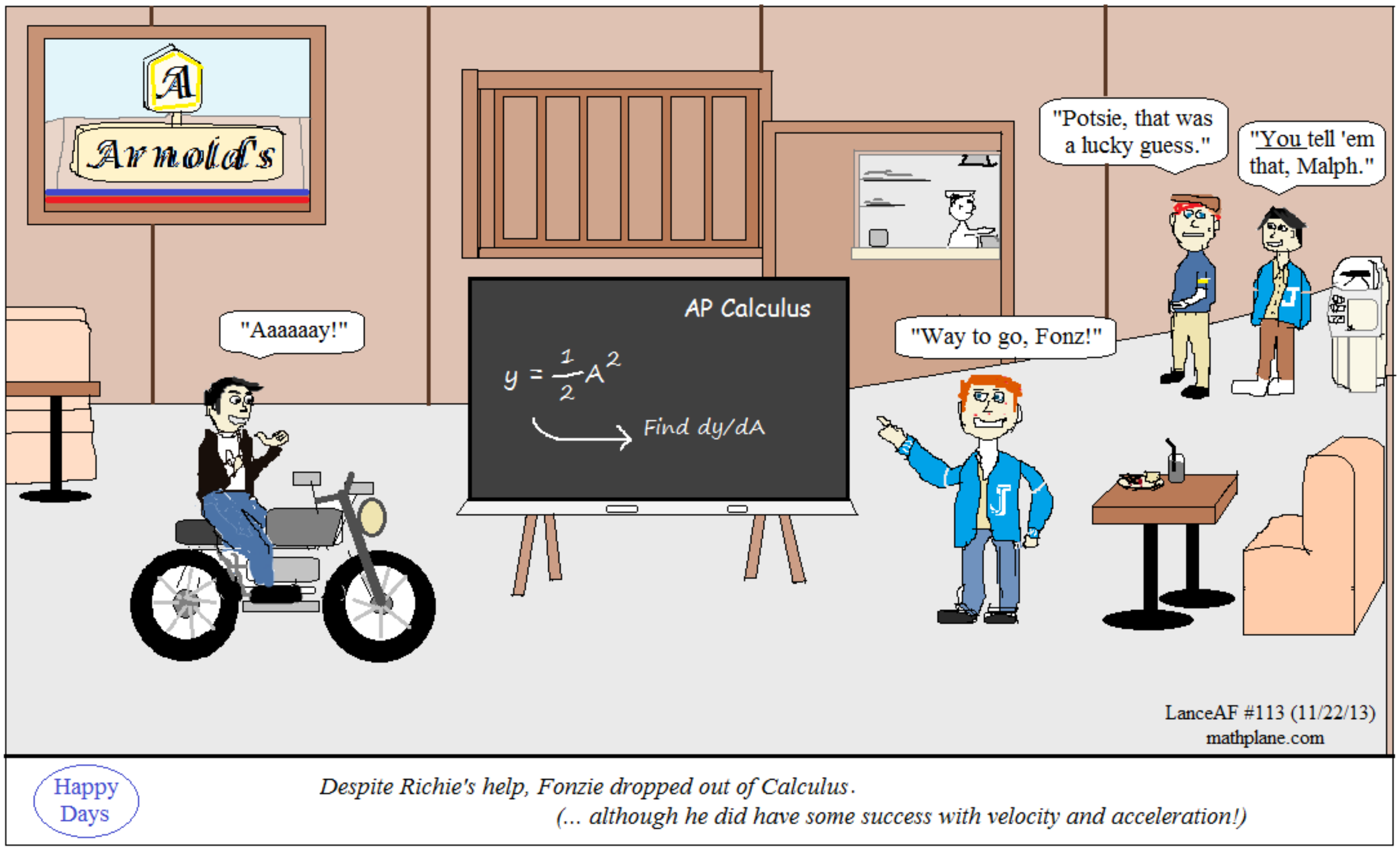
Calculus: Applications of Derivatives

Examples and explanations



Topics include linearization, velocity, sketching rational expressions, volume, slope, and more.

Mathplane.com

Projectile Motion: Calculus and Algebra Applications

A kid launches a water balloon from a balcony.
The trajectory is described by the function

$$h(t) = -16t^2 + 64t + 80 \quad \text{where}$$

t is time (in seconds)
h(t) is the height of the balloon (in feet)

Using properties of quadratics and Algebra:

1) What is the maximum height the water balloon reaches?

2) When does the balloon hit the ground?

3) What is the height of the balcony?

Answers and Graph:

Since the degree of the polynomial is 2, it's a quadratic (parabola facing down).

1) The maximum height will be at the vertex!

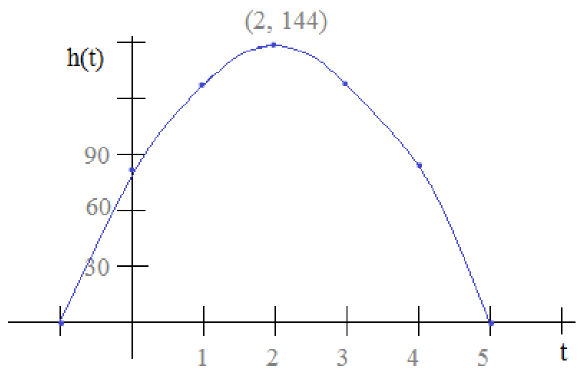
The vertex is $(-b/2a,\ h(-b/2a))$

$a = -16$
$b = 64$
$c = 80$

$$\frac{-b}{2a} = \frac{-64}{2(-16)} = 2$$

$$h(2) = -16(2)^2 + 64(2) + 80 = 144$$

The maximum height is 144 feet,
and it occurs at 2 seconds...



2) Balloon hits the ground when the height is 0.

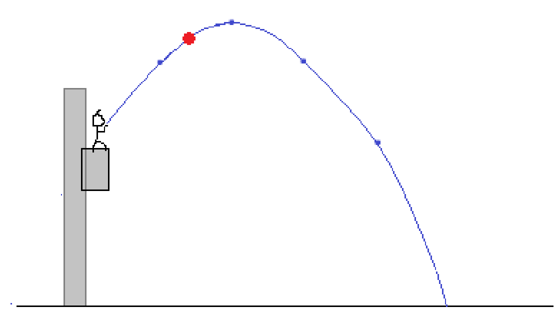
$h(t) = 0$

$-16t^2 + 64t + 80 = 0$

$-16(t^2 - 4t - 5) = 0$

$-16(t - 5)(t + 1) = 0$

The balloon hits the ground
at -1 and 5 seconds...
(since time cannot be negative,)
the answer is 5 seconds after launch.



