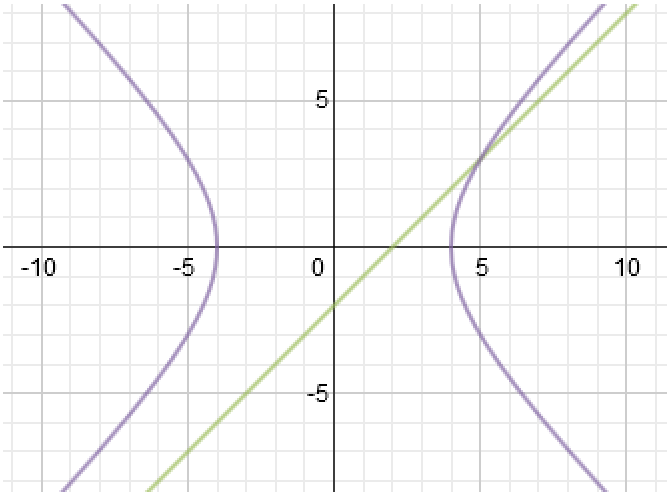
Conics IV: Systems

Examples and Practice Test (with solutions)



Topics include solving systems algebraically and graphically, word problems, completing the square, and more.

Conics Systems Examples and Applications

Here are examples that review properties of conics, completing the square, graphing, solving systems, and more...

Example: Solve and graph the following system: $9x^2 - 16y^2 = 144$

$x + y^2 = -4$

Step 1: Identify the 2 functions

$9x^2 - 16y^2 = 144$ hyperbola (x^2 and y^2 are different signs)

$x + y^2 = -4$ parabola (x^2 is missing)

Step 2: Solve algebraically

$9x^2 - 16y^2 = 144$ Use substitution method

$x + y^2 = -4$

$y^2 = -4 - x$

$$9x^2 - 16(-4 - x) = 144$$

$$9x^2 + 16x + 64 = 144$$

$$9x^2 + 16x - 80 = 0$$

$$(9x - 20)(x + 4) = 0$$

$$x = 9/2 \text{ and } -4$$

If $x = -4$: $(-4) + y^2 = -4$

$y = 0$

If $x = \frac{9}{2}$: $(\frac{9}{2}) + y^2 = -4$

$y^2 = -4 - \frac{9}{2}$

extraneous solution!
(because y^2 cannot be negative)

$(-4, 0)$ is the only solution
(i.e. the only point of intersection)

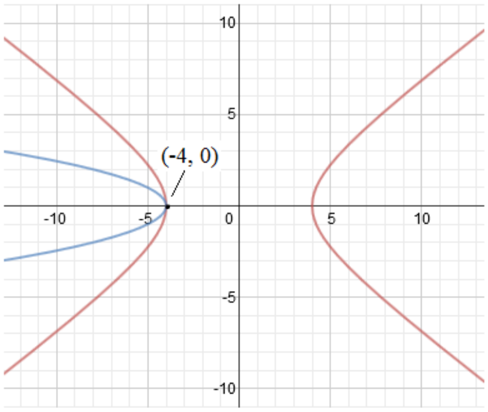
Step 3: Solve graphically

$9x^2 - 16y^2 = 144$

$$\boxed{\frac{x^2}{16} - \frac{y^2}{9} = 1}$$

horizontal hyperbola; center is (0, 0)
Vertices: (4, 0) and (-4, 0)
Foci: (5, 0) and (-5, 0)
asymptotes:

$y = \frac{3}{4}x$ $y = -\frac{3}{4}x$



$x + y^2 = -4$

$y^2 = -4 - x$

$$\boxed{y^2 = -(x + 4)}$$

Vertex: $(-4, 0)$
(opens to the left)

Focus: (-17/4, 0)
Directrix: x= -15/4

Horizontal Hyperbola:
(with center (h, k))
Standard Form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Horizontal Parabola:
(opens to the left or right)
Horizontal axis of symmetry

$$x = a(y-k)^2 + h$$

$$(x-h) = a(y-k)^2$$

where (h, k) is the vertex and

$$a = \frac{1}{4p}$$

OR,

$$(y-k)^2 = \frac{(x-h)}{a} \qquad 4p = \frac{1}{a}$$

Vertex at origin: $y^2 = 4px$

Example: Solve and Graph the following System:

$$x^2 + y^2 - 2x - 8 = 0$$

$$x^2 + y^2 + 6y + 5 = 0$$

Solve:

Since combination method won't eliminate both variables, we'll use substitution...

Change the first equation...

$$y^2 = -x^2 + 2x + 8$$

$$y = \pm\sqrt{-x^2 + 2x + 8}$$

Substitute into second equation...

$$x^2 + (-x^2 + 2x + 8) + 6\left(\pm\sqrt{-x^2 + 2x + 8}\right) + 5 = 0$$

$$6\left(\pm\sqrt{-x^2 + 2x + 8}\right) = -5 - 2x - 8$$

$$36(-x^2 + 2x + 8) = (-2x - 13)^2$$

$$-36x^2 + 72x + 288 = 4x^2 + 52x + 169$$

$$40x^2 - 20x - 119 = 0$$

Use Quadratic formula/Calculator:

$$x \approx -1.49 \quad \text{or} \quad 1.99$$

If $x = -1.49$:

$$y^2 = -x^2 + 2x + 8$$

$$y^2 = -(-1.49)^2 + 2(-1.49) + 8$$

$$y^2 = 2.8$$

$$y = \pm 1.67$$

~~(−1.49, 1.67)~~

(−1.49, −1.67)

Test the other equation!!

$$x^2 + y^2 + 6y + 5 = 0$$

$$(-1.49)^2 + (1.67)^2 + 6(1.67) + 5 > 0$$

Graph:

Each of the equations is a circle.
(coefficients of x^2 and y^2 are the same)

