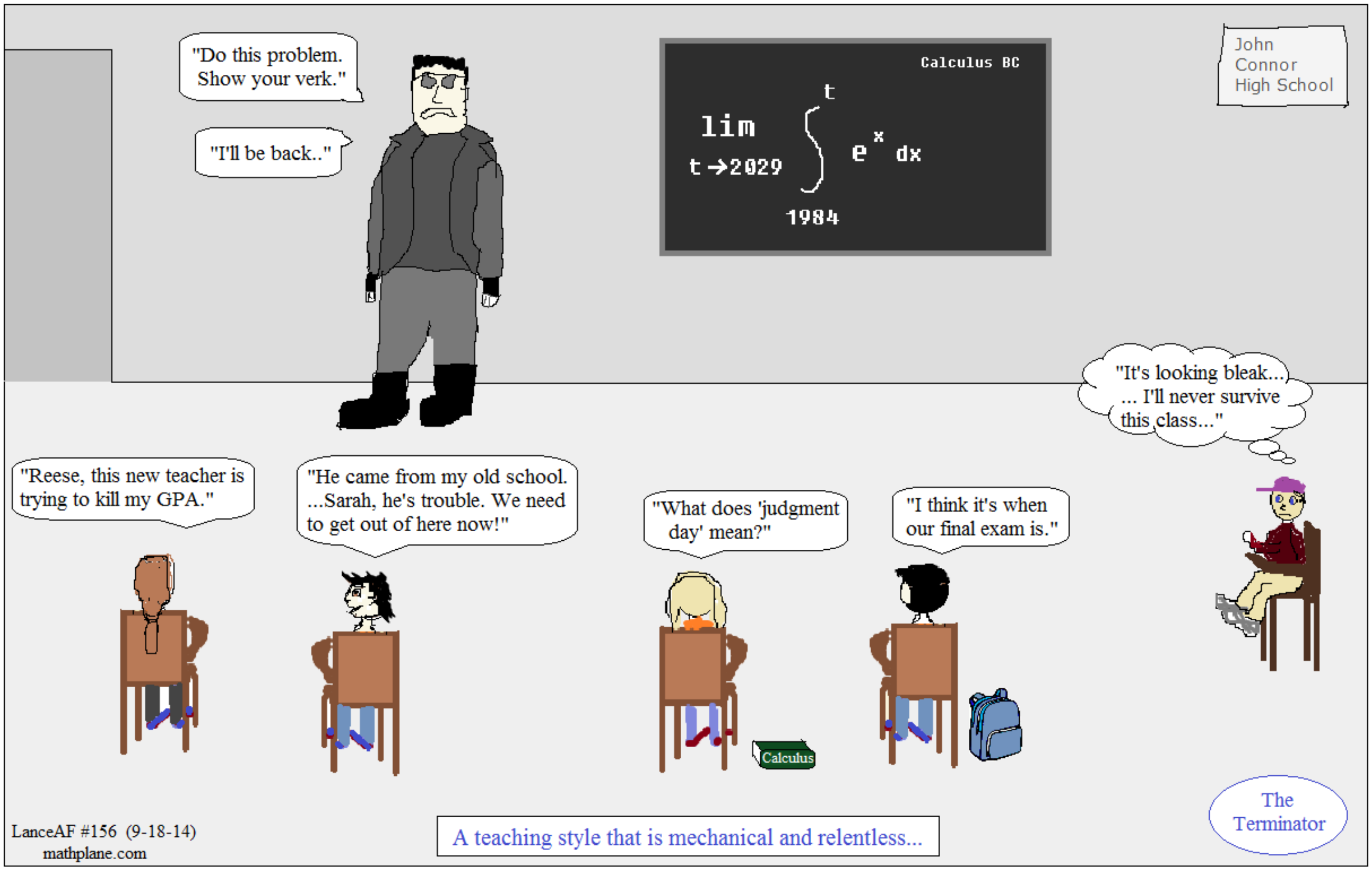
Calculus: AP Free Response-Type

AB Questions

Multi-part questions that help prepare for AP Test or a Calculus final.

Topics include volume of solids, related rates, particle movement, volume of solids, anti-derivatives, LRAM, and more.


"Do this problem. Show your verk."
"I'll be back.."
Calculus BC
$\lim_{t \to 2029} \int_{1984}^{t} e^x \, dx$
John Connor High School
"It's looking bleak... ... I'll never survive this class..."
"Reese, this new teacher is trying to kill my GPA."
"He came from my old school. ...Sarah, he's trouble. We need to get out of here now!"
"What does 'judgment day' mean?"
"I think it's when our final exam is."
Calculus
LanceAF #156 (9-18-14)
mathplane.com
A teaching style that is mechanical and relentless...
The Terminator

QUESTIONS-→

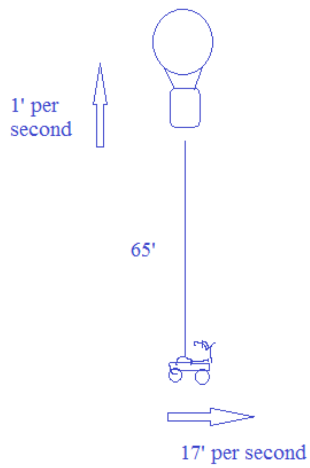
AP Calculus Free Response Example: Related Rates of Change

A balloon is 65 feet above the ground and rising at a rate of 1 foot per second.
Meanwhile, a bicycle rider is directly under the balloon, traveling on a flat road at a rate of 17 feet per second.

a) What is the distance between the balloon and the bicycle rider *3 seconds later?*

b) How fast is the distance between the balloon and the bicycle rider increasing *3 seconds later*?

