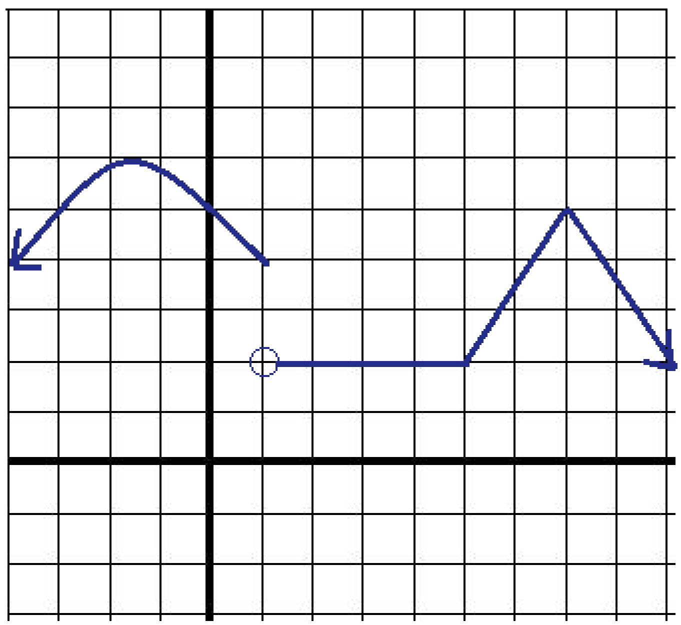
Calculus Introduction:

Continuity and Differentiability

Notes, Examples, and Practice Quiz (w/solutions)



Topics include definition of continuous, limits and asymptotes, differentiable function, and more.

Continuity/Discontinuity

Definition: A function $f(x)$ is continuous at point 'a' if

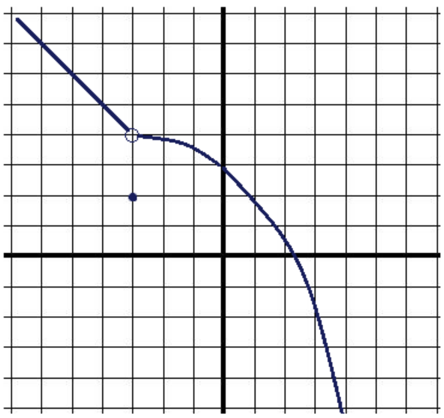
1) $f(a)$ is defined
2) $\lim_{x \to a} f(x)$ exists
3) $\lim_{x \to a} f(x) = f(a)$

A function is continuous if every point on the interval is continuous

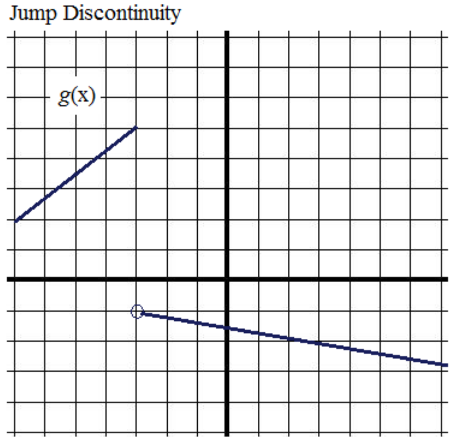
What is it? A function is continuous "if you can draw a graph without lifting your pencil off the paper"

DIScontinuity *Examples:*

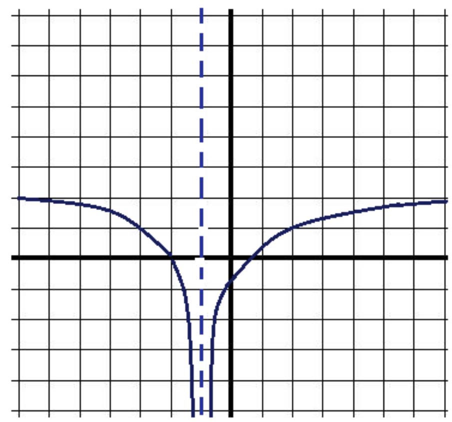
Removable Discontinuity "hole"



Jump Discontinuity



Vertical asymptote "undefined part"

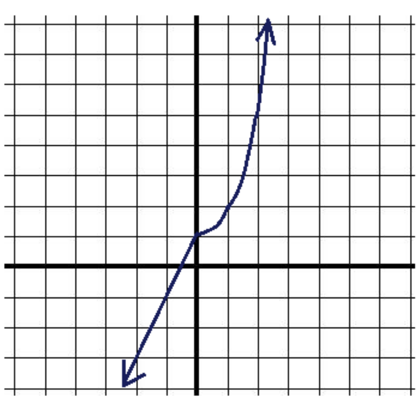


Note: An entire function may not be continuous, BUT it may contain "intervals" of continuity.
For example: $g(x)$ is not continuous, BUT the intervals [-7, -3] and (-3, 7] are continuous!

Can a piecewise function be continuous? Yes!

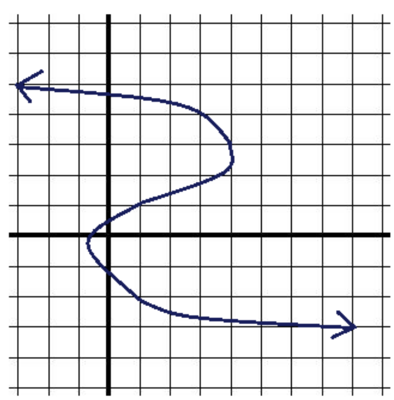
Example:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 0 \\ x^2 + 1 & \text{if } x \geq 0 \end{cases}$$



Since the pieces meet at the same spot, it's continuous!

Why is this not a continuous function?



Answer: It's not a function!! (violates vertical line test)

Continuity

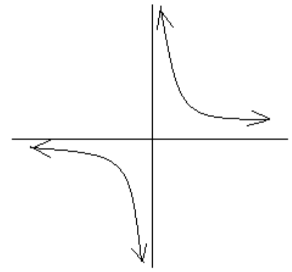
Example: Describe the discontinuity of each function at x = 0 a) $\frac{1}{x}$ b) $\frac{|x|}{x}$ c) $\frac{x}{x}$

a) $\frac{1}{x}$

$\lim_{x \to 0^-} = -\infty$

$\lim_{x \to 0^+} = +\infty$

limit does not exist; $f(0)$ is undefined;
asymptote, infinite discontinuity

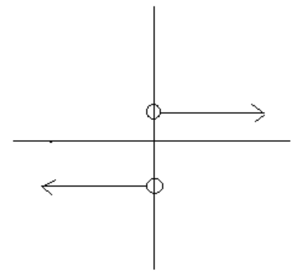


b) $\frac{|x|}{x}$

$\lim_{x \to 0^-} = -1$

$\lim_{x \to 0^+} = 1$

limit does not exist $f(0)$ is undefined;
jump discontinuity

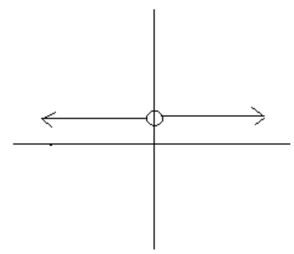


c) $\frac{x}{x}$

$\lim_{x \to 0^-} = 1$

$\lim_{x \to 0^+} = 1$

limit exists: equals 1 $f(0)$ is undefined
removable discontinuity (hole)



Examples: What values of a and b make the functions continuous?

$$f(x) = \begin{cases} 3b + a & \text{if } x \le -2 \\ x^2 + 5 & \text{if } -2 < x < 1 \\ 2x + a & \text{if } 1 \le x \end{cases}$$

$$g(x) = \begin{cases} 2ax - b & \text{if } x \le 1 \\ x^2 - 10 & \text{if } 1 < x \le 4 \\ a + bx & \text{if } 4 < x \end{cases}$$

$\lim_{x \to -1^-} = \lim_{x \to -1^+}$

$(1)^2 + 5 = 2(1) + a$

$6 = 2 + a$

$\boxed{4 = a}$

$\lim_{x \to -2^-} = \lim_{x \to -2^+}$

$3b + (4) = (-2)^2 + 5$

$\boxed{b = 5/3}$

"The equations must be equal at the break points"
(In other words, where one piece of the function stops, the next piece must resume in the same place.)

$\lim_{x \to -1^-} = \lim_{x \to -1^+}$

$2a(1) - b = (1)^2 - 10$

$2a - b = -9$

$\lim_{x \to 4^-} = \lim_{x \to 4^+}$

$(4)^2 - 10 = a + b(4)$

$6 = a + 4b$

then, solve the system: $2a - b = -9$, $a + 4b = 6$

$8a - 4b = -36$
$a + 4b = 6$

$9a = -30$

$\boxed{\begin{array}{l} a = -10/3 \\ b = 7/3 \end{array}}$

Differentiable function

A function that is diffentiable at every point in the domain. (A function that has a derivative)
A curve that is smooth and continous. (no discontinuities or cusps)

What is it? "If you can determine the instantaneous rate of change at any point, it's differentiable."

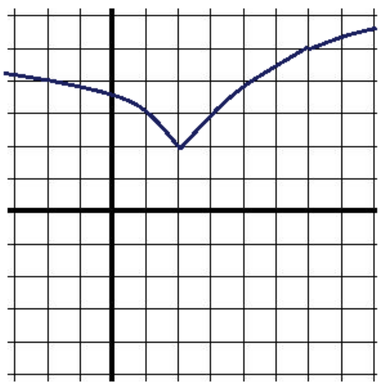
Comparison: If $\lim\limits_{x \to b^+} \neq \lim\limits_{x \to b^-}$ then $\lim\limits_{x \to b}$ does not exist

and,

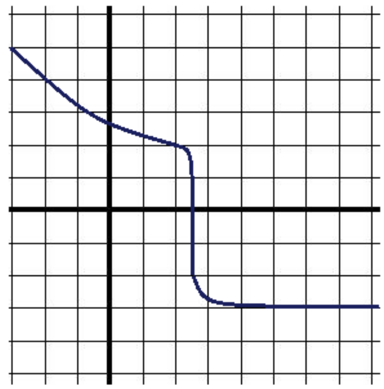
If the slope *from the left* is not equal to the slope *from the right*,
then the slope (instantaneous rate of change cannot be determined!)