Step 1: Complete the square (and change to standard form)

$$x^2 - 2x + y^2 = 8 \qquad\qquad x^2 + y^2 + 6y = -5$$

$$(x^2 - 2x + 1) + y^2 = 8 + 1 \qquad\qquad x^2 + (y^2 + 6y + 9) = -5 + 9$$

$$\boxed{(x - 1)^2 + y^2 = 9} \qquad\qquad \boxed{x^2 + (y + 3)^2 = 4}$$

Step 2: Identify parts and graph

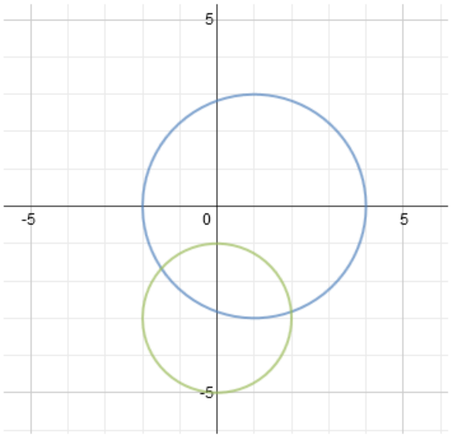
center: (1, 0)

radius: 3

center: (0, -3)

radius: 2

Step 3: Graph and confirm intersections



If x = 1.99:

$$y^2 = -x^2 + 2x + 8$$

$$y^2 = -(1.99)^2 + 2(1.99) + 8$$

$$y^2 = 8.0$$

$$y = \pm 2.83$$

~~(1.99, 2.83)~~

(1.99, −2.83)

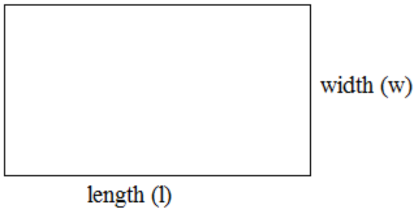
Again, +y will not satisfy the other equation!

Conics Systems Examples and Applications

Example: In my backyard, I built a rectangular area for my dog, using 200 yards of fencing. If the area is 2100 square yards, what are the dimensions?

Perimeter is 200 yards: $2l + 2w = 200$ yards

Area is 2100 square yards: $lw = 2100$ sq. yards



Solve the system:

$$l + w = 100 \qquad l + \left(\frac{2100}{l}\right) = 100$$

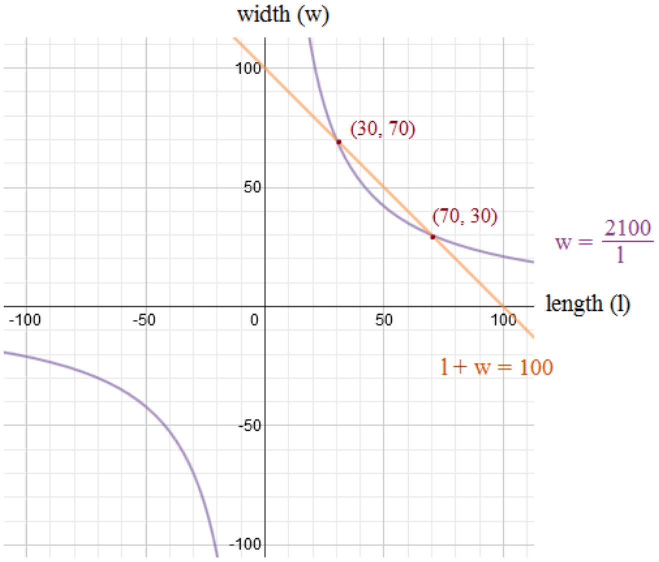
$$w = \frac{2100}{l} \qquad l^2 - 100l + 2100 = 0$$

$$(l - 30)(l - 70) = 0$$

$$l = 30 \text{ or } 70$$

then, $w = 70$ or 30

Dimensions are 30 yds x 70 yds



Example: Solve the system:

$$2xy + 3y^2 = 7$$
$$3xy - 2y^2 = 4$$

Using elimination method:

$$\begin{array}{lll} 2xy + 3y^2 = 7 & \times 3 & 6xy + 9y^2 = 21 \\ 3xy - 2y^2 = 4 & \times 2 & 6xy - 4y^2 = 8 \\ \hline & & 13y^2 = 13 \end{array}$$