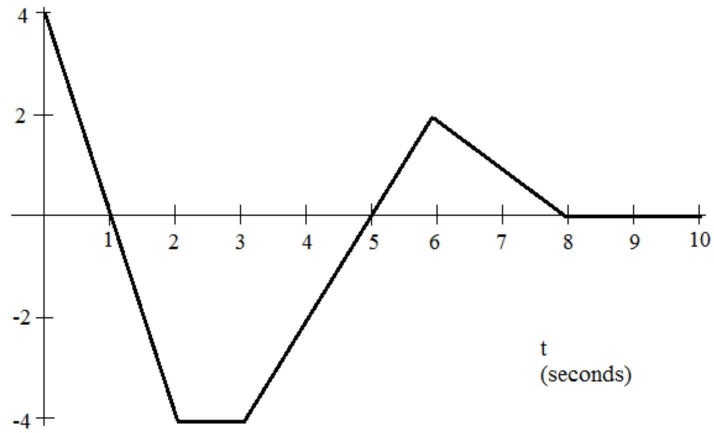
c) How fast is the angle between the ground and the line connecting the balloon and bicycle changing *3 seconds later?*



AP Calculus AB Free Response example: position, velocity, speed, acceleration, graphs

The figure shows the velocity of a particle.

V = *f*(t)



a) When does the particle

move forward?

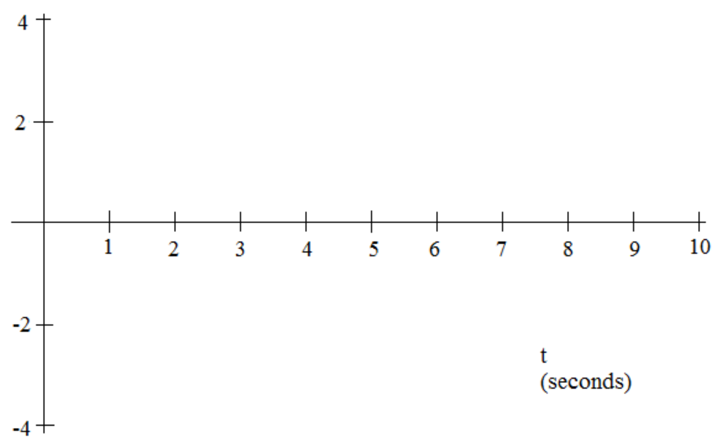
backward?

change direction?

b) When is the particle at rest?

c) When does the particle speed up? slow down?

d) Graph the particle's acceleration. Identify the intervals when its acceleration is positive, negative, or zero.



e) What is the particle's fastest speed?

AP Calculus Questions: Volume of Solids with various cross sections

The area of a region is bounded by $y = e^x$, x-axis, y-axis, and $x = 1$

a) Find the area of the region

b) Find the volume of the solid when the region is rotated around the x-axis

c) Find the volume of the solid where the cross sections perpendicular to the x-axis are squares

d) Find the volume of the solid where the cross sections perpendicular to the x-axis are semicircles

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a) Area of region

b) Rotated around x-axis

c) Square cross sections

d) Semicircle cross sections

AP Calculus Free Response Examples: Interpreting Tables

Water flows into a tank at varying rates.
The amount is measured in the time interval $0 < t < 6$,

given by the differentiable function $A(t)$

where t = time in minutes
$A(t)$ = gallons per minute at given time

t	0	1	2	3	4	5	6
A(t)	12	10	6	5	8	10	17

Selected values are in the table..

a) Use the table to approximate $A'(4)$

b) Is there a time c, $2 < c < 4$, where $A'(c) = 2$? Explain.

c) Use MRAM to estimate $\int_0^6 A(t)\,dt$

d) Is LRAM < or > the actual value of $\int_0^6 A(t)\,dt$ Justify.

