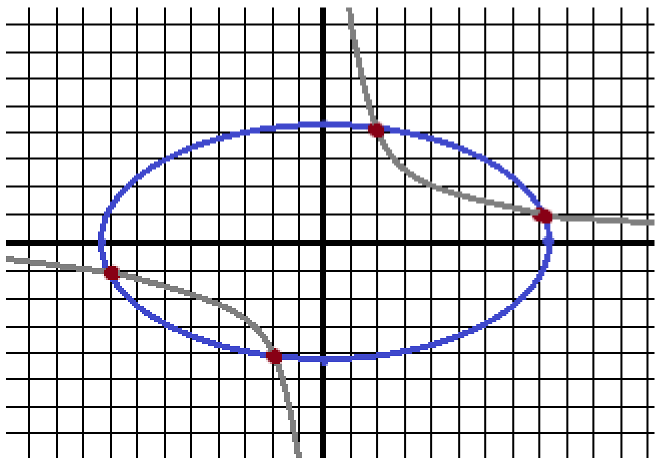
Conics III – Review

Examples, Applications, and Practice Exercises (w/Solutions)



Topics include solving systems, graphing, identifying conics, applications, and more.

Example: $x = .2y^2 - 2y - 3$

Identify:

Vertex:

Focus:

Directrix:

x-intercept(s)?

y-intercept(s)?

Then, graph....

Since there is no x^2 term, this is a parabola.

To reveal the properties, change to standard form.

$x = .2y^2 - 2y - 3$ multiply equation by 5 (to remove decimal)

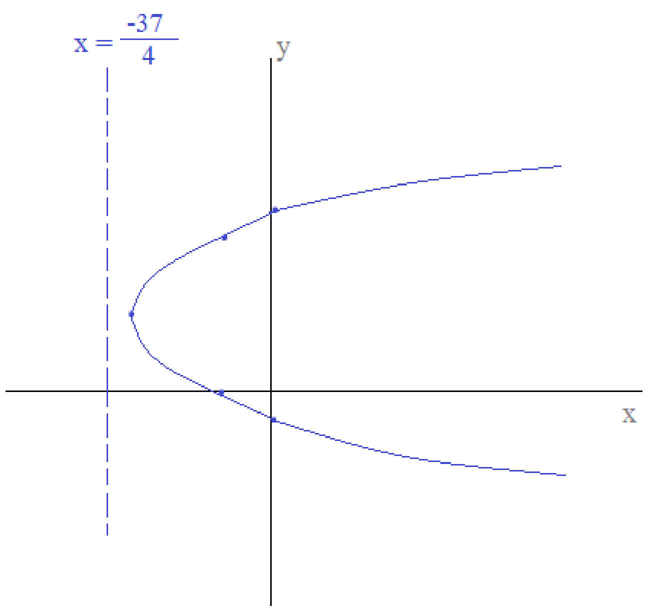
$5x = y^2 - 10y - 15$ complete the square

$5x = y^2 - 10y + 25 \quad - 15 - 25$

$5x = (y-5)^2 - 40$ collect terms, factor and simplify

$x = \frac{1}{5}(y-5)^2 - 8$

$$\boxed{x = a(y-k)^2 + h}$$



Vertex: $(h, k) = (-8, 5)$

since $a > 0$, parabola opens to the right

$p = \frac{1}{4a}$ so, $p = \frac{5}{4}$

So, the focus is $\left(\frac{-27}{4}, 5\right)$

And, the directrix is $x = \frac{-37}{4}$

x-intercept: $(x, 0)$

$x = .2y^2 - 2y - 3$ $(-3, 0)$

y-intercept: $(0, y)$

$x = \frac{1}{5}(y-5)^2 - 8$

$0 + 8 = \frac{1}{5}(y-5)^2$

$40 = y^2 - 10y + 25$ $(0, 5 + 2\sqrt{10})$

$y^2 - 10y - 15 = 0$ $(0, 5 - 2\sqrt{10})$

Example: $\dfrac{(x+4)^2}{16} - \dfrac{y^2}{16} = 1$

Identify:

Vertices:

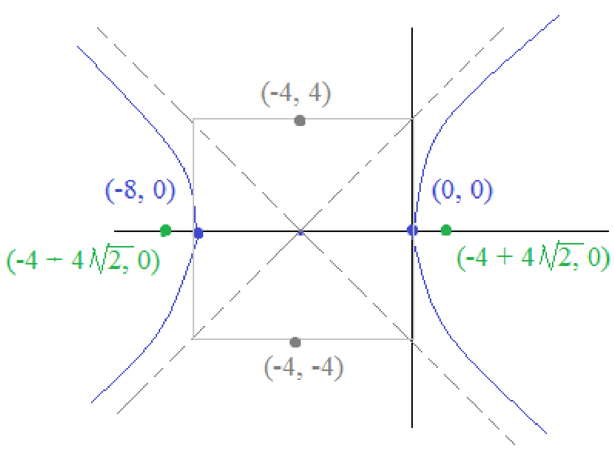
Foci:

Asymptotes:

Length of major axis:

Length of minor ('conjugate') axis:

Sketch a graph...



Since the coefficients are 1/16 and -1/16, this is a hyperbola.

The x term is positive; the y term is negative.
So, the hyperbola will open to the left and right.

The major axis will be horizontal.
The conjugate axis will be vertical.

$$\boxed{\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1}$$

center: (-4, 0)

$a = 4$

$b = 4$

vertices: (0, 0) (-8, 0)

$c^2 = a^2 + b^2$ $\quad c = 4\sqrt{2}$

foci: $(-4 + 4\sqrt{2}, 0)$

$(-4 - 4\sqrt{2}, 0)$

(asymptotes)

$y = x + 4$ $\quad$ asymptotes go through the center (-4, 0)

$y = -x - 4$ $\quad$ slope is $+a/b$ and $-a/b$

Length of major axis = 2a = 8 "semi-major axis" = a

Length of conjugate axis = 2b = 8 "semi-minor axis" = b

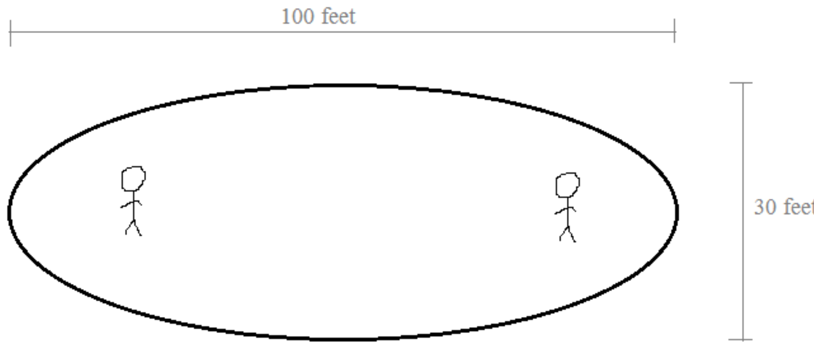
Example: Whispering Gallery

A Whispering Gallery (or, Whisper Chamber) is usually an elliptical enclosed space under a dome where whispers can be heard in different parts of the gallery.

The (ellipsoid) whisper chamber at the science museum is 100 feet long and 30 feet wide.
If my friend and I use it, how far apart do we stand?

To use the whisper chamber, each of us will stand on a focus.

