Calculus: Derivatives

Maximum/Minimum Word Problems

Topics include cost function, ellipse, distance, volume, surface area, and more.

Calculus: First Derivative Max/Min Applications

1) Revenue function: $R(x) = 6x$

Cost function: $C(x) = x^3 - 6x^2 + 15x$

Verify that the best your business can do is 'break even'

Profit = Revenue − Cost

$P(x) = 6x - (x^3 - 6x^2 + 15x)$

$= -x^3 + 6x^2 - 9x$

To find maximum/minimum profit, set first derivative equal to zero:

$P'(x) = -3x^2 + 12x - 9$

$-3x^2 + 12x - 9 = 0$ (divide by -3)

$x^2 - 4x + 3 = 0$ (factor)

$(x - 1)(x - 3) = 0$ (solve)

$x = 1$ and 3

Test each solution in the original equations!

At 1:

$R(1) = 6(1) = 6$

$C(1) = (1)^3 - 6(1)^2 + 15(1) = 10$

Cost (10) exceeds Revenue (6)

Business loses money.....

At 3,

$R(3) = 6(3) = 18$

$C(3) = (3)^3 - 6(3)^2 + 15(3)$

$= 27 - 54 + 45 = 18$

Cost (18) matches Revenue (18)

Business breaks even...

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2) What is the maximum area of a rectangle that is inscribed under $y = -x^2 + 9$ and above the x-axis.

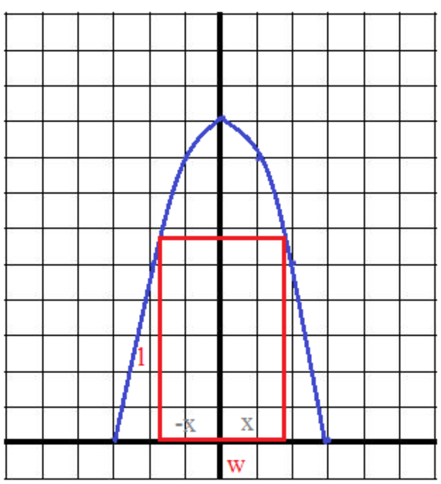
Step 1: Write variables and formulas

Area of rectangle = length x width

width = $x + |-x| = 2x$

length = $-x^2 + 9$

Step 1a: Draw a picture



Step 2: Find maximum of function
(take the derivative, and set = 0)

$A(x) = (-x^2 + 9)(2x)$

$= -2x^3 + 18x$

$A'(x) = -6x^2 + 18$

set equal to zero to find max/min

$-6x^2 + 18 = 0$

$x^2 = 3$

$x = \sqrt{3}, \cancel{-\sqrt{3}}$

Step 3: Solve/Answer the question

Since the maximum area occurs at $x = \sqrt{3}$

the length = $-(\sqrt{3})^2 + 9 = 6$

the width = $2(\sqrt{3}) = 2\sqrt{3}$

$\boxed{\text{Area} = 12\sqrt{3} \text{ or approximately } 20.8 \text{ square units}}$

Derivatives: Maximum/Minimum Examples

3) Find the absolute extremes:

$$f(x) = (x-3)^2 + 1 \qquad \text{over the domain } [2, 6]$$

Using Derivatives:

Find derivative of the function:

$$f'(x) = 2(x-3)^1 + 0$$
$$= 2x - 6$$

Then, to find the extremes, set $f'(x) = 0$

$$2x - 6 = 0$$
$$x = 3$$

Is $x = 3$ a minimum or a maximum?