NON-differentiable Examples

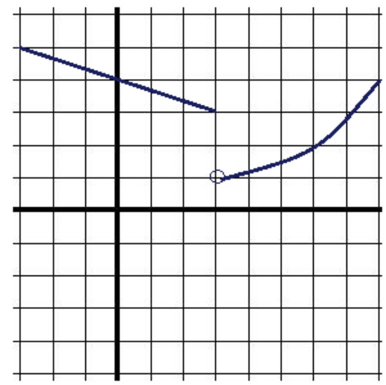
Cusp



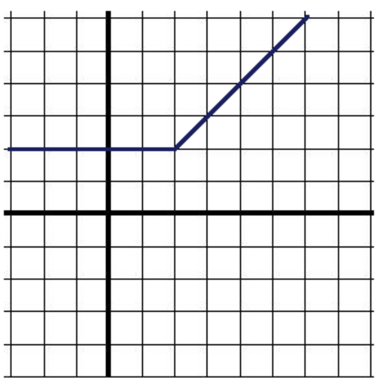
Undefined slope (not a function!)



Discontinuity



Corner



At x = 2, what is the slope?

From the <u>left</u>, the slope is 0
From the <u>right</u>, the slope is 1

The instantaneous rate of change at x = 2 is ambiguous.

Therefore, the derivative cannot be determined!!

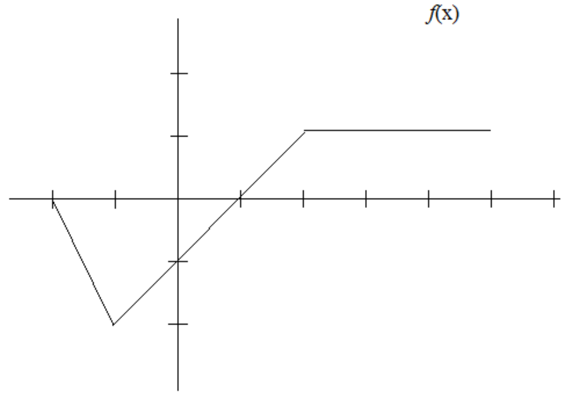
When is a function differentiable?
When you can determine the slope at every point on the given curve!

Important note: To be differentiable, the function must be continuous.

If a function is continuous, it <u>may or may not</u> be differentiable (at every point).
But,
if a function is differentiable, it <u>must</u> be continuous!

Example: Is the function $f(x)$ differentiable on the interval [-2, 5] ?

$f(x)$

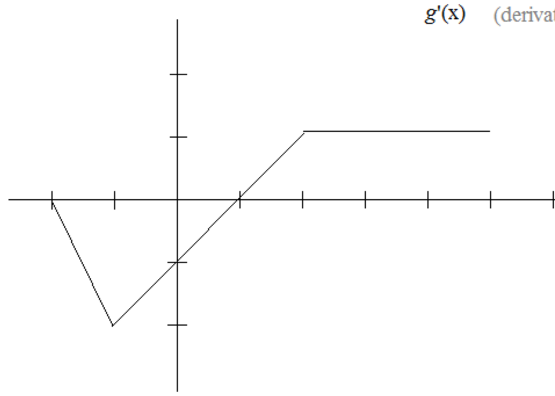


NO... It is not differentiable at $x = -1$ and $x = 2$

(because the IROC at each point is ambiguous..
at $x = -1$ from the left, the slope is -2, but
the slope at $x = -1$ from the right is 1..
then, at $x = 2$, the slope from the left is 1, but
the slope from the right is 0)

Example: Is the function $g(x)$ differentiable on the interval [-2, 5]?

$g'(x)$ (derivative of $g(x)$)



YES.. Because this graph displays the DERIVATIVE of $g(x)$.. In other words, it indicates the slope at every point on $g(x)$

at $x = -1$, the slope of $g(x)$ is -2

at $x = 2$, the slope of $g(x)$ is 1

Example: What values of a and b would make this function differentiable?

$$f(x) = \begin{cases} x^2 & \text{if } x < 10 \\ ax + b & \text{if } x \geq 10 \end{cases}$$

function must be continuous. Therefore,

$x^2 = ax + b$ at $x = 10$

$100 = 10a + b$

and, the function must have the same derivative at $x = 10$ (from the left and right)

from the left, derivative is $2x$...
so, 20

from the right, derivative is $a + 0$
therefore, a must be 20

and, if $a = 20$,
b must be -100

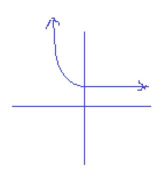
check: if $a = 20$
$b = -100$

at $x = 10$,
top is 100
bottom is $20(10) - 100 = 100$

top derivative is 20
and lower derivative is 20

Example:

$$g(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$$

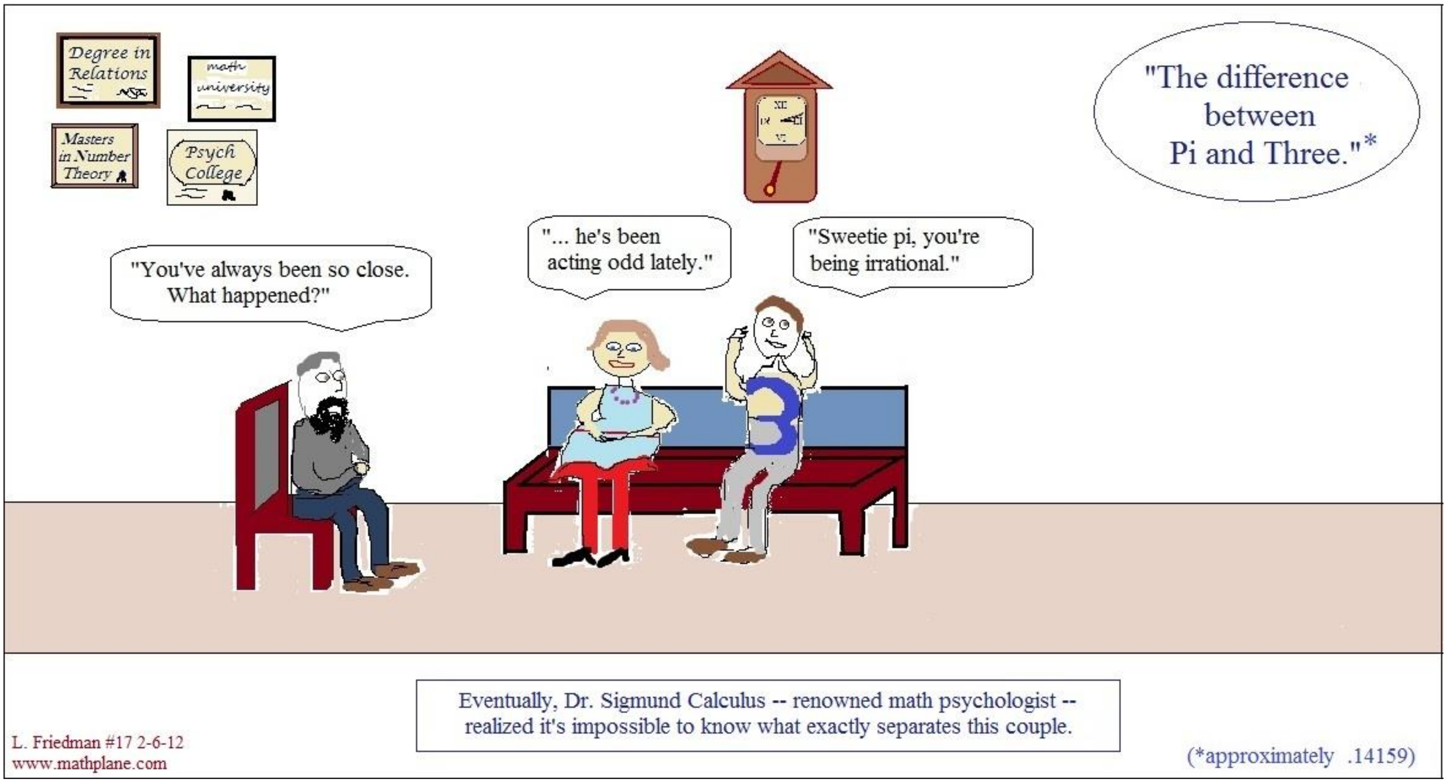


Note: this piecewise function is continuous and smooth...

Is this function continuous? Yes, because each piece meets at $x = 0$
at $x = 0$, $(0)^2 + 1 = 1$
and, the limit as x approaches 0 from the left also equals 1

differentiable? Yes, because the IROC (slope) at the "break point" is the same from the left and the right..

Both derivatives are 0 when $x = 0$

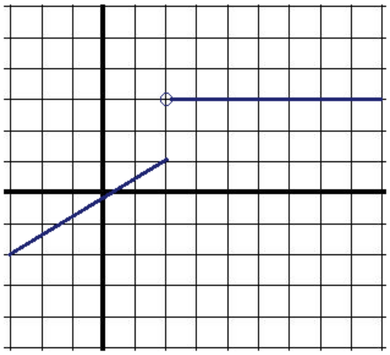


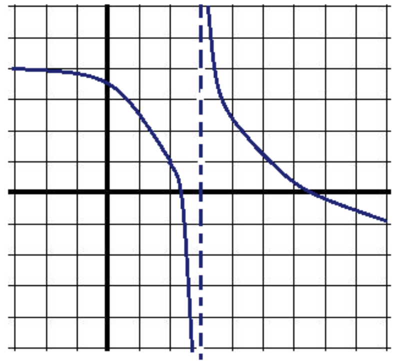
Continuity and Differentiation Exercises-→

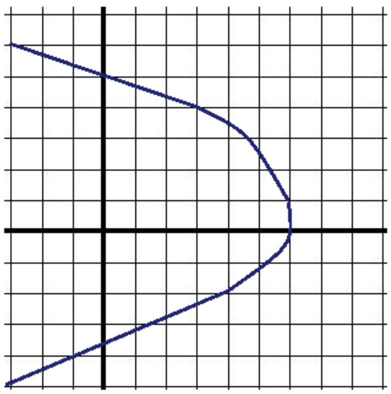
(with Solutions)