**Note: the domain is $0 \le t \le 5$

3) The height of the balcony would be the h(t) when t = 0. (assuming we omit the height of the kid!)

$h(0) = -16(0) + 64(0) + 80 = 80$ feet

Projectile Motion: Calculus and Algebra Applications

A kid launches a water balloon from a balcony.
The trajectory is described by the function $h(t) = -16t^2 + 64t + 80$ where t is time (in seconds)
h(t) is the height of the balloon (in feet)

Using Calculus applications

1) What is the initial velocity of the balloon?
2) What is the maximum height reached?
3) What is the acceleration of the balloon at 3 seconds?
4) What is the height of the balloon at 4 seconds?
5) What is the speed of the balloon at 4 seconds?
6) When is the balloon 100 feet high?
7) What is the AROC during the entire flight of the balloon?

h(t) describes the position of the balloon
$h(t) = -16t^2 + 64t + 80$ (feet)

h'(t) describes the velocity of the balloon
('instantaneous rate of change')

$h'(t) = -32t + 64$ (feet/second)

h''(t) describes the acceleration
(rate the velocity is changing)

$h''(t) = -32$ (feet/second^2)

|h'(t)| absolute value of velocity is the speed of the balloon

Answers:

1) The initial velocity is the rate of change at t = 0.

$h'(0) = -32(0) + 64 = 64$ feet/second

2) The maximum height occurs when the balloon changes direction.
(in other words, when the rate of change is zero, the balloon is at a max)

$h'(t) = -32t + 64$ When $h'(t) = 0$, $t = 2$

Since max height occurs at t = 2 seconds, the balloon is at $h(2) = 144$ feet...

3) The acceleration at 3 seconds can be found using the 2nd derivative.

$h''(t) = -32$, $h''(3) = -32$ feet/second^2

4) and 5) The height is "location", so use the function: $h(4) = -16(4)^2 + 64(4) + 80 = 80$

The velocity is "rate of change", so use the first derivative: $h'(4) = -32(4) + 64 = -64$ feet/second

The speed is the absolute value of the velocity: 64 feet/second

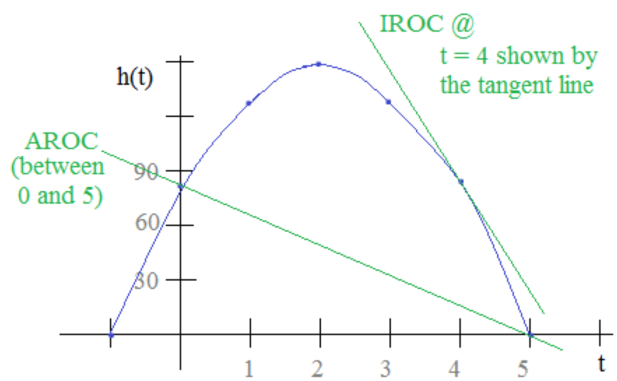
6) The "position/location" of the balloon is found using the function:

$100 = -16t^2 + 64t + 80$

$16t^2 - 64t + 20 = 0$

$4t^2 - 16t + 5 = 0$ t = 0.34 seconds and 3.66 seconds

(quadratic formula)



7) AROC (average rate of change) is the slope between 2 points.

The points in this case are (0, 80) and (5, 0)

$$\frac{h(5) - h(0)}{5 - 0} = \frac{0 - 80}{5 - 0} = -16 \text{ feet/second}$$

Example: Two particles that move along a horizontal axis have the following models:

$$x(t) = 3\cos\left(\frac{\pi}{4}t\right) \qquad s(t) = t^3 - 6t^2 + 9t + 4$$

On the interval $0 \le t \le 6$, when do the particles move in the same direction?

Find the intervals where each particle increases and decreases...

First derivative....

$$x'(t) = -3\sin\left(\frac{\pi}{4}t\right) \cdot \frac{\pi}{4} \qquad s'(t) = 3t^2 - 12t + 9 + 0$$

Then, set equal to zero (to find where particle changes direction)

$$-3\frac{\pi}{4}\sin\left(\frac{\pi}{4}t\right) = 0$$

$$\sin\left(\frac{\pi}{4}t\right) = 0$$

$t = 4\text{k}$ (where k is any integer)

So, in interval $[0, 6]$, 0 and 4

$$3t^2 - 12t + 9 = 0$$

$$3(t^2 - 4t + 3) = 0$$

$$(t-3)(t-1) = 0$$

$$t = 1 \text{ and } 3$$

Then, test each sub-interval to determine whether increasing or decreasing...

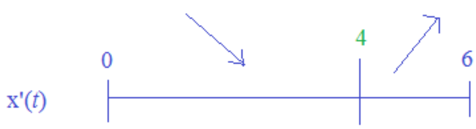
$$x'(1) = -3\frac{\pi}{4}\sin\left(\frac{\pi}{4}1\right) < 0$$

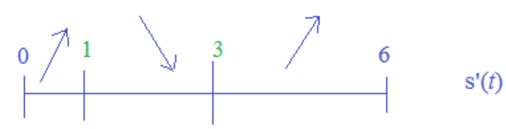
$$x'(5) = -3\frac{\pi}{4}\sin\left(\frac{\pi}{4}5\right) > 0$$

$$s'(1/2) = 3(1/2 - 3)(1/2 - 1) > 0$$

$$s'(2) = 2(2-3)(2-1) < 0$$

$$s'(4) = 2(4-3)(4-1) > 0$$

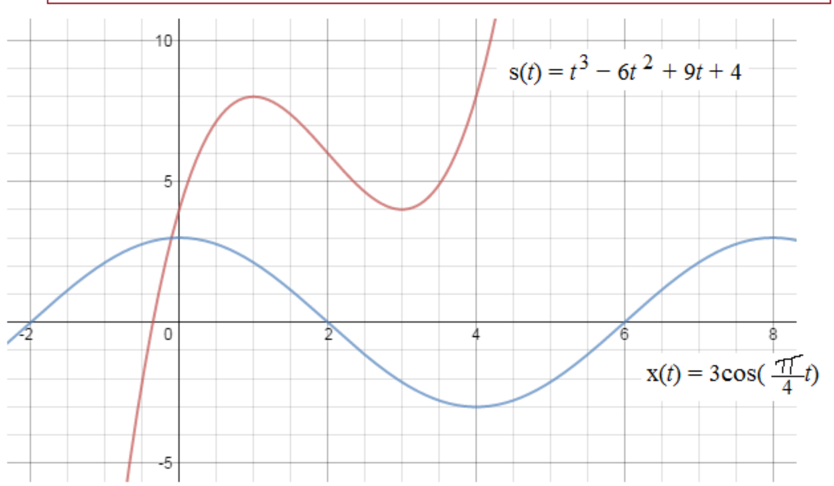




Finally, determine the sub-intervals where $x(t)$ and $s(t)$ move in the same direction....

