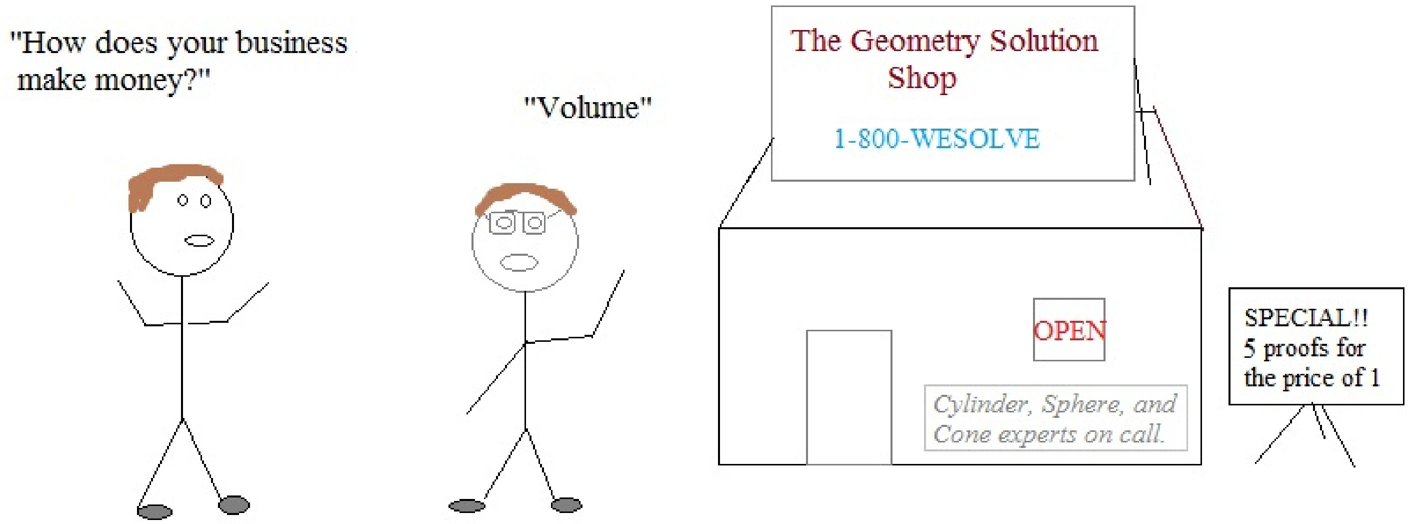
Using Calculus in Economics

Examples and Practice questions (with solutions)

Topics include Cost, Revenue, Max/min from derivatives, Volume, Surface area, and more.



Volume: A Math Guy's Business (Model)

LFriedman #1 10-21-11

Examples-→

Example: A math center charges $400 for a course, and they get 750 students. For every $25 increase in price, they lose 30 students. What price would maximize revenue?

$$(400 + 25x)(750 - 30x) = y$$

price   quantity   revenue

(x is the number of $25 increases)

Using Calculus:

$$f(x) = (400 + 25x)(750 - 30x)$$

$$f(x) = 300000 + 6750x - 750x^2$$

$$f'(x) = 6750 - 1500x$$

To find critical values, set first derivative equal to zero

$$6750 - 1500x = 0$$

$$x = 4.5$$

then, $f''(x) = -1500$

Since second derivative is < 0, it is concave down... (critical value is a maximum)

The optimal price would be $512.50

Using Algebra: recognizing this is a parabola where the vertex is the maximum

Find the vertex:

method 1: use midpoint of zeros

$400 + 25x = 0$ $x = -16$

$750 - 30x = 0$ $x = 25$

axis of symmetry of 4.5

method 2: -b/2a

$$300000 + 6750x - 750x^2$$

$$\frac{-6750}{2(-750)} = 4.5$$

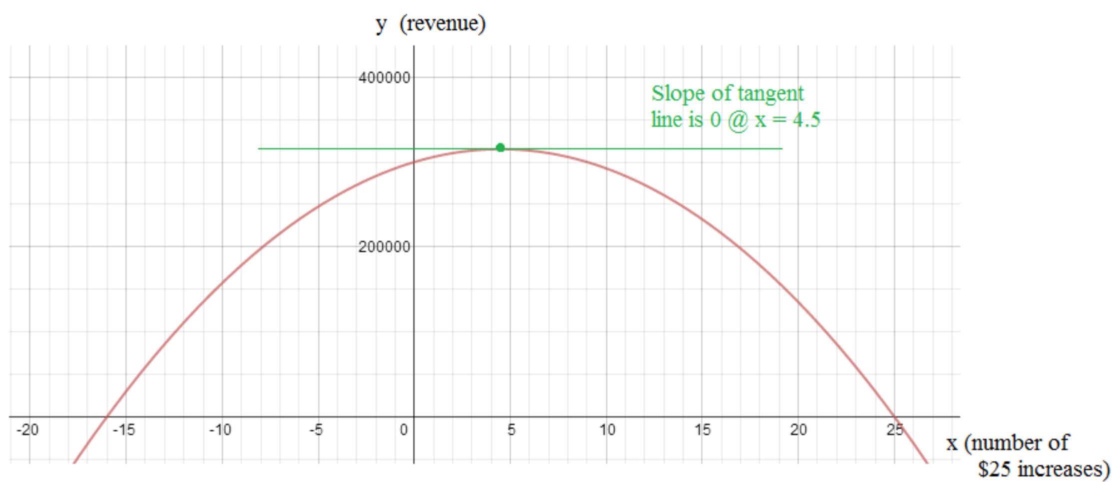
4.5 increases would lead to a price of

$(400 + 25(4.5)) = 512.50$

4.5 increases would lead to a quantity of

$(750 + 30(4.5)) = 615$

revenue = $315,187.5



Calculus: First Derivative Max/Min Applications

Example: Revenue function: $R(x) = 6x$

Cost function: $C(x) = x^3 - 6x^2 + 15x$

Verify that the best your business can do is 'break even'

Profit = Revenue − Cost

$P(x) = 6x - (x^3 - 6x^2 + 15x)$

$= -x^3 + 6x^2 - 9x$

To find maximum/minimum profit, set first derivative equal to zero:

$P'(x) = -3x^2 + 12x - 9$

$-3x^2 + 12x - 9 = 0$ (divide by -3)

$x^2 - 4x + 3 = 0$ (factor)

$(x-1)(x-3) = 0$ (solve)

$x = 1$ and 3

Test each solution in the original equations!

At 1:

$R(1) = 6(1) = 6$

$C(1) = (1)^3 - 6(1)^2 + 15(1) = 10$

Cost (10) exceeds Revenue (6)

Business loses money.....

At 3,

$R(3) = 6(3) = 18$

$C(3) = (3)^3 - 6(3)^2 + 15(3)$

$= 27 - 54 + 45 = 18$

Cost (18) matches Revenue (18)

Business breaks even...

Example: You operate a tour company with the following rates:

$200 per person

Your tour company accommodates 60 - 90 people.

If more than 60 people sign up, the rate drops *$2 per person for each additional person* after 60...

If less than 60 people sign up, the tour is cancelled.

The cost of each tour is $6000 plus $32 per person.

a) How many people would maximize your profit?

b) What is your maximum profit?

Step 1: Transform above description into math equations:

The domain is [60, 90]

Profit = Revenue − Cost

let p = # of people

Cost = $6000 + $32p

Revenue = p($200 − $2(p − 60))

Step 2: Maximize equation (profit)

Profit = p($200 − $2(p − 60)) − [$6000 + $32p]

= $200p − $2p² + $120p − $6000 − $32p

= $−2p² + $288p − $6000

Take derivative: $\frac{dProfit}{dp} = -4p + 288$

Set equal to zero to find max/min: $-4p + 288 = 0$ $\quad p = 72$

Step 3: answer questions

a) What is the optimal number of people: $\boxed{p = 72}$

b) What is your maximum profit ?

Revenue: $200 x 72 = $14,400

Discount: 12 people over 60 ---- $24 discount/person

$24 x 72 = $1728

Total revenue: $12,672

Cost: $6000 + $32(72 people) = $8304

Profit: $12,672 − $8304 = $\boxed{\$4,368}$

Step 4: Check your answer

71 tourists: Revenue: $200 x 71 = $14,200
Discount: (11 x $2) x 71 = −$1562
Cost: $6000 + ($32 x 71) = −$8272

Profit: $4366

72 tourists: Profit: $4368

73 tourists: Revenue: $200 x 73 = $14,600
Discount: (13 x $2) x 73 = $1898
Cost: $6000 + ($32 x 73) = $8336

Profit: $4366

Example: A person decides to start a math greeting card company.

An average cost function for a card producer is

$$\overline{C} = \frac{\text{Total Cost (C)}}{\text{Total Cards (x)}} = \frac{0.40x + 5000}{x} \qquad \text{domain: } x \geq 0$$

The machinery investment cost is $5000.