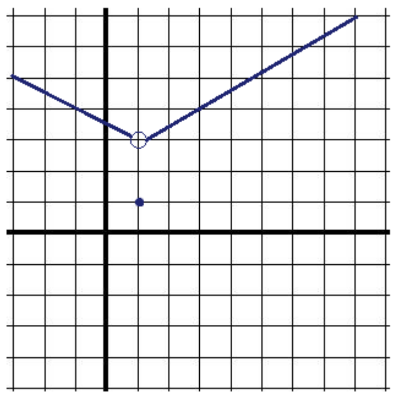
I. Explain why each is <u>not</u> a continuous function:

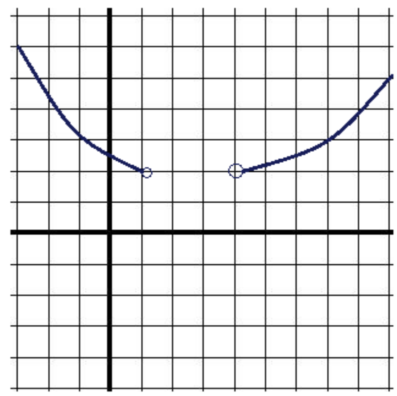
Exercise: Identifying Continuous & Differentiable Functions





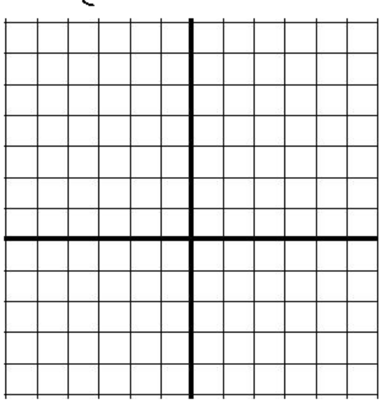




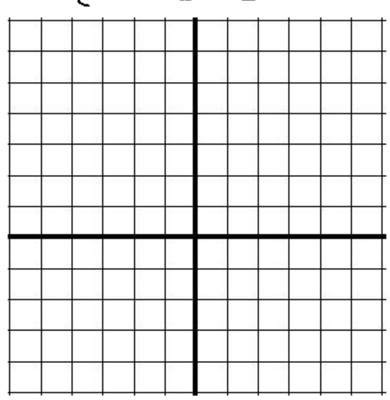


II. Determine if the following functions are continuous. Then, graph:

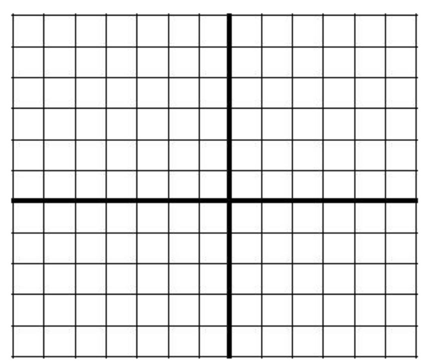
$$f(x) = \begin{cases} 3x + 2 & \text{if } x < 0 \\ x^2 - 2 & \text{if } x \geq 0 \end{cases}$$



$$g(x) = \begin{cases} 2x + 8 & \text{if } x < -3 \\ 2 & \text{if } -3 \leq x < 4 \\ 18 - 4x & \text{if } x \geq 4 \end{cases}$$



$$h(x) = \frac{1}{x + 3}$$

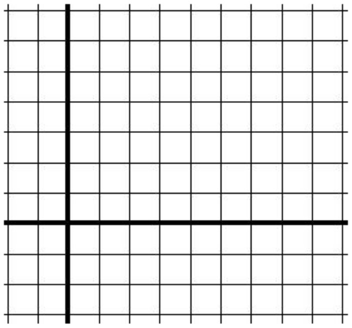


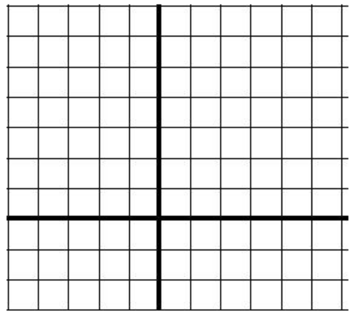
III. Determine where (and why) the functions are not differentiable. Then, sketch the graphs.

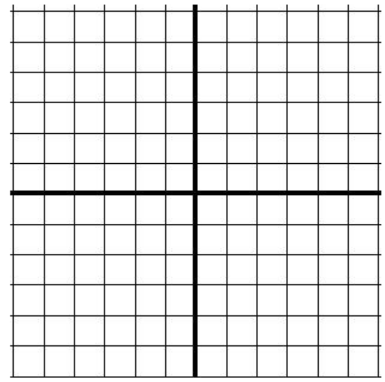
$f(x) = |x - 3| + 4$

$$g(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

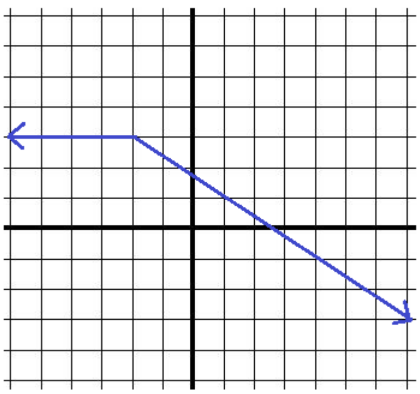
$h(x) = \dfrac{3}{x+2}$





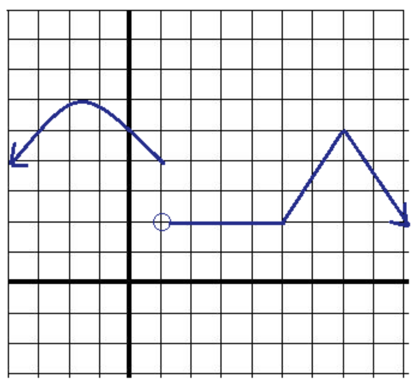


IV: Determine the intervals where the functions are a) continuous b) differentiable



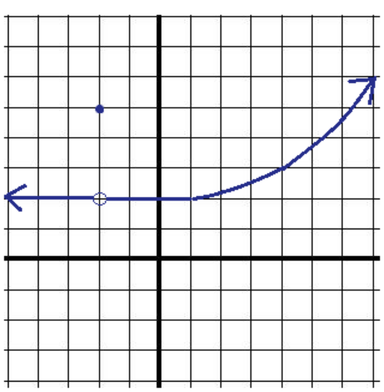
Continuous:

Differentiable:



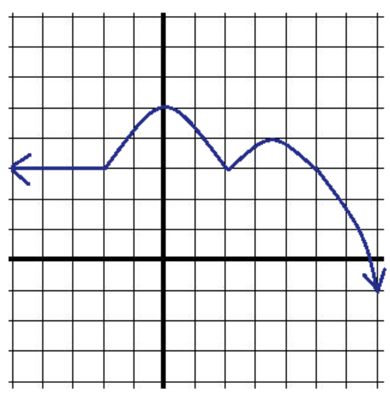
Continuous:

Differentiable:



Continuous:

Differentiable:



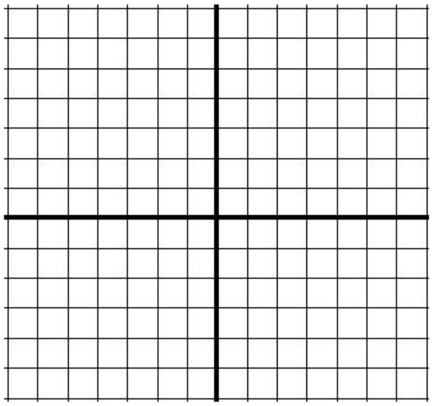
Continuous:

Differentiable:

V. Sketch a possible graph for the function f that has the given properties.

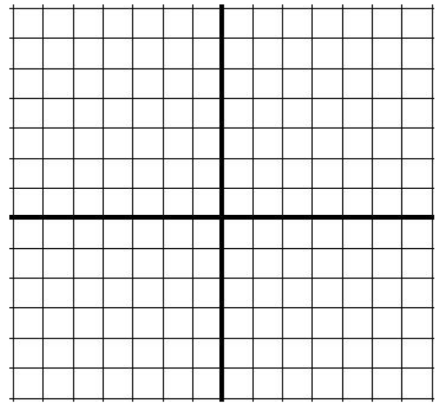
1) $f(3)$ exists

$\lim_{x \to 3} f(x)$ does not exist



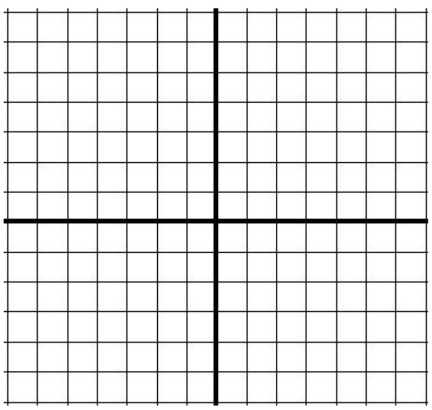
2) $f(-1)$ exists

$\lim_{x \to -1^+} f(x) = f(-1)$ $\qquad$ $\lim_{x \to -1^-} f(x)$ does not exist



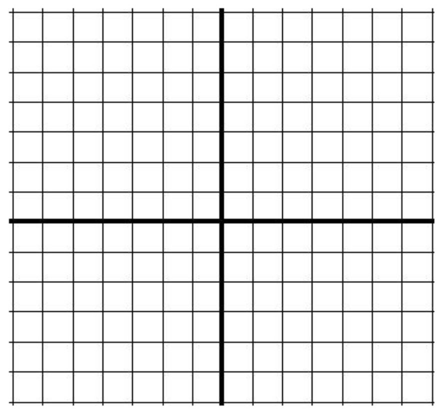
3) $f(5)$ exists

$\lim_{x \to 5} f(x)$ exists $\qquad$ $f(x)$ is not continuous



4) $f(2)$ does not exist $\qquad$ $f(5)$ exists

$\lim_{x \to 2} f(x)$ exists $\qquad$ f is not differentiable at $x = 5$



VI. What values would make the piecewise functions continuous?

$$g(x) = \begin{cases} \dfrac{x-4}{\sqrt{x}-2} & \text{if } x \neq 4 \\ \_\_\_\_\_\_\_\_ & \text{if } x = 4 \end{cases}$$

$$h(x) \begin{cases} \dfrac{2x^2+7x+3}{x+3} & \text{if } x \neq -3 \\ \_\_\_\_\_\_\_\_ & \text{if } x = -3 \end{cases}$$

VII. More Questions

1) $f(x) = 5 + \sqrt{x - 2}$

why is $f(x)$ not continuous at $x = 2$?

