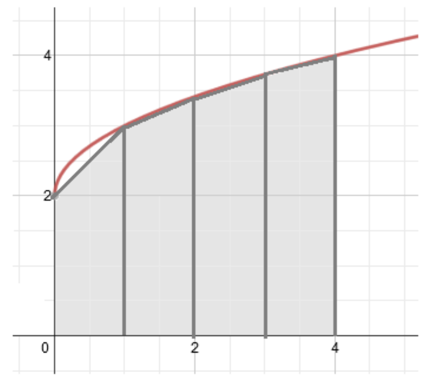
1) $\displaystyle\int_0^4 \sqrt{x} + 2 \; dx$ (Use n = 4 sub-intervals)

since the function's interval is 0 to 4, each equal sub-interval will be 1 unit.

Then, we evaluate the function at each sub-interval:

$f(0) = 2$
$f(1) = 3$
$f(2) = 3.41$
$f(3) = 3.73$
$f(4) = 4$

(these are the lengths of the bases of the 4 trapezoids)



sum of the 4 trapezoids:

$$\frac{1}{2}(2+3)(1) + \frac{1}{2}(3+3.41)(1) + \frac{1}{2}(3.41+3.73)(1) + \frac{1}{2}(3.73+4)(1) =$$

$$2.5 + 3.2 + 3.57 + 3.87 = \boxed{13.14}$$

$$\int_0^4 \sqrt{x} + 2\,dx = \frac{2}{3}x^{\frac{3}{2}} + 2x \Big|_0^4 = \frac{16}{3} + 8 - (0+0) = \boxed{13.\overline{33}}$$

Since the curve is concave down, the trapezoids will *underestimate* the actual value under the curve.

2) $\displaystyle\int_0^{\pi} \cos x + 2 \; dx$ (Use n = 4 sub-intervals)

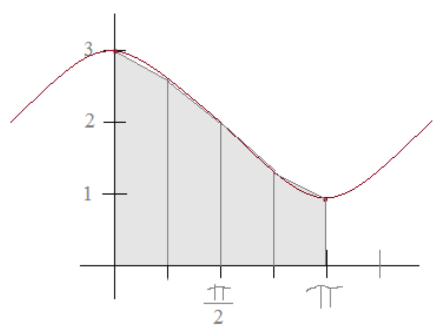
Since there are 4 sub-intervals, we need to find the following values:

Each sub-interval is $\frac{\pi}{4}$

$\cos(0) + 2 = 1 + 2 = 3$
$\cos(\frac{\pi}{4}) + 2 = \frac{1}{\sqrt{2}} + 2 = 2.707$
$\cos(\frac{\pi}{2}) + 2 = 0 + 2 = 2$
$\cos(\frac{3\pi}{4}) + 2 = \frac{-1}{\sqrt{2}} + 2 = 1.293$
$\cos(\pi) + 2 = -1 + 2 = 1$

The sum of the trapezoids:

$$\frac{1}{2} \cdot \frac{\pi}{4}\Big(3 + 2(2.707) + 2(2) + 2(1.293) + 1\Big) = \frac{\pi}{8} \cdot (16) = \boxed{2\pi}$$

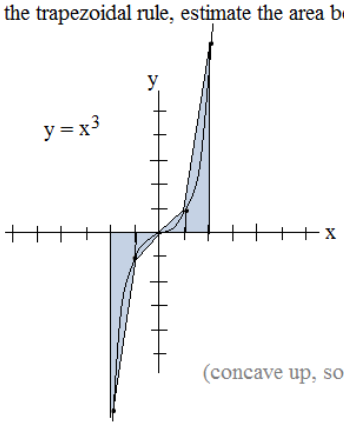


$$\int_0^{\pi} \cos x + 2\,dx = \sin x + 2x \Big|_0^{\pi} = \sin(\pi) + 2(\pi) - [\sin(0) + 2(0)]$$

$$\boxed{= 2\pi \text{ (or approx. 6.28)}}$$

NOTE: From 0 to $\frac{\pi}{2}$, the function is concave down. (so, trapezoids underestimate the area). But, from $\frac{\pi}{2}$ to π, the function is concave up. (so, trapezoids overestimate the area).

3) Using the trapezoidal rule, estimate the area between $y = x^3$ and the x-axis on the interval [-2, 2]
(Use n = 4 sub-intervals)



(concave up, so overestimates)

4 sub-intervals: 1 unit each

4 bases:

(-2, -8)	to (-2, 0)	8
(-1, -1)	to (-1, 0)	1
(0, 0)	to (0, 0)	0
(1, 1)	to (1, 0)	1
(2, 8)	to (2, 0)	8

base lengths: 8, 1, 0, 1, 8

areas must be positive values! $9/2 + 1/2 + 1/2 + 9/2 = \boxed{10}$

$$\int_0^2 x^3\,dx = \frac{x^4}{4}\Big|_0^2 = 4$$ so, exact area is 8...

SOLUTIONS

IV: Miscellaneous

1) Approximate the area enclosed by lines x = 3 and x = 6, the function $f(x) = -x^2 + 10x - 19$, and the x-axis.