Interval $(1, 3)$ where both are decreasing (i.e. moving to the left)

and, Interval $(4, 6]$ where both are increasing (i.e. moving to the right)



Derivatives and Linear Equations

Example: Find equation of a normal line to $f(x) = 3x^3 - 2x^2 + 5x - 3$ at $x = 2$

To find the equation of a line, we need a point and the slope.

To determine the slope, find the first derivative of $f(x)$.

$$f'(x) = 9x^2 - 4x + 5$$

Then, to find the slope of the tangent at $x = 2$: $f'(2) = 9(2)^2 - 4(2) + 5 = 33$

Since the slope of the tangent is 33, the slope of the normal (perpendicular) is -1/33

And, a point on the normal will be where $x = 2$:

$$f(2) = 3(2)^3 - 2(2)^2 + 5(2) - 3 = 23$$

Therefore, an equation with slope -1/33 going through (2, 23) is

$$y - 23 = \frac{-1}{33}(x - 2)$$

Example: Write the equation of a line tangent to $x^2 + 5x + 6$ at $x = 1$

Then, graph the equation and the tangent line.

If $y = x^2 + 5x + 6$, then

$$y' = 2x + 5$$

Therefore, the instantaneous rate of change at $x = 1$ is

$f'(1)$ $\qquad 2(1) + 5 = 7$

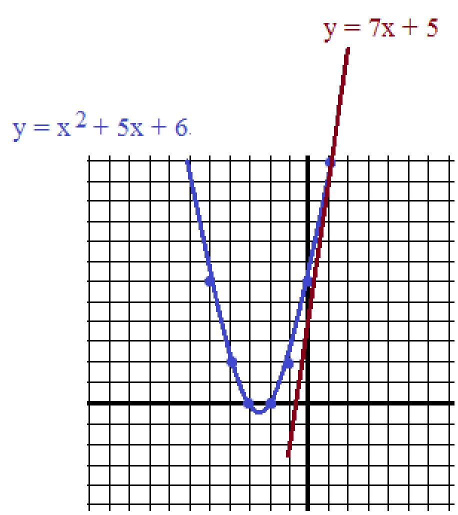
And, to find a point on the tangent line, we use $x = 1$

$f(1)$ $\qquad (1)^2 + 5(1) + 6 = 12$

So, the equation of the tangent line is

$$y - 12 = 7(x - 1)$$

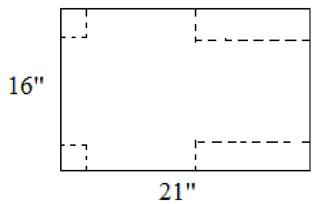
$$y = 7x + 5$$

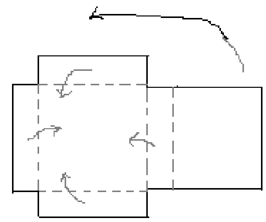


You are given a 16" x 21" cardboard sheet.
After cutting out the corners, you can fold up 3 of the sides.
Then, the fourth side will be folded up and extended over the other 3 to form a lid.

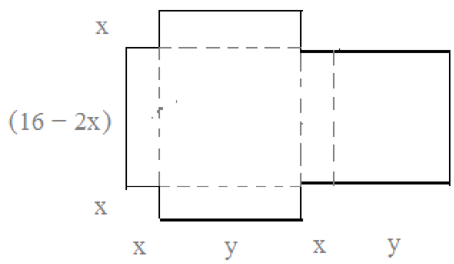
What are the dimensions of the enclosed box with the largest volume?

Step 1: Draw a diagram to visualize the question





Step 2: Label diagram, establish variables, and write equations



since $2x + 2y = 21$, $y = \dfrac{21 - 2x}{2}$

Volume = (length)(width)(height)

length = $(16 - 2x)$

height = x

width = $\dfrac{(21 - 2x)}{2}$

$$V = (16 - 2x)\left(\frac{(21-2x)}{2}\right)(x)$$

Step 3: Solve.

To find the maximum (or minimum) volume, find dV/dx and set it equal to 0...

$$V = (16x - 2x^2)\left(\frac{(21-2x)}{2}\right)$$

$$V = (8x - x^2)(21 - 2x)$$

$$V = 168x - 16x^2 - 21x^2 + 2x^3$$

$$V = 2x^3 - 37x^2 + 168x$$

$$\frac{dV}{dx} = 6x^2 - 74x + 168$$

then, set derivative equal to zero...

$$6x^2 - 74x + 168 = 0$$

$$3x^2 - 37x + 84 = 0$$

$$x = 3 \text{ or } 28/3$$

Step 4: Answer question and check solutions

If $x = 28/3$, height = 9.33
length = $(16 - 2(9.33)) = -2.66$

Impossible... $x \neq 28/3$

If $x = 3$, height = 3
length = $(16 + 2(3)) = 10$
width = $\dfrac{(21 + 2(3))}{2} = 7.5$