Test points:

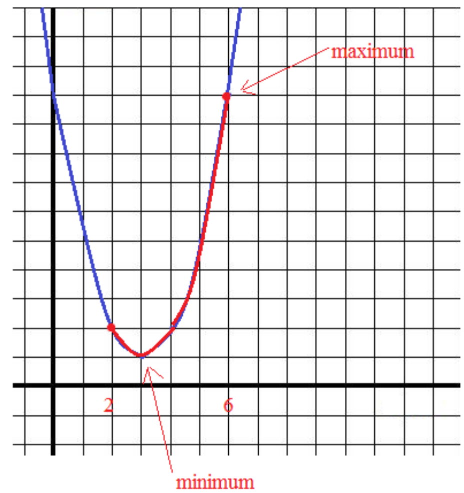
$x = 2$	$f(2) = 2$	
$x = 3$	$f(3) = 1$	MINIMUM
$x = 4$	$f(4) = 2$	

Note:
- $f'(2) = -2$ Since it is negative, the function decreases on the left
- $f'(3) = 0$
- $f'(4) = 2$ Since it is positive, the function increases on the right

$f''(x) = 2$ Since it is positive, the function is concave up (this implies the critical value is a minimum!)

Graphing: $f(x) = x^2 - 6x + 10$

Parabola: Axis of symmetry: $x = 3$
y-intercept: (0, 10)
x-intercepts: NONE
Vertex: (3, 1)



In the domain [2, 6],

the minimum is $x = 3$
maximum is $x = 6$

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4) Assume you have 60 feet of fencing. What is the maximum area you could enclose?
What are the dimensions of the enclosed area?

Step 1: Establish variables and formulas

Area = xy

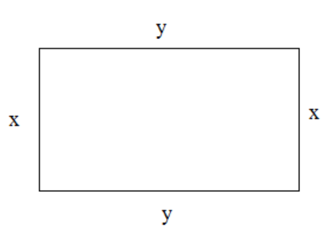
Perimeter = 60
Perimeter = 2x + 2y

$60 = 2x + 2y$

$30 = x + y$

$y = 30 - x$

Step 1a: Draw a diagram



Step 2: Find derivative of equation you want to maximize

We want to maximize the area:
(with respect to x)

$A(x) = x(30 - x)$

$A(x) = -x^2 + 30x$

$A'(x) = -2x + 30$

Step 3: Find critical values and solve

$A'(x) = 0$

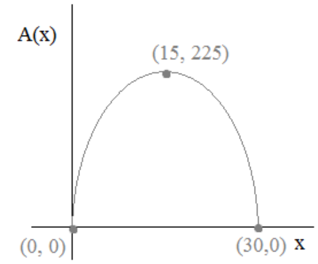
$-2x + 30 = 0$ $\qquad 2x + 2y = 60$

$x = 15$

Since $x = 15$, $y = 15$

Area = 225 square feet
dimensions: 15' x 15'

Graph of one fence side and the corresponding area



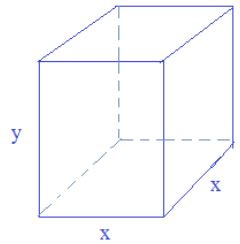
$A(x) = x(30 - x)$

Derivative Applications: maximum/minimum

5) You're contracted to build a *square-based* 600 cubic foot container made of steel. Assuming the construction is an *open-top* container,

a) What are the dimensions of the container that will minimize the weight?
b) What is the surface area of the container?

Step 1: Draw a diagram and label



(since it is square-based, the length and width are equal)

Step 2: Establish formulas

Volume = (length)(width)(height)

$V = x \cdot x \cdot y = x^2 y$

$\boxed{600\text{ft}^3 = x^2 y}$

Surface Area = $4 \cdot$(area of each side) + (area of bottom)

$\boxed{SA = 4xy + x^2}$

**Since we are trying to minimize surface area, we will try to set up SA in terms of one variable (x)

$y = \dfrac{600}{x^2}$ therefore, $\boxed{SA = 4x\left(\dfrac{600}{x^2}\right) + x^2}$

Step 3: Find minimum of function!

$SA = \dfrac{2400}{x} + x^2$

$SA' = \dfrac{-2400}{x^2} + 2x$

(set derivative equal to zero) to find critical values

$\dfrac{-2400}{x^2} + 2x = 0$ (multiply by x^2)

$2x^3 - 2400 = 0$

$x^3 = 1200$

$x = 2\sqrt[3]{150}$ feet

$x = 10.63$ feet (approximately)

Step 4: Answer the questions:

a) What are the dimensions?