What is the average cost to produce 50 cards? 100 cards? 1000 cards? 10,000 cards?

What is the rate the average cost changes?

What is the minimum cost per card?

$$C(x) = \frac{0.40x + 5000}{x}$$

$$C(50) = \frac{0.40(50) + 5000}{(50)} = 104 \qquad C(100) = \frac{0.40(100) + 5000}{(100)} = 54$$

$$C(1000) = \frac{0.40(1000) + 5000}{(1000)} = 5.4 \qquad C(10{,}000) = .90$$

$$C'(x) = \frac{(.40 + 0)(x) - (1)(.40x + 5000)}{x^2} = \frac{-5000}{x^2} \Leftarrow$$ the rate of change of the average cost...

$C'(x) < 0$

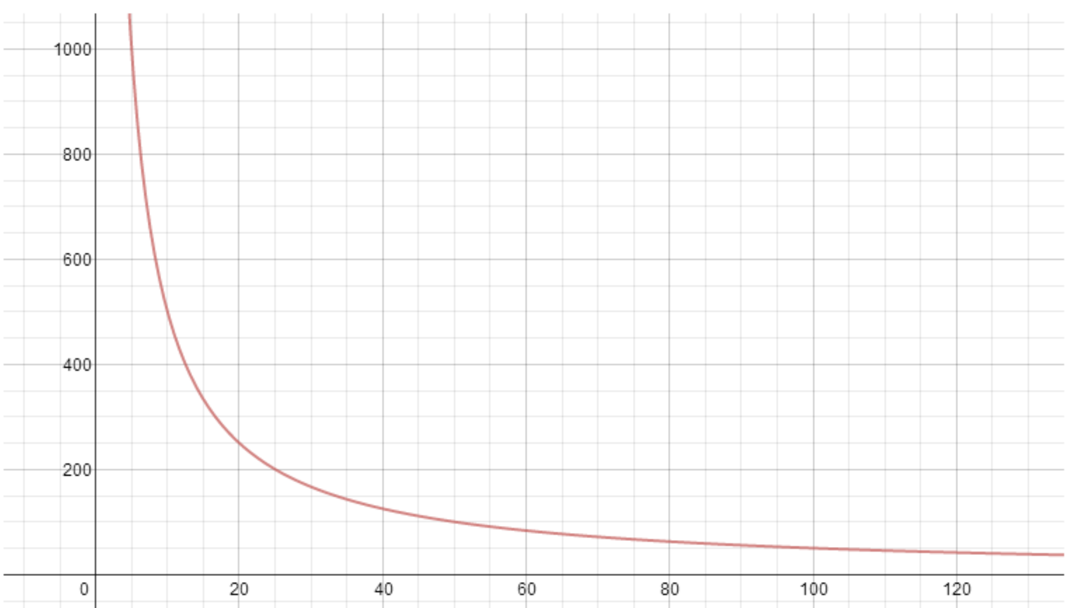
Since the derivative is negative for all x,
the average cost function is always decreasing!

Therefore, the theoretical minimum cost per card is

$$\lim_{x \to \infty} \frac{0.40x + 5000}{x} \Rightarrow \frac{\infty}{\infty}$$

L'Hospital's Rule: $\lim_{x \to \infty} \frac{0.40 + 0}{1} = .40$

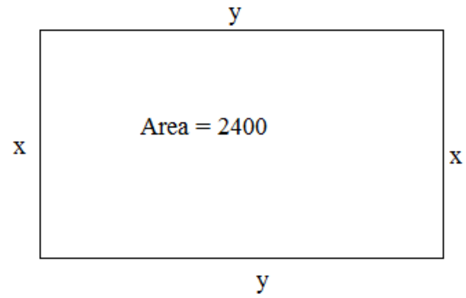
Average Cost C(x)



cards produced (x)

Example: Jerry wants to create a 2400 square foot garden in his backyard. He will use red bricks along the lengths that will cost \$5 per foot and white bricks along the widths that will cost \$3 per foot.

What dimensions would minimize the cost of the garden?



