Probability Introduction

Notes, Examples, and Practice Exercise (with Solutions)

Topics include independent events, mutually exclusive, conditional probability, tree diagram, Venn diagram, and more…

Introduction to Probability: Notes and Examples

Basic Definition: $p(A) = \dfrac{\text{\# of ways A can happen}}{\text{total possible outcomes}}$

probability of an event is $\dfrac{\text{'successes'}}{\text{'possibilities'}}$

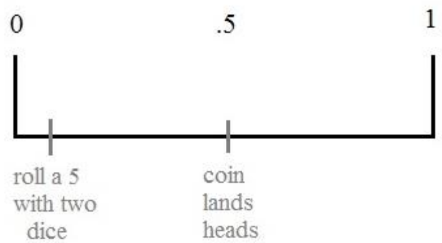
EXAMPLES

$p(\text{coin lands heads}) = \dfrac{1}{2}$ or .5

$p(\text{roll a 5 w/2 dice}) = \dfrac{4}{36} = \dfrac{1}{9}$ or .111

4 potential successes: 1|4 2|3 3|2 4|1
36 possibilities

The Range of Probability Values



0 is impossible
.5 is a 50/50 chance
1 is certainty

Mutually Exclusive

"If A happens, then B cannot simultaneously happen"

-- flipping a coin and getting heads or tails?
mutually exclusive

-- drawing a card and getting a queen or a king?
mutually exclusive

-- drawing a card and getting a queen or a red card?
NOT mutually exclusive
(queen of hearts or queen of diamonds)

If the events are <u>mutually exclusive</u>, then

$$P(A \text{ or } B) = p(A) + p(B)$$

p(rolling a 5 or 8 w/2 dice) = p(rolling a 5) + p(rolling an 8)

$$= \frac{4}{36} + \frac{5}{36} = \frac{1}{4}$$

(includes 1|4 2|3 3|2 4|1 and
2|6 3|5 4|4 5|3 6|2)

If the events are <u>NOT mutually exclusive</u>, then

$$P(A \text{ or } B) = p(A) + p(B) - P(AB)$$

(subtracting AB eliminates "double counting")

p(drawing a king or a spade) = p(king) + p(spade) − p(king & spade)

$$= \frac{\text{4 kings}}{\text{52 cards}} + \frac{\text{13 spades}}{\text{52 cards}} - \frac{\text{1 king of spades}}{\text{52 cards}}$$

$= \dfrac{16}{52}$ (includes 13 spades, king of hearts, king of diamonds, and king of clubs)

If you're unsure if the events are mutually exclusive and you want to avoid double counting, use the 2nd formula...

p(drawing a picture card or a four) = p(picture) + p(four) − p(picture/four)

$= \dfrac{12}{52} + \dfrac{4}{52} - \dfrac{0}{52}$ $\Longleftarrow$ (no card consists of a four AND a picture)

= 4/13 (includes all 4s, jacks, queens, and kings)

The probability of several events happening will be between 0 and 1.
Also, add all possible (mutually exclusive) events: the outcome is 1.

Complement: "Probability of NOT...."

Consider a bag of 10 marbles: 1 black, 2 red, 2 blue, 5 yellow

p(drawing green) = 0
p(drawing black) = 1/10
p(drawing red) = 2/10
p(drawing blue) = 2/10
p(drawing yellow) = 5/10
total probability = 1

Since p(black) + p(red) + p(blue) + p(yellow) = 1,
then the probability of NOT drawing a black marble is
1 – p(drawing black) = 9/10

Dependent vs. Independent Events (conditional probability)

"If 2 events affect each other, then they are dependent"
(more specifically, if event A affects event B, then the probability of B is dependent on the outcome of A)

"If events' outcomes don't affect each other, then they are independent."

-- flipping a coin 3 times?
Independent

-- drawing 2 cards from a deck (without replacement)?
Dependent

-- drawing 2 cards (with replacement)?
Independent

If the events are independent, then

p(A and B) = p(A)p(B)

p(flipping a coin 3 times & getting 3 heads) =
p(heads) * p(heads) * p(heads) =
1/2 x 1/2 x 1/2 = 1/8

'replacement' vs. 'without replacement'
replacement assumes you return the sample to its original set. EX: If you draw a card and put it back in the deck (replacement)

$$p(\text{3 heads}) = \frac{\text{'successes'}}{\text{'possibilities'}} = \frac{1}{8}$$

$\boxed{\text{HHH}}$

HHH	HHT	HTH	HTT
THH	THT	TTH	TTT

If 2 events are dependent, (i.e. B is dependent on the outcome of A), then

p(A and B) = p(A)p(B|A)

"probability of A times the probability of B, GIVEN A has happened"

Note: If A and B are independent, then
p(B) = p(B|A)
because the outcome of A doesn't affect the possibilities of B.

p(drawing 2 spades) = p(1st card is a spade) x p(2nd card is also a spade)

$$\frac{13}{52} \times \frac{12}{51} = \frac{156}{2652} = .059$$

"Dependent Events"

The chance of drawing a king is 4/52 p(king) = 4/52
But, what if you knew that the card chosen was a picture?
Now, what are the chances it's a king?

p(the picture card that you drew is a king) = p(king|it's a picture card)

$$= \frac{4 \text{ (possible kings)}}{12 \text{ (\# of picture cards)}}$$

"Conditional Probability"

Notice the difference between replacing a card (independent events) and not replacing a card (dependent events; 2nd draw depends on outcome of the 1st draw)

"Replacement"

vs.

