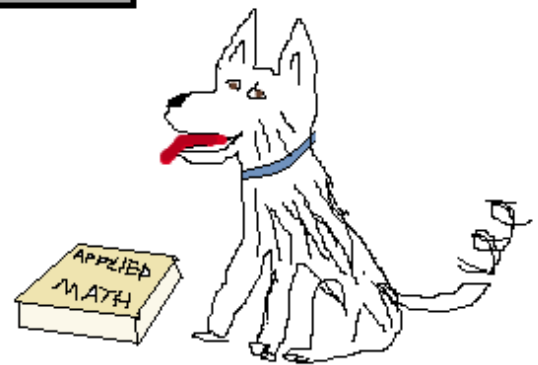
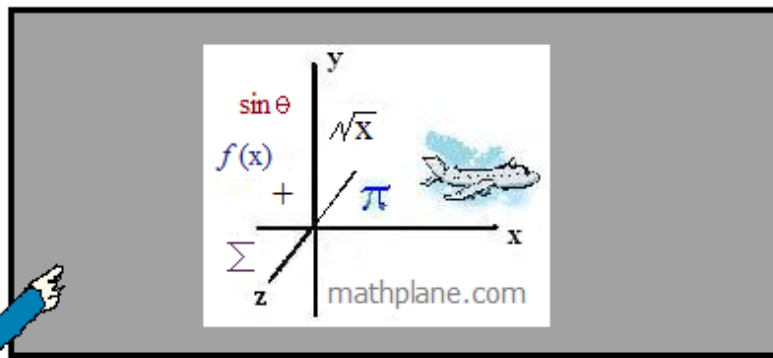


Additional Derivatives: Trigonometry, Logarithms, and Exponents

Formulas, Examples, and Practice Exercises (with Solutions)



Natural Logarithms

Definitions:

$$\frac{d}{dx} [\ln x] = \frac{1}{x}$$

If u is a differentiable function of x , then

$$\frac{d}{dx} [\ln u] = \frac{1}{u} \frac{du}{dx}$$

What does it mean? The derivative of a natural logarithm is $\frac{\text{'the derivative of the function'}}{\text{'the function'}}$

Examples: $f(x) = \ln(2x^2 + 4)$

$$u = 2x^2 + 4$$

(using the above definition)

$$\frac{du}{dx} = 4x$$

Note:

$$f'(x) = \frac{\text{'Derivative of function'}}{\text{'original function'}} = \frac{4x}{(2x^2 + 4)} = \frac{2x}{x^2 + 2}$$

$$\begin{aligned} \frac{d}{dx} [\ln(2x^2 + 4)] &= \frac{1}{2x^2 + 4} (4x) \\ &= \frac{2x}{x^2 + 2} \end{aligned}$$

$$g(x) = \ln \sqrt{x+1}$$

"original function": $\sqrt{x+1}$

"derivative of the function": $\frac{1}{2} (x+1)^{-1/2} = \frac{1}{2\sqrt{x+1}}$

$$g'(x) = \frac{\text{'derivative'}}{\text{'original'}} = \frac{\frac{1}{2\sqrt{x+1}}}{\sqrt{x+1}} = \frac{1}{2(x+1)}$$

Trig Functions

Definitions:

$$\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$$

$$\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$$

$$\frac{d}{dx} \csc u = -\csc u (\cot u) \frac{du}{dx}$$

$$\frac{d}{dx} \sec u = \sec u (\tan u) \frac{du}{dx}$$

$$\frac{d}{dx} \cot u = -\csc^2 u \frac{du}{dx}$$

Examples:

$$f(x) = \cos 3x^2$$

u "function" = $3x^2$

$$f'(x) = -\sin(3x^2)(6x) = -6x \sin(3x^2)$$

$\frac{du}{dx}$ "derivative of the function" = $6x$

Derivatives: Absolute Value and Exponents

Absolute Value Rule

Definition:

If $y = |u|$, where u is a differentiable function of x , then

$$\frac{d}{dx}[|u|] = \frac{u}{|u|} \frac{du}{dx}$$

wherever $u(x) \neq 0$

What does it mean?