The dimensions of the box (with lid) are

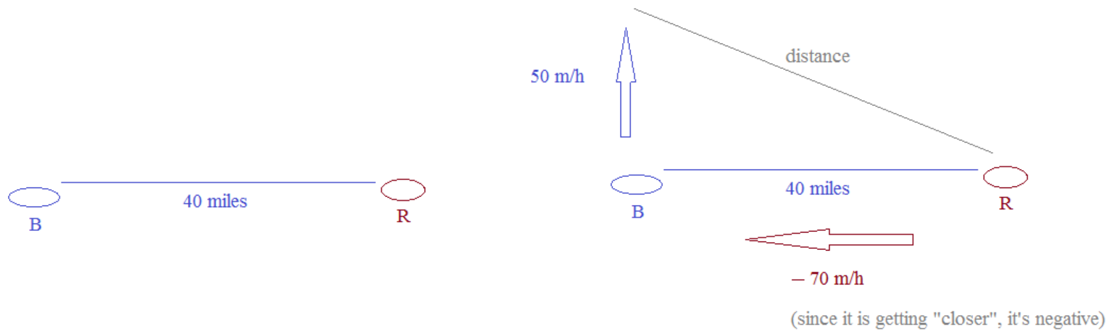
10" x 7.5" x 3"

Check:

If x = 2, then dimensions are 12" x 8.5" x 2" 204 cubic inches
If x = 2.5, then dimensions are 11" x 8" x 2.5" 220 cubic inches
If x = 3, then dimensions are 10" x 7.5" x 3" 225 cubic inches
If x = 3.5, then dimensions are 9" x 7" x 3.5" 220.5 cubic inches
If x = 4, then dimensions are 8" x 6.5" x 4" 208 cubic inches

Example: A red car is 40 miles due east of a blue car.
The red car is traveling west at a speed of 70 miles per hour.
At the same time, the blue car is traveling north at a speed of 50 miles per hour.

At what time will the cars be closest to each other?



We want to minimize distance...

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

or utilize the Pythagorean Theorem

$$x^2 + y^2 = d^2$$

$$(0 + 50t)^2 + (40 - 70t)^2 = d^2$$

$$\sqrt{(0 + 50t)^2 + (40 - 70t)^2} = d$$

$$d = \sqrt{2500t^2 + 1600 - 5600t + 4900t^2}$$

$$d = \sqrt{1600 - 5600t + 7400t^2}$$

(take 1st derivative and set equal to zero)

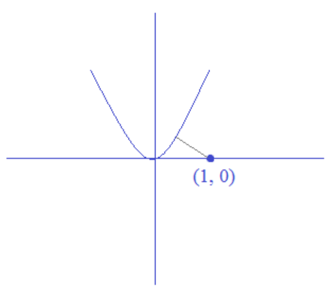
$$d' = \frac{1}{2}(1600 - 5600t + 7400t^2)^{-1/2} \cdot (-5600 + 14800t)$$

$$d' = \frac{1 \quad (-5600 + 14800t)}{2\,(1600 - 5600t + 7400t^2)^{1/2}}$$

$$\boxed{d' = 0 \quad \text{when} \quad t = .3783 \qquad 22.7 \text{ minutes}}$$

Example: Find the point on the curve $y = x^2$ that is closest to (1, 0)

quick sketch:



Using the distance formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(x - 1)^2 + (y - 0)^2}$$

$$= \sqrt{(x - 1)^2 + (x^2)^2}$$

$$= \sqrt{x^4 + x^2 - 2x + 1}$$

$$d' = \frac{1}{2}(x^4 + x^2 - 2x + 1)^{\frac{-1}{2}} (4x^3 - 2x + 2)$$

find where $(4x^3 - 2x + 2) = 0$

$$\boxed{\begin{array}{l} x = .59 \\ y = .35 \end{array}}$$

Linearization formula is similar to point slope form of line.

$(y - y_1) = m(x - x_1)$ point slope form where m is slope and (x_1, y_1) is a point

$$L(x) = f(a) + f'(x)(x - a)$$

$$L(x) - f(a) = f'(x)(x - a)$$

y y1 m (x x1)

Linearization Application:

Example: Approximate the value of $\sqrt{145}$

The general equation for square root is $f(x) = \sqrt{x}$

To approximate, we'll use the *nearby* point $\boxed{(144, 12)}$

To find the linear approximation, we need the slope (derivative)...

$$f'(x) = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(144) = \frac{1}{2\sqrt{144}} = \boxed{\frac{1}{24}}$$

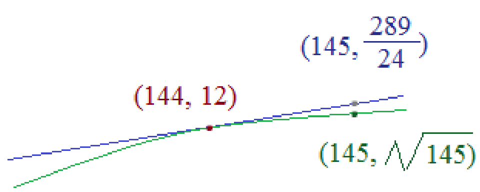
$(y - y_1) = m(x - x_1)$ $\quad L(x) - 12 = \frac{1}{24}(x - 144)$

$y = mx + b$ $\quad L(x) = \frac{1}{24}x + 6$

Approximate $\sqrt{145}$, using the linear equation:

$$L(145) = \frac{145}{24} + 6 = \frac{289}{24} \approx 12.04167$$

True value: $\sqrt{145} \approx 12.04159$ Very close!!



Example: $f(x) = \sqrt{1 - x}$

Using $x = 0$, find a linear approximation of $\sqrt{.9}$

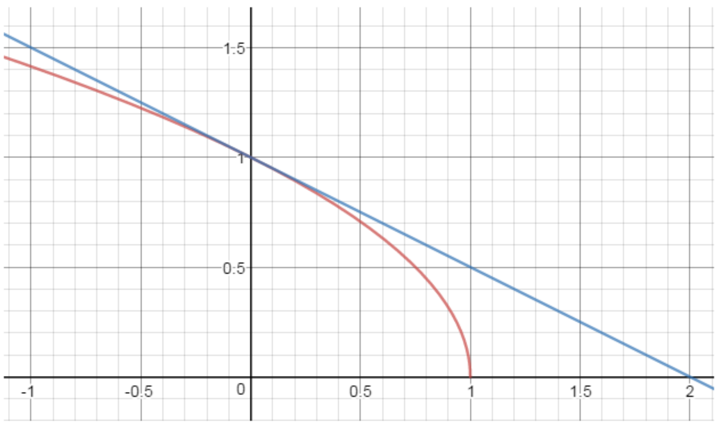
To find the tangent line at $x = 0$, we need a point and the slope....

point: @ $x = 0$, $f(0) = 1$ $(0, 1)$

slope: To find IROC, $f'(x) = -\frac{1}{2}(1 - x)^{-1/2}$

$$f'(0) = -1/2$$

Equation of the line: $y - 1 = \frac{-1}{2}(x - 0)$



Now, to find the approximation of $\sqrt{.9}$

we'll let $x = .1$ (because $f(.1) = \sqrt{1 - (.1)} = \sqrt{.9}$)

If $x = .1$, then $y - 1 = \frac{-1}{2}(.1 - 0)$

$$y = -.05 + 1$$

$$y = .95$$

true value: $\sqrt{.9} = .9487$

(Note: since the curve is concave down, the linear approximation *overestimates* the true value...)