"Without Replacement"

p(drawing 2 sevens w/o replacement) = 13/52 x 12/51

p(drawing 2 sevens with replacement) = 13/52 x 13/52

Each fraction is $\frac{\text{'possible successful outcomes'}}{\text{'total possible outcomes'}}$

Factorials: Counting Arrangements

Definition of Factorial: The product of an integer and all smaller positive integers.

$$x! = x(x-1)(x-2)\ldots 2 \times 1$$

5! = 5 x 4 x 3 x 2 x 1 = 120

Factorials are often used to 'count arrangements'

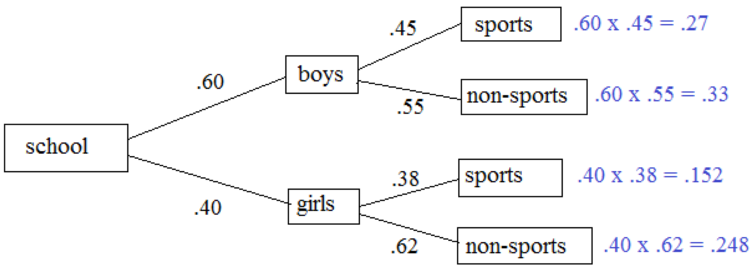
How many ways can you arrange 5 chairs?

In the first spot, you can choose from 5 chairs..
Then, in the second spot, you can choose from the 4 chairs not chosen..
Then, in the 3rd spot, you can choose from the 3 chairs remaining... Etc...

5! = 120 possible ways

Probability Tree Diagram

Example: 60% of the math school is male, and 40% of the school is female. If 45% of the boys play sports, and 38% of girls play sports, use a probability tree diagram to answer the following:



1) What is the probability of picking a boy who does not play sports?

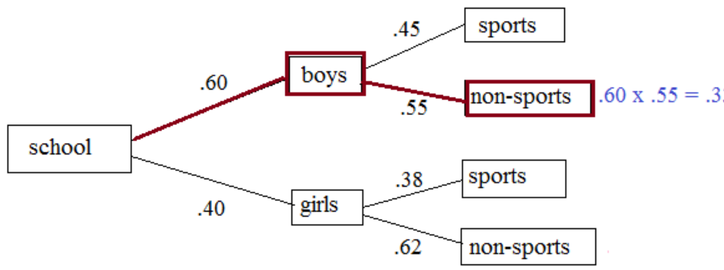
Follow the branches:

P(boy) = .60

P(not playing sports|boy) = .55

P(boy and non-sports) = P(boy) x P(not sports|boy)

= .60 x .55 = .33

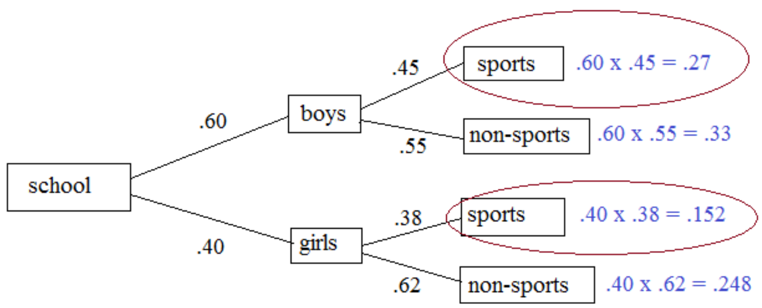


2) What percentage of students play sports?

% of students who play sports = boys who play sports + girls who play sports

= .27 + .152

.422

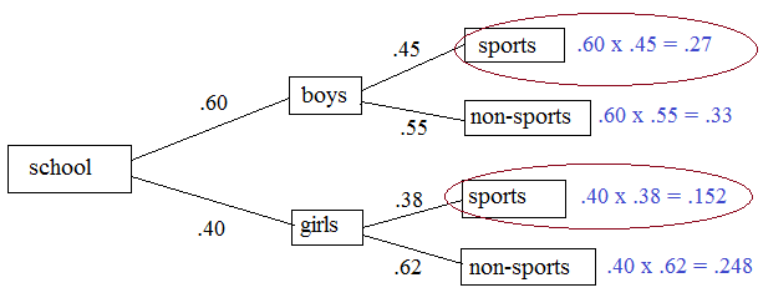


3) What percentage of athletes are girls?

percentage of athletes who are girls = $\frac{\text{girl athletes}}{\text{total athletes}}$

= $\frac{.152}{.422}$

= .36



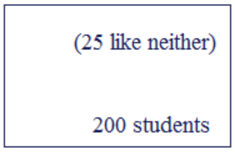
Venn Diagram Application

1) In a survey of 200 students, 115 like math, 80 like english, 25 like neither.