AP Calculus Free Response: Antiderivatives and Separable Differential Equations

Find the solutions:

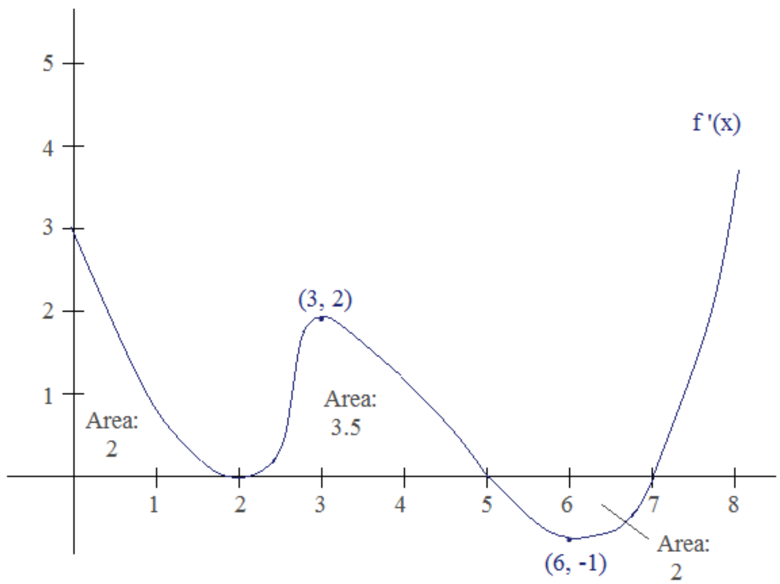
a) $\frac{ds}{dt} = -32t + 100$ $s = 50$ when $t = 0$

b) $\frac{dy}{dx} = \sin x + 2$ $y(2) = 5$ (answer to 3 decimals places)

c) Find the general solution: $\frac{dy}{dx} = 2xy$

AP Calculus Free Response example: interpreting a derivative graph

The following is a graph of the *derivative of* f(x):

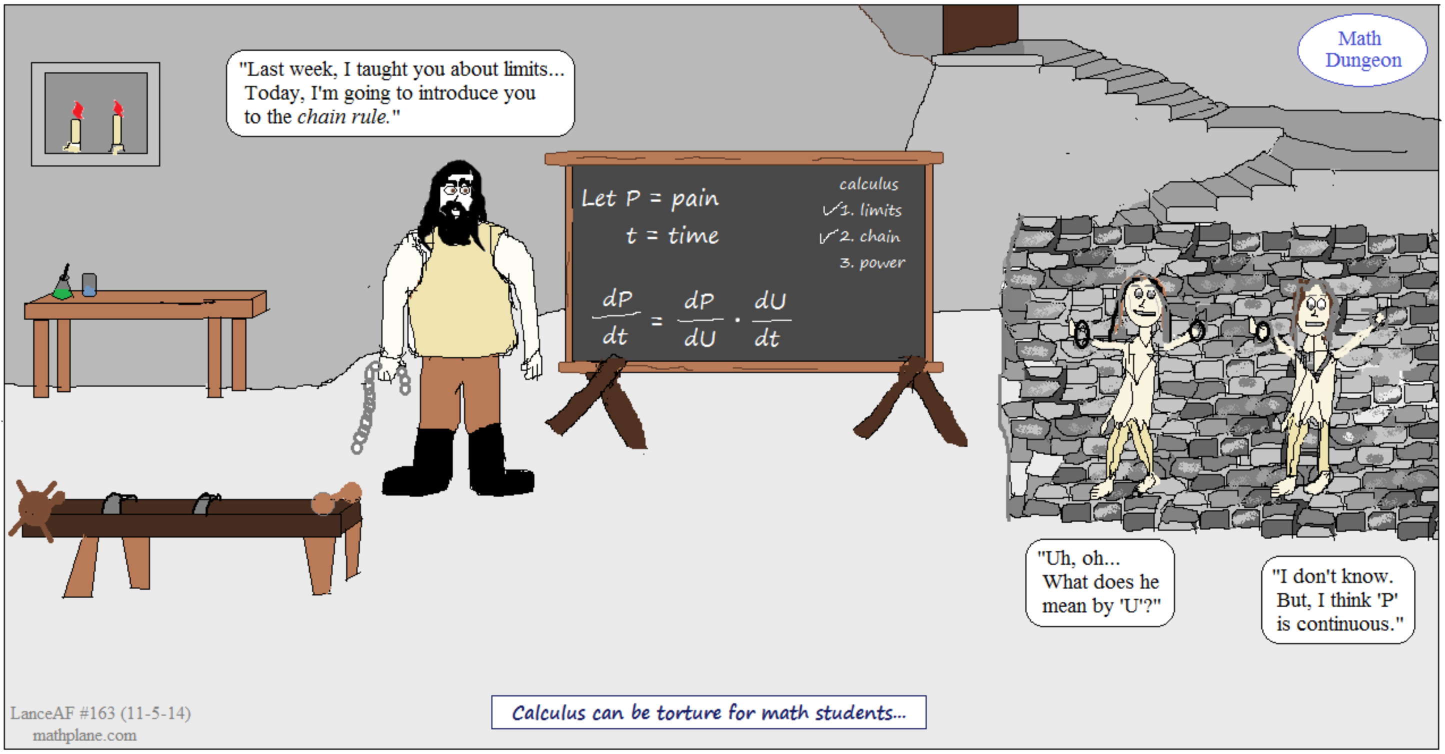


a) Where is the local minimum(s)?

b) Where is the absolute (global) minimum? Justify your answer.