So, where are the foci in this ellipse?



$$a = \frac{1}{2}100 = 50$$

$$b = \frac{1}{2}30 = 15$$

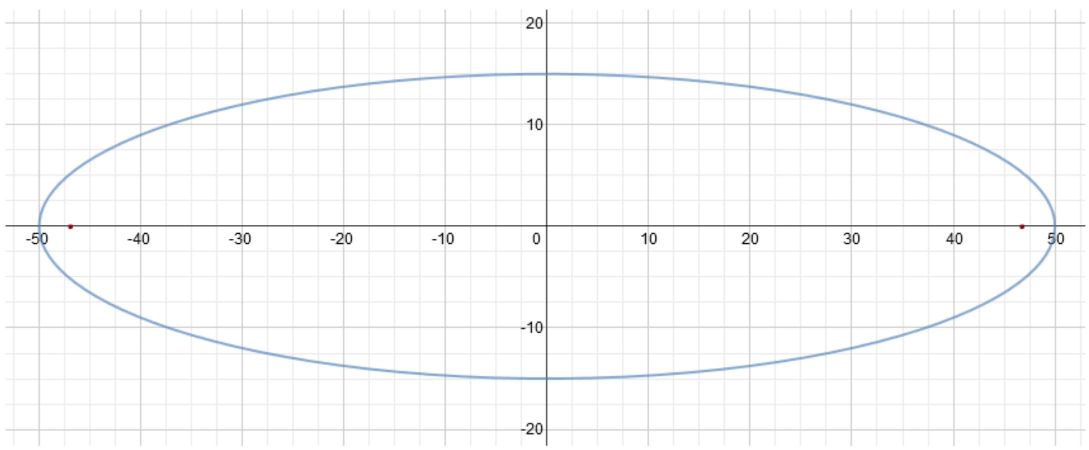
$$c^2 = a^2 - b^2$$

$$c^2 = 2500 - 225$$

$$c = \sqrt{2275} = 47.7$$

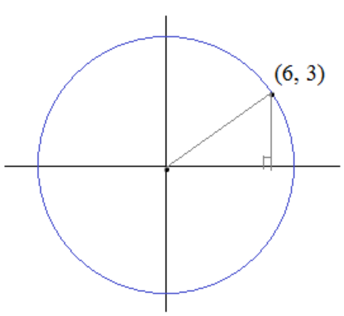
Each of us should stand 47.7 feet from the center..

Therefore, we will be 95.4 feet apart!



Example: What is the equation of a circle where the center is the origin and one of the points on the circle is (6, 3)?

First, draw a quick sketch:



$$x^2 + y^2 = 45$$

Second, recognize the standard form of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

where (h, k) is the center and r is the radius

We know the center is (0, 0)...
We need the radius, which is the distance between any point on the circle and the center....

Distance between points:

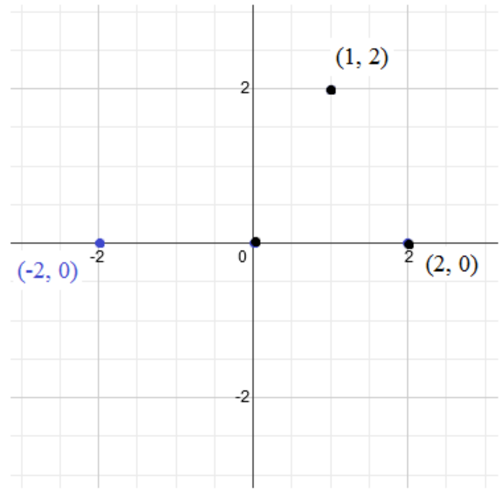
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(6 - 0)^2 + (3 - 0)^2}$$

$$= \sqrt{45}$$

Therefore, the radius is 45

Example:
Given: An ellipse with center (0, 0) and points (1, 2) and (2, 0).
Find: The equation of the ellipse.



Since the center is (0, 0), the vertices must be horizontal and vertical from that point..
(i.e. the vertices will be on the x-axis and y-axis)

Note: We're assuming this ellipse is not 'rotated'.
For example, (1, 2) and (-1, -2) could be vertices.
But, for convenience we'll use (2, 0)

Since (2, 0) is a point on the ellipse, it is one of the vertices. And, on the other side, (-2, 0) is another vertex.

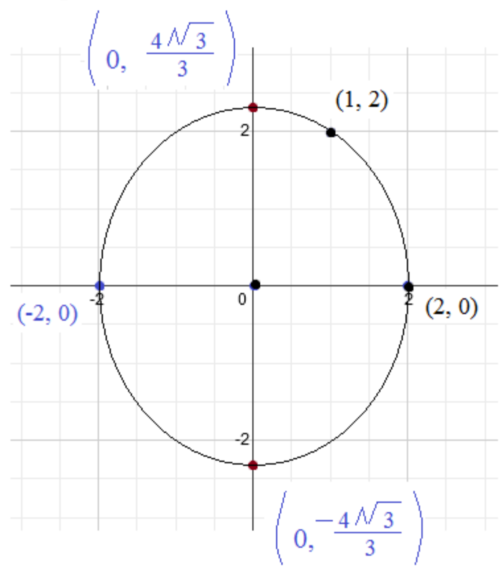
Standard form of an ellipse (vertical)

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

The distance from (0, 0) to (1, 2) = $\sqrt{5}$

The distance from (0, 0) to (2, 0) = 2

Since the distance to (1, 2) > (2, 0), the ellipse will be *vertical*.
(the major axis will be vertical)



To find a, use substitution w/ point (1, 2):