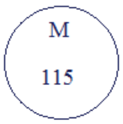
A) What is the probability that a selected student likes *both* english and math?

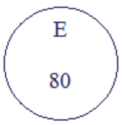
B) What is the probability that a selected student likes *either* math or engish?

An effective method of solving is to use a Venn Diagram:

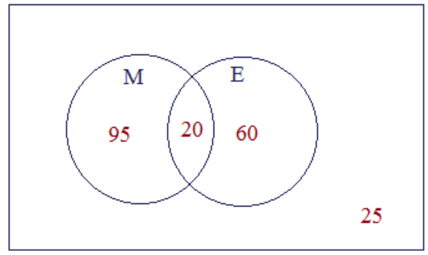


Since 25 like neither,
175 must like either
math or english.
200 − 25 = 175





Since 115 like math
and 80 like english,
there is an overlap of 20
(195 − 175 = 20)



Math only = 115
English only = 80
Math AND English 20
Neither = 25

A) P(both M and E) = $\frac{20}{200} = 10\%$

B) P(either M or E) = $\frac{175}{200} = \frac{7}{8} = 87.5\%$

or, $1 - \frac{25}{200}$

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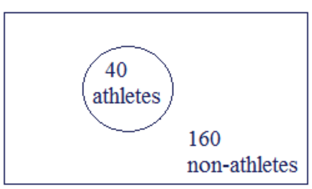
2) At the local high school, 20% of the students are athletes that play a sport.
Of the athletes, 25% play football, 10% play ONLY basketball, and 5% play football and basketball.
(The rest of the athletes play other sports.)

A) What percent of athletes play sports other than football or basketball?

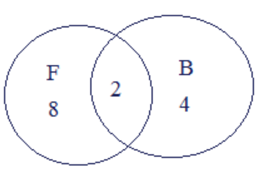
B) If I pick a random student, what is the probability that he plays basketball?

To simplify, let's assume the high school has 200 students



20% athletes
80% non-athletes

Then, let's break up the athletes

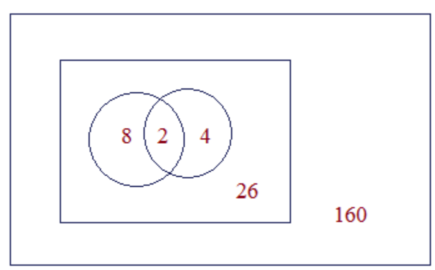


8 football only
4 basketball only
2 football/basketball

A) Therefore, 40 athletes − 14 basketball/football = 26
26 out of 40 play a different sport!

$\frac{26}{40} = 65\%$

Assuming 200 students:



In the diagram, there are 200 students.
And, 6 play basketball
(4 play only basketball; 2 play basketball and football)

B) P(student plays basketball) = $\frac{6}{200} = 3\%$

Example: What is the probability of drawing a club, then a face card (WITHOUT replacement)?

There are 52 cards in a standard deck. There are 4 suits --- 13 clubs.. and, there are 3 face cards per suit (12 total)
Jack, Queen, King

the answer is NOT $\frac{13}{52} \cdot \frac{12}{51}$ because these are not completely independent events....

SOLUTION

CASE 1: First card is non-face club

$$\frac{10}{52} \cdot \frac{12}{51} = \frac{120}{2652}$$

CASE 2: First card is face club

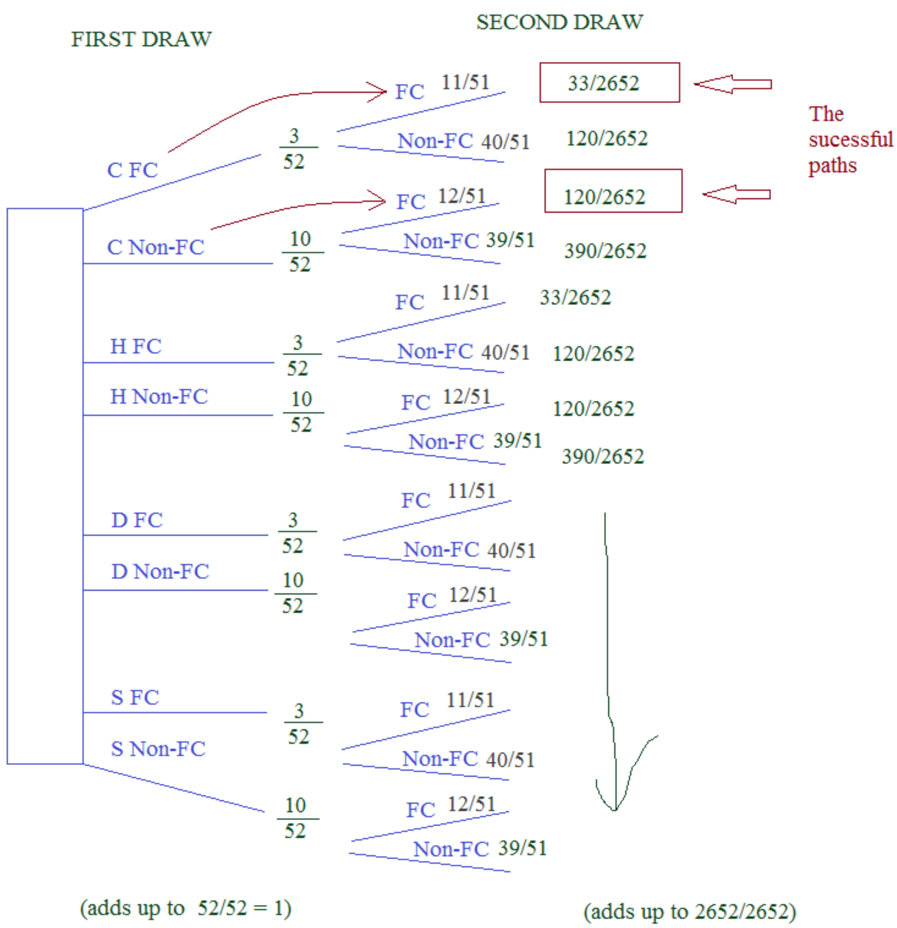
$$\frac{3}{52} \cdot \frac{11}{51} = \frac{33}{2652}$$