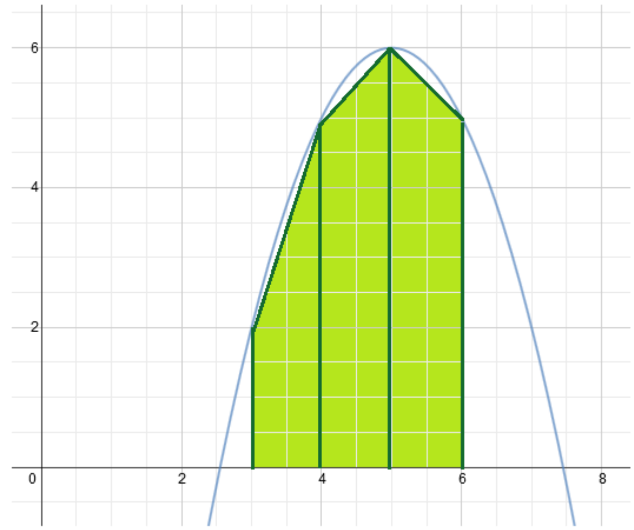
Using 3-subintervals (n = 3),
The bases of the trapezoids will extend from the x-axis to

$f(3) = 2 \quad f(4) = 5 \quad f(5) = 6 \quad f(6) = 5$

The height of each trapezoid will be 1. $\triangle x = \frac{b-a}{n} = \frac{6-3}{3} = 1$

Area of 3 trapezoids $= \frac{\triangle x}{2}\left[f(3) + 2f(4) + 2f(5) + f(6)\right] =$

$\frac{1}{2}[2 + 10 + 12 + 5] =$ **14.5**



Using Integration to find exact area:

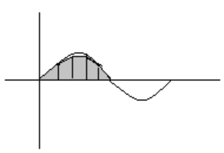
$$\int_3^6 -x^2 + 10x - 19\,dx \qquad \left.\frac{-x^3}{3} + 5x^2 - 19x\right|_3^6 =$$

$$\frac{-216}{3} + 180 - 114 - [-9 + 45 - 57] = -6 - [-21] = 15$$

2) Determine whether the trapezoid rule would overestimate, underestimate, or equal the area under the function $h(x)$ for the given intervals:

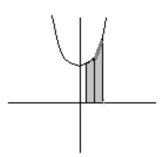
A) $y = \sin x \quad [0, \pi]$

concave down -- **underestimate**



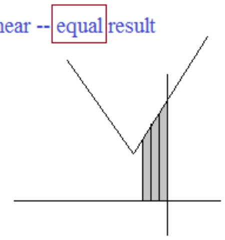
B) $f(x) = 3x^2 + 7 \quad [1, 4]$

concave up -- **overestimate**



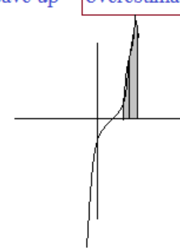
C) $y = 2|x + 5| + 7 \quad [-4, 0]$

linear -- **equal** result



D) $g(x) = (x - 3)^3 \quad [5, 10]$

concave up -- **overestimate**



3) Using the trapezoid rule, approximate the shaded area:

There are many approximations...
For example, divide the area into 6 equal sub-intervals...

		base lengths
At y = 6,	base extends from 1 to 3	2
y = 5,	base extends from 1 to 3.5	2.5
y = 4,	base extends from 1 to 2.75	1.75
y = 3,	base extends from 1 to 2.75	1.75
y = 2,	base extends from 1 to 3.5	2.5
y = 1,	base extends from 1 to 6.75	5.75
y = 0,	base extends from 1 to 7	6

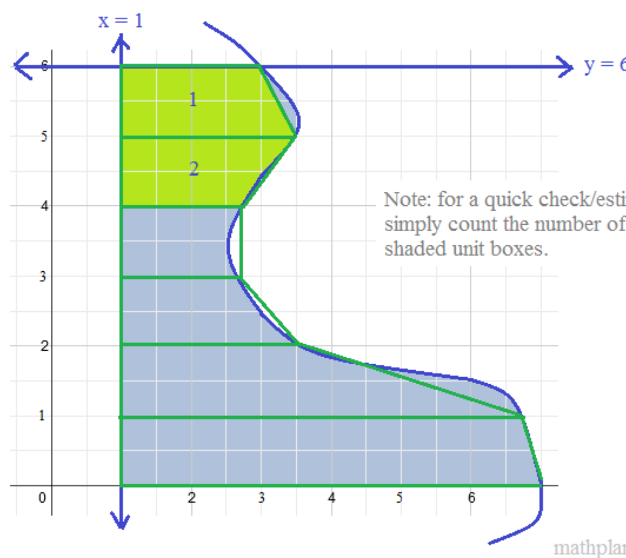
trapezoid 1: $\frac{1}{2}(2 + 2.5)(1) = 2.25$

trapezoid 2: $\frac{1}{2}(2.5 + 1.75)(1) = 2.125$

etc...

approximate area $= \frac{1}{2}(1)[2 + 2(2.5) + 2(1.75) + 2(1.75) + 2(2.5) + 2(5.75) + 6]$

$\frac{1}{2}(1)[36.5] =$ **18.25**



Note: for a quick check/estimate, simply count the number of shaded unit boxes.

Thanks for checking out this packet. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers, LAF

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