$$\frac{(1-0)^2}{2^2} + \frac{(2-0)^2}{a^2} = 1$$

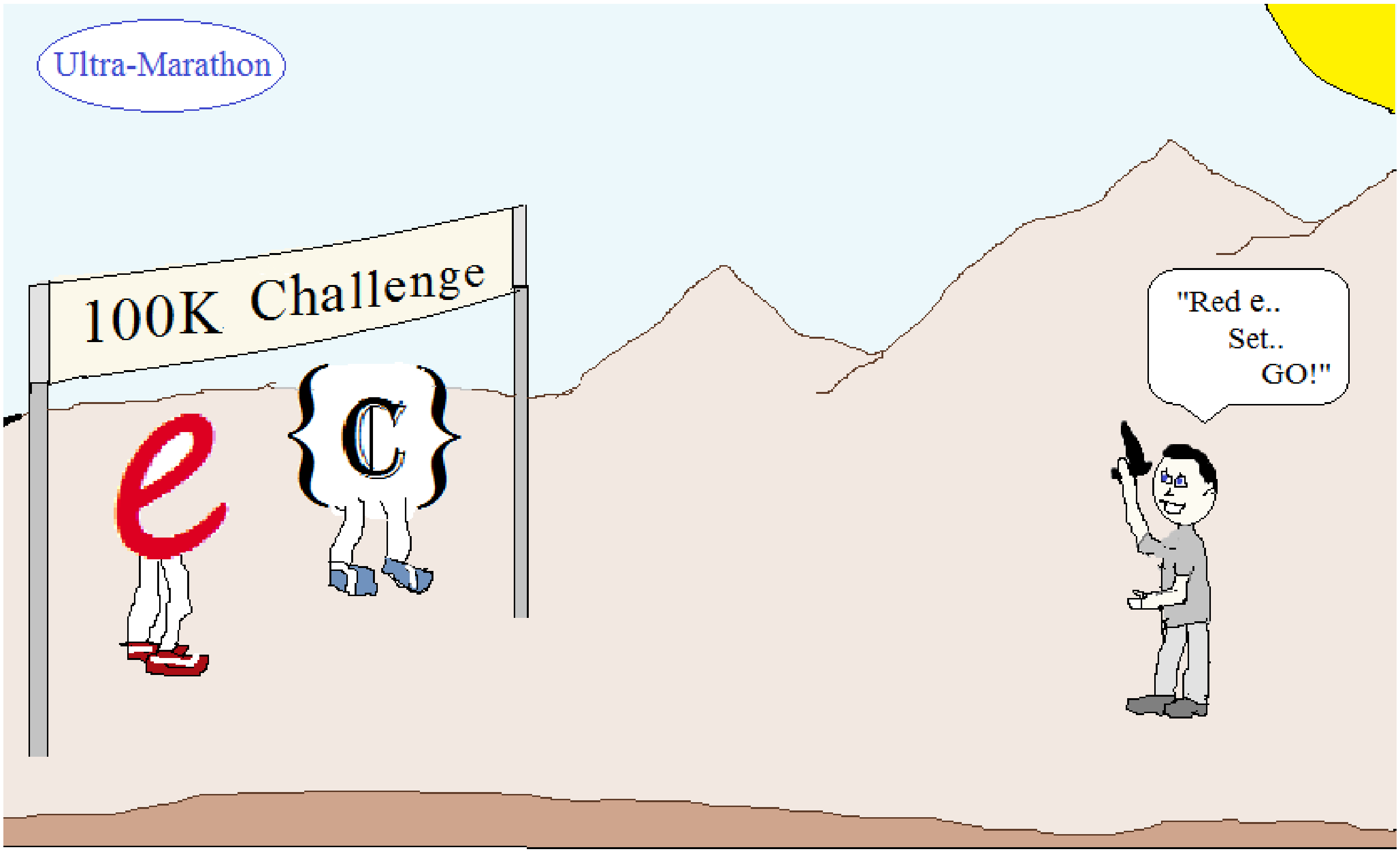
$$\frac{1}{4} + \frac{4}{a^2} = 1$$

$$\frac{4}{a^2} = \frac{3}{4}$$

$$a^2 = \frac{16}{3}$$

$$a = \pm\frac{4\sqrt{3}}{3} \quad \text{or approx. } \pm 2.31$$

$$\frac{x^2}{4} + \frac{y^2}{\frac{16}{3}} = 1 \longrightarrow \boxed{\frac{x^2}{4} + \frac{3y^2}{16} = 1}$$



Testing the limits of endurance,
these math figures will run on and on...

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Practice Exercises (and Solutions) -→

Conics Questions

1) $9x^2 - y^2 - 90x + 4y + 302 = 0$

Find: a) center

b) vertices

c) foci

d) asymptotes

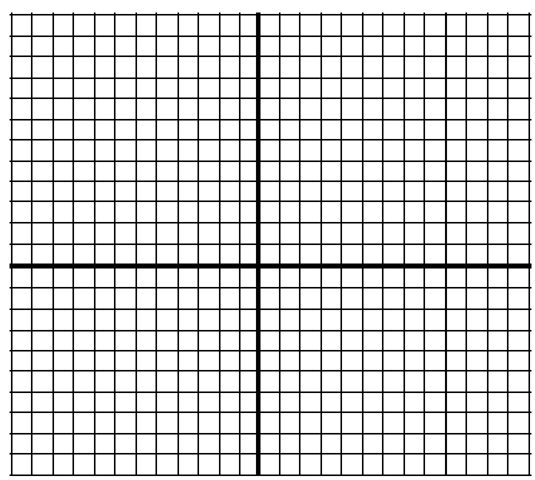
2) Write an equation for an ellipse that passes through (2, 5) and (6, 3)

Bonus: Write an equation for <u>another</u> ellipse that passes through (2, 5) and (6, 3)

3) Solve the system <u>algebraically</u>.
Then, sketch to verify the solutions <u>graphically</u>.

$xy = 8$

$x^2 + 4y^2 = 68$



4) sketch examples of the following systems:

a) 2 conics; two solutions

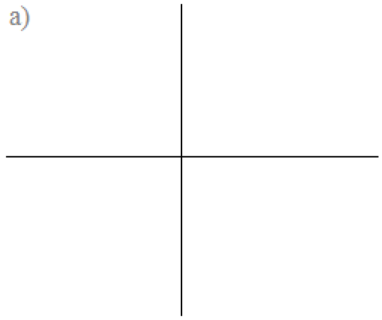
b) 2 conics; three solutions

c) circle and ellipse; four solutions

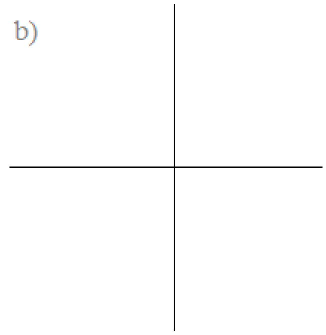
d) line and hyperbola; one solution

e) hyperbola and parabola; no solutions

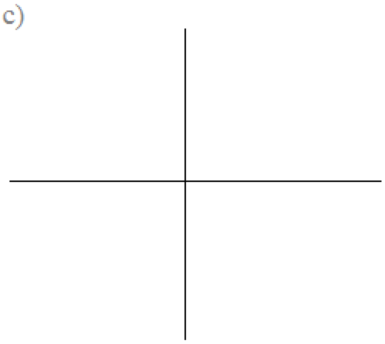
a)



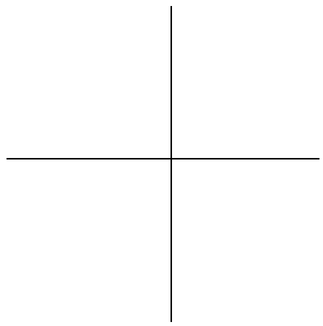
b)



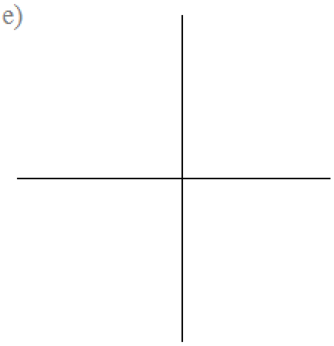
c)



d)



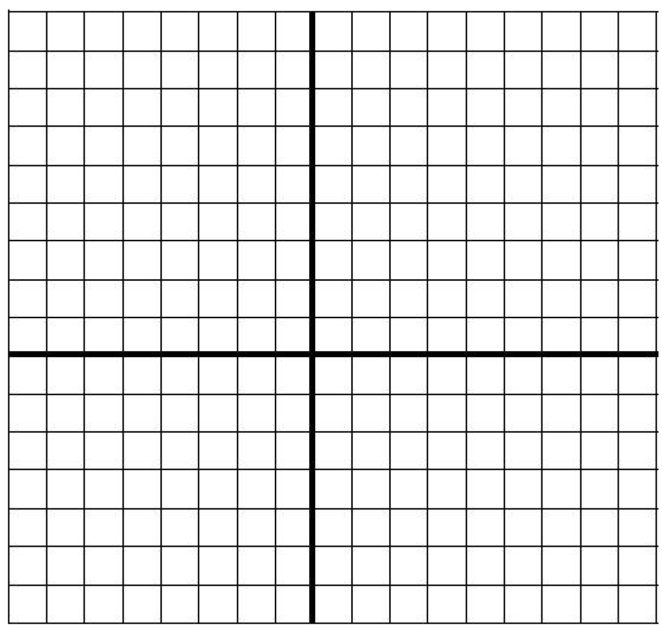
e)



5) Solve the following system (algebraically).
Then, sketch the system and verify the solutions.

$$x^2 + y^2 + 8x = -15$$

$$9x^2 + 25y^2 = 225$$



Conics Questions

6) Given: eccentricity $e = 4$
vertices: $(-2, 7)$ and $(-2, 3)$

Find the equation of the conic.