Together, there are $\frac{153}{2652} = \frac{3}{52}$

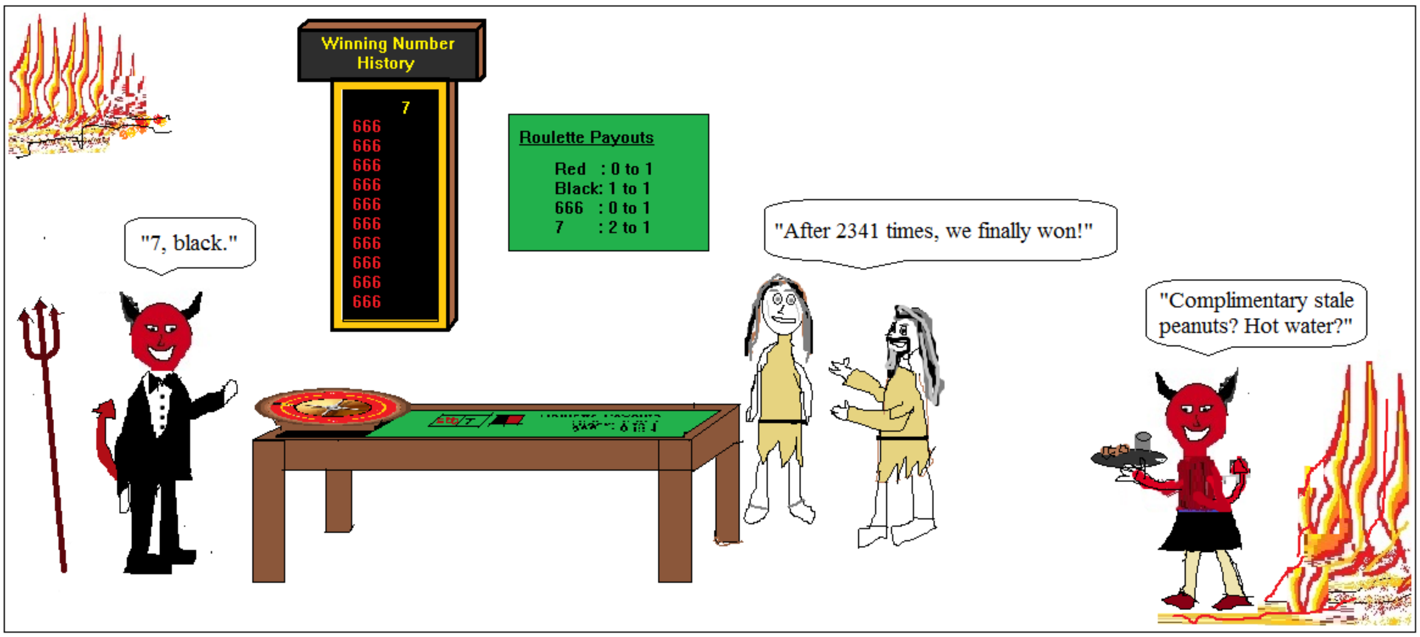
This can be illustrated with a probability tree diagram:

Club = C
Heart = H
Diamond = D
Spade = S

FC = Face card (Jack, Queen, King)
Non-FC = Non-Face Card



Roulette
in
Hell



Two poor souls try their luck at this game of (no) chance...

What are the odds of success? Slim to none!

LanceAF #61 (12-1-12)
www.mathplane.com

Practice Exercises-→

Conditional Probability Quick Quiz

I. The following questions refer to a drawn card from a standard 52 card deck.

P(7 of spades) =

P(7 of spades|spade) =

P(7 of spades|seven) =

P(7 of spades|black card) =

Are "Kings" and "diamonds" independent?

Are "Kings" and "face cards" independent?

II.