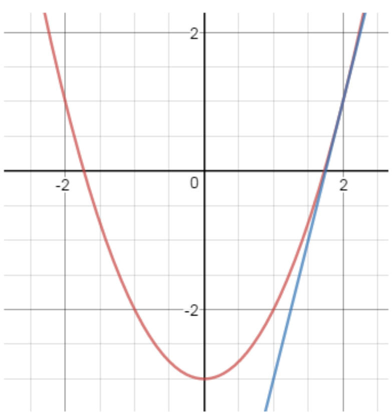
Example: $f(x) = x^2 - 3$ Find $f(1.7)$

Calculate the error from using a linear approximation at $x = 2$

$f(2) = 1$ so, the point is $(2, 1)$

$f'(2) = 2(2) - 0 = 4$

the linearization of the curve is $y - 1 = 4(x - 2)$



The true value: $f(1.7) = (1.7)^2 - 3 = -.11$

The linear approximation: $y - 1 = 4((1.7) - 2)$

$$y = -1.2 + 1 = -.20$$

The error is $-.09$...

(since the curve is concave up, it makes sense that the approximation underestimates...)

Example: Using a linear approximation, estimate the value of $\cos\left(\frac{2\pi}{7}\right)$

The function is $f(x) = \cos(x)$

We'll let $x = \frac{\pi}{3}$ for a comparison point.. $\cos(\frac{\pi}{3}) = 1/2 \Rightarrow$ point: $(\frac{\pi}{3}, \frac{1}{2})$

Now, we need the slope: $f'(x) = -\sin(x)$ slope is $-\sin(\frac{\pi}{3}) = \frac{-\sqrt{3}}{2}$

linearized model in point-slope form: $y - \frac{1}{2} = \frac{-\sqrt{3}}{2}(x - \frac{\pi}{3})$

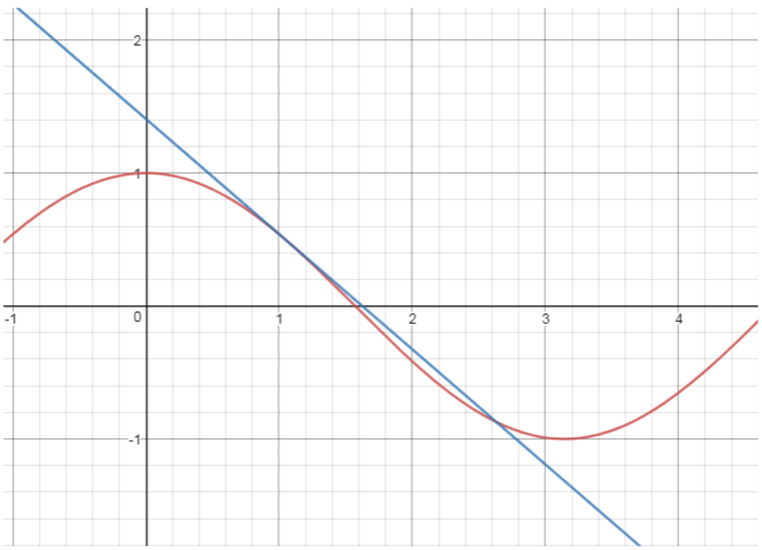
Now, let's estimate $\cos\left(\frac{2\pi}{7}\right)$ $y - \frac{1}{2} = \frac{-\sqrt{3}}{2}(\frac{2\pi}{7} - \frac{\pi}{3})$

$$y - \frac{1}{2} = \frac{-\sqrt{3}}{2}\left(-\frac{\pi}{21}\right)$$

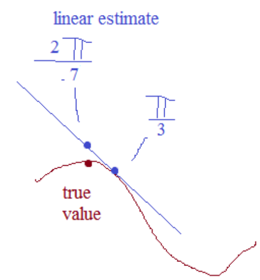
$$y = \frac{\sqrt{3}\,\pi}{42} + \frac{1}{2}$$ approx: .62955

True value:

$\cos\left(\frac{2\pi}{7}\right)$ approx: .62351

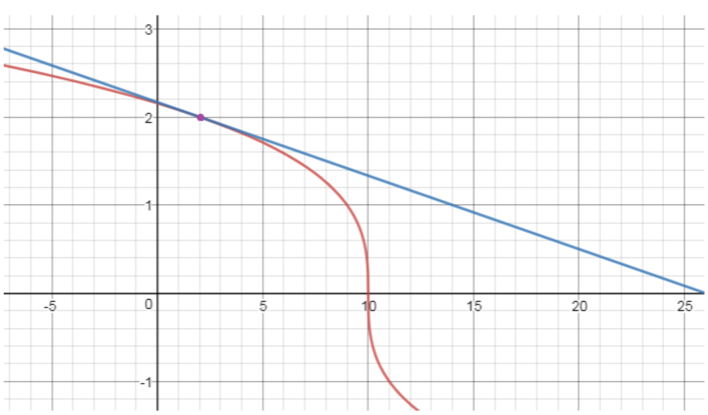


Since the line is above the curve when $x = \frac{2\pi}{7}$ we'd expect our estimate to be greater than the true value!



Example: $f(x) = \sqrt[3]{10 - x}$ $x = 2$

Find approx. of $\sqrt[3]{8.3}$



point: (2, 2)

slope: $\frac{-1}{3}(10 - x)^{\frac{-2}{3}}$ at $x = 2$, ths slope is $\frac{-1}{12}$

Linearized model: $y - 2 = \frac{-1}{12}(x - 2)$

To approximate, let x = 1.7...

true value is approx. 2.0247

linear approximation: 2.025

Since the curve is concave down,
the linear approximation overestimates!

Finding derivatives of inverses and slope of inverses

Method 1: (If possible), find the inverse and take the derivative...

Example: $f(x) = x^3 + 7$

If $f^{-1}(x)$ is the inverse, find the slope of the curve at $f^{-1}(15)$

Find the inverse...