Since $x = 10.63$, $y = \dfrac{600}{(10.63)^2} = 5.31$ (approximately)

$\boxed{10.63' \text{ x } 10.63' \text{ x } 5.31'}$ → slightly more than 600 cubic feet

b) What is the surface area of the container?

$SA = 4xy + x^2$

$= 4(10.63)(5.31) + (10.63)^2$

$= \boxed{338.78 \text{ square feet}}$

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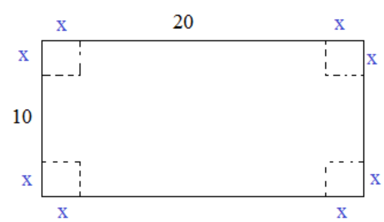
6) Suppose you have a 10' x 20' piece of cardboard. If you wanted to make an open rectangular box (by cutting out the corners & folding up the sides),

a) what dimensions would create a box with the largest volume?
b) what is the maximum volume?

Step 1: Label the diagram and write formulas



(note: the cut-out corners must be squares; otherwise, the top of the open box will be uneven)

Area of cardboard box = (length)(width)

(original length)(original width) = 200 sq. ft.

Volume of open box = (length)(width)(height)

$(20 - 2x)(10 - 2x)(x)$

Step 2: Establish function

Since we want to maximize volume,

$V = (20 - 2x)(10 - 2x)(x)$

$V = (20 - 2x)(10x - 2x^2)$

$V = 200x - 40x^2 - 20x^2 + 4x^3$

$V = 4x^3 - 60x^2 + 200x$

(To find critical value -- max/min --- set first derivative equal to zero)

$V' = 12x^2 - 120x + 200 = 0$

$3x^2 - 30x + 50 = 0$

(quadratic formula)

$x = \dfrac{30 \pm \sqrt{900 - 600}}{6}$

$= 5 + \dfrac{5\sqrt{3}}{3} = \cancel{7.9}$ extraneous! $2 \text{ x } 7.9 = 15.8$; $15.8 >$ the width!

$5 - \dfrac{5\sqrt{3}}{3} = 2.1$

Step 3: Answer the questions

a) What dimension creates the maximum volume?

$x = 2.1$ $\boxed{\text{(cut out 2.1' x 2.1' in each corner)}}$

$\boxed{\begin{array}{l} \text{length} = 20 + 2x = 15.8 \text{ feet} \\ \text{width} = 10 + 2x = 5.9 \text{ feet} \end{array}}$

b) Volume = length x width x height

$15.8' \text{ x } 5.9' \text{ x } 2.1' = \boxed{195.8 \text{ cubic feet}}$

7) You operate a tour company with the following rates: \$200 per person

Your tour company accommodates 60 - 90 people.

If more than 60 people sign up, the rate drops *\$2 per person for each additional person* after 60...

If less than 60 people sign up, the tour is cancelled.

The cost of each tour is \$6000 plus \$32 per person.

a) How many people would maximize your profit?

b) What is your maximum profit?

Step 1: Transform above description into math equations:

The domain is [60, 90]

let p = # of people

Profit = Revenue − Cost

Cost = \$6000 + \$32p

Revenue = p(\$200 − \$2(p − 60))

Step 3: answer questions

a) What is the optimal number of people: $\boxed{p = 72}$

b) What is your maximum profit?