	Freshmen	Sophomores	Juniors	Seniors	Totals
Algebra	52	32	16	0	100
Geometry	28	44	20	6	98
Trigonometry	17	20	59	20	116
Calculus	3	11	19	53	86
Totals	100	107	114	79	400

What is the probability that a randomly chosen student is a senior?

What is the probability that a random junior is taking calculus?

What is the probability that a senior is taking geometry?

What is the probability that a geometry student is a senior?

P(Calculus) =

P(freshmen or sophomore) =

P(geometry and trigonometry) =

P(senior|algebra) =

P(freshman|trigonometry) =

P(trigonometry|freshman) =

Hidden Message

Clue: "A Sure Thing"

Answer the questions below. Then, convert the numbers to letters to reveal the solution.

Letter Key:

0	1	2	3	4	5	6	7	8	9
A	B	E	I	L	O	P	R	S	Y

1) Number of ways to roll "doubles" with 2 dice. $\square \rightarrow$ ___

2) Chance of drawing a club, diamond, or heart (from a deck of 52 cards). $.\square 5 \rightarrow$ ___

3) Chances that the red face card drawn is a diamond. $.\square 0 \rightarrow$ ___

4) If the odds of success are 7:3, how many successes would you expect in 30 independent trials? $2\square \rightarrow$ ___

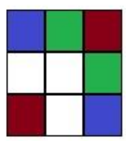
5) Probability of an impossible event? $\square \rightarrow$ ___

6) p('event A') + p(not 'event A') = $\square \rightarrow$ ___

7) If the probability of X is 30%, the probability of Y is 100%, and X and Y are independent, what is the probability of X and Y occurring? $\square 0\% \rightarrow$ ___

8) If it rains 60% of the time in February, what are the chances it doesn't rain on Valentine's day? $.\square 0 \rightarrow$ ___

9) Probability of randomly selecting a white square:



$3\square.\overline{3}\% \rightarrow$ ___

10) FREE ENTRY — free entry $\rightarrow$ T

11) Outcomes M, N, and O are mutually exclusive. If p(M) = .3 p(N) = .6 p(O) = .1 , what is the p(M or N)? $.\square \rightarrow$ ___

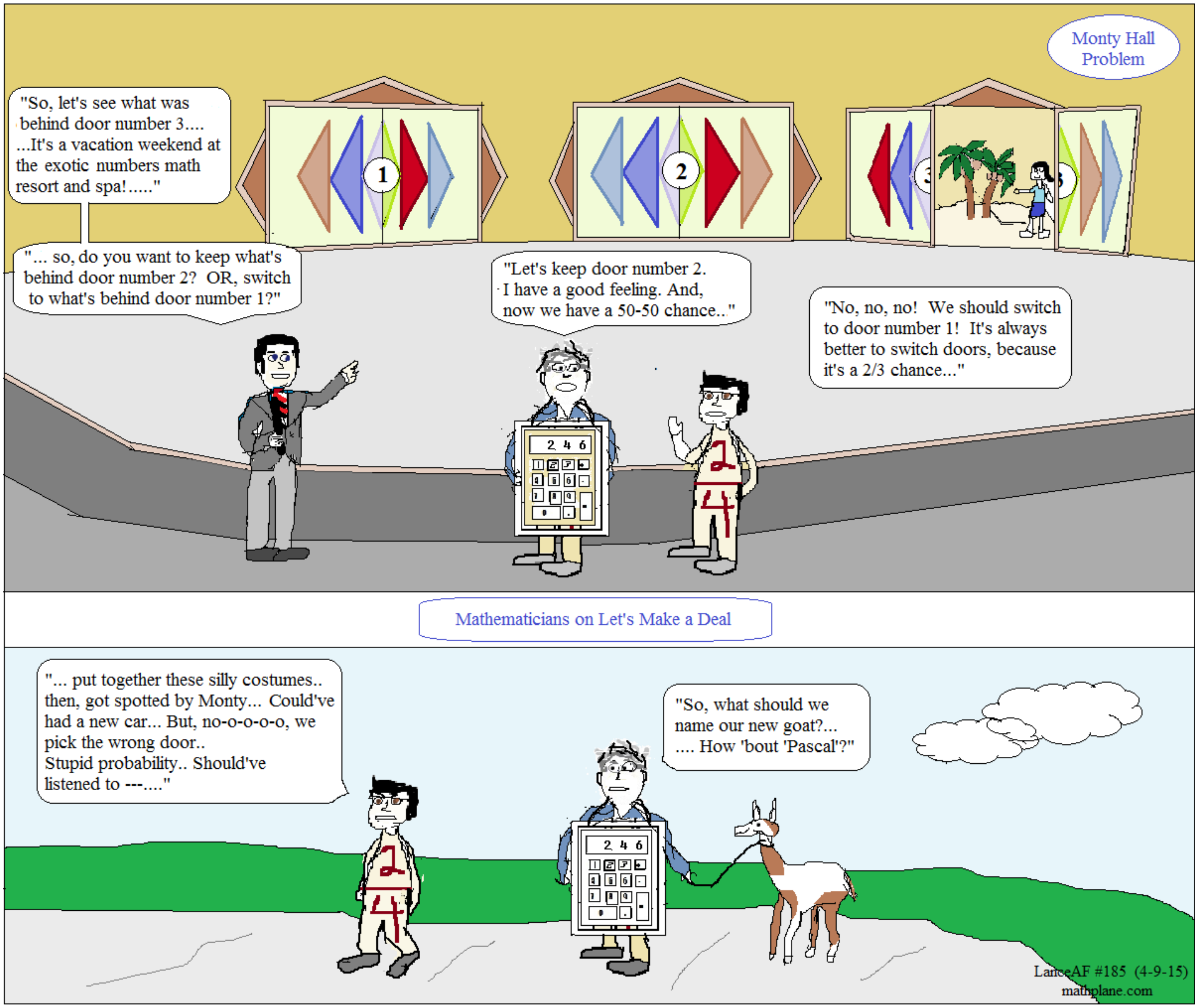
12) A bag contains 3 green blocks, 5 red blocks, and 2 blue blocks. Each time you draw, you put the block back. If you draw a green one three times in a row, what is the probability the next draw is green? $\frac{\square}{10} \rightarrow$ ___