$y = x^3 + 7$ switch the x's and y's...

$x = y^3 + 7$ solve for y...

$x - 7 = y^3$

$\sqrt[3]{x-7} = y$

Take the derivative...

$f^{-1}(x) = \sqrt[3]{x-7}$

The derivative is $\frac{1}{3}(x-7)^{-2/3}$

So, the slope at 15 is

$\frac{1}{3}(15-7)^{-2/3} = \frac{1}{12}$

Unfortunately, sometimes it's difficult to find the inverse.

Method 2: $g'(x) = \dfrac{1}{f'(g(x))}$

Example: $f(x) = x^3 + 3x + 6$

If $g(x)$ is the inverse, find $g'(2)$

NOTE: Since $f(x)$ and $g(x)$ are inverses, their coordinates are flipped...

$f(x)$ has a coordinate (? , 2)
and
$g(x)$ has a coordinate (2, ?)

What is $g(2)$?

$2 = x^3 + 3x + 6$

$x^3 + 3x + 4 = 0$

(Using a calculator or Rational Root/Factor Theorem, we can find that (-1) is a solution!)

Possible rational roots: 1, 2, 4, -1, -2, -4

$(-1)^3 + 3(-1) + 4 = 0$ ✓

The relevant coordinate is (-1, 2) $f(x)$
(2, -1) $g(x)$

The slope of tangent at (-1, 2) is

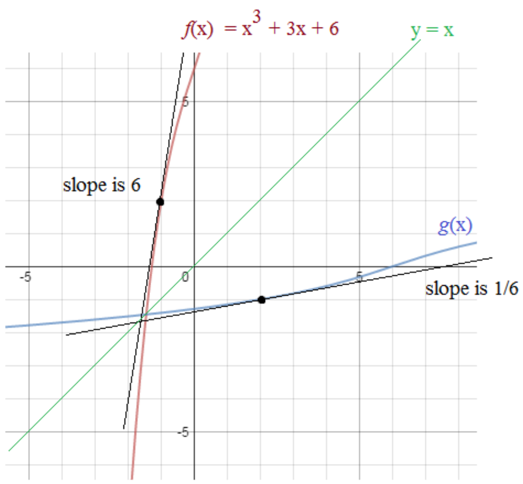
$3(-1)^2 + 3(-1) + 6 = 6 \longleftarrow f'(g(x))$

***Therefore the slope of tangent at (2, -1) must be $\frac{1}{6}$

$g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(-1)} = \dfrac{1}{6}$

NOTE: inverse functions reflect over the line y = x... Therefore, the slopes of mirror points are reciprocals!

Graph of $f(x)$ and $g(x)$...
Note the symmetry/reflection over y = x



Method 3: Flip the x and y; Use implicit differentiation

$x = y^3 + 3y + 6$

$1 = 3y^2\frac{dy}{dx} + 3\frac{dy}{dx} + 0$

$1 = \frac{dy}{dx}(3y^2 + 3)$

$\dfrac{dy}{dx} = \dfrac{1}{3y^2 + 3}$

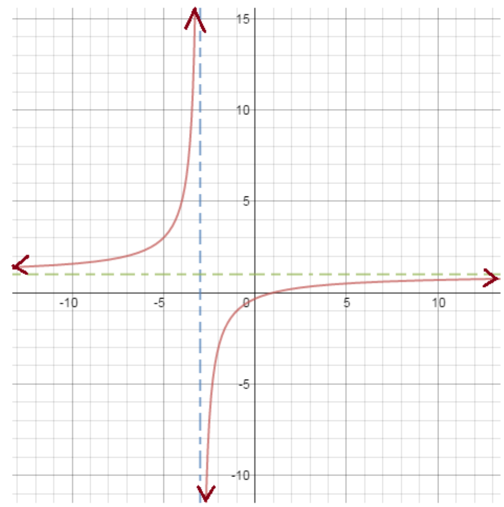
at (2, -1), the slope is $\frac{1}{6}$

Derivative Application: Sketching Rational Expressions

Example: $f(x) = \frac{x-1}{x+3}$

Use algebra concepts to graph the function.
Then, use calculus concepts to verify the shape!

Vertical Asymptotes: $x = -3$	Where the function is undefined. (denominator = 0 and numerator $\neq$ 0)
Horizontal Asymptote: $y = 1$	End behavior (since degree of numerator is same as degree of denominator, use the lead coefficients)
"Holes": None	
x-intercept: (1, 0)	x-intercept: when y = 0
y-intercept: $(0, \frac{-1}{3})$	y-intercept: when x = 0



Now, let's use derivatives to verify the shape:

Use quotient rule

$$f'(x) = \frac{(1)(x+3) - (1)(x-1)}{(x+3)^2} = \frac{2}{(x+3)^2}$$

Critical values: Since $f'(x)$ never equals 0, there are no extrema (no max or min)

The derivative $f'(x)$ is undefined at x = -3: an asymptote

Test -4: $\frac{2}{(-4+3)^2} > 0$ increasing in the interval $(-\infty, -3)$

Test -2: $\frac{2}{(-2+3)^2} > 0$ increasing in the interval $(-3, \infty)$

$$f''(x) = \frac{(0)(x+3)^2 - 2(x+3)^1 \cdot (2)}{\left((x+3)^2\right)^2} = \frac{-4(x+3)}{(x+3)^4} = \frac{-4}{(x+3)^3}$$

Concavity: Since $f''(x)$ never equals 0, there are no points of inflection

The second derivative has a critical value at x = -3 (where it is undefined)

Test -4: $\frac{-4}{(-4+3)^3} > 0$ concave up between $(-\infty, -3)$

Test -2: $\frac{-4}{(-2+3)^3} < 0$ concave down between $(-3, \infty)$

Example: $g(x) = \frac{x^2-2}{x+1}$ Find the relative extrema.
Determine the concavity and inflection points (if any).

Sketch the graph.

$$g'(x) = \frac{2x(x+1) - 1(x^2-2)}{(x+1)^2} = \frac{x^2+2x+2}{(x+1)^2}$$

g'(x) will never equal zero,
so no max or min...

Note: the discriminant of $x^2 + 2x + 2$ is -4...
So, the two solutions are imaginary!