Establish the variables and relevant equations:

let x = width of the rectangle
y = length of the rectangle

$$\text{Area} = (\text{width})(\text{length}) = xy$$

$$\text{Cost} = \$3(2x) + \$5(2y)$$

To find the minimum, we'll take the derivative...

$$C = 6x + 10y \qquad \text{and, we know} \quad \text{Area:} \quad 2400 = xy$$

$$\text{so, } y = \frac{2400}{x}$$

$$C = 6x + 10 \cdot \frac{2400}{x}$$

$$C = 6x + \frac{24000}{x}$$

$$\frac{dC}{dx} = 6 + 24000 \cdot \frac{-1}{x^2}$$

$$\boxed{37.95 \;\; x \;\; 63.25}$$

Then, set the derivative equal to zero...

$$0 = 6 - \frac{24000}{x^2}$$

$$6x^2 - 24000 = 0$$

$$x^2 = 4000$$

$$x = 63.25, \; \cancel{-63.25}$$

The minimum cost occurs when the <u>width</u> is approx. 63.25

and, $2400 = (63.25)(\text{length})$ when the length is 37.95

Quick check: cost of 35 x 68.6

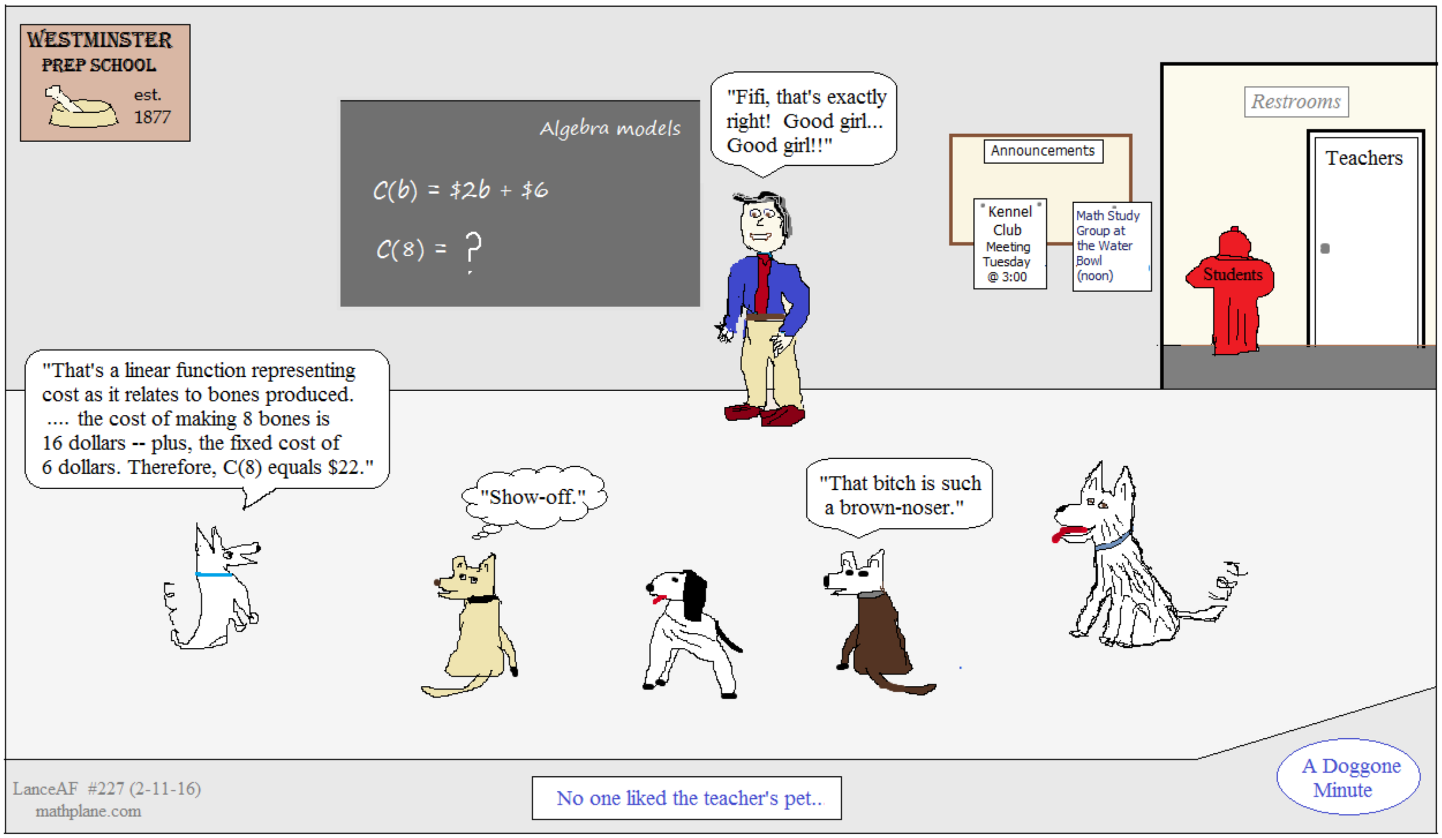
L 35 x \$5 = \$175
W 68.6 x \$3 = \$205.8
\$380.8