13) Odds of a coin landing heads 3 times in a row. $1/\square \rightarrow$ ___

14) A bag has 4 marbles: 3 yellow and 1 blue. What are the chances of reaching in the bad and pulling 2 yellow marbles out? $.\square 0 \rightarrow$ ___

15) FREE ENTRY — free entry $\rightarrow$ N

16) Total number of different blackjack hands that can possibly be dealt. $265\square \rightarrow$ ___


Monty Hall Problem
"So, let's see what was behind door number 3.... ...It's a vacation weekend at the exotic numbers math resort and spa!....."
"... so, do you want to keep what's behind door number 2? OR, switch to what's behind door number 1?"
"Let's keep door number 2. I have a good feeling. And, now we have a 50-50 chance.."
"No, no, no! We should switch to door number 1! It's always better to switch doors, because it's a 2/3 chance..."
Mathematicians on Let's Make a Deal
"... put together these silly costumes.. then, got spotted by Monty... Could've had a new car... But, no-o-o-o-o, we pick the wrong door.. Stupid probability.. Should've listened to ---...."
"So, what should we name our new goat?... .... How 'bout 'Pascal'?"
LanceAF #185 (4-9-15)
mathplane.com

SOLUTIONS-→

Conditional Probability Quick Quiz

SOLUTIONS

I. The following questions refer to a drawn card from a standard 52 card deck.

P(7 of spades|seven) = "Probability that the card I'm holding is the 7 of spades, given that you are told it's a seven"

P(7 of spades) = $\frac{1}{52}$ (7 of spades) (total cards)

P(7 of spades|spade) = $\frac{1}{13}$ (the 7 of spades) (13 total spades)

P(7 of spades|seven) = $\frac{1}{4}$ (the 7 of spades) (4 total 7s)

A side note: P(7 of spades|red card) = 0

"the probability that a red card is the 7 of spades is 0: impossible."

P(7 of spades|black card) = $\frac{1}{26}$ (7 of spades) (# of black cards)

Are "Kings" and "diamonds" independent? P(K) = 4/52 = 1/13 P(K|D) = 1/13 Yes

Definition of independent: P(B|A) = P(B), then A and B are independent...

What is the probability the card I'm holding is a king? 4/52.. Now, what if I told you the card is a diamond? The probability it is a king remains 1/13.

Are "Kings" and "face cards" independent? P(K) = 1/13 P(K|FC) = 4/12 = 1/3 No

Also, P(FC) = 12/52 = 3/13 P(FC|K) = 1 No

What is the probability that the card I'm holding is a king? 1/13.. But, what if I revealed that the card I'm holding is a face card? Now, the probability it is a king is 4/12...

II.

	Freshmen	Sophomores	Juniors	Seniors	Totals
Algebra	52	32	16	0	100
Geometry	28	44	20	6	98
Trigonometry	17	20	59	20	116
Calculus	3	11	19	53	86
Totals	100	107	114	79	400

What is the probability that a randomly chosen student is a senior? $\frac{79}{400}$ (seniors) (total students)

What is the probability that a random junior is taking calculus? P(Calculus|Junior) = $\frac{19}{114}$ "If I select a junior, what is the probability that he/she is taking calculus?"

What is the probability that a senior is taking geometry? P(Geometry|Senior) = $\frac{6}{79}$

What is the probability that a geometry student is a senior? P(Senior|Geometry) = $\frac{6}{98}$ or $\frac{6 \text{ Senior Geometry students}}{98 \text{ Total Geometry students}}$

P(Calculus) = $\frac{86}{400}$

P(senior|algebra) = 0

P(freshmen or sophomore) = $\frac{207}{400}$

P(freshman|trigonometry) = $\frac{17}{116}$ "If I already know the student is in trigonometry, what is the probability he/she is a freshman?"