c) Where is the function concave up AND increasing on the interval [0, 8]?

d) $g(x) = (f(x))^2$ If $f(3) = 8$, find the slope of the line tangent to the graph g(x) at $x = 3$


Math Dungeon
"Last week, I taught you about limits... Today, I'm going to introduce you to the *chain rule.*"
Let P = pain
t = time
calculus
1. limits
2. chain
3. power
dP/dt = dP/dU · dU/dt
"Uh, oh... What does he mean by 'U'?"
"I don't know. But, I think 'P' is continuous."
Calculus can be torture for math students...
LanceAF #163 (11-5-14)
mathplane.com

SOLUTIONS-→

AP Calculus Free Response Example: Related Rates of Change

SOLUTIONS

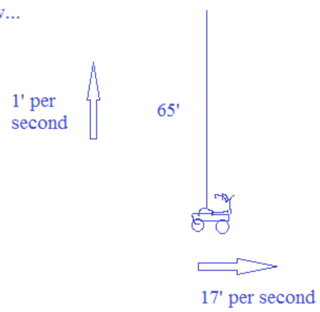
A balloon is 65' above the ground and rising at a rate of 1' per second.
Meanwhile, a bicycle rider is directly under the balloon, traveling on a flat road at a rate of 17' per second.

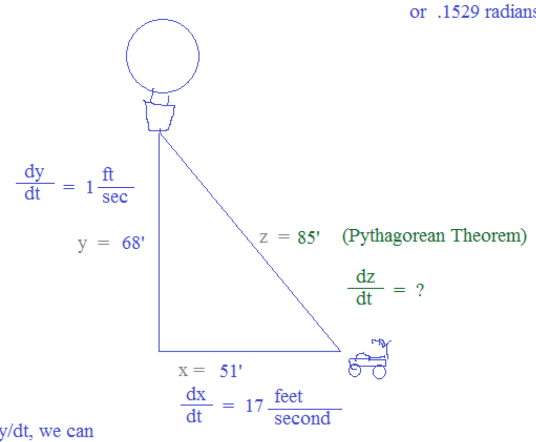
a) What is the distance between the balloon and the bicycle rider *3 seconds later?* 85 feet

b) How fast is the distance between the balloon and the bicycle rider increasing *3 seconds later*? 11 feet per second

c) How fast is the angle between the ground and the line connecting the balloon and bicycle changing *3 seconds later?* decreasing 8.76 degrees per second or .1529 radians per second

Step 1: Draw diagram and establish variables





Step 2: Write the basic equation (showing the relationship between variables)

$x^2 + y^2 = z^2$

Since we know x, y, z, and we know the dx/dt and dy/dt, we can find the related rates.. (i.e. the change in distance related to time)

Step 3: Find related rate (e.g. implicit differentiation)

The rates of change *with respect to time (t)* $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

Then, substitute... (after the units cancel,) $2(51)(17) + 2(68)(1) = 2(85)(\frac{dz}{dt})$

$$1734 + 136 = 170\left(\frac{dz}{dt}\right)$$

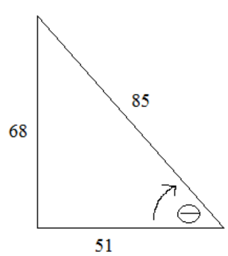
$$11 = \frac{dz}{dt}$$

After 3 seconds, the distance between the bicycle and the balloon is increasing at a rate of 11 feet/second

Step 4: Check for reasonableness...

Snapshots:	now	1 second later	2 seconds later	3 seconds later	4 seconds later
	distance apart: 65'	distance apart: 68.15	distance apart: 75.13	distance apart 85	distance apart: 96.88
	65' high 0' along	66' high 17' along	67' high 34' along	68' high 51' along	69' high 68' along

average rate between 0 and 1: 3.15
average rate between 1 and 2: 6.98
average rate between 2 and 3: 9.87
IROC at 3 seconds: 11
average rate between 3 and 4: 11.88



One equation that relates the angle to the sides: $\tan\theta = \frac{y}{x}$ (opposite) (adjacent)

Then, the derivative that relates the rates to time: (using implicit diff. and the quotient rule)

$$\sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

$\sec\theta = \frac{85}{51}$

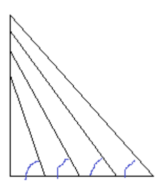
$$\left(\frac{85}{51}\right)^2\frac{d\theta}{dt} = \frac{51(1) - 68(17)}{51^2}$$

$$\frac{d\theta}{dt} = -.1529 \text{ radians or } -8.76 \text{ degrees/second}$$

quick check:

Now: angle is undefined
1 second later: 75.5 degrees
2 seconds later: 63.1 degrees
3 seconds later: 53.1 degrees
4 seconds later: 45.4 degrees