If you have an equation inside an absolute value, then the derivative is:

$$\frac{\text{equation without the absolute value}}{\text{equation with the absolute value}} \quad (\text{derivative of equation})$$

(and, no zeros in the denominator)

Example: $f(x) = |4 - x^2|$

$$f'(x) = \frac{4 - x^2}{|4 - x^2|} (-2x)$$

$$f'(x) = (1)(-2x) = -2x \quad \text{when } 4 - x^2 > 0$$

$$f'(x) = (-1)(-2x) = 2x \quad \text{when } 4 - x^2 < 0$$

And, the derivative does not exist at $x = \pm 2$

Exponential Functions

Definitions:

$$\begin{aligned} \frac{d}{dx}[e^x] &= e^x & \frac{d}{dx}[a^x] &= a^x (\ln a) \\ \frac{d}{dx}[e^u] &= e^u \frac{du}{dx} & \frac{d}{dx}[a^u] &= a^u (\ln a) \frac{du}{dx} \end{aligned}$$

What does it mean?

The derivative of 'e to an exponent' is "e to the exponent times the derivative of the exponent."

The derivative of 'number to an exponent' is "number to an exponent times $\ln(\text{number})$ times the derivative of the exponent."

Examples:

$$f(x) = e^{3x-1}$$

$$f'(x) = e^{3x-1} (3) = 3e^{3x-1}$$

$$g(x) = 10^{x^2}$$

$$g'(x) = 10^{x^2} \ln(10) (2x)$$

Reminder: $e \approx 2.718$
 $\ln(e) = \log_e e = 1$
 $\ln(4) = \log_e 4 = (\text{approx.}) 1.386$

Derivatives of Trig Functions

Examples:

$$g(x) = \sin \sqrt{2x}$$

function: $\sqrt{2x}$

derivative of the function: $\frac{1}{2} (2x)^{-\frac{1}{2}} (2) = \frac{1}{\sqrt{2x}}$

$$g'(x) = \cos \sqrt{2x} \left(\frac{1}{\sqrt{2x}} \right) = \frac{\cos \sqrt{2x}}{\sqrt{2x}}$$

$$h(x) = \sec^3(2x)$$

Let $u = 2x$

$u' = 2$

$$h(x) = \sec(u)^3$$

Note: according to the power rule, derivative of $[\sec(u)]^3$

$$h'(x) = 3\sec(u)^2 \cdot \sec(u)\tan(u)u'$$

is $3[\sec(u)]^2 \cdot$ ("derivative of $\sec(u)$ ")

$$= 3\sec(2x)^2 \cdot \sec(2x)\tan(2x)(2)$$

$$= 6\tan(2x)\sec^3(2x)$$

Find the relative extrema of $f(x) = \sin 2x + 4$ on the interval $[0, 2\pi]$

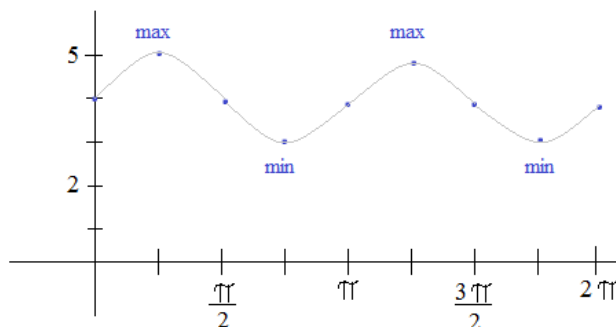
Find $f'(x)$: $\cos(2x)(2) + 0 = 2\cos(2x)$

To find the max/min, set $f'(x) = 0$

$$2\cos(2x) = 0$$

$$\cos(2x) = 0$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$



Also, $f''(x) = 0$ will help identify concavity and points of inflection.