2) $g(x) = x + \| \cos(\pi x) \|$

why is $g(x)$ not continuous at $x = 2$?

3) $h(x) = \begin{cases} kx^2, & \text{if } x \leq 3 \\ kx + 3, & \text{if } x > 3 \end{cases}$

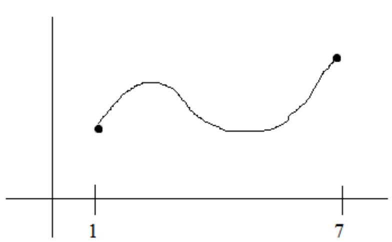
If $h(x)$ is a continuous function, what is k?

4) $f(x) = \begin{cases} 2, & \text{if } x \leq -1 \\ ax + b, & \text{if } -1 < x < 3 \\ -2, & \text{if } x \geq 3 \end{cases}$

If $f(x)$ is continuous, what are a and b?

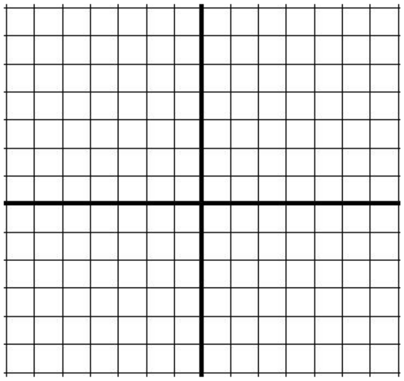
5) Where is the interval differentiable?

(1, 7) or [1, 7] ?

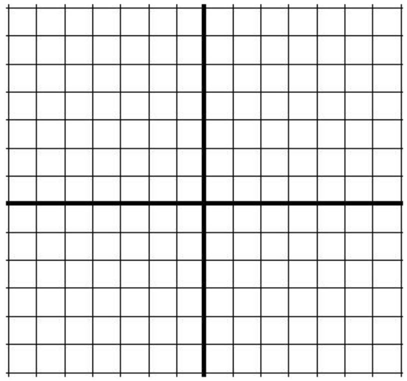


Sketch examples of functions containing the following features. (BONUS: write a possible equation)

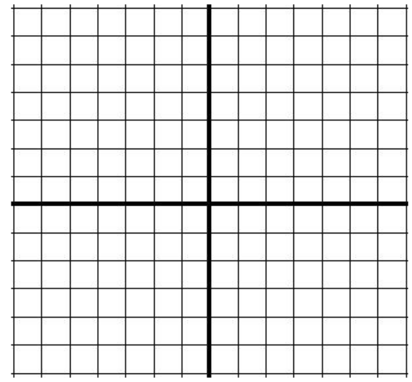
1) Vertical asymptote @ $x = -4$



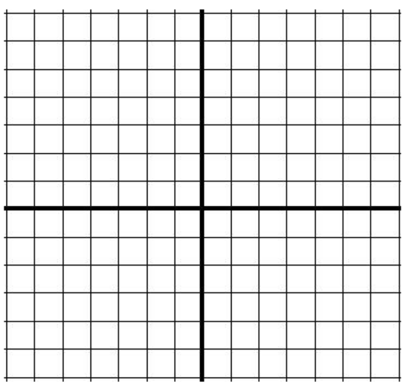
2) Continuous @ $x = 1$; a "corner" at $x = 1$



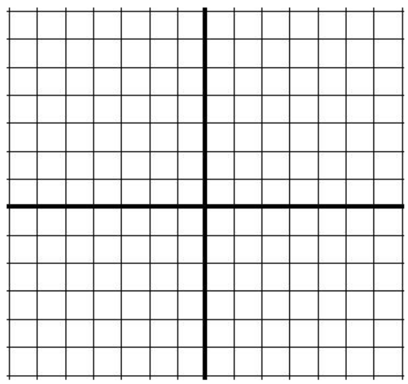
3) $\lim_{x\to 3} f(x)$ does exist; $f(3) \neq \lim_{x\to 3} f(x)$



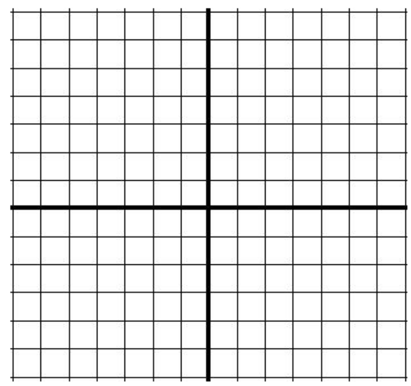
4) Infinite discontinuity @ $x = -5$



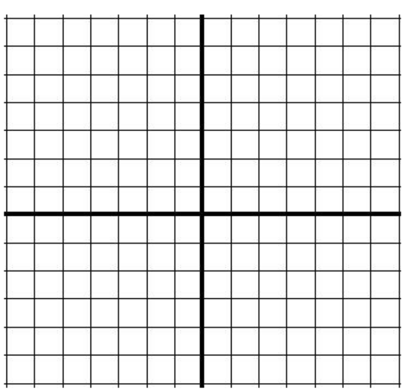
5) Jump discontinuity @ $x = 3$



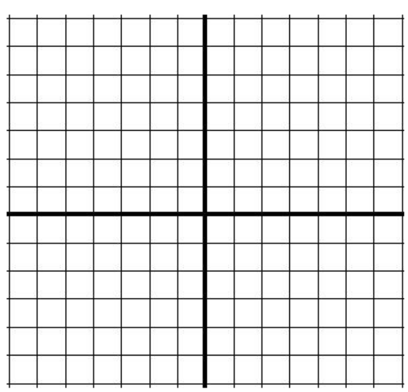
6) Step discontinuity @ $x = 5$



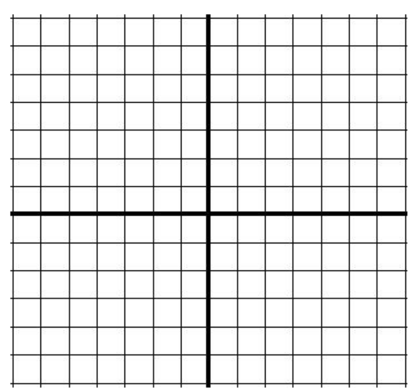
7) Kink @ $(-1, 5)$



8) Cusp @ $(4, 5)$



9) $\lim_{x\to 2}$ exists, but $f(2)$ has no value



Exercise: Identifying Continuous & Differentiable Functions

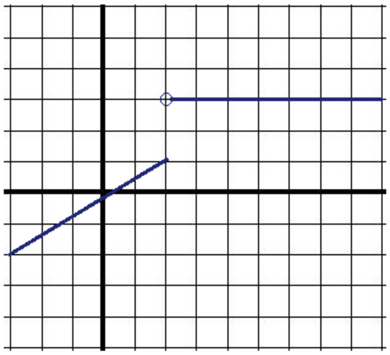
I. Explain why each is not a continuous function: SOLUTIONS

Definition: A function $f(x)$ is continuous at point 'a'

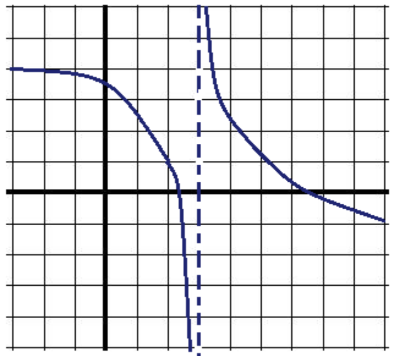
if 1) $f(a)$ is defined

2) $\lim_{x \to a} f(x)$ exists

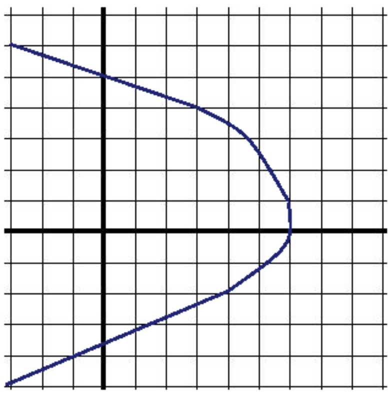
3) $\lim_{x \to a} f(x) = f(a)$



"jump" discontinuity --
limit does not exist at x = 2

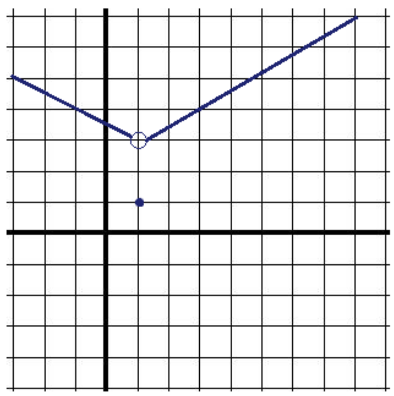


vertical asymptote
function is not defined at x = 3;
limit $x \to 3$ DNE



Not a function!

(fails "vertical line test")

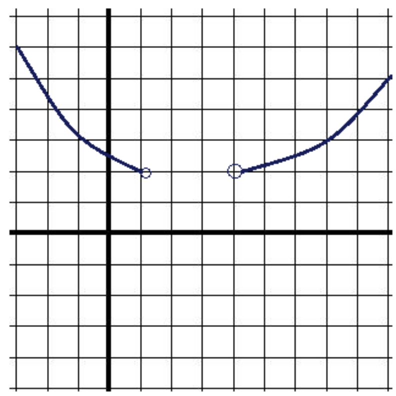


$f(1) = 1$ so, it is defined

$\lim_{x \to 1} f(x) = 3$ so, the limit exists

HOWEVER, $f(1) \neq \lim_{x \to 1} f(x)$

(removable discontinuity/"hole")



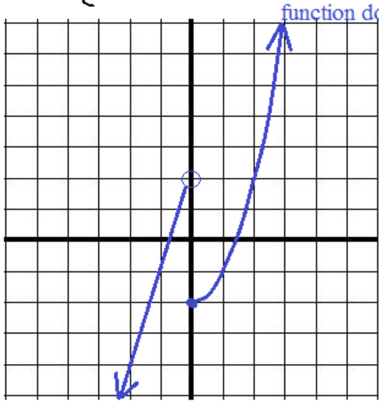
Although each interval is continuous,
the entire function in not because of the gap.

i.e. $f(x)$ is undefined from 1 to 4

II. Determine if the following functions are continuous. Then, graph:

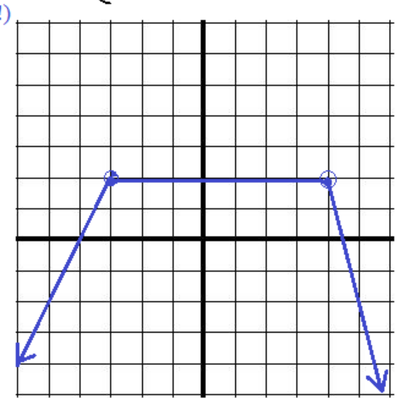
$$f(x) = \begin{cases} 3x + 2 & \text{if } x < 0 \\ x^2 - 2 & \text{if } x \geq 0 \end{cases}$$

NO ('Jump' discontinuity -- at x = 0, the 2 parts of the piecewise function don't meet!)