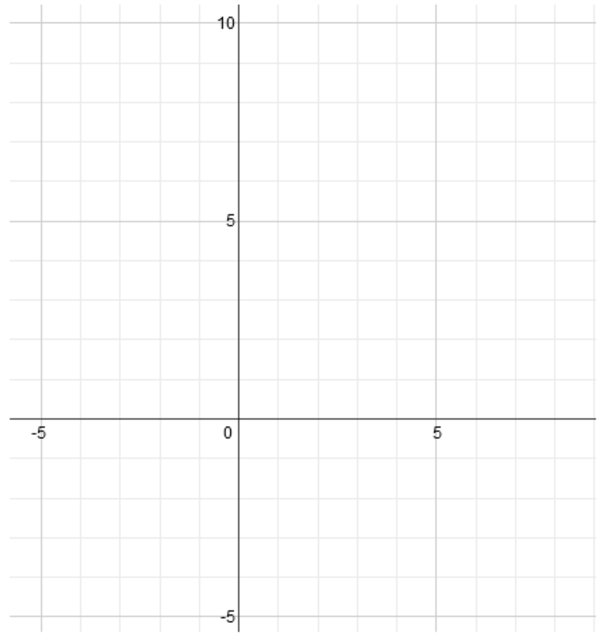
7) Given: Vertices $(3, -8)$ and $(3, -2)$
Asymptotes $y = 3x - 14$
$y = -3x + 4$

What is the equation of the hyperbola?

8) A hyperbola with vertices $(4, 0)$ and $(-4, 0)$ contains the point $(-5, 3)$.
What is the equation (in general form) of the hyperbola?

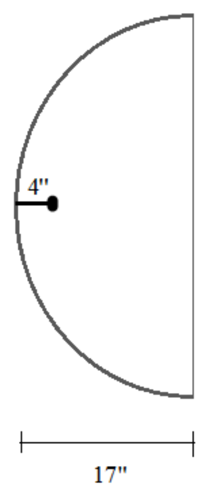
9) For the equation $\frac{(x+1)^2}{9} + \frac{(y-3)^2}{16} = 1$ (Calculator necessary) Conics Questions

What are the y-intercept(s) and x-intercept(s)?
Sketch a graph and label the intercepts.



10) A microphone has a parabolic cross-section that is 17 inches deep with a focus that is 4 inches from the vertex.

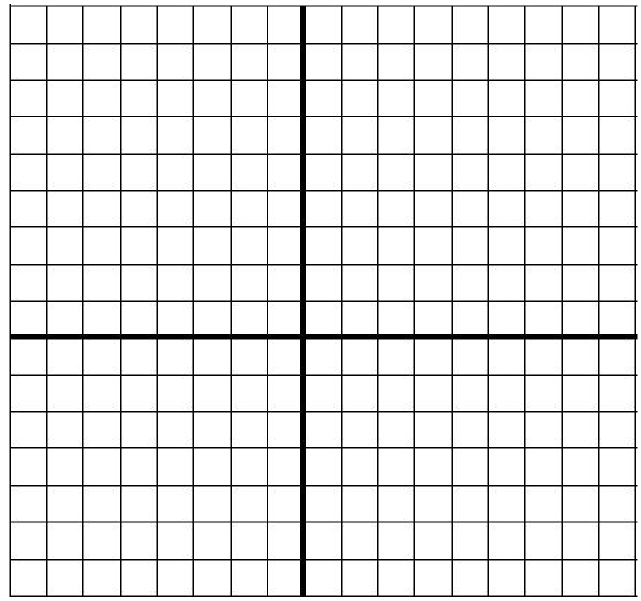
What is the diameter of the microphone opening?



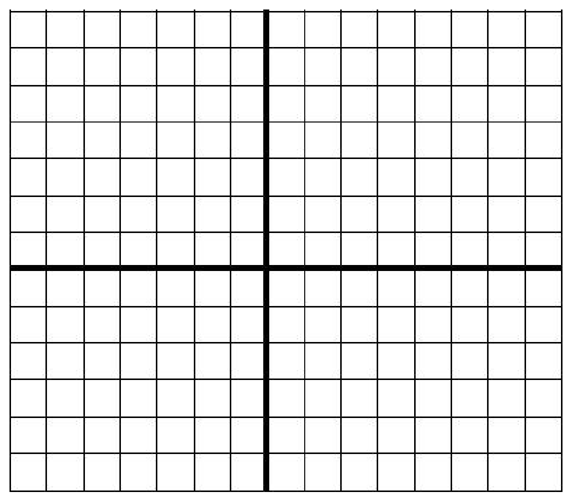
Graphing/Identifying Conics Exercise

Sketch the following conics:

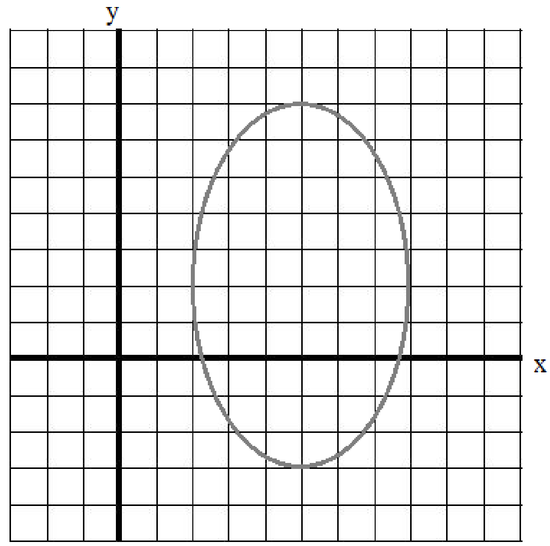
1) $x^2 - y^2 + 4x + 8y - 21 = 0$



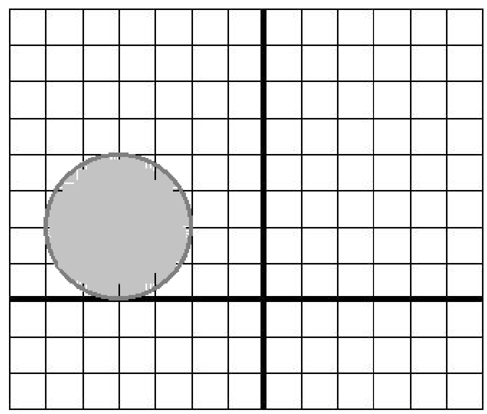
2) $x^2 + (y - 3)^2 = 4$



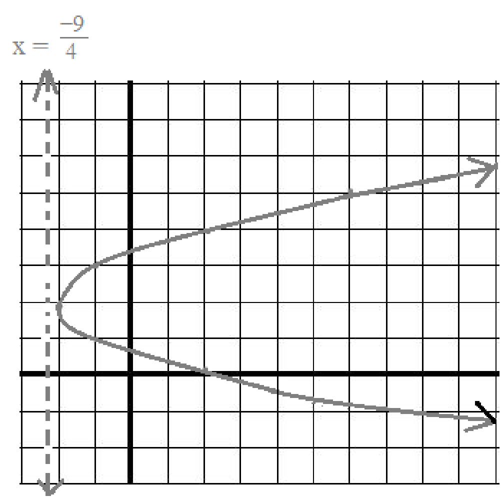
3) Determine the equation in the graph:



4) Write the equation represented in the graph:



5) Write the equation for the parabola:



Describe the figure:

1) $3x^2 + 5y^2 = 17$

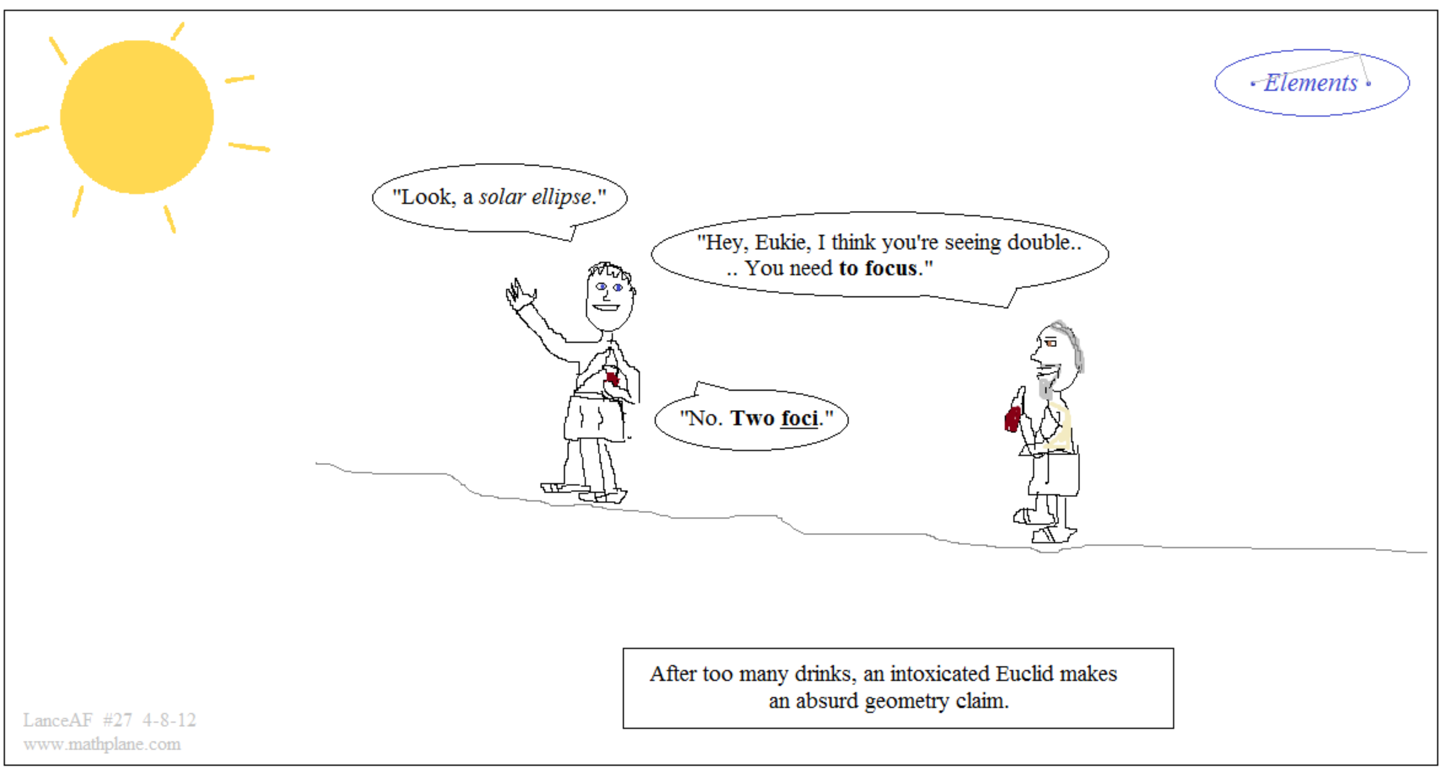
2) $3x^2 - 5y^2 = 17$

3) $3x^2 + 3y^2 = 17$

4) $3x - 3y = 17$

5) $3x^2 + y = 17$

6) $3x^2 - 3y^2 = 17$



Conics Questions

SOLUTIONS

1) $9x^2 - y^2 - 90x + 4y + 302 = 0$

Find: a) center $(5, 2)$

b) vertices $(5, 11)$ and $(5, -7)$

c) foci $(5, 2 + \sqrt{90})$ and $(5, 2 - \sqrt{90})$

d) asymptotes $y - 2 = 3(x - 5)$

$y - 2 = -3(x - 5)$

complete the square to put into standard form:

$$9(x^2 - 10x \quad) - 1(y^2 - 4y \quad) = -302$$

$$9(x^2 - 10x + 25) - 1(y^2 - 4y + 4) = -302 + 225 - 4$$

$$9(x-5)^2 - 1(y-2)^2 = -81$$

(divide by *negative* 81)

$$\frac{(y-2)^2}{81} - \frac{(x-5)^2}{9} = 1$$

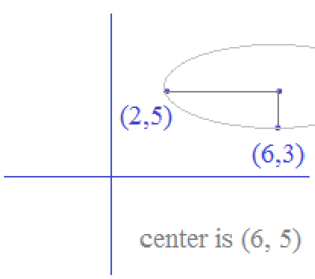
$(h, k) = (5, 2)$
$a = 3$
$b = 9$
$c = \sqrt{90}$

(note: this is a *vertical* hyperbola)

slope of asymptotes: 3 and -3

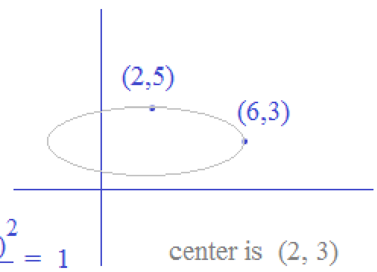
2) Write an equation for an ellipse that passes through (2, 5) and (6, 3)

Bonus: Write an equation for <u>another</u> ellipse that passes through (2, 5) and (6, 3)



$$\frac{(x-6)^2}{16} + \frac{(y-5)^2}{4} = 1$$

center is (6, 5) 'a' = 4 'b' = 2



$$\frac{(x-2)^2}{16} + \frac{(y-3)^2}{4} = 1$$

center is (2, 3)
'a' = 4
'b' = 2

3) Solve the system <u>algebraically</u>.
Then, sketch to verify the solutions <u>graphically</u>.

$xy = 8 \qquad y = \frac{8}{x}$

$x^2 + 4y^2 = 68$

Use substitution:

$$x^2 + 4\left(\frac{8}{x}\right)^2 = 68$$

$$x^2 + \frac{256}{x^2} = 68 \qquad \text{multiply by } x^2$$

$$x^4 - 68x^2 + 256 = 0$$

$$(x^2 - 4)(x^2 - 64) = 0$$

$$x = -2, 2, -8, 8$$

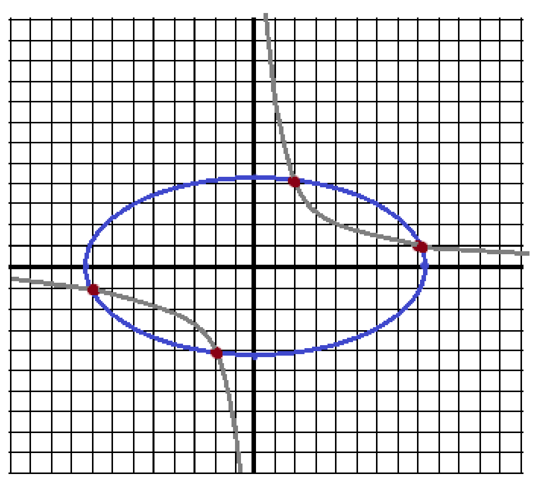
substitute to find y

$\boxed{(-2, -4) \ (2, 4) \ (-8, 1) \ (8, 1)}$

xy = 8 is a ("tilted") hyperbola

$$\frac{x^2}{68} + \frac{y^2}{17} = 1$$

(center: origin)
semi-major axis length (a) : $\sqrt{68}$
semi-minor axis length (b) : $\sqrt{17}$