Revenue: \$200 x 72 = \$14,400

Discount: 12 people over 60 ---- \$24 discount/person

\$24 x 72 = \$1728

Total revenue: \$12,672

Cost: \$6000 + \$32(72 people) = \$8304

Profit: \$12,672 − \$8304 = $\boxed{\$4,368}$

Step 2: Maximize equation (profit)

$$\text{Profit} = p(\$200 - \$2(p - 60)) - [\$6000 + \$32p]$$

$$= \$200p - \$2p^2 + \$120p - \$6000 - \$32p$$

$$= \${-2}p^2 + \$288p - \$6000$$

Take derivative: $\dfrac{d\text{Profit}}{dp} = -4p + 288$

Set equal to zero to find max/min: $-4p + 288 = 0$

$p = 72$

Step 4: Check your answer

71 tourists: Revenue: \$200 x 71 = \$14,200
Discount: (11 x \$2) x 71 = −\$1562
Cost: \$6000 + (\$32 x 71) = −\$8272
Profit: \$4366

72 tourists: Profit: \$4368

73 tourists: Revenue: \$200 x 73 = \$14,600
Discount: (13 x \$2) x 73 = \$1898
Cost: \$6000 + (\$32 x 73) = \$8336
Profit: \$4366

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8) The quantity $Q = 2x^2 + 3y^2$ is subject to the constraint $x + y = 5$.
What is the minimum quantity of Q?

Since Q is a function of x and y, let's change to 1 variable....

$x + y = 5 \longrightarrow y = 5 - x$ then, substitute into the main equation...

$$Q = 2x^2 + 3(5 - x)^2$$

$$Q = 2x^2 + 75 - 30x + 3x^2$$

$Q = 5x^2 - 30x + 75$ find derivative of Q...

$$Q' = 10x - 30$$

$10x - 30 = 0$ so,

minimum occurs at $x = 3$

and, therefore, $y = 2$

because $x + y = 5$

$$Q = 2(3)^2 + 3(2)^2$$

$\boxed{Q = 30}$

If $x = 2$ and $y = 3$,

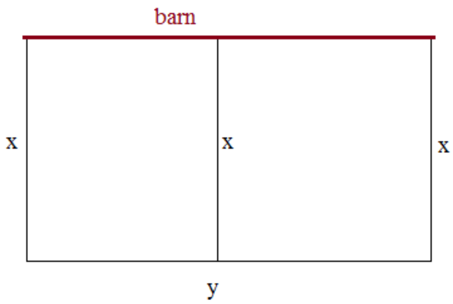
then, $Q = 35$

If $x = 4$ and $y = 1$

then, $Q = 35$

9) A farmer is going to build a pen using 240 feet of wood. One side of the pen will border a barn, and there will be a wooden divider to separate the pen into 2 parts.

What is the maximum area of the pen?



Area = xy 'Main Function' that we want to maximize

$240 = 3x + y$ 'Constraint Function'

Using substitution, we make Area *as a function of x*

Area $= x(240 - 3x)$

$A = 240x - 3x^2$

$A' = 240 - 6x$

$0 = 240 - 6x$

Maximum occurs when $x = 40$ feet

and, then when $y = 120$ feet...

$\boxed{\text{Maximum area: 4800 square feet}}$

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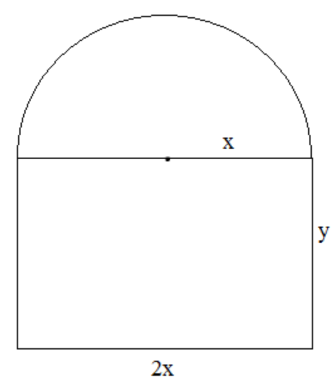
10) A building's window frame shaped as a rectangle with a semicircle is constructed with 28 feet of wood.

Which dimensions would maximize the light shining inside the building?



To maximize the light, we need to maximize the area of the window.

'Main function to optimize': $\text{Area} = \underbrace{2xy}_{\text{rectangle}} + \underbrace{\frac{1}{2}\pi x^2}_{\text{semicircle}}$

'Constraint function' $\text{Perimeter} = 2x + 2y + \frac{1}{2} \cdot 2\pi x$

$28' = 2x + 2y + \pi x$

$y = \dfrac{28' - 2x - \pi x}{2}$

Use substitution and combine the equations

$\text{Area} = 2x\left(\dfrac{28 - 2x - \pi x}{2}\right) + \frac{1}{2}\pi x^2$

$\text{Area} = 28x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$

$A' = 28 - 4x - 2\pi x + \pi x$

To maximize, set derivative equal to zero....