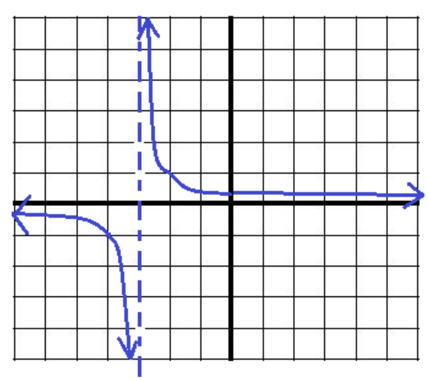
$$g(x) = \begin{cases} 2x + 8 & \text{if } x < -3 \\ 2 & \text{if } -3 \leq x < 4 \\ 18 - 4x & \text{if } x \geq 4 \end{cases}$$

YES (pencil never leaves the paper)



$$h(x) = \frac{1}{x + 3}$$

NO (vertical asymptote)

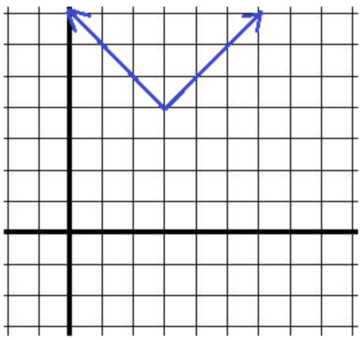


III. Determine where (and why) the functions are not differentiable. Then, sketch the graphs.

SOLUTIONS

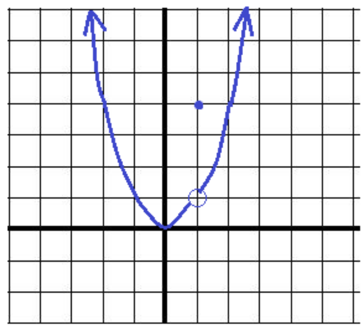
$f(x) = |x - 3| + 4$

there is a "corner" at (3, 4)
(i.e. the 'slope from the left' is -1
and, the 'slope from the right' is 1)



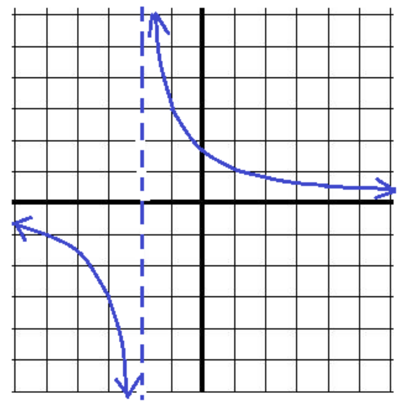
$$g(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

$g(x)$ is not continuous at $x = 1$
removable discontinuity

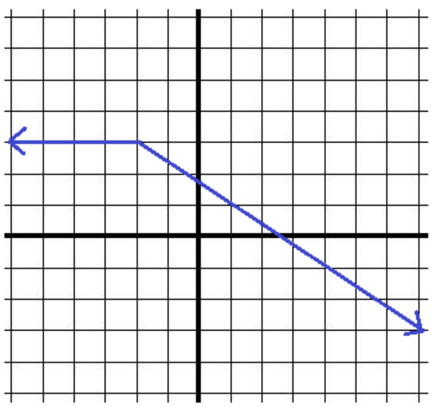


$$h(x) = \frac{3}{x + 2}$$

$h(x)$ is undefined (and not continuous)
at $x = -2$

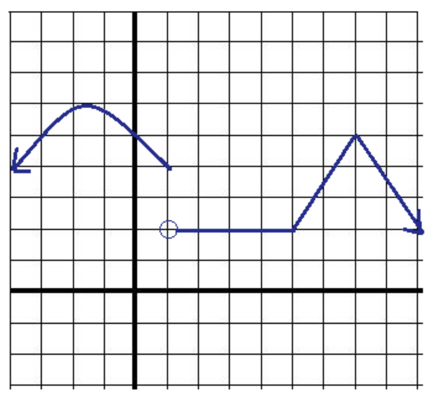


IV: Determine the intervals where the functions are a) continuous b) differentiable



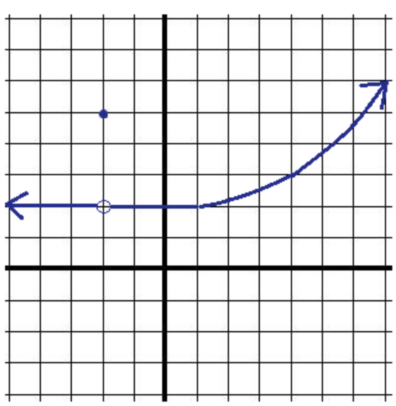
Continuous: all real numbers

Differentiable: $(-\infty, -2) \cup (-2, +\infty)$
anywhere except $x = -2$



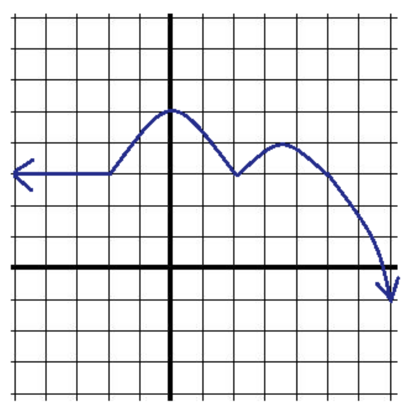
Continuous: $(-\infty, 1] \cup (1, +\infty)$

Differentiable: $(-\infty, 1] \cup (1, 5) \cup (5, 7) \cup (7, +\infty)$
All real numbers except 1, 5, or 7



Continuous: all real numbers except $x = -2$

Differentiable: $(-\infty, -2) \cup (-2, +\infty)$
anywhere except $x = -2$



Continuous: all real numbers

Differentiable: all real numbers, except $x = -2$ or 2
("cusps" or "kinks" at $x = -2$ and $x = 2$)

V. Sketch a possible graph for the function f that has the given properties.

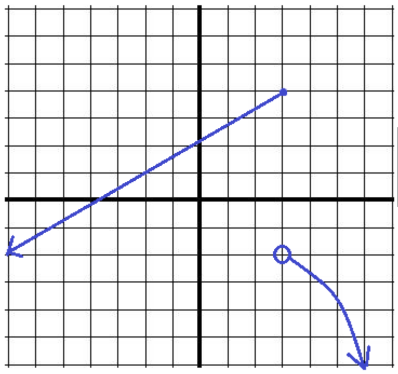
SOLUTIONS

1) $f(3)$ exists

$\lim_{x \to 3} f(x)$ does not exist

$f(3) = 4$

limit from the left is 4
limit from the right is -2..
so limit DNE

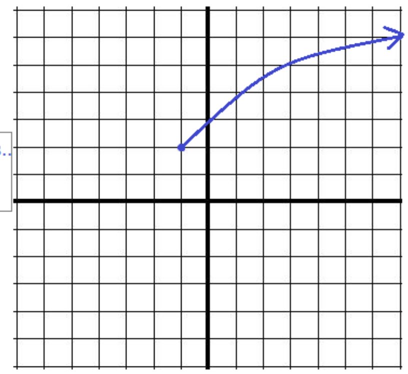


These are possible graphs..
Infinitely many other answers exist...

2) $f(-1)$ exists

$\lim_{x \to -1^+} f(x) = f(-1)$

$\lim_{x \to -1^-} f(x)$ does not exist



$f(-1) = 2$

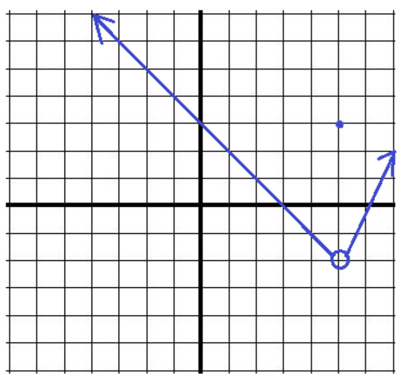
limit from the right is 2...

There is no limit from the left..

3) $f(5)$ exists

$\lim_{x \to 5} f(x)$ exists

$f(x)$ is not continuous



$f(5) = 3$

limit as x approaches 5 is -2..