4) sketch examples of the following systems:

SOLUTIONS

a) 2 conics; two solutions

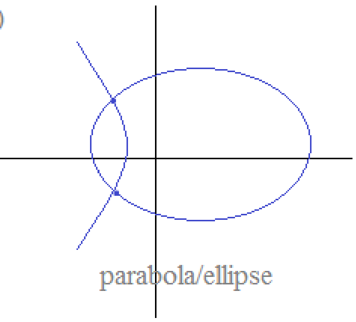
b) 2 conics; three solutions

c) circle and ellipse; four solutions

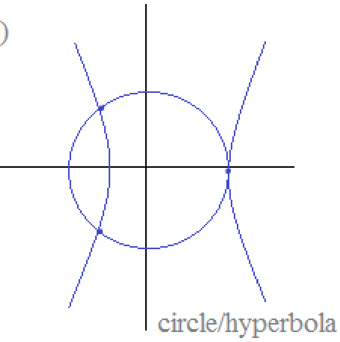
d) line and hyperbola; one solution

e) hyperbola and parabola; no solutions

a)



b)



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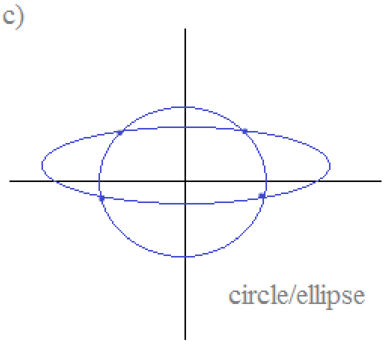

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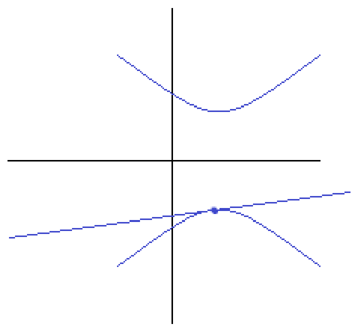

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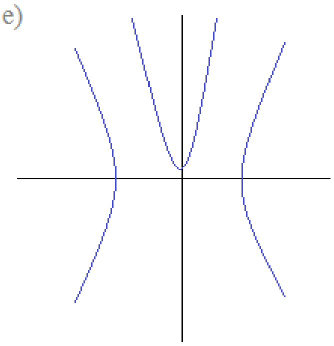


5) Solve the following system (algebraically).
Then, sketch the system and verify the solutions.

$$x^2 + y^2 + 8x = -15$$

$$9x^2 + 25y^2 = 225$$

$$y^2 = -x^2 - 8x - 15$$

Use substitution:

$$9x^2 + 25(-x^2 - 8x - 15) = 225$$

$$-16x^2 - 200x - 375 = 225$$

$$16x^2 + 200x + 600 = 0$$

$$2x^2 + 25x + 75 = 0$$

$$(2x + 15)(x + 5) = 0$$

$$x = -5 \text{ or } \cancel{-15/2}$$

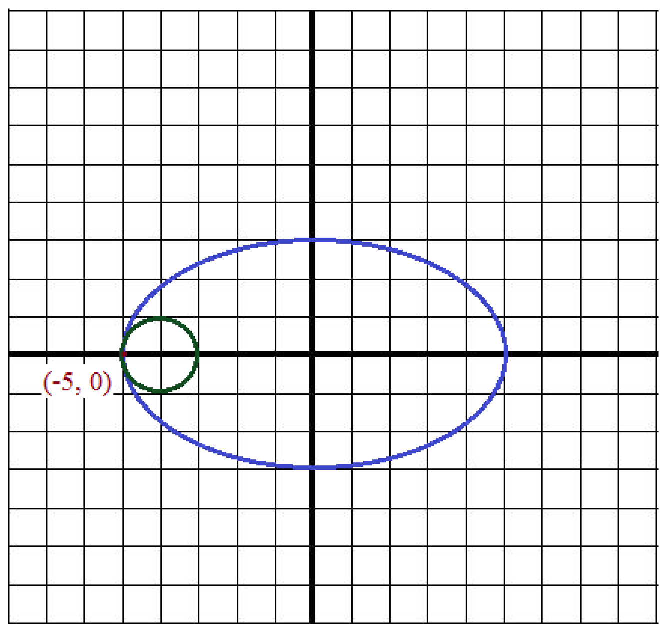
(-5, 0) is the only real solution

If $x = -5$, then $y = 0$

But, if $x = -15/2$ then y ????

$$225/4 + y^2 - 60 = -15$$

$$y^2 = -45/4 \quad \text{EXTRANEOUS!!}$$



Ellipse:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Circle:

$$x^2 + 8x + 16 + y^2 = -15 + 16$$

$$(x + 4)^2 + y^2 = 1$$

Conics Questions

SOLUTIONS

$e = \frac{c}{a}$ (horizontal hyperbola)

$e = \frac{c}{b}$ (vertical hyperbola)

6) Given: eccentricity $e = 4$
vertices: $(-2, 7)$ and $(-2, 3)$

Find the equation of the conic.

Since the eccentricity is 4 > 0, it is a hyperbola

Since the vertices are (-2, 7) and (-2, 3),

the center is (-2, 5) midpoint of the vertices
and
it's a 'vertical hyperbola'

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

$\frac{c}{b} = 4$ $b = 2$ then, $c = 8$

$a^2 + b^2 = c^2$

$a^2 + (2)^2 = (8)^2$

$a^2 = 60$

center: (-2, 5)

vertices: (-2, 7) and (-2, 3)

$a^2 = 60$

$b^2 = 4$

$$\boxed{\frac{(y-5)^2}{4} - \frac{(x+2)^2}{60} = 1}$$

7) Given: Vertices (3, -8) and (3, -2)
Asymptotes $y = 3x - 14$
$y = -3x + 4$

(vertical hyperbola)

What is the equation of the hyperbola?

The center of the hyperbola is the midpoint of the vertices
(or, the intersection of the asymptotes)

(3, -5)

the slopes of asymptotes are $\frac{b}{a}$ and $\frac{-b}{a}$

$b = 3$

therefore, $a = 1$

$$\boxed{\frac{(y+5)^2}{9} - \frac{(x-3)^2}{1} = 1}$$

8) A hyperbola with vertices (4, 0) and (-4, 0) contains the point (-5, 3).
What is the equation (in general form) of the hyperbola?

The center is the origin (0, 0) (the midpoint of the vertices)

And, it is a horizontal hyperbola, where a = 4

$$\frac{(x-0)^2}{4^2} - \frac{(y-0)^2}{b^2} = 1$$

To find b, just plug in the given point!

$$\frac{(-5-0)^2}{4^2} - \frac{(3-0)^2}{b^2} = 1$$

$$\frac{25}{16} - \frac{9}{b^2} = 1 \qquad b = \pm 4$$

center: (0, 0)

$a^2 = 16$

$b^2 = 16$

$\frac{x^2}{16} - \frac{y^2}{16} = 1$ Standard Form

$\boxed{x^2 - y^2 = 16}$ General Form

SOLUTIONS

9) What are the y-intercept(s) and x-intercept(s)? Sketch a graph and label the intercepts.

$$\frac{(x+1)^2}{9} + \frac{(y-3)^2}{16} = 1$$

Any y-intercept will have the coordinate (0, ?)