subtract

$$y = 1 \text{ or } -1$$

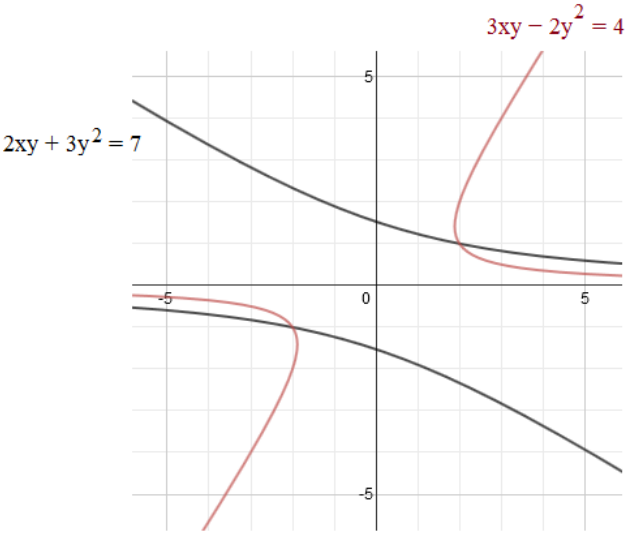
If $y = 1$, then $2x(1) + 3(1)^2 = 7$

$$x = 2$$

If $y = -1$, then $2x(-1) + 3(-1)^2 = 7$

$$x = -2$$

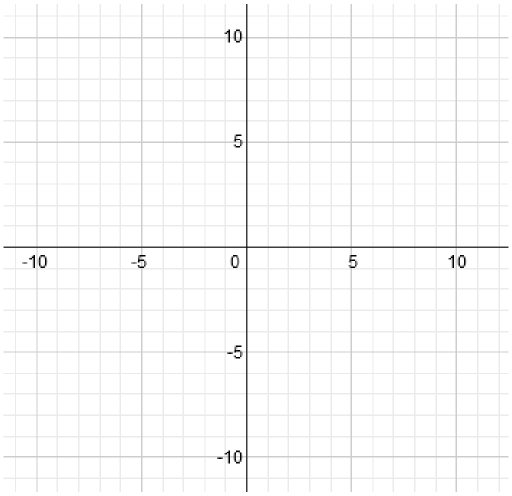
$(2, 1)$ and $(-2, -1)$



Conics Systems Test

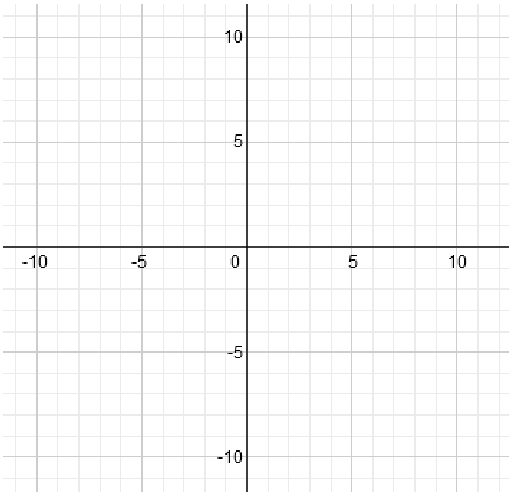
1) Graph and solve the following system: $x^2 - y^2 = 16$

$x - y = 2$



2) Solve and graph: $x^2 + y^2 = 25$

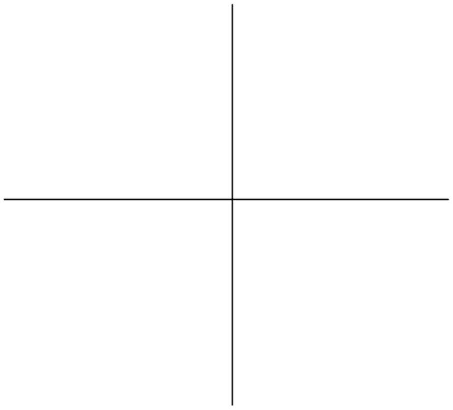
$xy = 12$



3) A rectangular painting with perimeter 82 inches has a diagonal of 29 inches. What are the dimensions of the painting?

4) Solve and Graph the following System: $9x^2 + y^2 - 2y = 80$

$x^2 + y^2 - 10y = 0$

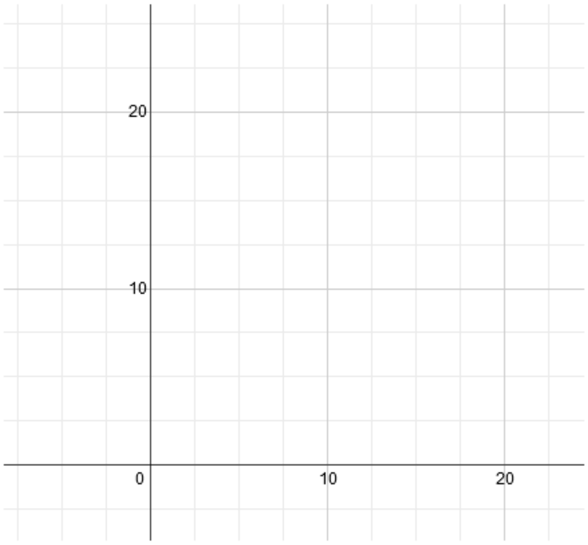


5) A friend on the 2nd floor throws a ball off the balcony with a trajectory of $y = -.4x^2 + 2x + 20$

Simultaneously, below him, I throw a ball in the same direction with a trajectory path of $y = -.2x^2 + 3x + 4$

(x is the number of feet away from the building, and y is the number of feet above the ground)

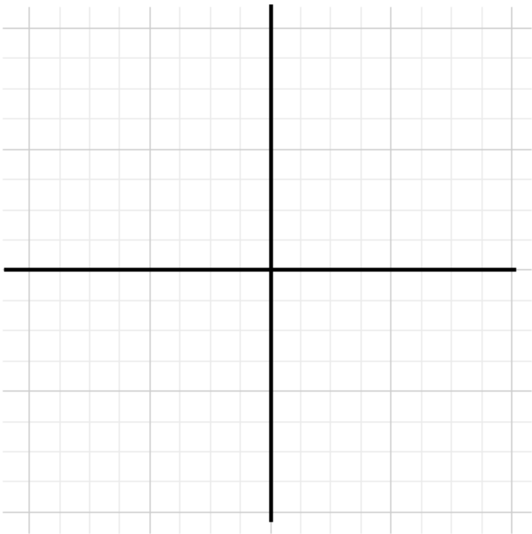
Is it possible that the balls can collide? Explain.



6) Find the points of intersection. (Calculator)
Then, sketch a graph of the system to confirm your answer.

$$4x^2 + 16y^2 = 64$$

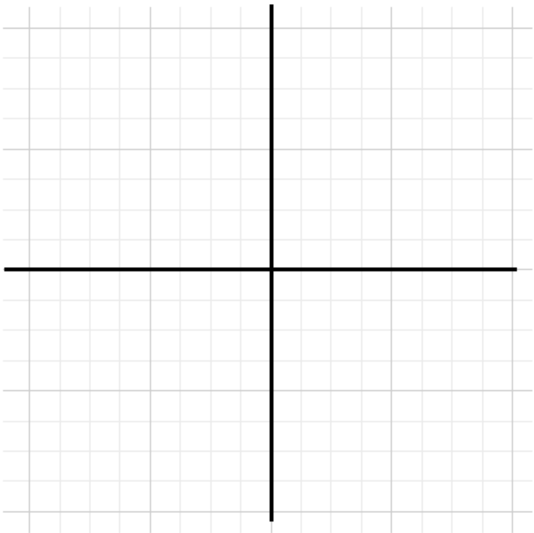
$$16x^2 + 4y^2 = 64$$



7) ***Challenge: Determine the intersection(s) of

$$y = |x + 2| \quad \text{and} \quad x^2 + y^2 = 25$$

(Hint/Suggestion: Sketch the equations)