$0 = 28 - 4x - \pi x$

$28 = 4x + \pi x$

$x = \dfrac{28}{(4 + \pi)} = 3.92$ feet

$y = \dfrac{28' - 2(3.92') - \pi(3.92')}{2}$

$= \dfrac{28' - 7.84' - 12.32'}{2} = 3.92$ feet

Bottom: 7.84 feet Left side: 3.92 feet Right side: 3.92 feet Arch: 3.92 π feet	total: 28 feet

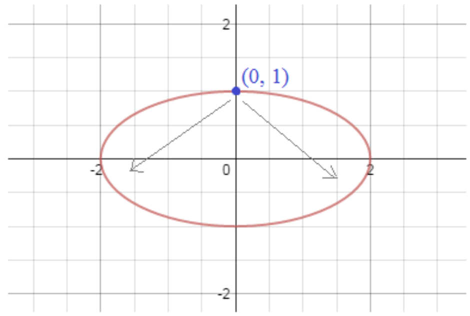
11) Find the point(s) on the ellipse $x^2 + 4y^2 = 4$ farthest from the point (0, 1).

Step 1: Sketch a diagram

The ellipse in standard form: $\frac{x^2}{4} + \frac{y^2}{1} = 1$

center: (0, 0)
major semi-axis: 2
minor semi-axis: 1

horizontal ellipse...



Step 2: Identify main function you wish to maximize

We're looking for the farthest point ---> distance formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

since we want the distance from (0, 1):

$$d = \sqrt{(x - 0)^2 + (y - 1)^2}$$ where (x, y) is a point from the ellipse..

Step 3: Establish a function of *one variable*

Rewriting the ellipse: $x^2 = 4 - 4y^2$

then, substitute into distance formula

Note: we could substitute for y, but substituting for x is much easier...

$$d = \sqrt{(x)^2 + (y - 1)^2}$$

$$d(y) = \sqrt{4 - 4y^2 + (y - 1)^2}$$

$$d(y) = \sqrt{5 - 2y - 3y^2}$$

$$\boxed{\left(\frac{-4\sqrt{2}}{3}, \frac{-1}{3}\right) \quad \left(\frac{4\sqrt{2}}{3}, \frac{-1}{3}\right)}$$

Note: since the maximum of A is the same as the maximum of $\sqrt{A}$, we could have ignored the radical!

Step 4: Find the maximum value

$$d'(y) = \frac{1}{2}\left(5 - 2y - 3y^2\right)^{\frac{-1}{2}}(-2 - 6y)$$

$$d'(y) = \frac{(-1 - 3y)}{\sqrt{5 - 2y - 3y^2}}$$ then, set equal to zero

$$y = \frac{-1}{3}$$

Step 5: Answer the question

If $y = \frac{-1}{3}$ then $x^2 + 4(-1/3)^2 = 4$

$$x^2 = \frac{32}{9}$$

$$x = \frac{4\sqrt{2}}{3} \quad -\frac{4\sqrt{2}}{3}$$

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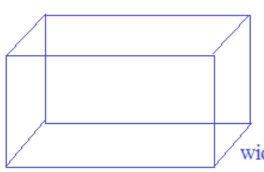
12) An open-top rectangular storage container must have volume 10 cubic feet. Also, the length of the base must be twice the width of the base. If the cost of the base is \$10 per square ft. and the cost of each side is \$6 per square ft., what is the minimum cost of a container?