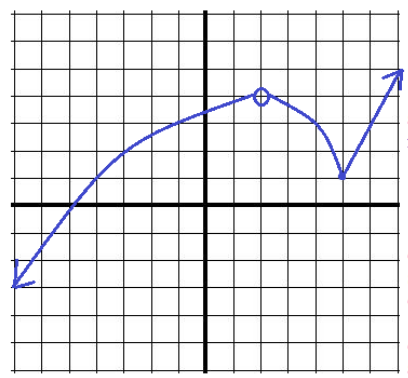
$f(x)$ is not continuous because of the removeable discontinuity

4) $f(2)$ does not exist

$\lim_{x \to 2} f(x)$ exists

$f(5)$ exists

f is not differentiable at $x = 5$



$f(2)$ DNE

limit as x approaches 2 is 4

$f(5) = 1$

due to the 'cusp' or 'kink', the slope is not defined, so $f(x)$ is not differentiable at $x = 5$

VI. What values would make the piecewise functions continuous?

$$g(x) = \begin{cases} \dfrac{x-4}{\sqrt{x}-2} & \text{if } x \neq 4 \\ 4 & \text{if } x = 4 \end{cases}$$

find $\displaystyle\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$

if we substitute 4 into the function, it is indeterminate $\dfrac{0}{0}$

so, multiply by the conjugate of the denominator

$$\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \frac{(x-4)(\sqrt{x}+2)}{x-4}$$

$$= \lim_{x \to 4} \sqrt{x}+2 = 4$$

$$h(x) \begin{cases} \dfrac{2x^2+7x+3}{x+3} & \text{if } x \neq -3 \\ -5 & \text{if } x = -3 \end{cases}$$

the rational expression is a line with a 'hole'

To fill that hole, we find the limit as x approaches -3

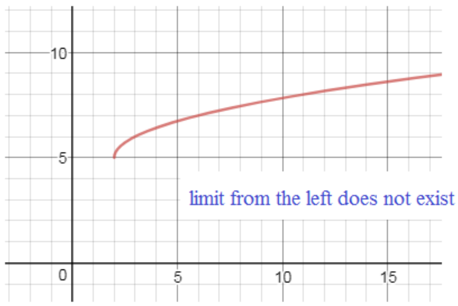
$$\lim_{x \to -3} \frac{(2x+1)(x+3)}{(x+3)} \qquad \lim_{x \to -3} (2x+1) = -5$$

VII. More Questions

SOLUTIONS

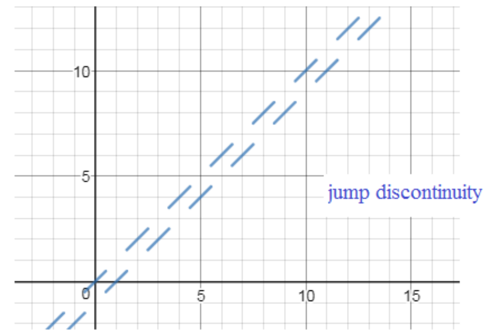
1) $f(x) = 5 + \sqrt{x-2}$

why is f(x) not continuous at x = 2?



2) $g(x) = x + \| \cos(\pi x) \|$

why is g(x) not continuous at x = 2?



3) $$h(x) = \begin{cases} kx^2, & \text{if } x \le 3 \\ kx + 3, & \text{if } x > 3 \end{cases}$$

If h(x) is a continuous function, what is k?

a) $h(3)$ must exist

b) $\lim_{x\to 3} h(x)$ must exist

c) $\lim_{x\to 3} h(x) = h(3)$

$$\lim_{x\to 3^+} h(x) = k(3) + 3 = 3k + 3$$

$$\lim_{x\to 3^-} h(x) = k(3)^2 = 9k$$

for the limit to exist,
the limit from the right
must equal
the limit from the left...

$3k + 3 = 9k$

$\boxed{k = 1/2}$

4) $$f(x) = \begin{cases} 2, & \text{if } x \le -1 \\ ax + b, & \text{if } -1 < x < 3 \\ -2, & \text{if } x \ge 3 \end{cases}$$

If f(x) is continuous, what are a and b?

$$\lim_{x\to -1^-} f(x) = 2$$

$$\lim_{x\to -1^+} f(x) = a(-1) + b$$

$-a + b$

$-a + b = 2$

$$\lim_{x\to 3^-} f(x) = a(3) + b$$

$$\lim_{x\to 3^+} f(x) = -2$$

$3a + b = -2$

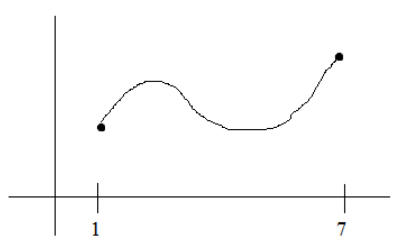
$-a + b = 2$

$3a + b = -2$

solve by elimination

$-4a = 4$

$\boxed{a = -1 \quad b = 1}$



Is

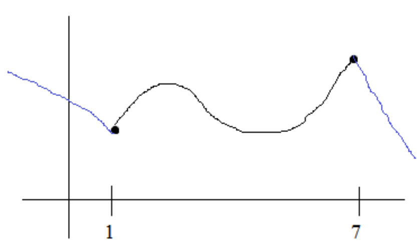
$$\lim_{x\to 7^+} f'(x) = \lim_{x\to 7^-} f'(x) \; ???$$

We don't know....

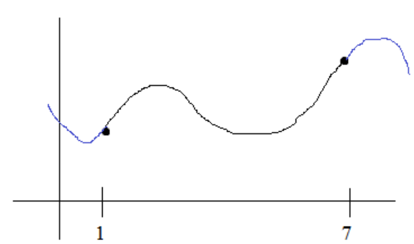
Where is the interval differentiable?

(1, 7) or [1, 7] ?

The answer: (1, 7), because the rates of change of 1 and 7 aren't confirmed...



function at x = 1 and x = 7 are
NOT differentiable...

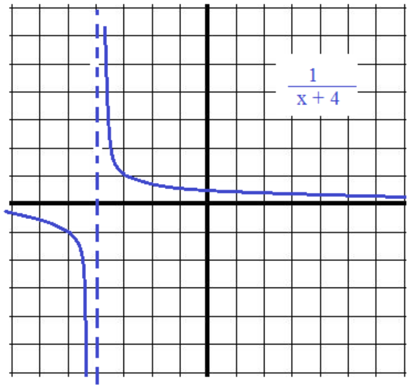


Function at x = 1 and x = 7 are
differentiable!

Sketch examples of functions containing the following features. (BONUS: write a possible equation)

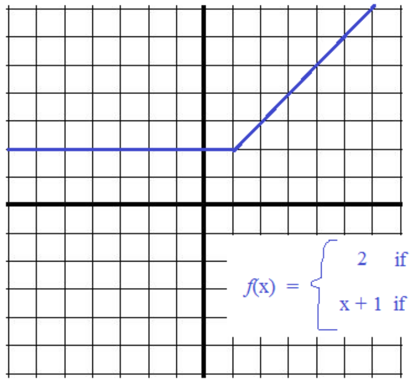
(Possible) SOLUTIONS

1) Vertical asymptote @ $x = -4$



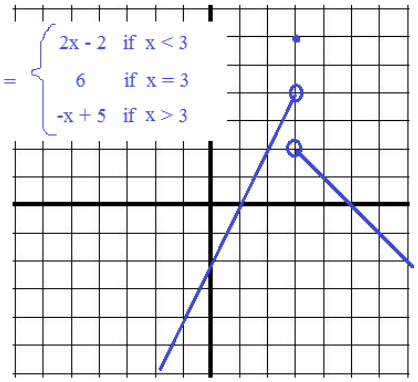
$\frac{1}{x+4}$

2) Continuous @ $x = 1$; a "corner" at $x = 1$



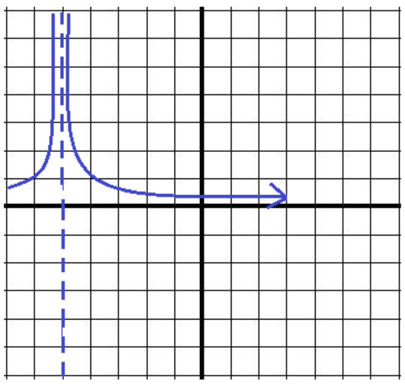
$f(x) = \begin{cases} 2 & \text{if } x \le 1 \\ x+1 & \text{if } x > 1 \end{cases}$

3) $\lim_{x \to 3} f(x)$ does exist; $f(3) \neq \lim_{x \to 3} f(x)$

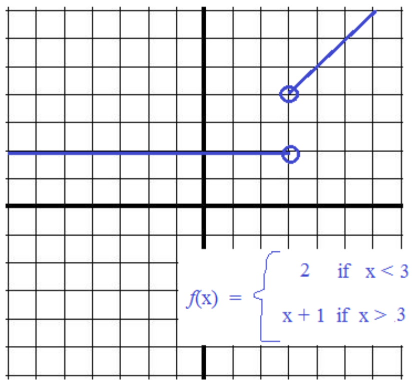


$f(x) = \begin{cases} 2x - 2 & \text{if } x < 3 \\ 6 & \text{if } x = 3 \\ -x + 5 & \text{if } x > 3 \end{cases}$

4) Infinite discontinuity @ $x = -5$ $\quad \frac{1}{(x+5)^2}$



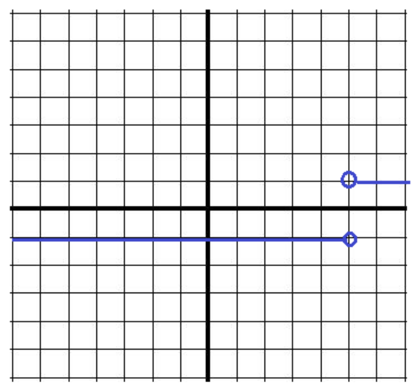
5) Jump discontinuity @ $x = 3$



$f(x) = \begin{cases} 2 & \text{if } x < 3 \\ x+1 & \text{if } x > 3 \end{cases}$

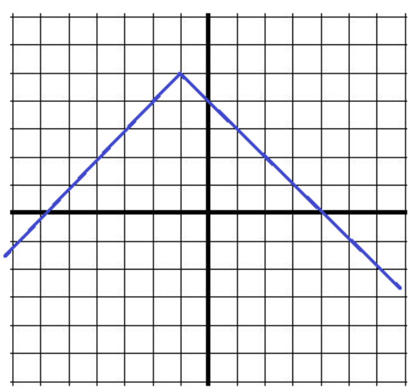
(Step and Jump are same)

6) Step discontinuity @ $x = 5$

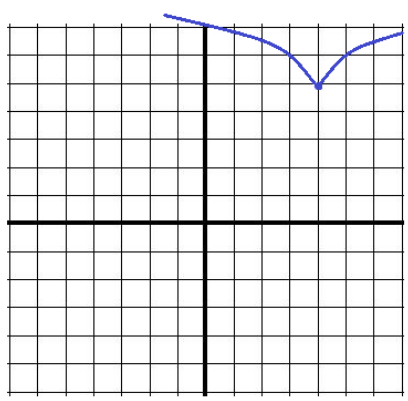


$\frac{x-5}{|x-5|}$ limit from left is -1
limit from right is 1

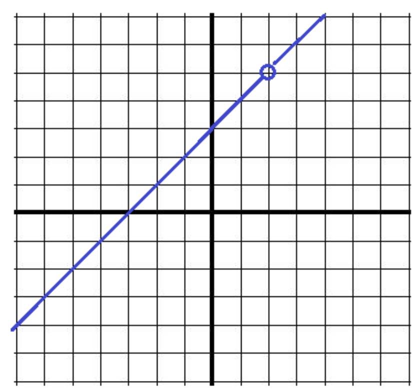
7) Kink @ (-1, 5) $\quad y = -|x+1| + 5$