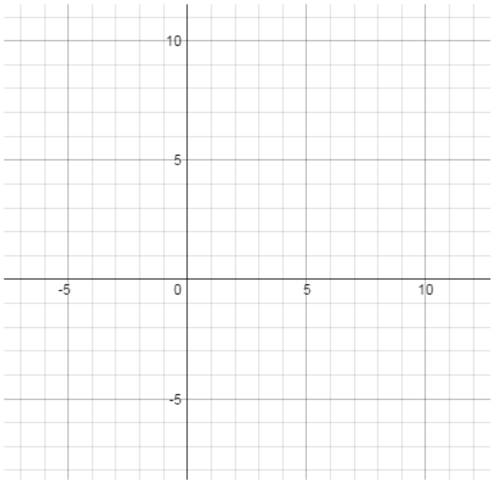


(Solve and Graph the Systems)

Conics Systems Test

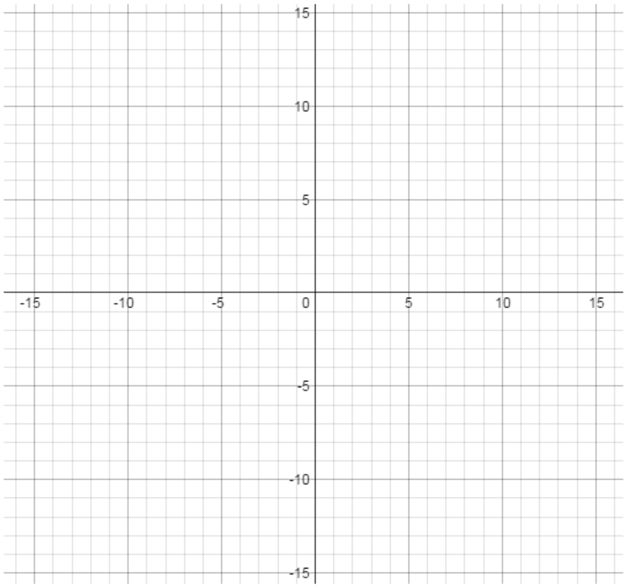
8) $25x^2 + 9y^2 - 250x - 36y + 436 = 0$

$x^2 + y^2 - 4y = 0$



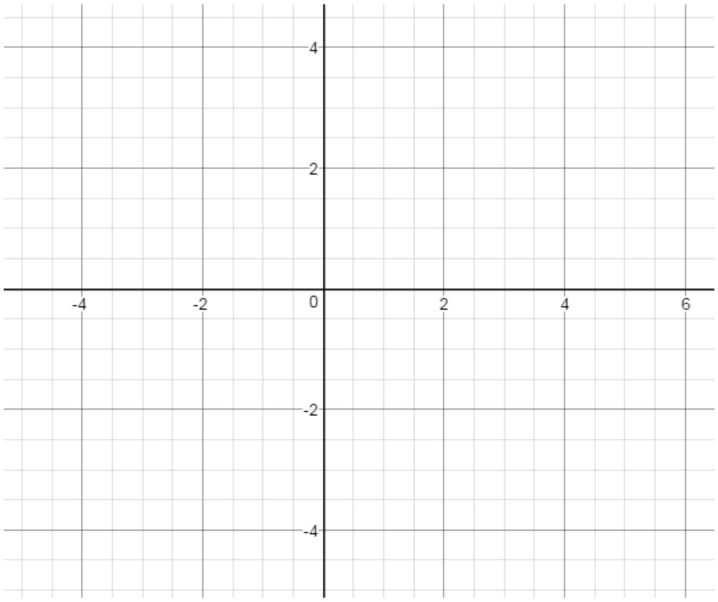
9) $y^2 - x^2 = 36$

$xy = 24$

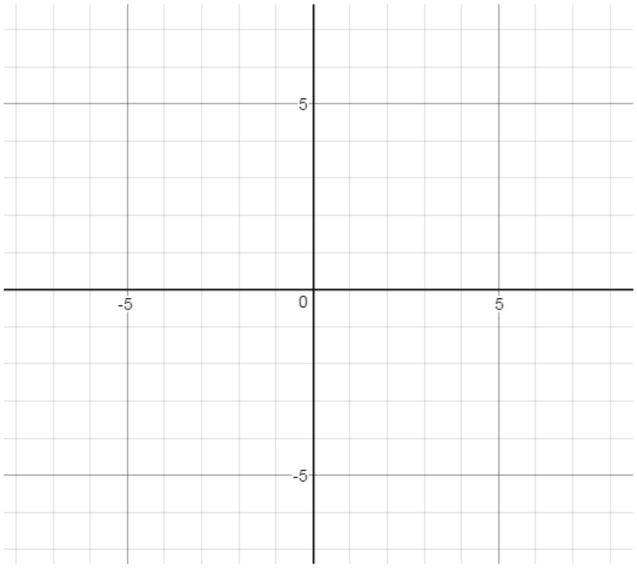


Conics Systems Test

10) $4x^2 + 9y^2 = 36$
$3y + 2x = 6$



11) $y^2 = x + 5$
$x^2 + y^2 = 25$




Elements
"Look, a *solar ellipse*."
"Hey, Eukie, I think you're seeing double.. .. You need **to focus**."
"No. **Two foci**."
After too many drinks, an intoxicated Euclid makes an absurd geometry claim.
LanceAF #27 4-8-12
www.mathplane.com

Solutions -→

Conics Systems Test

SOLUTIONS

1) Graph and solve the following system: $x^2 - y^2 = 16$

$x - y = 2$

Use Substitution: $x = 2 + y$

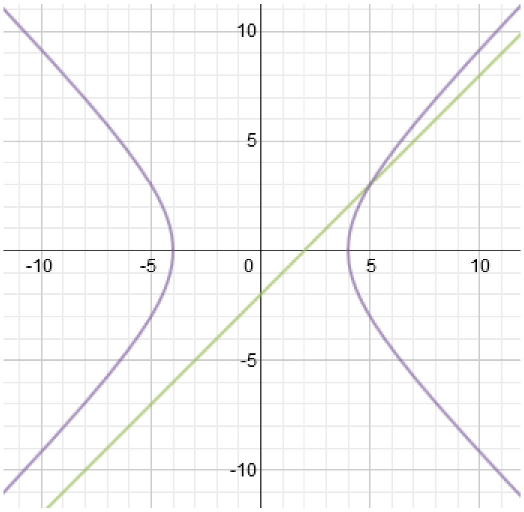
$(2 + y)^2 - y^2 = 16$

$y^2 + 4y + 4 - y^2 = 16$

$4y = 12$ $\quad y = 3$

$x = 5$

There is only one answer: (5, 3)