AROC (1 - 2) -12.4
AROC (2 - 3) -10
IROC -8.76 degrees
AROC (3 - 4) -7.7



Note: You may use another trig function such as $\sin\theta = \frac{y}{z}$

$$\cos\theta\frac{d\theta}{dt} = \frac{z\frac{dy}{dt} - y\frac{dz}{dt}}{z^2} \qquad \frac{51}{85}\frac{d\theta}{dt} = \frac{85(1) - 68(11)}{85^2}$$

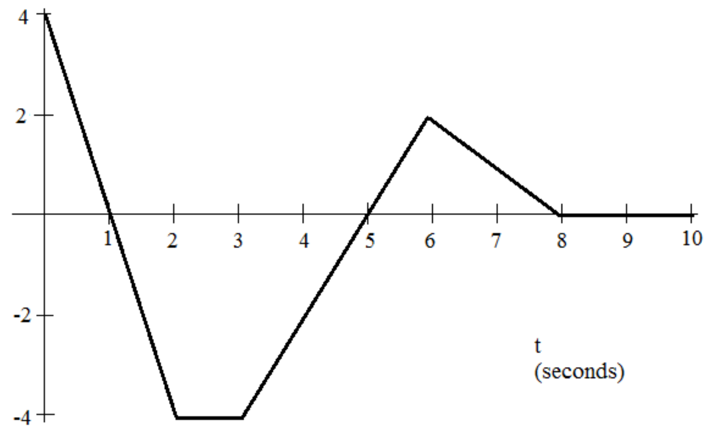
$$\frac{d\theta}{dt} = -.1529 \text{ radians or } -8.76 \text{ degrees/second}$$ ✓

AP Calculus AB Free Response example: position, velocity, speed, acceleration, graphs

SOLUTION

The figure shows the <u>velocity</u> of a particle.

V = *f*(t)



a) When does the particle

move forward? when derivative is positive (above x-axis)

(0, 1) ∪ (5, 8)

backward? when derivative is negative (below x-axis)

interval (1, 5)

change direction?

at t = 1 and t = 5

b) When is the particle at rest?

when velocity = 0 at t = 1 t = 5 and interval [8, 10]

c) When does the particle speed up? slow down?

Particle is speeding up when velocity is positive AND increasing
OR velocity is negative AND decreasing...

(1, 2) and (5, 6)

Particle is slowing down when velocity is positive AND decreasing
OR velocity is negative AND increasing
(velocity and acceleration in opposite directions)

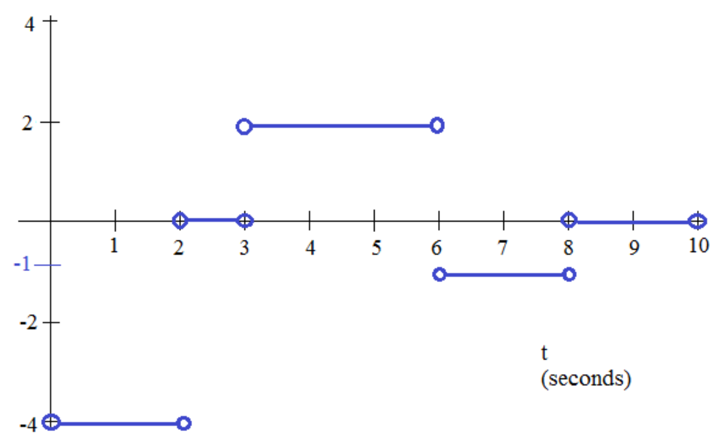
(0, 1) and (3, 5) and (6, 8)

d) Graph the particle's acceleration. Identify the intervals when its acceleration is positive, negative, or zero.

a(t) > 0 in the interval (3, 6)

a(t) < 0 in the intervals (0, 2) and (6, 8)

a(t) = 0 in the intervals (2, 3) and (8, 10)



e) What is the particle's fastest <u>speed?</u>

Speed is the absolute value of velocity...

So, the fastest speed occurs at t = 0
and between 2 and 3 seconds

The fastest speed is 4..

AP Calculus Questions: Volume of Solids with various cross sections

SOLUTIONS

The area of a region is bounded by $y = e^x$, x-axis, y-axis, and $x = 1$

a) Find the area of the region

b) Find the volume of the solid when the region is rotated around the x-axis

c) Find the volume of the solid where the cross sections perpendicular to the x-axis are squares

d) Find the volume of the solid where the cross sections perpendicular to the x-axis are semicircles

---

a) Find the area of the region

Step 1: Determine the span of the integral..