There is a critical value at -1, where the derivative is undefined...

-2	-1	0
+	undef.	+

Then, test a point on left and right of -1... increasing for all x, except -1

$$g''(x) = \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x+2)}{(x+1)^4}$$

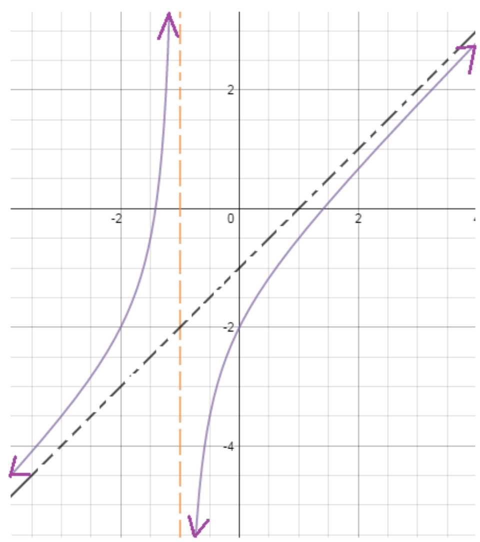
$$= \frac{(2x+2)(x+1) - 2(x^2+2x+2)}{(x+1)^3} = \frac{-2}{(x+1)^3}$$

g''(x) never equals zero, so there are no points of inflection.
But, the 2nd derivative is undefined at x = -1

-2	-1	0
+	undef.	–

After testing points on the left and right of -1, we find its concavity...

concave up: $(-\infty, -1)$ concave down: $(-1, \infty)$



Vertical Asymptote: x = -1

Horizontal Asymptote: None

"Slant" (Oblique) Asymptote: $y = x - 1$

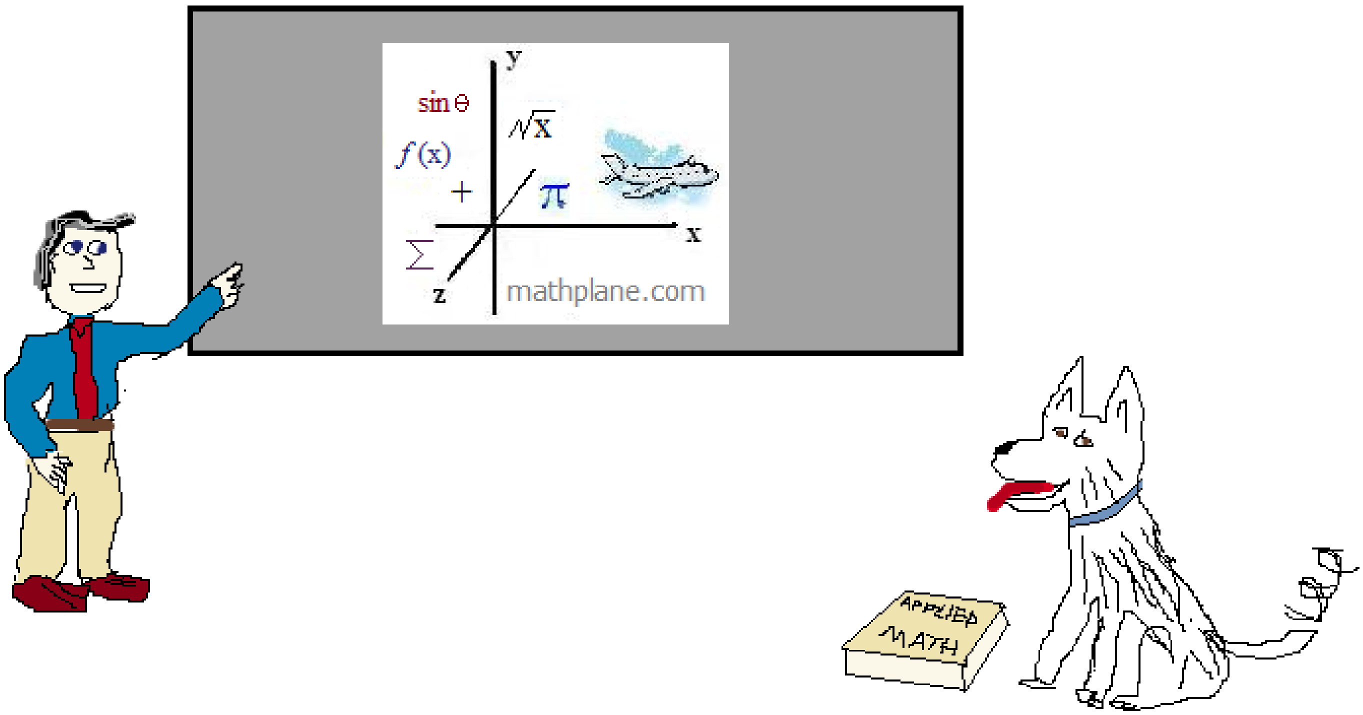
x-intercepts: $(-\sqrt{2}, 0)$ $(\sqrt{2}, 0)$

y-intercept: (0, -2)