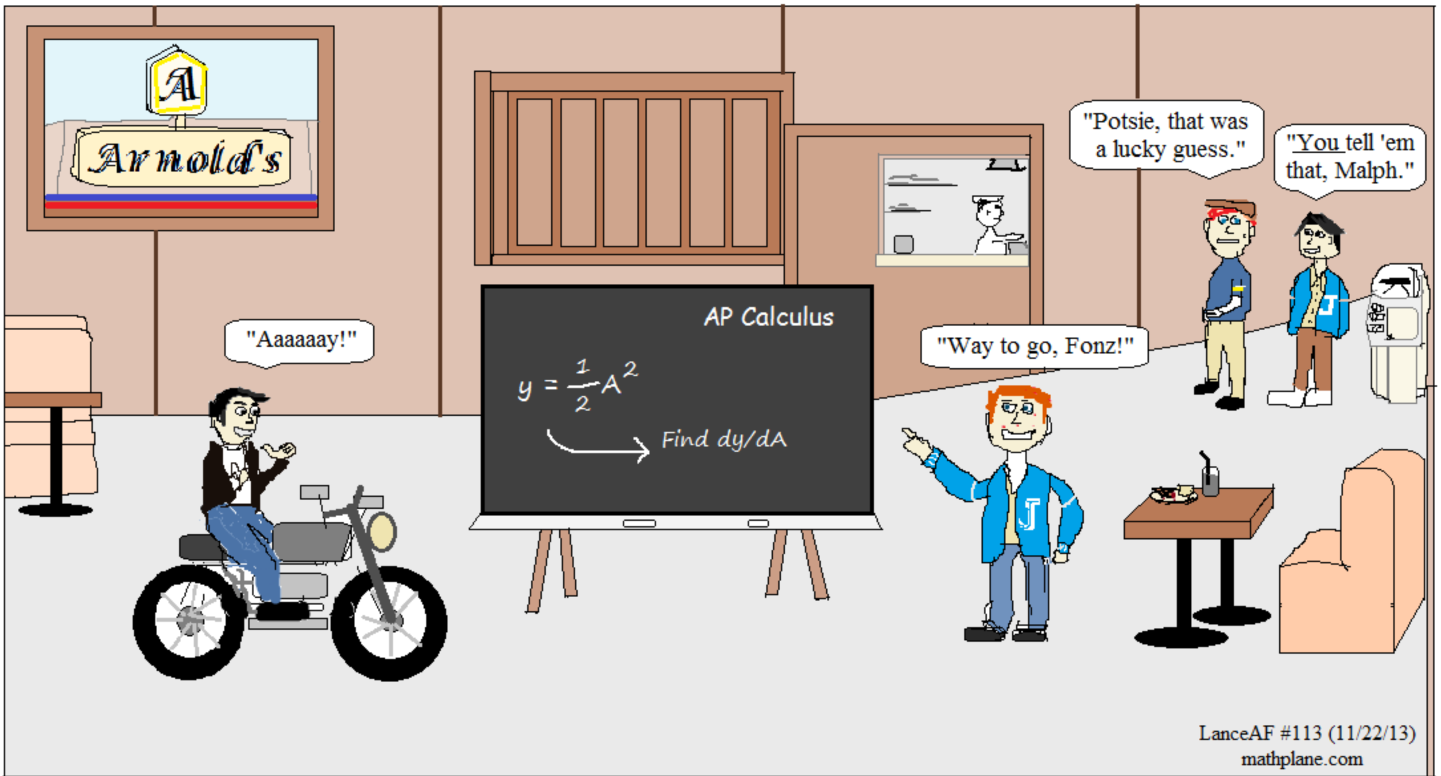
8) Cusp @ (4, 5) $\quad \sqrt{|x-4|} + 5$



9) $\lim_{x \to 2}$ exists, but $f(2)$ has no value



$f(x) = \frac{(x+3)(x-2)}{(x-2)}$ (hole)



Happy Days

Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

Limits, Asymptotes, and Continuity Questions

(w/ answers)

Limits, Asymptotes, and Continuity

I. Identify the vertical asymptote(s).
Then, describe the behavior of $f(x)$
to the left and right of each asymptote.

1) $f(x) = \dfrac{x^2 - 1}{2x + 4}$

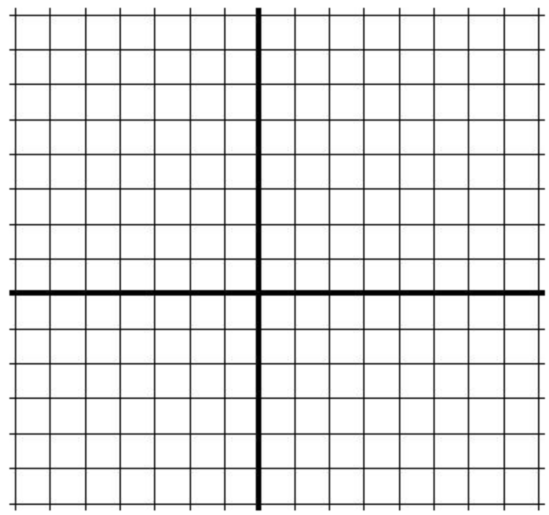
2) $f(x) = \tan(x)$

3) $f(x) = \begin{cases} \dfrac{x-2}{x-1} & x \le 0 \\ \dfrac{1}{x} & x > 0 \end{cases}$

4) $f(x) = \dfrac{x^2 + 4x + 3}{x^2 - 9}$

II. Sketch a graph of the function $g(x) = \dfrac{x^2 - 4}{2x - 1}$

Identify the *slant asymptote.*



III. Find and describe each discontinuity:
(e.g. Jump, infinite, removable,...)

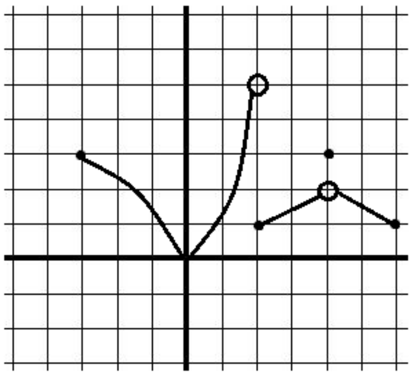
1) $f(x) = \dfrac{x + 5}{(x-2)(x-3)}$

2) $g(x) = \dfrac{|x|}{x}$

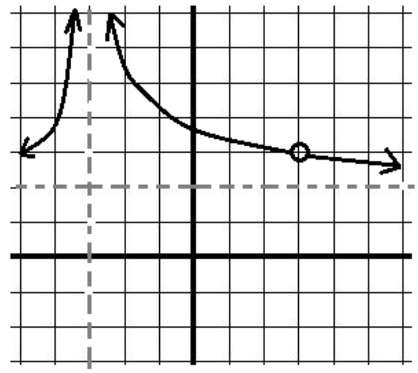
3) $h(x) = \begin{cases} 3 - x & x < 2 \\ 4 & x = 2 \\ \dfrac{x}{2} & x > 2 \end{cases}$

IV. Find and describe each discontinuity:
(e.g. Jump, infinite, removable,...)

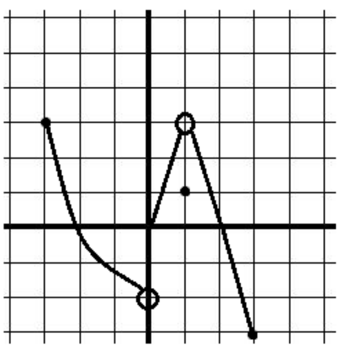
1)



2)



V. Applying the definition of continuous:



a) Does $f(1)$ exist?

b) Does $\lim_{x \to 1} f(x)$ exist?

c) Does $f(1) = \lim_{x \to 1} f(x)$?

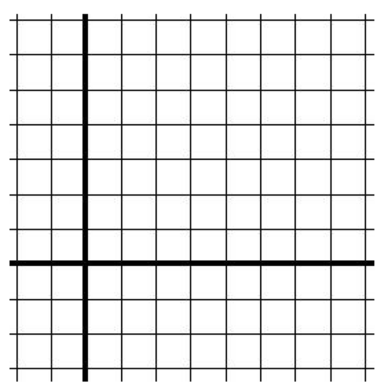
d) Is $f(x)$ continuous at 1 ?