The boundaries go from $x = 0$ to $x = 1$

$$\int_0^1$$

Step 2: Determine the function to evaluate

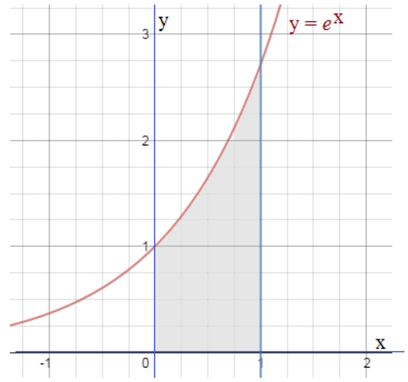
The upper boundary is $y = e^x$ and

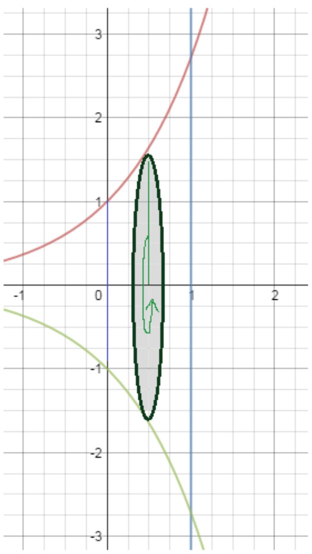
the lower boundary is $y = 0$

$$\int_0^1 e^x - 0 \, dx$$

Step 3: Evaluate

$$e^x \Big|_0^1 = e^1 - e^0 = \boxed{e - 1 \text{ or } 1.72}$$





b) Find the volume of the solid when the region is rotated around the x-axis

area of a circle: π (radius)^2

(the radius is the length of the function)

$$\int_0^1 \pi (e^x)^2 \, dx$$

radius of each partition

$$\int_0^1 \pi e^{2x} \, dx = \pi \int_0^1 e^{2x} \, dx$$

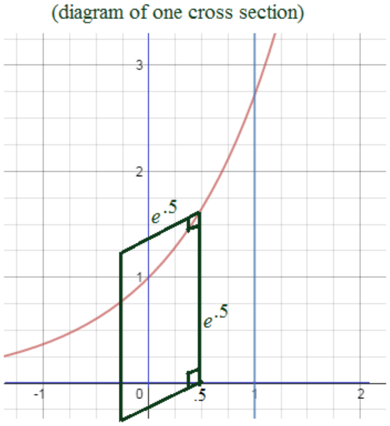
$$= \frac{1}{2} \pi \int_0^1 2 e^{2x} \, dx = \frac{1}{2} \pi \cdot e^{2x} \Big|_0^1$$

$$\frac{1}{2} \pi \left(e^2 - e^0 \right) = \boxed{\frac{1}{2} \pi (e^2 - 1)}$$

approx. 10.03

c) cross sections perpendicular to the x-axis are squares

(diagram of one cross section)



area of a square: (side)^2

(the side is the length of the function)

$$\int_0^1 (e^x)^2 \, dx$$

side of each partition

$$\int_0^1 e^{2x} \, dx$$

$$\frac{1}{2} \int_0^1 2 e^{2x} \, dx$$

$$\frac{1}{2} \cdot e^{2x} \Big|_0^1 = \boxed{\frac{1}{2} (e^2 - 1)}$$

approx. 3.19

d) cross sections perpendicular to the x-axis are semicircles

(diagram of one cross section)

$$\int_0^1 \frac{1}{2} \pi \left(\frac{1}{2} e^x \right)^2 \, dx$$

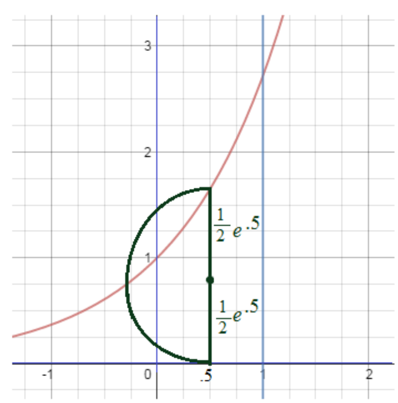
radius of each partition

$$\frac{1}{2} \pi \int_0^1 \frac{1}{4} e^{2x} \, dx$$

$$\frac{1}{16} \pi \int_0^1 2 e^{2x} \, dx$$

$$\frac{1}{16} \pi \cdot e^{2x} \Big|_0^1$$

$$\frac{1}{16} \pi \left(e^2 - e^0 \right) = \boxed{\text{approx. } 1.25}$$



area of a semicircle: $\frac{1}{2} \pi$ (radius)^2

(the radius is 1/2 the length of the function)

Water flows into a tank at varying rates.
The amount is measured in the time interval $0 < t < 6$,

given by the differentiable function A(t)

where t = time in minutes
A(t) = gallons per minute at given time

SOLUTIONS

t	0	1	2	3	4	5	6
A(t)	12	10	6	5	8	10	17

Selected values are in the table..

a) Use the table to approximate A'(4)

Since we don't know the function, we'll use estimates of rates of change from the left of 4 and right of 4...

Rate of change from 3 to 4: $\frac{8-5}{4-3} = 3$

Rate of change from 4 to 5: $\frac{10-8}{5-4} = 2$

The average of the two values is 2.5

A'(4) is approximately 2.5

(The gallons per minute water flow is increasing by 2.5)

b) Is there a time c, $2 < c < 4$, where $A'(c) = 2$? Explain.