P(geometry and trigonometry) = 0

P(trigonometry|freshman) = $\frac{17}{100}$ "If I already know the student is a freshman, what is the probability he/she is in trigonometry?"

Hidden Message

Clue: "A Sure Thing"

Answer the questions below.
Then, convert the numbers to letters to reveal the solution.

SOLUTIONS

Letter Key:

0	1	2	3	4	5	6	7	8	9
A	B	E	I	L	O	P	R	S	Y

1) Number of ways to roll "doubles" with 2 dice.

six possible ways:
1|1 2|2 3|3 4|4 5|5 6|6

6 → P

2) Chance of drawing a club, diamond, or heart (from a deck of 52 cards).

$\frac{\text{\# of successes}}{\text{\# of possibilities}} = \frac{39}{52}$ = 3/4 or .75

.75 → R

3) Chances that the red face card drawn is a diamond.

Since we know it is a RED card, it can either be hearts or diamonds. Therefore, the chances are 50/50..

.50 → O

4) If the odds of success are 7:3, how many successes would you expect in 30 independent trials?

If the odds are 7:3, this implies that for every 10 trials, 7 would be successful.

$\frac{7}{10} = \frac{x}{30}$

x = 21

21 → B

5) Probability of an impossible event?

If it's impossible, then the chance of success is zero.

0 → A

6) p('event A') + p(not 'event A') = 1

EX: p(A) = .40 then, p(not A) would be .60

(all possibilities will always be 1)

1 → B

7) If the probability of X is 30%, the probability of Y is 100%, and X and Y are independent, what is the probability of X and Y occurring?

(since independent)
p(X and Y) = p(X)p(Y)
= .30 x 1.00 = .3

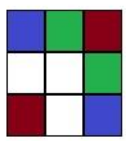
30% → I

8) If it rains 60% of the time in February, what are the chances it doesn't rain on Valentine's day?

p(not raining in Feb) = 1 - p(rain)
.40 = 1 - .60

.40 → L

9) Probability of randomly selecting a white square:



$\frac{3 \text{ white}}{9 \text{ total}} = \frac{1}{3}$

$33.\overline{3}\%$ → I

10) FREE ENTRY

free entry → T

11) Outcomes M, N, and O are mutually exclusive. If p(M) = .3 p(N) = .6 p(O) = .1, what is the p(M or N)?

(since mutually exclusive) p(M) + p(N) = p(M or N)

.3 + .6 = .9

.9 → Y

12) A bag contains 3 green blocks, 5 red blocks, and 2 blue blocks. Each time you draw, you put the block back. If you draw a green one three times in a row, what is the probability the next draw is green?

Since you keep putting the blocks back into the bag, each draw is independent!
p(green) = 3/10

$\frac{3}{10}$ → I

13) Odds of a coin landing heads 3 times in a row.

8 possibilities (2 x 2 x 2): HHH HHT HTH HTT THH THT TTH TTT

(1 successful outcome)

1/8 → S

14) A bag has 4 marbles: 3 yellow and 1 blue. What are the chances of reaching in the bad and pulling 2 yellow marbles out?

(dependent events!)

3/4 x 2/3 = 6/12 = 1/2
(1st marble) (2nd marble)

.50 → O

15) FREE ENTRY

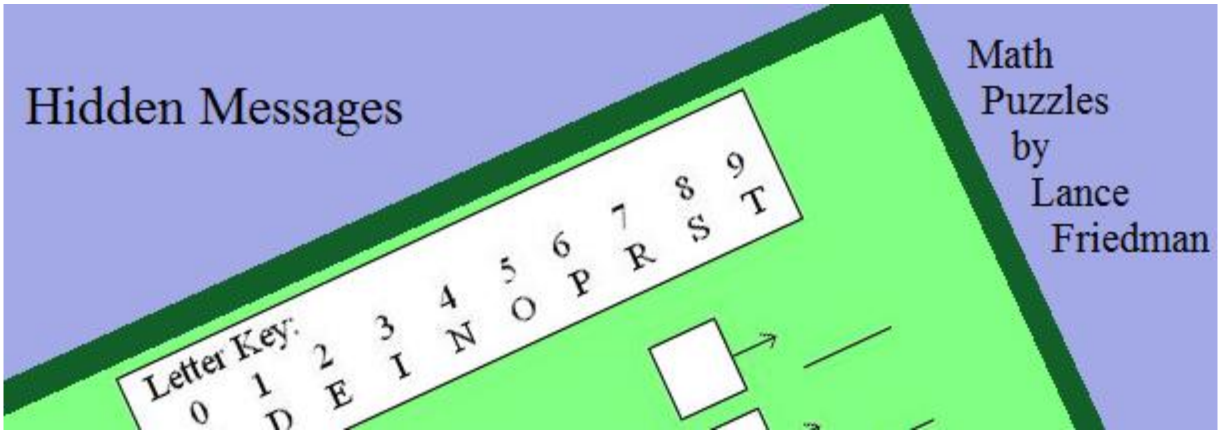
free entry → N

16) Total number of different blackjack hands that can possibly be dealt.