cost of 37.95 x 63.25

L 37.95 x \$5 = 189.75
W 63.25 x \$3 = 189.75
\$379.5

cost of 40 x 60

L 40 x \$5 = \$200
W 60 x \$3 = \$180
\$380



Practice Questions-→

Using Derivatives in Economics:
Differentiation rules, maximum, and minimum

Some questions found in "Calculus: with Analytic Geometry" by Larson & Hostetler" (1979)

Solve the following word problems:

1) A manufacturer of lighting fixtures has daily production costs of $C = 800 - 10x + \frac{1}{4}x^2$

How many fixtures x should he produce each day to minimize costs?

2) A power station is on one side of a river that is 1/2 mile wide, and a factory is 3 miles downstream on the other side. It costs $5 per foot to run power lines overland and $8 per foot to run the lines underwater.

Find the most economical path for the transmission line from the power station to the factory.

3) Assume that the amount of money deposited in a bank is proportional to the square of the interest rate the bank pays on this money. Also, the bank can invest this money at 12%.

Find the interest rate the bank should pay to maximize profit. (Use simple interest)

4) A company manufactures soup containers that have a volume of 250 cubic inches.
They wish to minimize the cost by using the least amount of materials.
What dimensions (radius and height) will minimize the surface area?

5) Suppose the metal needed to produce the wall costs .03 cents per square inch, and the metal used to produce the top and bottom costs .05 cents per square inch.
Now, which dimensions will minimize costs?

6) A fence surrounds a rectangular plot of land that is 4800 square feet.

Due to the owner's demands, the front must use wood that costs \$8/foot.
And, the other 3 sides will use wood that costs \$5/foot.

What dimensions would minimize the cost of the fence?
And, how much would it cost?

7) A manufacturer's production plant has the following cost function: $C(x) = x^3 - 19x^2 - 160x + 4750$

A) Find the output (x) to minimize

i) total cost

ii) average cost

B) What is the minimum

i) total cost?

ii) average cost?

Using Derivatives in Economics:
Differentiation rules, maximum, and minimum

SOLUTIONS

Solve the following word problems:

1) A manufacturer of lighting fixtures has daily production costs of $C = 800 - 10x + \frac{1}{4}x^2$

How many fixtures x should he produce each day to minimize costs?

The cost function is $C = 800 - 10x + \frac{1}{4}x^2$

Therefore, the marginal cost function is the derivative of C:

$$C' = 0 - 10 + \frac{2}{4}x$$

To find the optimal number, we must find where marginal costs are minimum.

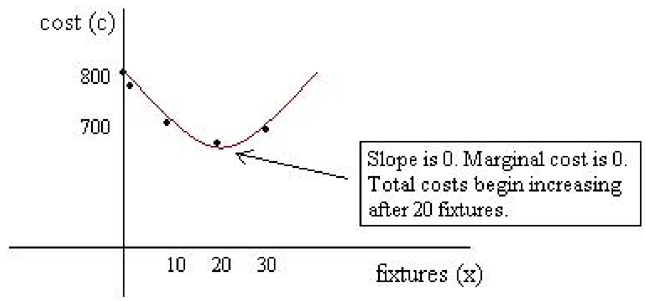
---> Set $C' = 0$

$$0 = -10 + \frac{2}{4}x$$

$$10 = 1/2\,x$$

$$\boxed{x = 20}$$

x	c
0	800
3	772.25
10	725
20	700
30	725

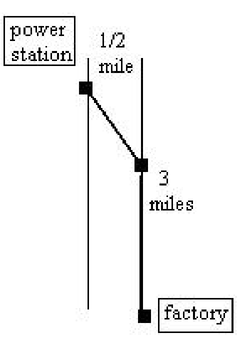


2) A power station is on one side of a river that is 1/2 mile wide, and a factory is 3 miles downstream on the other side. It costs \$5 per foot to run power lines overland and \$8 per foot to run the lines underwater.

Find the most economical path for the transmission line from the power station to the factory.