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

Cheers

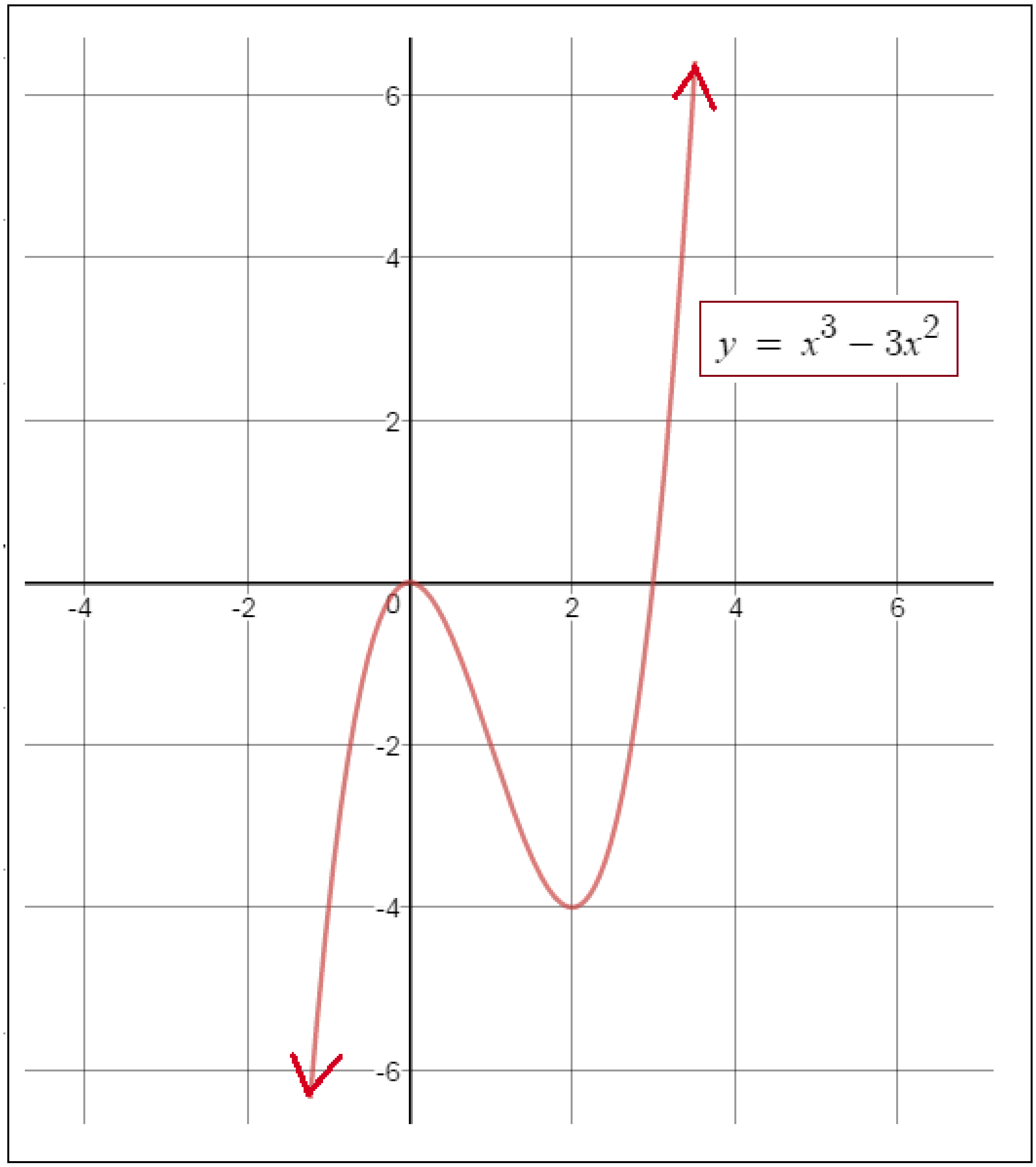


And, Mathplane *Express* for mobile at mathplane.ORG

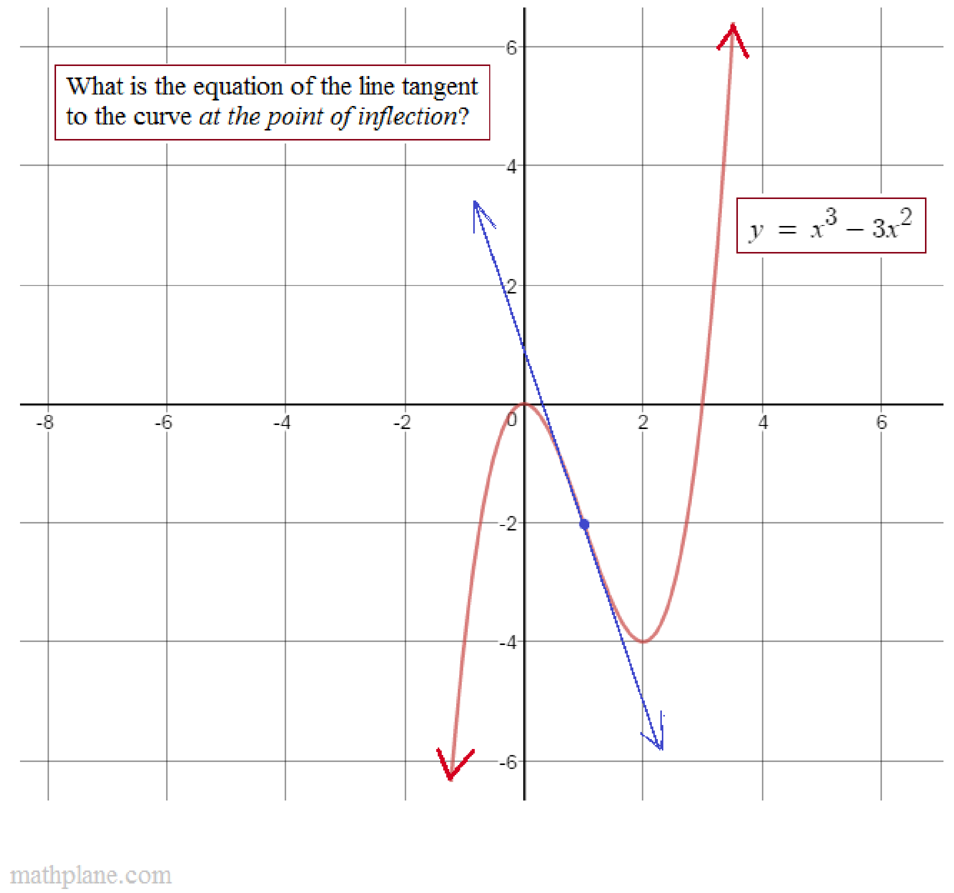
Also, at Facebook, Google+, Pinterest, TES, and TeachersPayTeachers

One more question-→

What is the <u>equation of the line</u> tangent to the curve *at the point of inflection*?



ANSWER-→



ANSWER

First, where is the point of inflection?

Where 2nd derivative equals zero.

$y' = 3x^2 - 6x$

$y'' = 6x - 6$

$y'' = 0$ when $x = 1$

Therefore, point of inflection is (1, -2)

$-2 = (1)^3 - 3(1)^2$

Now, find the slope at $x = 1$

$y' = 3x^2 - 6x$

$y' = 3(1)^2 - 6(1) = -3$

Equation of the line:

slope: -3 point: (1, -2)

$$\boxed{\begin{array}{c} y + 2 = -3(x - 1) \\ \text{or} \\ y = -3x + 1 \end{array}}$$