Find derivative of $2\cos(2x)$: $2 \cdot [-\sin(2x)(2)] = -4\sin(2x)$

To find points of inflection, set second derivative equal to zero.

$$-4\sin(2x) = 0$$

$$\sin(2x) = 0 \quad \text{points of inflection at } x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

Derivatives: Natural Logarithms

Example: $y = xe^2 - e^x$ Find y'

(product rule)

$$y' = (1)e^2 + (0)e^2(x) - e^x$$

$$y' = e^2 - e^x$$

derivative of $e^u =$ "derivative of u " $\cdot e^u$

Example: $y = \ln \frac{1}{x}$ Find dy/dx

method 1:

$$\frac{dy}{dx} = \frac{-1(x)^{-2}}{\frac{1}{x}} = \frac{-x}{-x^2} = \frac{-1}{x}$$

derivative of \ln : $\frac{\text{"derivative of equation"}}{\text{"equation"}}$

method 2: (applying log rules)

$$y = \ln(x)^{-1}$$

$$y = -\ln(x) \quad (\text{logarithm power rule})$$

$$\frac{dy}{dx} = \frac{-1}{x}$$

Example: "The speed of a rumor"

A rumor can be modeled by the function $p(t) = \frac{340}{1 + 2^{4-t}}$

where t is time (hours)
and $p(t)$ is the number of people who
know about the rumor

a) How many students originally heard the rumor?

$$p(0) = \frac{340}{1 + 2^{4-0}} = \frac{340}{17} = 20 \text{ students}$$

t	0	1	2	3	4	5	6
$p(t)$	20	38	68	113	170	227	272

b) How fast is the rumor spreading after 4 hours?

Find $p'(t)$ to find the rate of change.

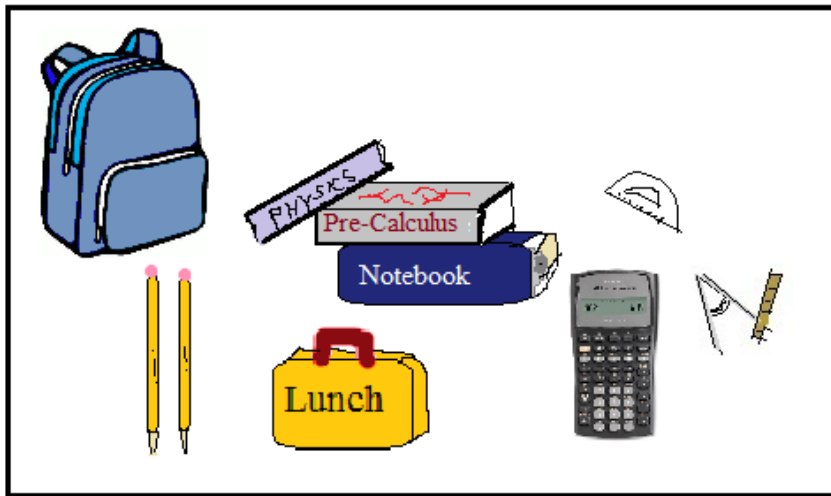
Using Quotient rule:

$$p'(t) = \frac{0 - 2^{4-t} (\ln 2)(-1)(340)}{(1 + 2^{4-t})^2} \quad \text{Then, } p'(4) = \frac{0 - 2^0 (\ln 2)(-1)(340)}{(1 + 2^0)^2} = \frac{(\ln 2)(340)}{4} = 58.9$$

Approximately, 59 students/hour

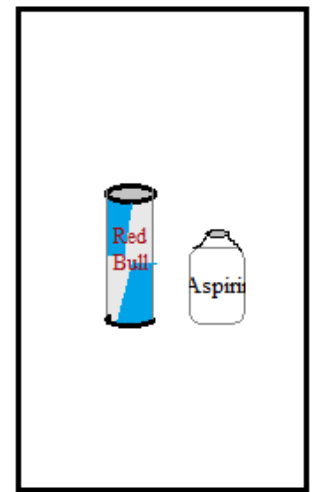
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Practice Exercises →

Differentiation: Trigonometry, logarithm, and exponent functions

Find the 1st derivative of each function.

I. Trigonometry:

a) $f(x) = 2x(\cos x)$

b) $y = \tan^4(3 + 4x)$

c) $g(x) = \frac{1 - \sin x}{1 + \sin x}$

II. Logarithms:

a) $y = \ln(x^2)$

b) $h(x) = \frac{\ln x}{x^2}$

c) $f(x) = \ln \sqrt{x^2 - 4}$

III. Exponents:

a) $y = x^2 e^x$

b) $y = 3^{x-1}$

c) $f(x) = 4e^{2x}$

IV: More Questions

Differentiation: Trigonometry, logarithm, and exponent functions

a) Find the 1st and 2nd derivatives of the following:

$$\sin^2 x$$

$$\sin(x^2)$$

$$\sin \sqrt{1+x^2}$$

b) $f(x) = 2\cos^3\left(\frac{3x}{4} + \frac{\pi}{12}\right)$

$$f'\left(\frac{11\pi}{9}\right) =$$

c) $y = e^x(\sin x + \cos x)$

$$\frac{dy}{dx} =$$

d) If $y = (\cos x)(\sin x)$, what is the slope of the graph at $\frac{\pi}{3}$

e) $y = x^n + n^x$

find dy/dx

f) Find the equation of the line tangent to $y = \frac{e^x}{2x+1}$ @ $x=0$

g) Find the equation of the line tangent to $f(x) = \cos^2(2x^3)$ @ $x = \frac{\pi}{3}$

h) $y = \sin(x^2)\cos(y)$ $dy/dx =$

i) $y = \sec^2(5x)$ $y' =$

j) $y = 5^{(2x-3)}$ $dy/dx =$

k) $y = \ln(\cos 2x)$ find the first derivative.

Answers

I. Trigonometry:

a) $f(x) = 2x(\cos x)$

(product rule) $u = 2x$
 $u' = 2$
 $v = \cos x$
 $v' = -\sin x$

$$f'(x) = 2 \cdot (\cos x) + (-\sin x) \cdot 2x$$

$$= 2\cos x - 2x(\sin x)$$

$$= \boxed{2(\cos x - x\sin x)}$$

b) $y = \tan^4(3 + 4x)$

$$= [\tan(3 + 4x)]^4$$

(power rule) $u = \tan(3 + 4x)$
 $u' = \sec^2(3 + 4x)(4)$

$$\frac{dy}{dx} = 4\tan^3(3 + 4x) \cdot [\sec^2(3 + 4x)(4)]$$

$$= \boxed{16\tan^3(3 + 4x) \sec^2(3 + 4x)}$$

c) $g(x) = \frac{1 - \sin x}{1 + \sin x}$

(quotient rule) $u = 1 - \sin x$
 $u' = -\cos x$
 $v = 1 + \sin x$
 $v' = \cos x$

$$g'(x) = \frac{(-\cos x)(1 + \sin x) - (\cos x)(1 - \sin x)}{(1 + \sin x)^2}$$

$$= \frac{-\cos x - \cos x \sin x - \cos x + \cos x \sin x}{(1 + \sin x)^2}$$

$$= \boxed{\frac{-2\cos x}{(1 + \sin x)^2}}$$

II. Logarithms:

a) $y = \ln(x^2)$

derivative of natural log: $\frac{\text{"derivative of function"}}{\text{"original function"}}$

$$u = x^2$$

$$u' = 2x$$

$$y' = \frac{2x}{x^2} = \boxed{\frac{2}{x}}$$

b) $h(x) = \frac{\ln x}{x^2}$

(quotient rule) $u = \ln x$
 $u' = 1/x$
 $v = x^2$
 $v' = 2x$

$$h'(x) = \frac{\frac{1}{x} \cdot x^2 - 2x(\ln x)}{(x^2)^2}$$

$$= \frac{x - 2x(\ln x)}{x^4}$$

$$= \boxed{\frac{1 - 2(\ln x)}{x^3}}$$

c) $f(x) = \ln \sqrt{x^2 - 4}$

$$u = \sqrt{x^2 - 4}$$

$$\frac{du}{dx} = \frac{1}{2} (x^2 - 4)^{-\frac{1}{2}} (2x) = \frac{x}{\sqrt{x^2 - 4}}$$

$$f'(x) = \frac{\frac{x}{\sqrt{x^2 - 4}}}{\sqrt{x^2 - 4}} = \boxed{\frac{x}{x^2 - 4}}$$