To begin, let's label variables, construct equations, and draw a picture.

$$C_{total} = C_{water} + C_{overland}$$

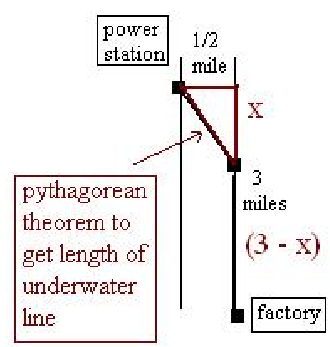


$$C_{water} = d_{water} \cdot \$8/\text{foot}$$

$$C_{overland} = d_{overland} \cdot \$5/\text{foot}$$

$$C_{total} = \left[\sqrt{\left(\frac{1}{2}\right)^2 + x^2}\right] \cdot 8 + \left[(3 - x)\right] \cdot 5$$

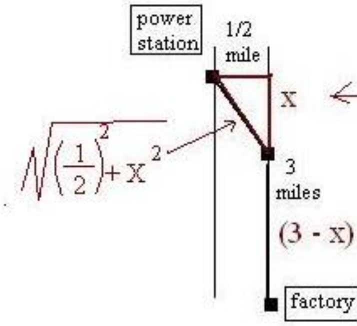
Note: Since we are seeking a minimum path -- rather than the cost -- we can omit the 'foot' measurements (we only need the ratio)



Using Derivatives in Economics:
Differentiation rules, maximum, and minimum

(SOLUTIONS Continued)

$$C_{total} = \left[\sqrt{\left(\frac{1}{2}\right)^2 + x^2}\right] \cdot 8 + \left[(3 - x)\right] \cdot 5$$



The "most economical path" would be the one with the minimum cost. To find the minimum, we take the first derivative and set the equation equal to 0.

$$C_{total} = 8\sqrt{\frac{1}{4} + x^2} + (15 - 5x)$$

$$C'_{total} = 8 \cdot \frac{1}{2}\left(\frac{1}{4} + x^2\right)^{-\frac{1}{2}}\left(2x\right) + (0 - 5)$$

$$C'_{total} = \frac{8x}{\left(\frac{1}{4} + x^2\right)^{\frac{1}{2}}} - 5$$

set derivative equal to 0 $\quad \dfrac{8x}{\left(\frac{1}{4} + x^2\right)^{\frac{1}{2}}} - 5 = 0$

add 5 to both sides.. then square.. $\quad \dfrac{64x^2}{\left(\frac{1}{4} + x^2\right)} = 25$

combine terms and solve.. $\quad \dfrac{1}{4} + x^2 = \dfrac{64x^2}{25}$

$$\frac{1}{4} = \frac{39x^2}{25} \qquad x = \frac{5}{\sqrt{156}} \text{ miles}$$ (downstream)

3) Assume that the amount of money deposited in a bank is proportional to the square of the interest rate the bank pays on this money. Also, the bank can invest this money at 12%.

Find the interest rate the bank should pay to maximize profit. (Use simple interest)

P = Profit
X = interest rate
M = Money deposited

$$P = \left(\begin{matrix}\text{interest}\\ \text{collected}\end{matrix}\right) - \left(\begin{matrix}\text{interest}\\ \text{paid}\end{matrix}\right)$$

$$P = .12M - XM$$

"amount of money ... is proportional to square of interest rate the bank pays..."

$$\longrightarrow \quad \frac{M}{X^2} = 1 \quad \text{or} \quad M = X^2$$

To find X that would maximize profit, take the derivative of P $\quad P = .12M - XM$

To find $\frac{dP}{dX}$ let's "substitute X's for M's" $\quad P = .12X^2 - X^3$

$$P' = .24X - 3X^2$$

Then, to find relative extrema, set equation equal to 0 $\quad 0 = .24X - 3X^2$

0 is the minimum
.08 is the maximum

$$X = 0 \text{ or } \boxed{.08}$$

$$\boxed{\text{Bank should pay 8\% interest}}$$

4) A company manufactures soup containers that have a volume of 250 cubic inches. They wish to minimize the cost by using the least amount of materials. What dimensions (radius and height) will minimize the surface area?

SOLUTIONS

Using derivatives in Economics

Step 1: Identify formulas and variables

Since we want to minimize surface area, the 'main function' is $SA = \underbrace{2(\pi r^2)}_{\text{area of bases}} + \underbrace{(2\pi rh)}_{\text{lateral area (area of wall)}}$

And, the 'constraint function' is $V = \pi r^2 h$

$250 = \pi r^2 h$

Step 2: Set up main function

The surface area has 2 variables, r and h...
If we substitute the second function into the first, we can eliminate a variable...