Step 1: Sketch diagram and label variables

height (h)



width (w)

length (l) = 2w

Step 2: Identify functions and constraints

Volume = lwh = $2w^2h = 10$

Cost = $\$10(2w^2) + 2 \times \$6(2wh) + 2 \times \$6(wh)$

bottom cost | front/back cost | left/right cost

Step 3: Establish function (of one variable) that you wish to minimize

Cost = $20w^2 + 24wh + 12wh = 20w^2 + 36wh$

since (volume) $2w^2 h = 10$ $\quad h = \frac{5}{w^2}$

Substitute into cost function: $C(x) = 20w^2 + \frac{180}{w}$

Step 4: find the minimum

$$C'(x) = 40w - \frac{180}{w^2}$$

set C'(x) = 0,

$40w = \frac{180}{w^2}$ $\quad 40w^3 = 180$ $\quad w = 1.65$

so, width = 1.65, length = 3.30
and,
10 = (1.65)(3.30)(h) so height = 1.84

Step 5: Answer the question

Cost of bottom: \$10(1.65)(3.3)
= \$54.45

Cost of left/right: 2x \$6(1.65)(1.84)
= \$36.43

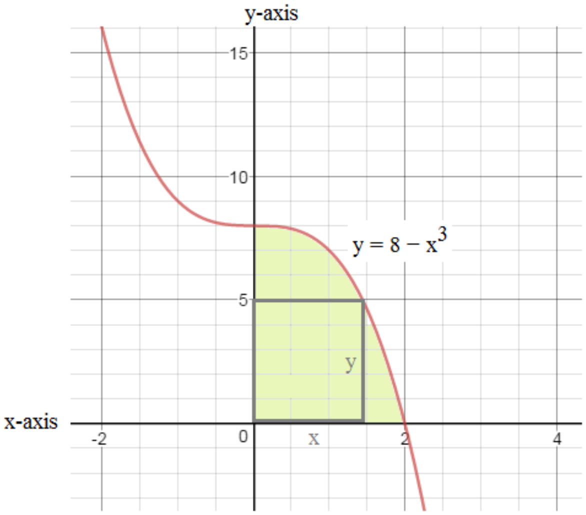
Cost of front/back: 2x \$6(3.3)(1.84)
= \$72.86

$\boxed{\text{Total: } \$163.74}$

SOLUTIONS

13) Find the rectangle with the largest area inscribed in the region bounded by the x-axis, y-axis, and $y = 8 - x^3$

Step 1: Sketch a graph



Step 2: Identify the 'optimization' function

We're trying to find the "rectangle with the *largest* area"

Area = xy

Since we have 2 variables, we'll substitute for y..

$A(x) = x(8 - x^3)$ where A(x) is the area as a function of x

Step 3: Find max/min from derivative

$A(x) = 8x - x^4$

$A'(x) = 8 - 4x^3$

$0 = 8 - 4x^3$

$x = \sqrt[3]{2}$

Step 4: Answer question

If $x = \sqrt[3]{2}$ then $y = 6$

so, the dimensions of the rectangle are $\boxed{\sqrt[3]{2} \text{ by } 6}$

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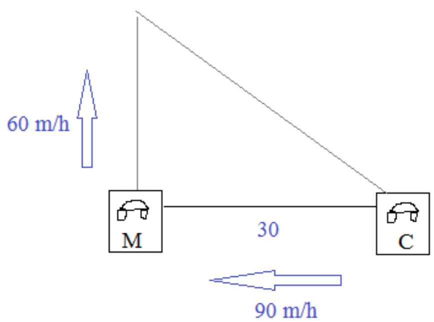
14) At noon, a corvette is 30 miles due East of a mustang. The corvette goes west at 90 miles per hour. Meanwhile, the mustang goes north at 60 miles per hour. What is the minimum distance between the 2 cars? (When does this occur?)