2) Solve and graph: $x^2 + y^2 = 25$

$xy = 12$ $\quad y = \frac{12}{x}$

substitution:

$$x^2 + \left(\frac{12}{x}\right)^2 = 25$$

$$x^2 + \frac{144}{x^2} - 25 = 0$$

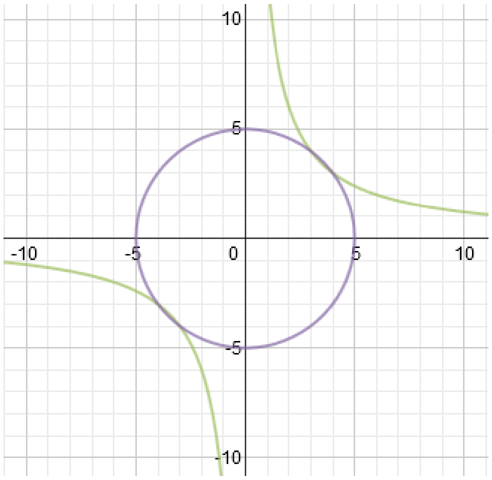
$$x^4 - 25x^2 + 144 = 0$$

$$(x^2 - 16)(x^2 - 9) = 0$$

$$x = -4, 4, -3, 3$$

Then, to find y, substitute into xy = 12 and check...

Solutions: (-3, -4) (3, 4)
(4, 3) (-4, -3)



3) A rectangular painting with perimeter 82 inches has a diagonal of 29 inches. What are the dimensions of the painting?

perimeter: 2(length) + 2(width) = 82

diagonal: $(\text{length})^2 + (\text{width})^2 = 29^2$

$1 + w = 41$

$1^2 + w^2 = 841$

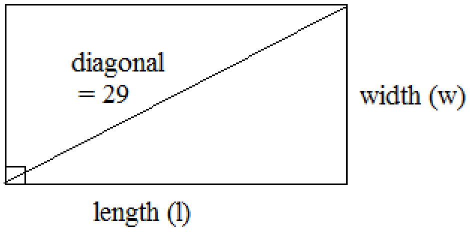
$(41 - w)^2 + w^2 = 841$

$2w^2 - 82w + 1681 = 841$

$w^2 - 41w + 420 = 0$

$(w - 20)(w - 21) = 0$

$w = 20$ or 21



dimensions are 20" x 21"
(or, 21" x 20")

SOLUTIONS

4) Solve and Graph the following System: $9x^2 + y^2 - 2y = 80$

$x^2 + y^2 - 10y = 0$

(complete the square)

$9x^2 + y^2 - 2y = 80$

$9x^2 + y^2 - 2y + 1 = 80 + 1$

$9x^2 + (y - 1)^2 = 81$

$$\frac{x^2}{9} + \frac{(y-1)^2}{81} = 1$$

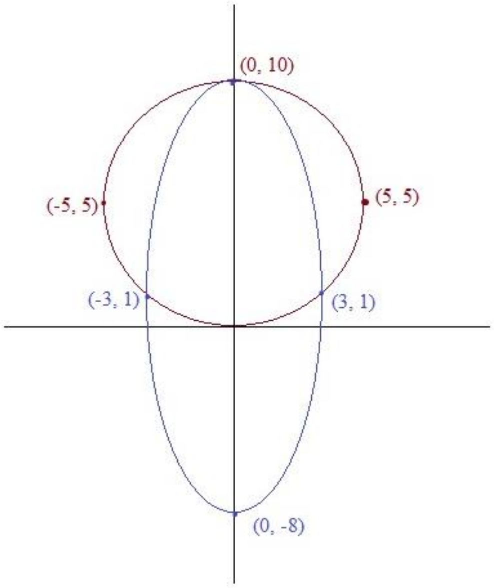
Ellipse: minor axis x, a = 3
major axis y, b = 9
Center: (0, 1)

$x^2 + y^2 - 10y = 0$

$x^2 + y^2 - 10y + 25 = 0 + 25$

$x^2 + (y - 5)^2 = 25$

Circle: Radius = 5
Center: (1, 5)



Algebraic Solution:

$9x^2 + y^2 - 2y = 80$ $\quad$ $x^2 + y^2 - 10y = 0$

$x^2 = -y^2 + 10y$ (isolate x^2)

(Use substitution) $9(-y^2 + 10y) + y^2 - 2y = 80$

$-9y^2 + 90y + y^2 - 2y = 80$

$-8y^2 + 88y - 80 = 0$

(factor) $8y^2 - 88y + 80 = 0$

$y^2 - 11y + 10 = 0$

(solve) $(y - 10)(y - 1) = 0$

$y = 1, 10$

(using the y solutions, find x)

$x^2 + y^2 - 10y = 0$

$x^2 + (1)^2 - 10(1) = 0$

$x^2 - 9 = 0$

$x = 3, -3$

$x^2 + (10)^2 - 10(10) = 0$

$x^2 + 100 - 100 = 0$

$x = 0$

Solutions are (0, 10)
(3, 1)
(-3, 1)

To check, plug solutions into other equation:

$9x^2 + y^2 - 2y = 80$

(0, 10) $\quad 9(0)^2 + (10)^2 - 2(10) = 80$

$0 + 100 - 20 = 80$ ✓

(3, 1) $\quad 9(3)^2 + (1)^2 - 2(1) = 80$

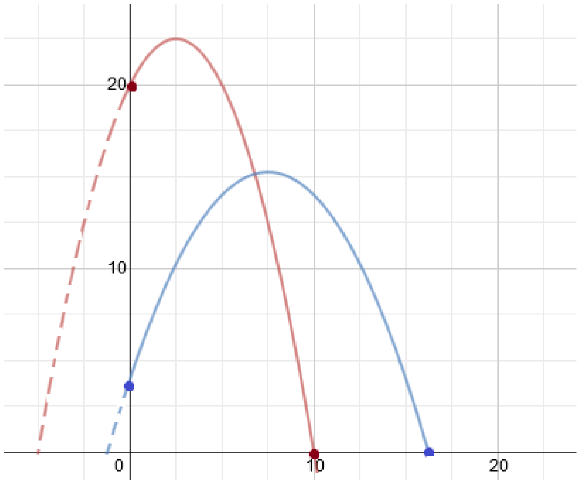
$81 + 1 - 2 = 80$ ✓

(-3, 1) $\quad 9(-3)^2 + (1)^2 - 2(1) = 80$

$81 + 1 - 2 = 80$ ✓

5) A friend on the 2nd floor throws a ball off the balcony with a trajectory of $y = -.4x^2 + 2x + 20$
Simultaneously, below him, I throw a ball in the same direction with a trajectory path of $y = -.2x^2 + 3x + 4$
(x is the number of feet away from the building, and y is the number of feet above the ground)
Is it possible that the balls can collide? Explain.

If the timing is right, the balls could collide...



$$y = -.4x^2 + 2x + 20$$

$$y = -.2x^2 + 3x + 4$$

$$-.2x^2 + 3x + 4 = -.4x^2 + 2x + 20$$

$$.2x^2 + x - 16 = 0$$

$$x^2 + 5x - 80 = 0$$

(quadratic formula)

$$x = \frac{-5 \pm \sqrt{25 - 4(1)(-80)}}{2(1)} = \frac{-5 \pm \sqrt{345}}{2}$$

approximately $\cancel{-11.79}$ and 6.79

Since the ball is presumably thrown forward, we'll eliminate the negative term.
Also, the negative intersection occurs when y is negative (i.e. underground).

If x = 6.79, then $y = -.4(6.79)^2 + 2(6.79) + 20 \approx 15.14$ ✓

$y = -.2\,(6.79)^2 + 3\,(6.79) + 4 \approx 15.14$ ✓

If the timing were right, the balls would collide 6.79 feet from the building and 15.14 feet above the ground!

(Calculator) SOLUTIONS

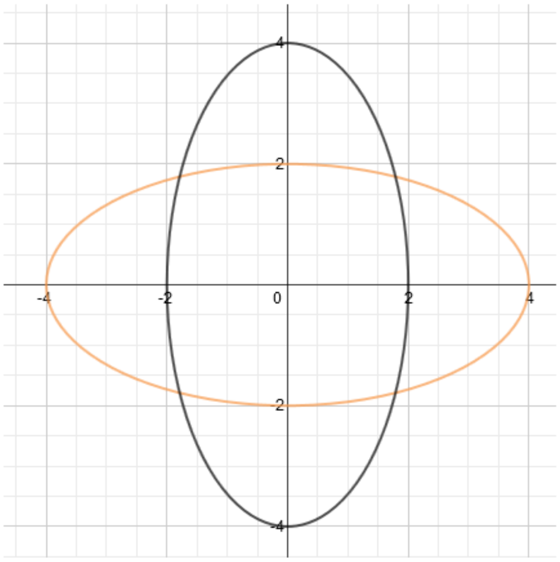
6) Find the points of intersection.
Then, sketch a graph of the system to confirm your answer.

$$4x^2 + 16y^2 = 64$$
$$16x^2 + 4y^2 = 64$$

To find points of intersection (algebraically), use combination/elimination method:

$$-4\left(4x^2 + 16y^2 = 64 \right)$$
$$16x^2 + 4y^2 = 64$$
$$+ \;\; -16x^2 - 64y^2 = -256$$
$$-60y^2 = -192$$
$$y = \pm\sqrt{3.2} \quad \text{(approx. } \pm 1.79)$$
$$4x^2 + 16y^2 = 64$$
$$4x^2 + 16(3.2) = 64$$
$$x^2 = 3.2$$
$$x = \pm\sqrt{3.2} \quad \text{(approx. } \pm 1.79)$$

$(\sqrt{3.2},\ \sqrt{3.2})$
$(\sqrt{3.2},\ -\sqrt{3.2})$
$(-\sqrt{3.2},\ \sqrt{3.2})$
$(-\sqrt{3.2},\ -\sqrt{3.2})$



7) ***Challenge: Determine the intersection(s) of

$$y = |x + 2| \quad \text{and} \quad x^2 + y^2 = 25$$

(Hint/Suggestion: Sketch the equations)

$y = |x + 2|$ is composed of 2 lines: $y = x + 2$ (where $x > -2$)
$y = -x - 2$ (where $x < -2$)

So, we'll use these lines for to find the intersections...

$y = x + 2$ (use substitution)

$$x^2 + y^2 = 25 \longrightarrow x^2 + (x + 2)^2 = 25$$
$$2x^2 + 4x - 21 = 0$$

(quadratic formula)

$$x \approx -4.39 \text{ and } 2.39$$

$y = x + 2 \longrightarrow$ ~~if x = -4.39, then y = −2.39~~ (since it is absolute value, y cannot be negative)

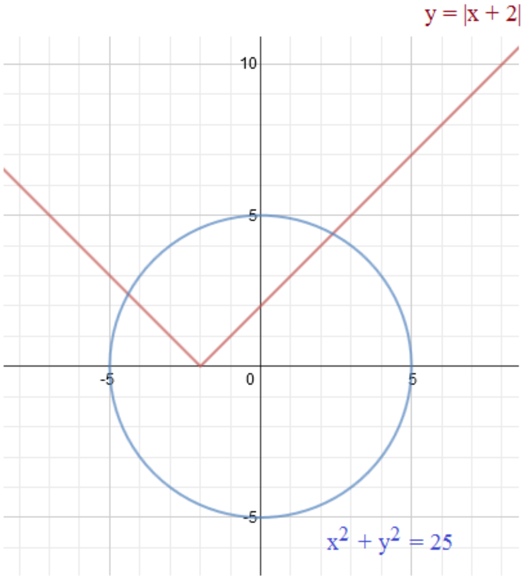
$\boxed{\text{if } x = 2.39\text{, then } y = 4.39}$