VI. Find the value for a, so that the function is continous:

$$f(x) = \begin{cases} x^2 - 4 & x < 3 \\ 2ax & x \geq 3 \end{cases}$$

VII: Sketch a possible graph:

- $f(3)$ exists
- $\underset{x \to 3^+}{\text{limit}} f(x) = f(3)$
- $\underset{x \to 3^-}{\text{limit}} f(x)$ does not exist



I. Identify the vertical asymptote(s).
Then, describe the behavior of $f(x)$
to the left and right of each asymptote.

SOLUTIONS

Limits, Asymptotes, and Continuity

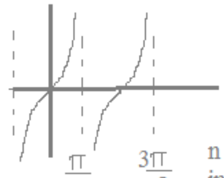
1) $f(x) = \dfrac{x^2 - 1}{2x + 4}$

$f(x) = \dfrac{(x + 1)(x - 1)}{2(x + 2)}$

zeros: -1 and 1
y-intercept: (0, -1/4)
vertical asymptote: $x = -2$
(function is undefined at $x = -2$)

limit to -2 from the left: $-\infty$
limit to -2 from the right: $+\infty$
(test -2.0001 and -1.9999)

2) $f(x) =$ tan(x)



$\frac{\pi}{2}$ $\frac{3\pi}{2}$ n is any integer

vertical asymptotes: $\dfrac{n\pi}{2}$

behavior (left): $+\infty$ behavior (right): $-\infty$

3) $f(x) = \begin{cases} \dfrac{x-2}{x-1} & x \le 0 \\ \dfrac{1}{x} & x > 0 \end{cases}$

vertical asymptote: $x = 0$

behavior on right of 0:
goes toward $+\infty$

behavior on left of 0:
goes toward 2

4) $f(x) = \dfrac{x^2 + 4x + 3}{x^2 - 9}$

$f(x) = \dfrac{(x + 3)(x + 1)}{(x + 3)(x - 3)}$

x-intercept: (-1, 0)
y-intercept: (0, -1/3)
vertical asymptote: $x = 3$
"Hole": $x = -3$

approaching 3 (left): $-\infty$
approaching 3 (from the right): $+\infty$

II. Sketch a graph of the function $g(x) = \dfrac{x^2 - 4}{2x - 1}$

Identify the *slant asymptote.*

factor the expression: $\dfrac{(x - 2)(x + 2)}{2x - 1}$

reveals x-intercepts: (2, 0) (-2, 0)

y-intercept: (0, 4) $g(0) = 4/1 = 4$

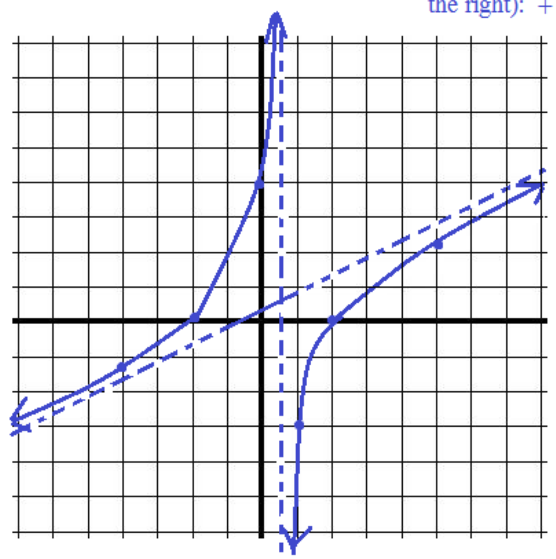
vertical asymptote: $x = 1/2$

long division to find slant asymptote

$$\begin{array}{r} x/2 + 1/4 + \dfrac{17/4}{2x-1} \\ 2x - 1 \overline{)\, x^2 + 0x + 4} \\ -\ x^2 - x/2 \\ \hline x/2 + 4 \\ -\ x/2 - 1/4 \\ \hline 17/4 \end{array}$$

since degree of numerator (2) is one more than degree of denominator (1), there is a slant asymptote...

slant asymptote: $y = \dfrac{1}{2}x + \dfrac{1}{4}$



x	g(x)
-7	-3
-4	-4/3
-2	0
0	4
1	-3
2	0
5	7/3

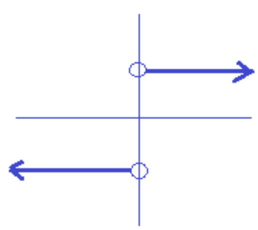
III. Find and describe each discontinuity:
(e.g. Jump, infinite, removable,...)

1) $f(x) = \dfrac{x + 5}{(x - 2)(x - 3)}$

(there are vertical asymptotes at $x = 2$ and $x = 3$)

"infinite discontinuity" at 2 and 3

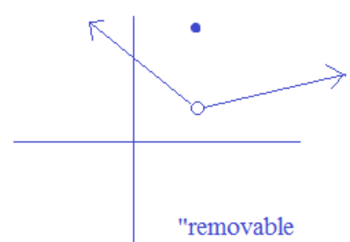
2) $g(x) = \dfrac{|x|}{x}$



"jump discontinuity" at $x = 0$

limit from left is -1 and
limit from right is 1

3) $h(x) = \begin{cases} 3 - x & x < 2 \\ 4 & x = 2 \\ \dfrac{x}{2} & x > 2 \end{cases}$

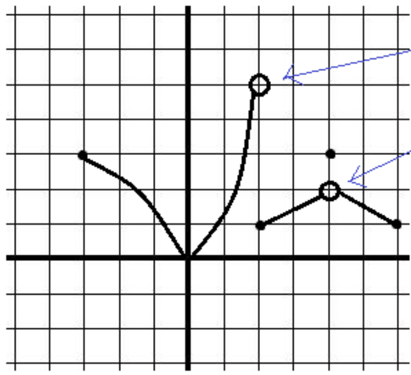


"removable discontinuity" at $x = 2$

IV. Find and describe each discontinuity:
(e.g. Jump, infinite, removable,...)

SOLUTIONS

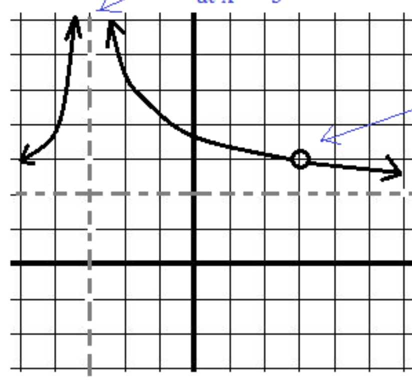
1)



jump discontinuity at $x = 2$

removable discontinuity at $x = 4$

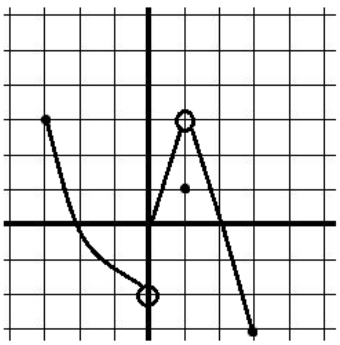
2)



infinite discontinuity at $x = -3$

"hole" (removable discontinuity) at $x = 3$

V. Applying the definition of continuous:



a) Does $f(1)$ exist? yes... $f(1) = 1$

b) Does $\lim_{x\to 1} f(x)$ exist? yes... the limit as x approaches 1 is 3

c) Does $f(1) = \lim_{x\to 1} f(x)$? no... $1 \neq 3$

d) Is $f(x)$ continuous at 1 ?

no..
by definition: since c) is not satisfied, the function is not continuous...
by graph: since you would "lift your pencil off the paper" at x = 1 and x= 0, the function is not continuous...

VI. Find the value for a, so that the function is continous:

$$f(x) = \begin{cases} x^2 - 4 & x < 3 \\ 2ax & x \geq 3 \end{cases}$$

to be continuous, the function must meet at x = 3..

at $x = 3$, $x^2 - 4 = 5$

therefore, at $x = 3$, $2ax$ must equal 5....

$2a(3) = 5 \quad a = 5/6$

VII: Sketch a possible graph:

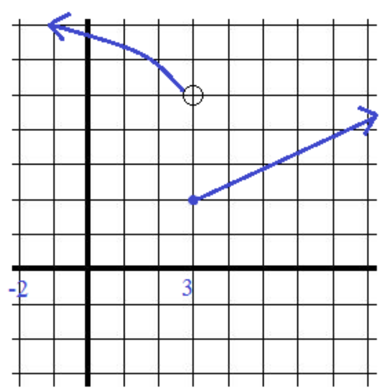
- $f(3)$ exists
- $\lim_{x\to 3^+} f(x) = f(3)$
- $\lim_{x\to 3^-} f(x)$ does not exist

A piecewise function with jump discontinuity would satisfy the conditions..

$f(3) = 2$

$\lim_{x\to 3^+} f(x) = 2$

$\lim_{x\to 3^-} f(x)$ is undefined



Thanks for visiting. (Hope it helped!)

If you have suggestions, questions, or requests, let us know.

Cheers,

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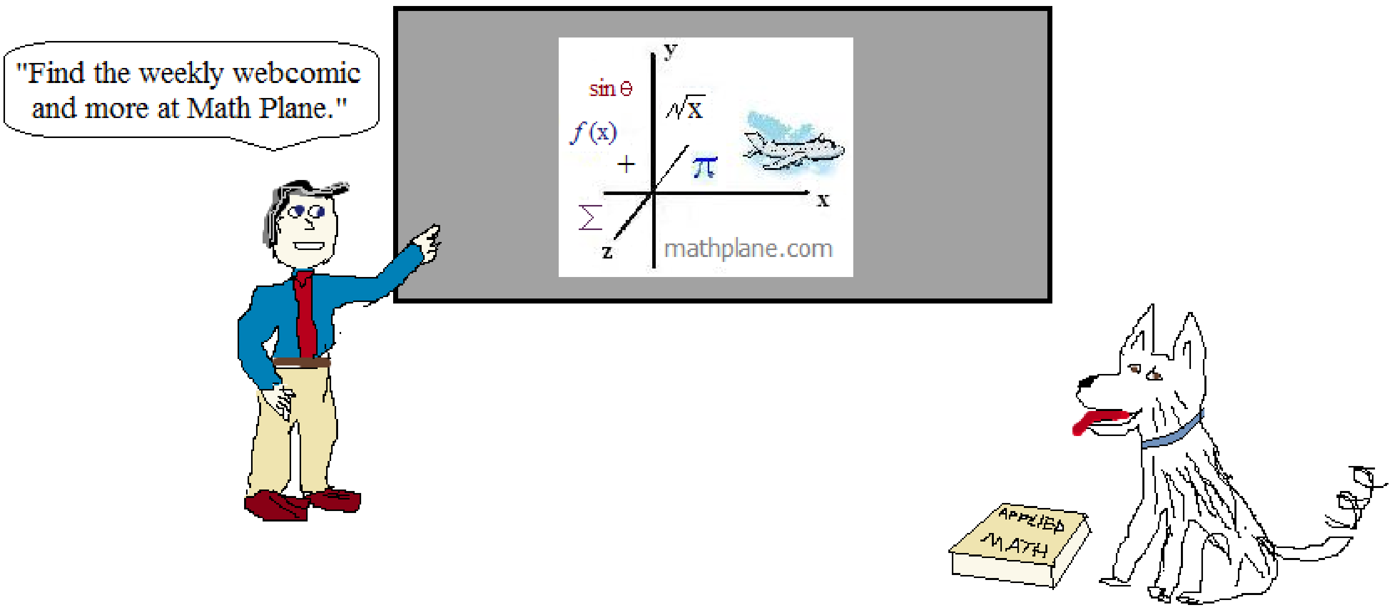

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