The Average Rate of Change (AROC) from 2 to 3 is $\frac{5-6}{3-2} = -1$

and, the AROC from 3 to 4 is $\frac{8-5}{4-3} = 3$

And, we know the function is differentiable (and continuous).. Therefore, at some point, the Instantaneous Rate of Change (IROC) must be between -1 and 3.. (Mean Value Theorem)

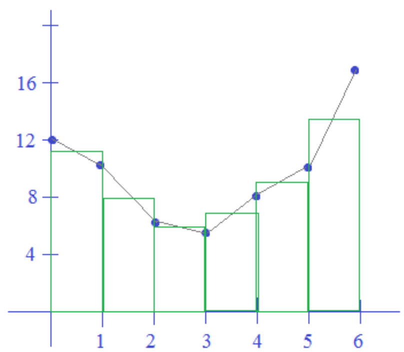
c) Use MRAM to estimate $\int_0^6 A(t)\,dt$

	midpoint	△t	
0 to 1	11 x 1 = 11		
1 to 2	8 x 1 = 8		
2 to 3	5.5 x 1 = 5.5		
3 to 4	6.5 x 1 = 6.5		
4 to 5	9 x 1 = 9		
5 to 6	13.5 x 1 = 13.5		

total estimate:

53.5 gallons

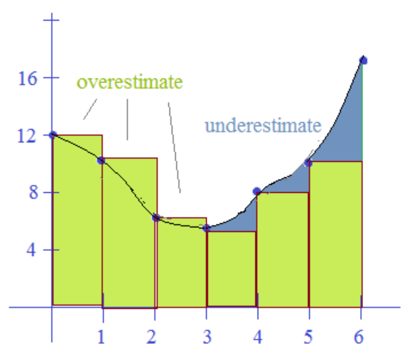
(amount of water that flowed into the tank)



d) Is LRAM < or > the actual value of $\int_0^6 A(t)\,dt$ Justify.

While the rate is declining from 0 to 3, LRAM is overestimating the value..
then, when the rate is increasing from 3 to 6, LRAM is underestimating the value...

Since the increase from 3 to 6 appears larger than 0 to 3, it's more likely LRAM underestimates the actual value.



AP Calculus Free Response: Antiderivatives and Separable Differential Equations

SOLUTIONS

Find the solutions:

a) $\frac{ds}{dt} = -32t + 100$ $\quad$ $s = 50$ when $t = 0$

Find the antiderivate using integration: $s = -\frac{32t^2}{2} + 100t + C$

$s = -16t^2 + 100t + C$

Then, find constant C by substitution:

$(50) = -16(0)^2 + 100(0) + C$ $\quad$ $C = 50$

$\boxed{s = -16t^2 + 100t + 50}$

b) $\frac{dy}{dx} = \sin x + 2$ $\quad$ $y(2) = 5$ $\quad$ (answer to 3 decimals places)

Find the indefinite integral: $y = -\cos x + 2x + C$

Then, recognizing that when $y = 5$, when $x = 2$, we can use substitution to find C...

$(5) = -\cos(2) + 2(2) + C$

$5 = -(-.416) + 4 + C$

$5 - .416 - 4 = C$

$C = .584$

$\boxed{y = -\cos x + 2x + .584}$

c) Find the general solution: $\frac{dy}{dx} = 2xy$

$\frac{dy}{dx} = \frac{2xy}{1}$ $\quad$ (cross multiply)

$1\, dy = 2xy\, dx$ $\quad$ separate the variables

$\frac{1}{y}\, dy = 2x\, dx$ $\quad$ integrate

$\int \frac{1}{y}\, dy = \int 2x\, dx$

$\ln|y| + C = x^2 + C$

$\ln|y| = x^2 + C$ $\quad$ (convert log function into exponential form)

$y = e^{x^2 + C}$

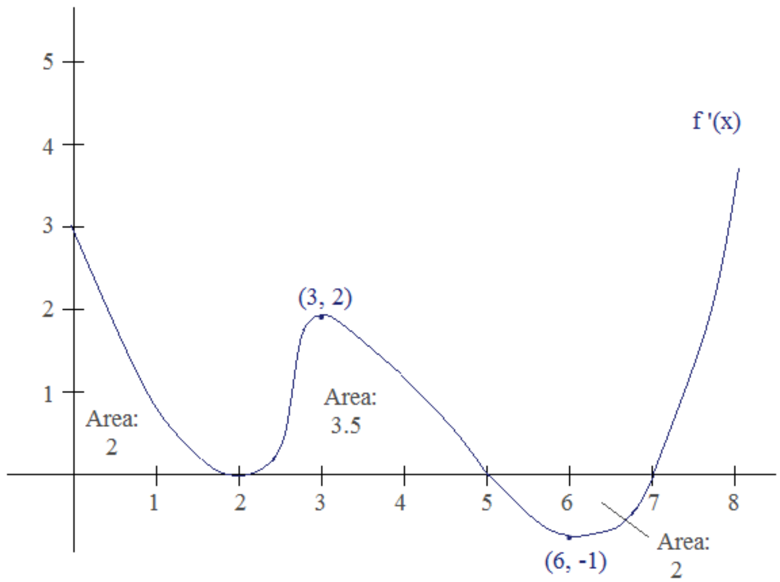
$y = e^{x^2} \cdot e^{C}$ $\quad$ (exponent laws) $\quad$ (note: since C can be any number, presumably e^C can be any constant.)