Step 1: Sketch a diagram and label variables



Step 2: Identify the optimization function

We're trying to find the *minimum distance.*

$$d = \sqrt{x^2 + y^2}$$

$$d = \sqrt{(30 - 90t)^2 + (0 + 60t)^2} \quad \text{where t is time in hours}$$

$$d = \sqrt{900 - 5400t + 8100t^2 + 3600t^2} = \sqrt{900 - 5400t + 11700t^2}$$

$$d' = \frac{1}{2}(900 - 5400t + 11700t^2)^{\frac{-1}{2}}(-5400 + 23{,}400t)$$

$$d' = \frac{(-5400 + 23{,}400t)}{2\sqrt{900 - 5400t + 11700t^2}}$$

When is d' = 0? It occurs when the numerator $-5400 + 23{,}400t = 0$

$t = .2308$ hours
or 13.84 minutes

Step 3: Answer the questions

$\boxed{\text{The minimum distance between cars occurs at 12:14}}$

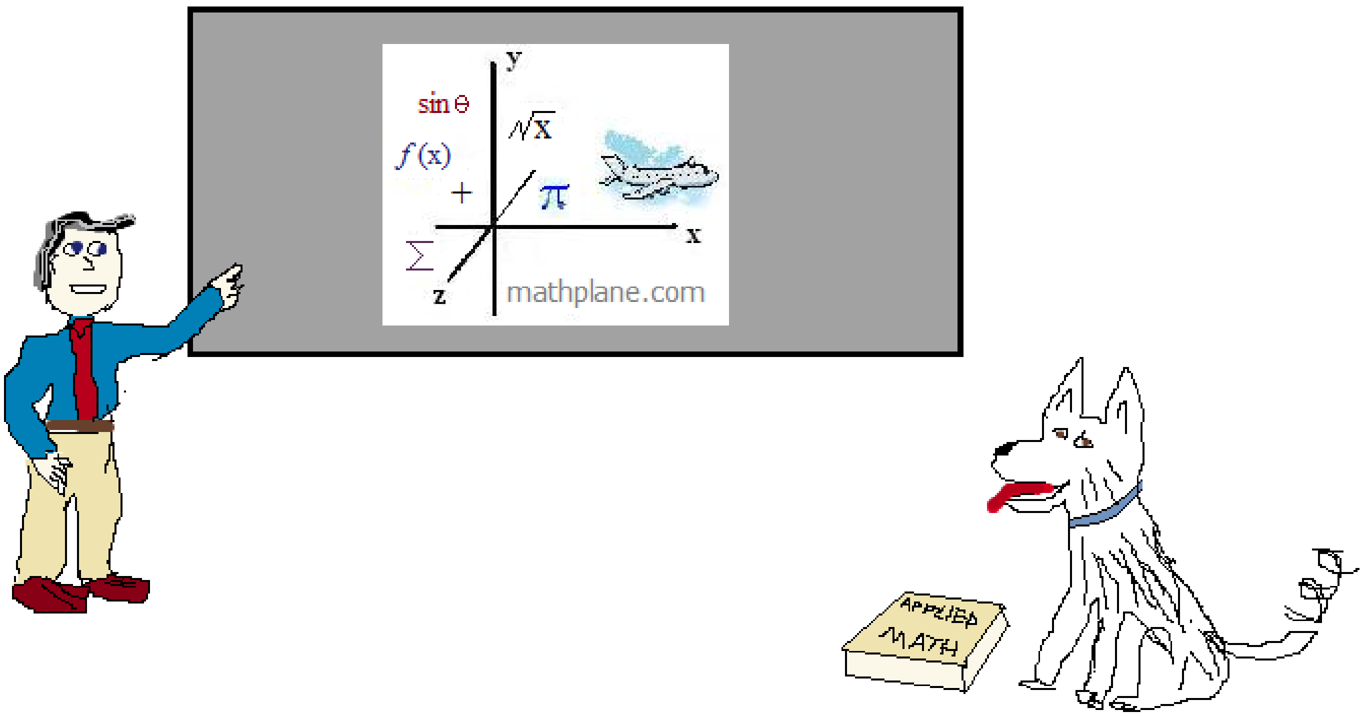
$$d = \sqrt{(30 - 90t)^2 + (0 + 60t)^2} \quad \text{at } t = .2308$$