(possible 1st card) x (possible 2nd card) = possibilities
52 x 51 (remaining) = 2652

2652 → E

"A Sure Thing"? PROBABILITY IS ONE



Find more Hidden Message Puzzles at Mathplane.com…

(Throughout the site and in the 'travel log collection')

One more probability question:

"Seating Assignment"

You and 2 friends receive invitations to the math awards banquet.

Each guest table is round and seats 10 people.

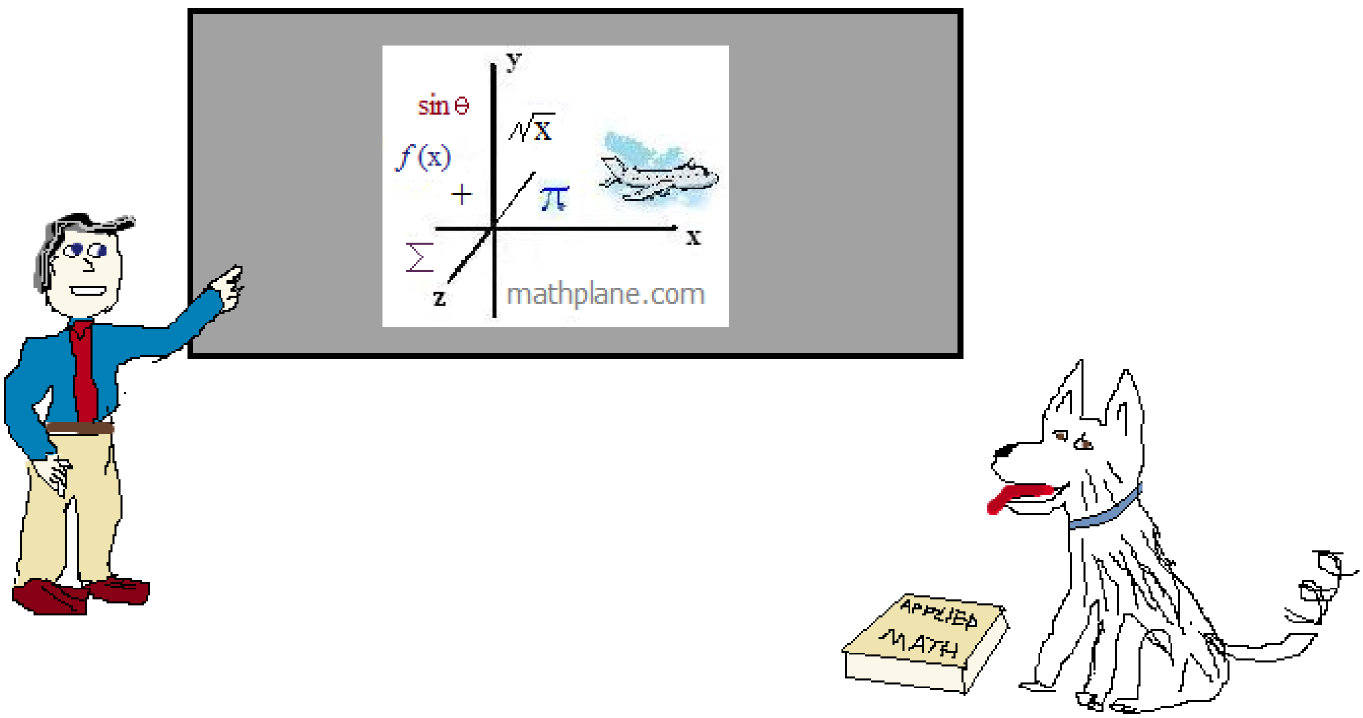
If you and your friends were randomly assigned seats at the same table, what is the probability that you are seated next to both friends?

Solution at the end of the packet…

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

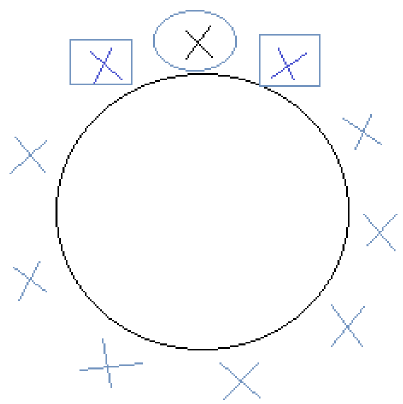
Cheers



Also, at Facebook, Google+, Pinterest, and TeachersPayTeachers

Seating Assignment Question:

You and 2 friends receive invitations to the math awards banquet.
Each guest table is round and seats 10 people.
If you and your friends were randomly assigned seats at the same table,
what is the probability that you are seated next to both friends?



Your seat does not matter....

Now, consider the first friend... What is the probability that he/she is seated next to you?
There are 9 seats left... And, there are 2 seats that are next to you. so the probability is 2/9.

Then, consider the second friend... What is the probability that he/she is seated next to you --- *assuming the first friend got one of the 2 seats*? 1/8

So, the probability that both friends get the two seats next to you is 2/9 x 1/8 = 1/36

$$\boxed{{}_9C_2 = \frac{9!}{2!7!} = 36}$$