$y = -x - 2$ (substitution)

$$x^2 + y^2 = 25 \longrightarrow x^2 + (-x - 2)^2 = 25$$
$$2x^2 + 4x - 21 = 0$$
$$x \approx -4.39 \text{ and } 2.39$$

$y = -x - 2 \longrightarrow$ $\boxed{\text{if } x = -4.39\text{, then } y = 2.39}$

~~if x = 2.39, then y = -4.39~~



(Solve and Graph the Systems)

SOLUTIONS

Conics Systems Test

8) $25x^2 + 9y^2 - 250x - 36y + 436 = 0$

$x^2 + y^2 - 4y = 0$

To graph, convert each equation into Standard Form...

$25x^2 - 250x + 9y^2 - 36y = -436$ (Ellipse)

$25(x^2 - 10x + 25) + 9(y^2 - 4y + 4) = -436 + 625 + 36$ Complete the square

$25(x-5)^2 + 9(y-2)^2 = 225$

$$\frac{(x-5)^2}{9} + \frac{(y-2)^2}{25} = 1$$

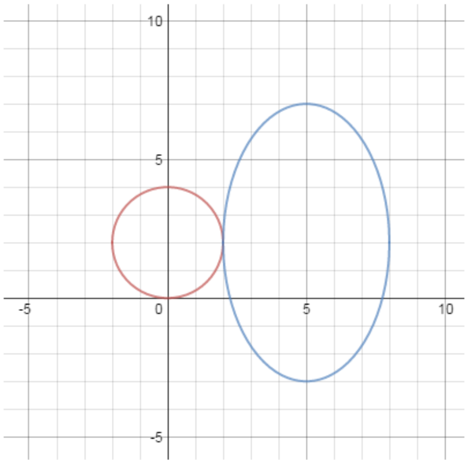
Center: (5, 2)
Major Axis: 10
Minor Axis: 6

$x^2 + y^2 - 4y = 0$ (Circle)

$x^2 + y^2 - 4y + 4 = 0 + 4$ Complete the square

$x^2 + (y-2)^2 = 4$

Center: (0, 2)
Radius: 2



Intersection at (2, 2)

To confirm Algebraically...

$25x^2 + 9y^2 - 250x - 36y + 436 = 0 \Longrightarrow 25x^2 - 250x \boxed{+ 9y^2 - 36y} + 436 = 0$

$x^2 + y^2 - 4y = 0 \Longrightarrow 9x^2 \boxed{+ 9y^2 - 36y} = 0$

since $9y^2 + 36y = -9x^2$

$25x^2 - 250x - 9x^2 + 436 = 0$

$16x^2 - 250x + 436 = 0$

$8x^2 - 125x + 218 = 0$

$(x-2)(8x-109) = 0$

$\boxed{x = 2}$ or $x = \cancel{109/8}$ $\boxed{(2, 2)}$

109/8 is extraneous
(it cannot be substituted
into the equations)

9) $y^2 - x^2 = 36$

$xy = 24$

To graph, change first equation into Standard Form

$$\frac{y^2}{36} - \frac{x^2}{36} = 1$$ (Vertical Hyperbola)

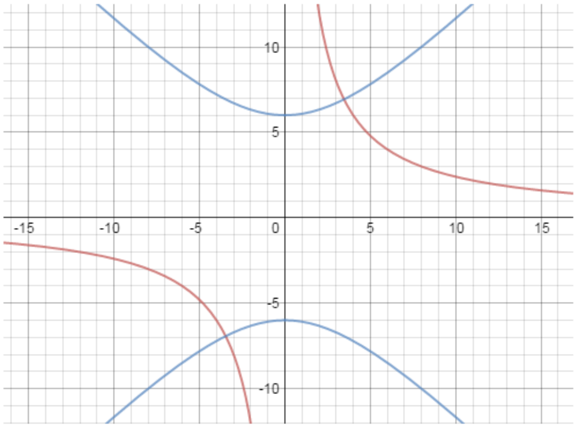
"a" value is 6
"b" value is 6
"c" value is $6\sqrt{2}$

Center: (0, 0)

Vertices: (0, 6) (0, -6)

Foci: $(0, 6\sqrt{2})$ $(0, -6\sqrt{2})$

Asymptotes: y = x and y = -x



Recognize the second equation is a reciprocal function

$y = \frac{24}{x}$ y = 1/x that is 'stretched by 24'

(2, 12)	(-2, -12)
(3, 8)	(-3, -8)
(4, 6)	(-4, -6)
(6, 4)	(-6, -4)
(8, 3)	(-8, -3)
(12, 2)	(-12, -2)

To solve system, use substitution:

$y^2 - x^2 = 36$

$y = \frac{24}{x}$

$\left(\frac{24}{x}\right)^2 - x^2 = 36$

$\frac{576}{x^2} - x^2 = 36$

$x^4 + 36x^2 - 576 = 0$

$(x^2 + 48)(x^2 - 12) = 0$

$\cancel{(x^2 + 48) = 0}$

$(x^2 - 12) = 0$

x = 3.46 and -3.46

(and, y = 6.93 and -6.93)

$$\boxed{\begin{matrix}(\sqrt{12}, \sqrt{48}) \\ (-\sqrt{12}, -\sqrt{48})\end{matrix}}$$