$$= \sqrt{(9.228)^2 + (13.84)^2} = \boxed{\text{16.6 miles apart (approx)}}$$

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

Cheers



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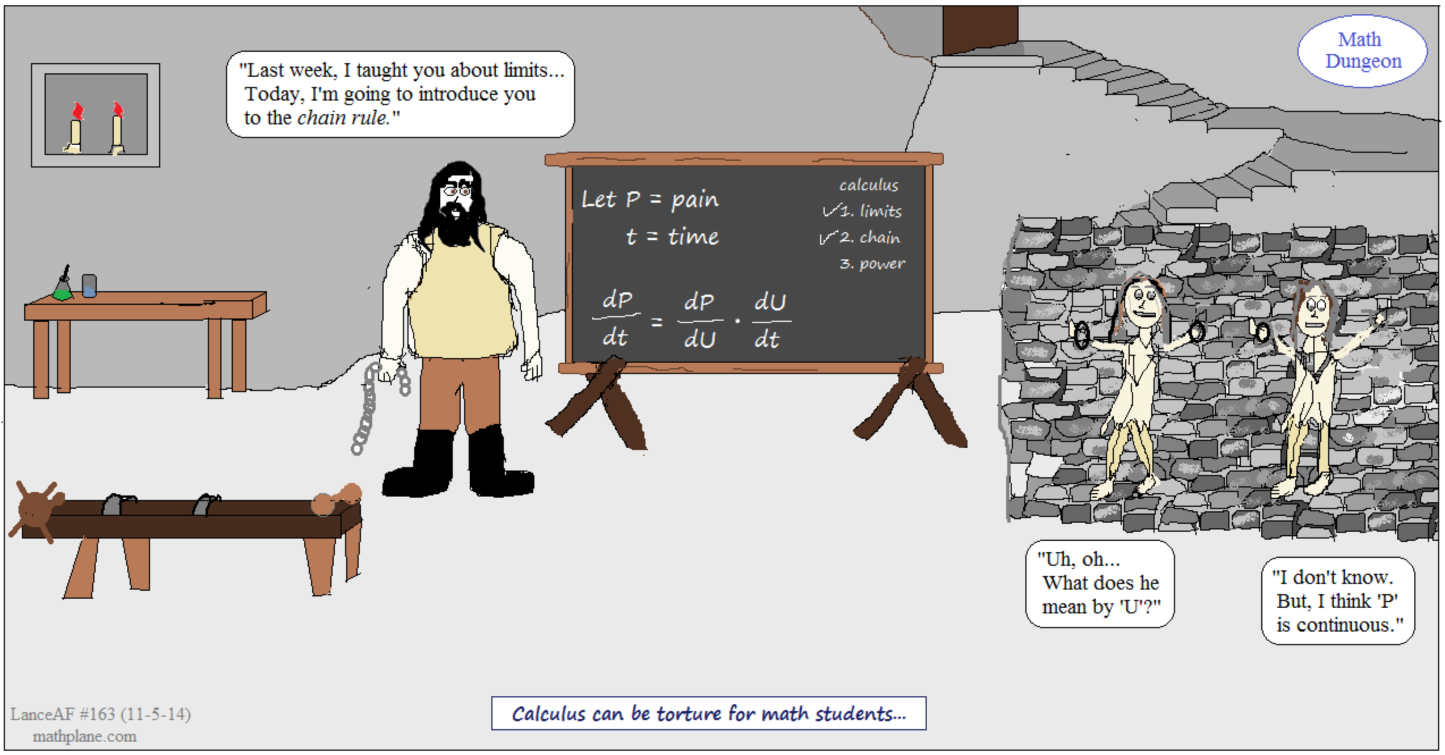
Math Dungeon
"Last week, I taught you about limits... Today, I'm going to introduce you to the *chain rule.*"
Let P = pain
t = time
calculus
✓1. limits
✓2. chain
3. power
$\frac{dP}{dt} = \frac{dP}{dU} \cdot \frac{dU}{dt}$
"Uh, oh... What does he mean by 'U'?"
"I don't know. But, I think 'P' is continuous."
Calculus can be torture for math students...
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