$SA = 2(\pi r^2) + (2\pi rh)$

$250 = \pi r^2 h \quad h = \dfrac{250}{\pi r^2}$

$$SA = 2(\pi r^2) + \left(2\pi r\left(\frac{250}{\pi r^2}\right)\right)$$

$$SA = 2\pi r^2 + \frac{500}{r}$$

Step 3: Find derivative and minimum/maximum

$$SA' = 4\pi r - \frac{500}{r^2}$$

Then, to find max/min, set equal to zero...

$$0 = 4\pi r - \frac{500}{r^2}$$

$$\frac{500}{r^2} = 4\pi r$$

$$500 = 4\pi r^3 \qquad r = 3.414$$

(***This is a minimum, because
if $r = 1$, SA' is negative (decreasing)..
if $r = 5$, SA' is positive (increasing)...)

Step 4: Answer the question

$\boxed{\text{The radius is 3.414...}}$

Then, $250 = \pi r^2 h$

$250 = \pi (3.414)^2 (h)$

$6.828 = h$

$\boxed{\text{The height is 6.828 inches...}}$

5) Suppose the metal needed to produce the wall costs .03 cents per square inch, and the metal used to produce the top and bottom costs .05 cents per square inch. Now, which dimensions will minimize costs?

Since the bases cost more, we would expect the shape of the can would be taller (a longer wall with smaller bases).... Let's see how much!

The main cost function is $\text{Cost} = \underbrace{.05(2\pi r^2)}_{\text{cost of bases}} + \underbrace{.03(2\pi rh)}_{\text{cost of wall (side)}}$

The constraint is still $250 = \pi r^2 h$

$$C = .05(2\pi r^2) + .03\left(2\pi r \cdot \frac{250}{\pi r^2}\right)$$

$$C = .1\pi r^2 + \frac{15}{r}$$

derivative $C' = .2\pi r - \dfrac{15}{r^2}$

Set equal to zero to find minimum.... $0 = .2\pi r - \dfrac{15}{r^2}$

$$\frac{15}{r^2} = .2\pi r$$

$$r^3 = \frac{15}{.2\pi}$$

$\boxed{\text{radius = 2.88 inches...}}$

then, $250 = \pi (2.88)^2 h$

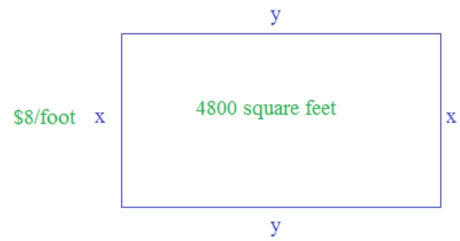
$\boxed{\text{height = 9.59 inches...}}$

6) A fence surrounds a rectangular plot of land that is 4800 square feet.

Due to the owner's demands, the front must use wood that costs \$8/foot.
And, the other 3 sides will use wood that costs \$5/foot.

What dimensions would minimize the cost of the fence?
And, how much would it cost?

SOLUTIONS



Step 1: Draw a diagram and develop formulas

Area = (length)(width) = xy

Cost = \$8(x) + \$5(x + 2y)

Step 2: Express the function in terms of one variable

Since the area 4800 = xy, $y = \dfrac{4800}{x}$

We're trying to minimize the cost: C = \$8x + \$5(x + 2y)

= \$13x + \$10y

$$C = 13x + 10\left(\frac{4800}{x}\right)$$

Step 3: Evaluate

To find the minimum cost, find the derivative and set C' = 0

$$C = 13x + 48{,}000x^{-1}$$

$$C' = 13 + (-1)48{,}000x^{-2}$$

$$C' = 13 - \frac{48{,}000}{x^2}$$

then, set equal to 0 $\quad 0 = 13 - \dfrac{48{,}000}{x^2}$

$$\frac{48{,}000}{x^2} = 13$$

$$13x^2 = 48{,}000$$

$$x = 60.76$$

(obviously, length x cannot be negative)

Step 4: Answer the questions

The front will be 60.76 feet...

And, the other sides will be 60.76, 79, and 79...

$\boxed{60.76 \text{ x } 79}$

The cost will be \$8 x 60.76 + \$5 x (60.76 + 79 + 79) = \$486.08 + \$1093.80 = $\boxed{\$1579.88}$

7) A manufacturer's production plant has the following cost function: $C(x) = x^3 - 19x^2 - 160x + 4750$

A) Find the output (x) to minimize

i) total cost 16 units

ii) average cost 17.37 units

B) What is the minimum

i) total cost? 1422

ii) average cost? 85.15/unit

Minimum total cost:

$$C'(x) = 3x^2 - 38x - 160$$

$$0 = (3x + 10)(x - 16)$$

$$x = \cancel{-10/3} \text{ or } \boxed{16}$$

Since the output cannot be negative, 16 is the only possibility..