SOLUTIONS

Conics Systems Test

10) $4x^2 + 9y^2 = 36$
$3y + 2x = 6$

$3y = 6 - 2x$

square both sides

$9y^2 = (6 - 2x)^2$

substitution

$4x^2 + (6 - 2x)^2 = 36$

$4x^2 + 36 - 24x + 4x^2 = 36$

$8x^2 - 24x = 0$

$8x(x - 3) = 0$

$x = 0, 3$

looking at the linear equation

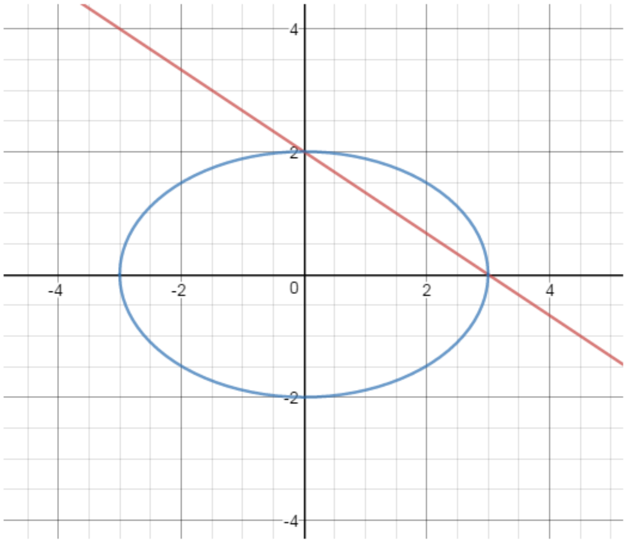
if $x = 0$, then $y = 2$ (0, 2)

if $x = 3$, then $y = 0$ (3, 0)

graphing a line and an ellipse...

$2x + 3y = 6$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$



11) $y^2 = x + 5$
$x^2 + y^2 = 25$

Direct substitution!

$x^2 + (x + 5) = 25$

$x^2 + x - 20 = 0$

$(x + 5)(x - 4) = 0$

$x = -5, 4$

looking at the parabola..

if $x = -5$, then $y^2 = -5 + 5$ (-5, 0)

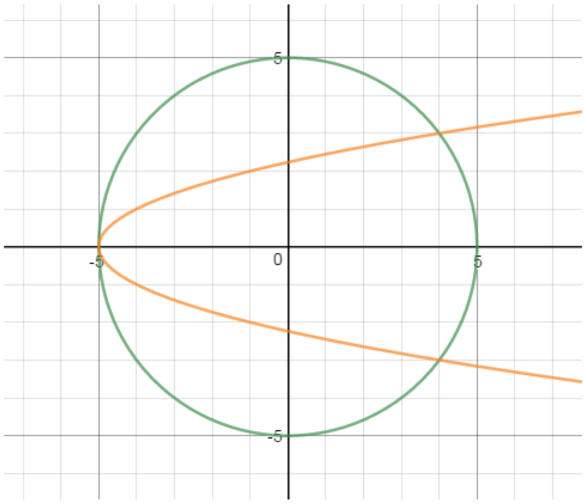
if $x = 4$, then $y^2 = 4 + 5$ (4, -3) and (4, 3)

graphing a parabola

opens out to the right
vertex: (-5, 0)
focus: (-4.75, 0)
directrix: x = -5.25

and, a circle

radius: 5
center: (0, 0)



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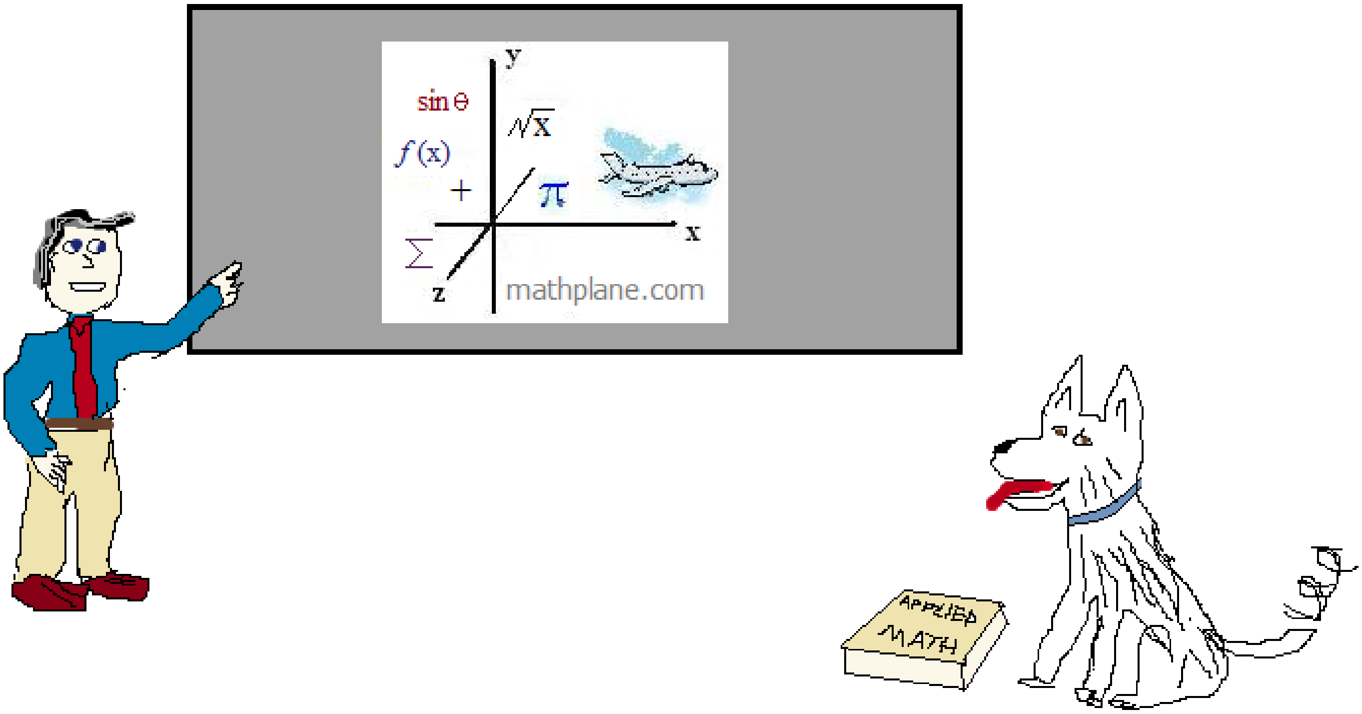
Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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