Substitute 0 for x:

$$\frac{(0+1)^2}{9} + \frac{(y-3)^2}{16} = 1$$

$$\frac{(y-3)^2}{16} = \frac{8}{9}$$

$$128 = 9(y-3)^2$$

$$128 = 9y^2 - 54y + 81$$

$$9y^2 - 54y - 47 = 0$$

Use quadratic formula (or calculator)

$$y \approx -.77 \quad \text{and} \quad 6.77$$

(0, -.77)

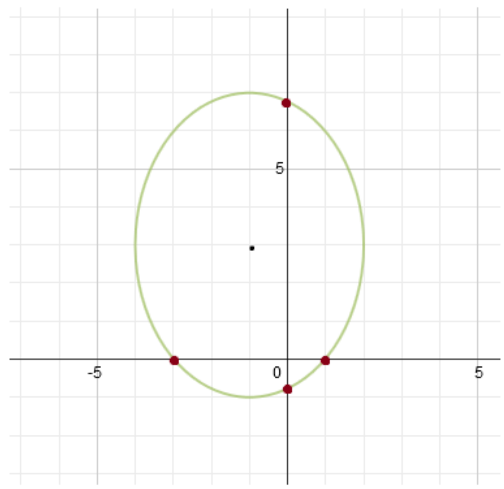
(0, 6.77)

The figure is an ellipse:

Center: (-1, 3)

Vertices: (-1, 7) (-1, -1)
(major axis)

co-Vertices: (-4, 3) (2, 3)
(minor axis)



Any x-intercept will have the coordinate (?, 0)

Substitute 0 for y:

$$\frac{(x+1)^2}{9} + \frac{(0-3)^2}{16} = 1$$

$$\frac{(x+1)^2}{9} = \frac{7}{16}$$

$$63 = 16(x+1)^2$$

$$63 = 16x^2 + 32x + 16$$

$$16x^2 + 32x - 47 = 0$$

$$x \approx -2.98 \quad \text{and} \quad .98$$

(-2.98, 0)

(.98, 0)

10) A microphone has a parabolic cross-section that is 17 inches deep with a focus that is 4 inches from the vertex.

What is the diameter of the microphone opening?

Since it opens up to the right, we'll use the standard form (assuming the vertex is (0, 0))

$y^2 = 4px$ or $x = \frac{1}{4p} y^2$

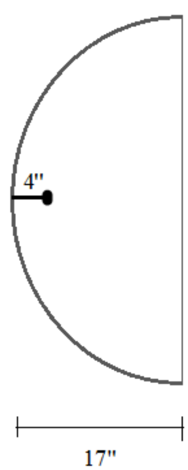
since p (distance from vertex to focus) = 4,

$y^2 = 16x$ or $x = \frac{1}{16} y^2$

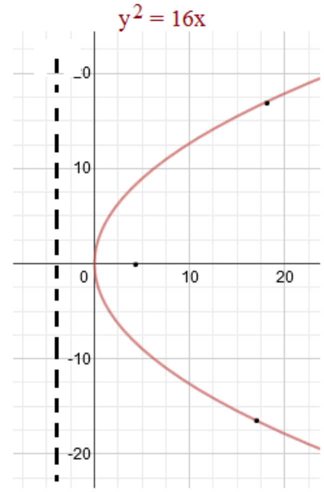
Once we have the equation, let x = 17 to find the y values (span of the opening)

$$y^2 = 16(17)$$

$$y = 16.49 \quad \text{and} \quad -16.49$$



$y^2 = 16x$



Therefore, the diameter (span) of the opening is 32.98 inches

Graphing/Identifying Conics Exercise

SOLUTIONS

Sketch the following conics:

1) $x^2 - y^2 + 4x + 8y - 21 = 0$

(hyperbola)
put into standard form:

$(x^2 + 4x \quad) - 1(y^2 - 8y \quad) = 21$

$(x^2 + 4x + 4) - 1(y^2 - 8y + 16) = 21 + 4 - 16$

$(x+2)^2 - (y-4)^2 = 9$

$$\frac{(x+2)^2}{9} - \frac{(y-4)^2}{9} = 1$$

horizontal hyperbola

a = 3
b = 3
$c = 3\sqrt{2}$

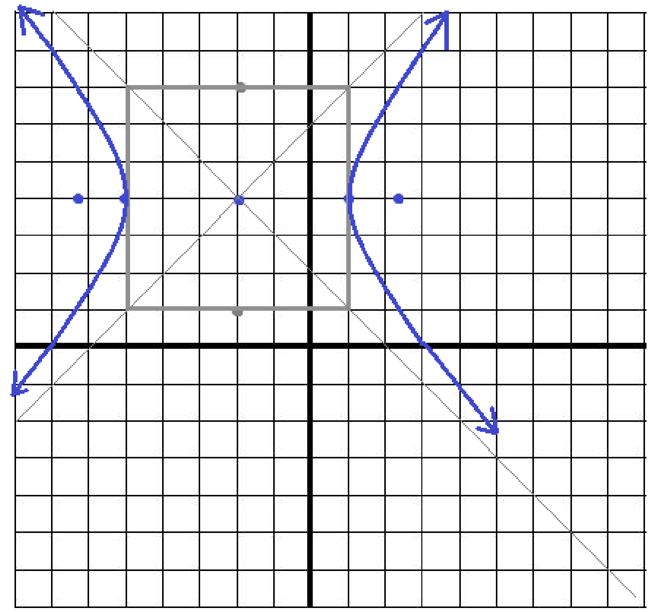
center: (-2, 4)

vertices: (1, 4) and (-5, 4)

foci: $(-2 \pm 3\sqrt{2}, 4)$

(-6.24, 4) and (2.24, 4)

"co-vertices": (-2, 7) and (-2, 1)



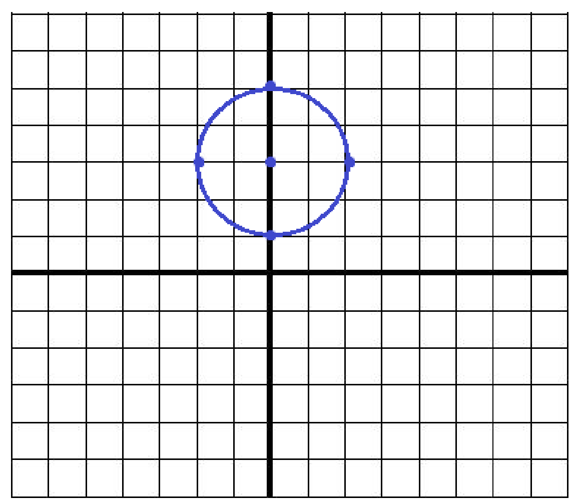
asymptotes: $y = x + 6$
$y = -x + 2$

2) $x^2 + (y-3)^2 = 4$

(circle)

center: (0, 3)

radius: 2



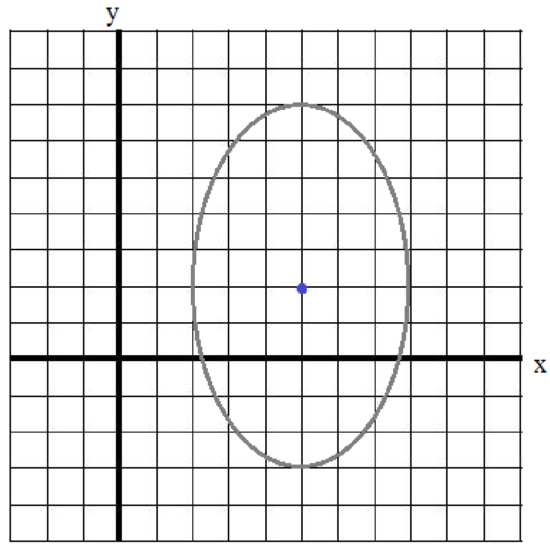
To check, try a few points on the circle:

(2, 3): $(2)^2 + ((3) - 3)^2 = 4$
$4 + 0 = 4$ ✓

(0, 5): $(0)^2 + ((5) - 3)^2 = 4$
$0 + 4 = 4$ ✓

SOLUTIONS

3) Determine the equation in the graph:



Find the properties: (vertical ellipse)

center: (5, 2)

length of semi-major axis, (a): 5 units

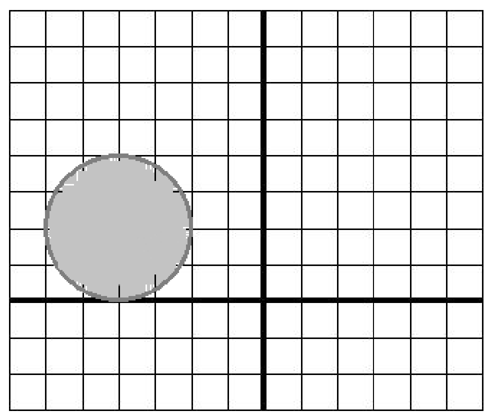
length of semi-minor axis, (b): 3 units

Then, placed in standard form:

$$\boxed{\frac{(x-5)^2}{9} + \frac{(y-2)^2}{25} = 1}$$

To check, substitute various points on the ellipse into the equation

4) Write the equation represented in the graph:



Find the properties: (circle inequality)

Center: (-4, 2)

radius: 2 units

Then, in standard form:

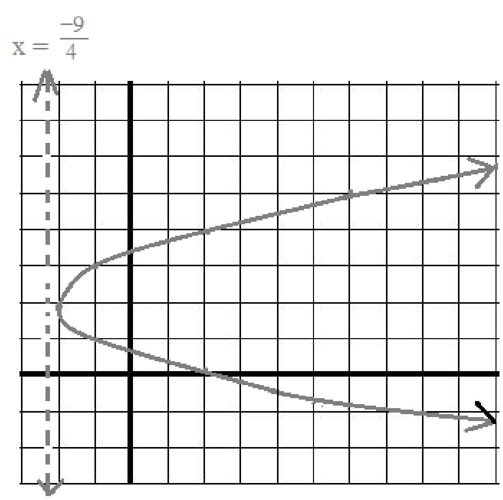
$(x+4)^2 + (y-2)^2 = 4$ is the equation of the circle..

finally, test a point in the shaded region!

ex: (-3, 2): $((-3)+4)^2 + ((2)-2)^2 = 1$ 1 is less than the radius 4

$$\boxed{(x+4)^2 + (y-2)^2 \leq 4}$$

5) Write the equation for the parabola:



The distance between vertex and directrix is 1/4

$a = \frac{1}{4p}$ $a = \frac{1}{4(1/4)} = 1$

vertex: (h,k) = (-2, 2)

standard form of *horizontal* parabola *facing/opening to the right*:

$x = a(y-k)^2 + h$

$x = 1(y-2)^2 + (-2)$

$$\boxed{x = (y-2)^2 - 2}$$

Describe the Figures

SOLUTIONS

1) $3x^2 + 5y^2 = 17$
2) $3x^2 - 5y^2 = 17$
3) $3x^2 + 3y^2 = 17$
4) $3x - 3y = 17$
5) $3x^2 + y = 17$
6) $3x^2 - 3y^2 = 17$

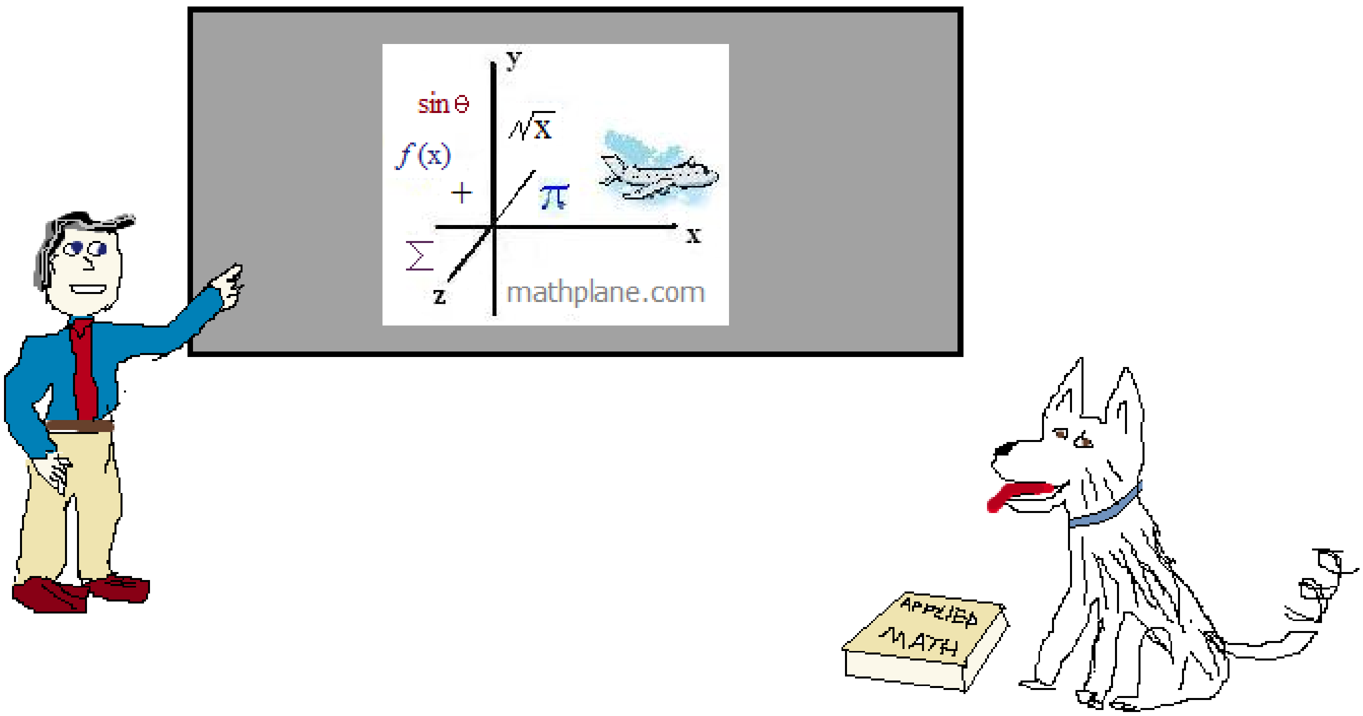
1) ellipse
2) hyperbola
3) circle
4) line
5) parabola
6) hyperbola

Thanks for visiting. (Hope it helped!)

Find more lessons about conics and other algebra II topics at mathplane.com

If you have questions, suggestions, or requests, let us know.

Enjoy



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