$\boxed{y = Ce^{x^2}}$

AP Calculus Free Response example: interpreting a derivative graph

SOLUTIONS

The following is a graph of the *derivative of* f(x):



a) Where is the local minimum(s)?

In the graph, a local minimum occurs at x = 7...
Why? Because the function f(x) is increasing from 0 to 5... then, f(x) is decreasing from 5 to 7...
Then, f(x) is increasing from 7 to 8... If f(x) goes from decreasing to increasing, then it's a minimum!

b) Where is the absolute (global) minimum? Justify your answer.

We know x= 7 is a minimum... But, is it the *absolute* minimum?
No... x = 0 is the absolute minimum... The function is increasing from 0 to 5...
Then, it decreases from 5 to 7... However, the increasing area (5.5) is more than the decreasing area (2)...

Therefore, $f(0) < f(7)$

c) Where is the function concave up AND increasing on the interval [0, 8]?

Function is increasing when $f'(x) > 0$... [0, 2) (2, 5) (7, 8]

Function f(x) is concave up when $f''(x) > 0$ OR when the derivative of f '(x) is increasing...
In other words, when the above graph has a slope that is positive!

f(x) is concave up: (2, 3) (6, 8]

(2, 3) and (7, 8]

d) $g(x) = (f(x))^2$ If f(3) = 8, find the slope of the line tangent to the graph g(x) at x = 3

To find the slope of tangent line, we need to find the instantaneous rate of change (derivative) of g(x)...

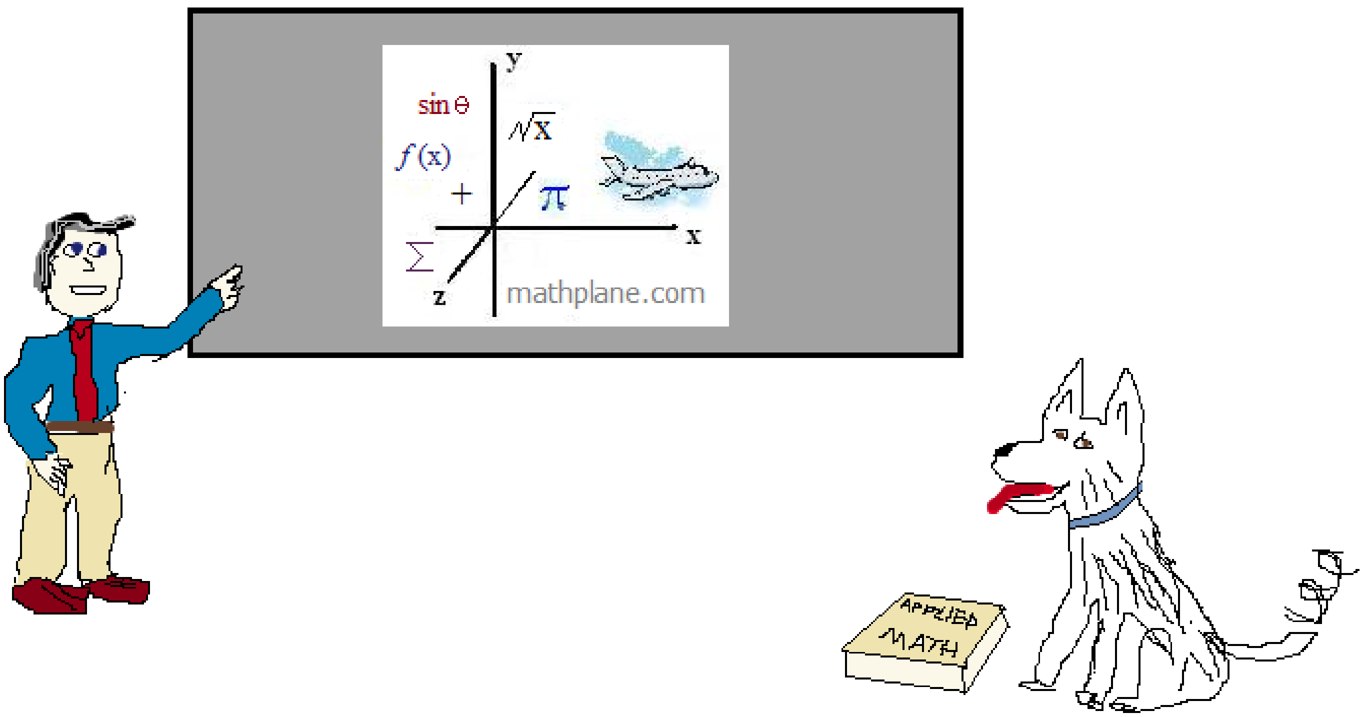
$g'(x) = 2f(x)^1 \cdot f'(x)$ (power rule and chain rule)

$g'(3) = 2f(3) \cdot f'(3) = 2(8) \cdot 2 = 32$

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

Good luck on the test!



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