Is 16 a max or a min?

$$C''(x) = 6x - 38$$

C''(16) > 0 so, concave up..
therefore, a minimum...

So, the minimum total cost is

$$C(16) = 16^3 - 19(16)^2 - 160(16) + 4750 = \boxed{1422}$$

Minimum AVERAGE cost:

$$\overline{C}(x) = \frac{x^3 - 19x^2 - 160x + 4750}{x}$$

$$\overline{C}(x) = x^2 - 19x - 160 + \frac{4750}{x}$$

$$\overline{C}'(x) = 2x - 19 - 0 - \frac{4750}{x^2}$$

$$0 = 2x^3 - 19x^2 - 4750$$

$$\boxed{x = 17.37}$$

$$\overline{C}''(x) = 6x^2 - 38x$$

$\overline{C}''(17.37) > 0$ (concave up ---> minimum)

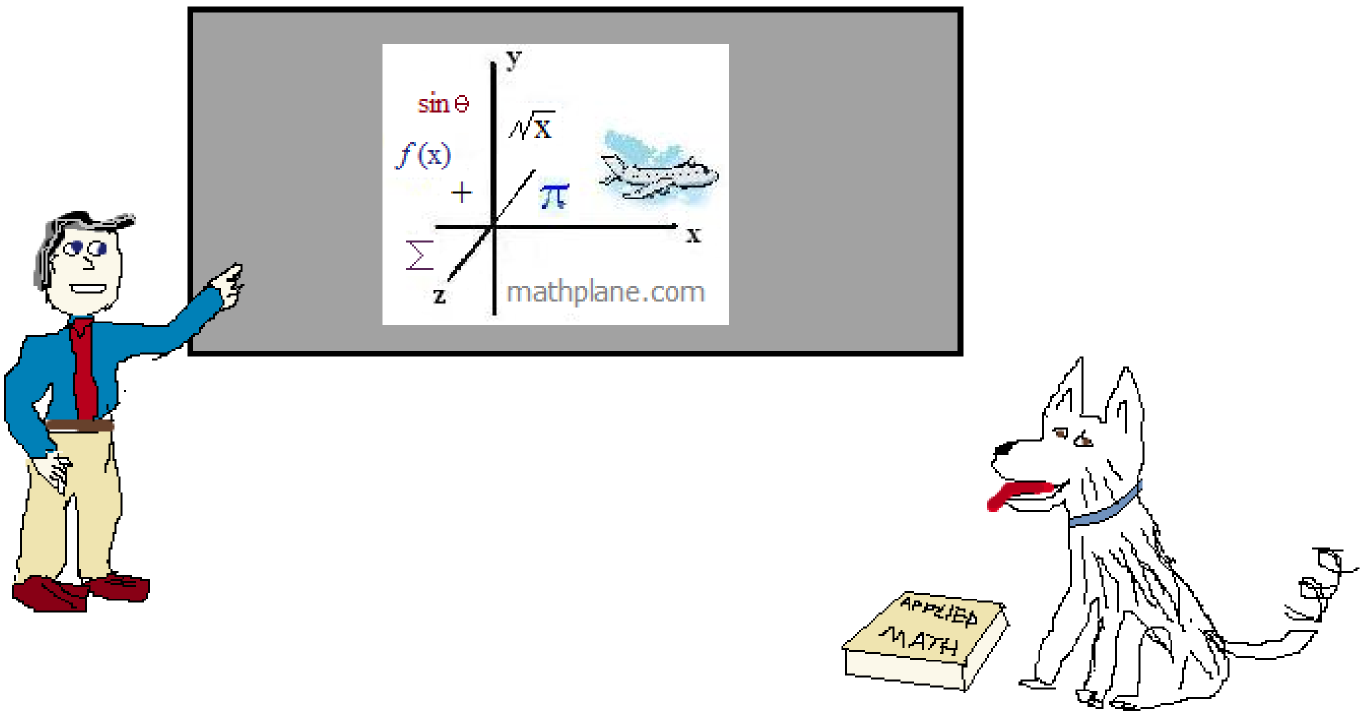
So, the minimum average cost is

$$\overline{C}(17.37) = \boxed{85.15}$$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



Also, at TES, TeachersPayTeachers, Facebook, Google+, Pinterest