III. Exponents:

a) $y = x^2 e^x$

(product rule)

$$u = x^2$$

$$u' = 2x$$

$$v = e^x$$

$$v' = e^x$$

$$\frac{dy}{dx} = 2x(e^x) + e^x(x^2)$$

$$= \boxed{x e^x (x + 2)}$$

b) $y = 3^{x-1}$

$$a = 3$$

$$u = x - 1$$

$$u' = 1$$

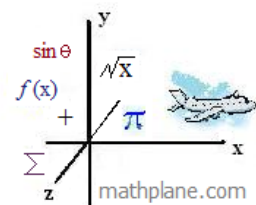
$$y' = 3^{x-1} (\ln 3)(1)$$

$$= \boxed{\ln 3 (3^{x-1})}$$

c) $f(x) = 4e^{2x}$

$$f'(x) = 4 [e^{2x}] (2)$$

$$= \boxed{8e^{2x}}$$



IV: More Questions

SOLUTIONS

Differentiation: Trigonometry, logarithm, and exponent functions

a) Find the 1st and 2nd derivatives of the following:

$$\sin^2 x$$

$$\begin{aligned} & (\sin x)^2 \\ y' &= 2(\sin x)^1 \cdot \cos x \\ &= \sin 2x \\ y'' &= 2\cos 2x \end{aligned}$$

also, $y' = 2\sin x \cos x$

$$\begin{aligned} y'' &= 2[\cos x \cos x + (-\sin x) \sin x] \\ &= 2[\cos^2 x - \sin^2 x] \\ &= 2\cos 2x \end{aligned}$$

$$\sin(x^2)$$

$$\begin{aligned} y' &= \cos(x^2) \cdot 2x \\ &= 2x \cos(x^2) \\ y'' &= 2 \cdot \cos(x^2) + -\sin(x^2) 2x \cdot 2x \\ &= 2 \cos(x^2) - 4x^2 \sin(x^2) \end{aligned}$$

$$\sin \sqrt{1+x^2}$$

$$\begin{aligned} y' &= \cos(\sqrt{1+x^2}) \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x \\ &= \frac{x \cos(\sqrt{1+x^2})}{\sqrt{1+x^2}} \\ y'' &= 1 \cdot \cos(\sqrt{1+x^2}) - x \cdot \left(-\sin(\sqrt{1+x^2}) \cdot \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \cdot 2x \right) \\ &= \frac{\cos(\sqrt{1+x^2}) + x^2 \sin(\sqrt{1+x^2}) (1+x^2)^{-\frac{1}{2}}}{1+x^2} \\ &= \frac{\cos(\sqrt{1+x^2}) + \frac{x^2 \sin(\sqrt{1+x^2})}{\sqrt{1+x^2}}}{1+x^2} \end{aligned}$$

b) $f(x) = 2\cos^3\left(\frac{3x}{4} + \frac{\pi}{12}\right)$

$$\begin{aligned} f'\left(\frac{11\pi}{9}\right) &= \\ f(x) &= 2\left[\cos\left(\frac{3x}{4} + \frac{\pi}{12}\right)\right]^3 \\ f'(x) &= 6\cos^2\left(\frac{3x}{4} + \frac{\pi}{12}\right) \cdot \left[-\sin\left(\frac{3x}{4} + \frac{\pi}{12}\right)\right] \cdot \frac{3}{4} \\ f'\left(\frac{11\pi}{9}\right) &= 6\cos^2\left(\frac{33\pi}{36} + \frac{\pi}{12}\right) \cdot \left[-\sin\left(\frac{33\pi}{36} + \frac{\pi}{12}\right)\right] \cdot \frac{3}{4} \\ &= 6 \cdot (-1)^2 \cdot 0 \cdot \frac{3}{4} = 0 \end{aligned}$$

d) If $y = (\cos x)(\sin x)$, what is the slope of the graph at $\frac{\pi}{3}$

$$\begin{aligned} y' &= (-\sin x)(\sin x) + (\cos x)(\cos x) \\ & \text{(product rule)} \end{aligned}$$

at $\frac{\pi}{3}$ slope is $(-\sin \frac{\pi}{3})(\sin \frac{\pi}{3}) + (\cos \frac{\pi}{3})(\cos \frac{\pi}{3})$

$$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1$$

c) $y = e^x(\sin x + \cos x)$

Use the product rule:

$$\begin{aligned} \frac{dy}{dx} &= \\ y' &= e^x(\sin x + \cos x) + e^x(\cos x + (-\sin x)) \\ &= u' \cdot v + u \cdot v' \\ &= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x \\ &= 2e^x \cos x \end{aligned}$$

e) $y = x^n + n^x$

$$\frac{dy}{dx} = n(x)^{n-1} + \ln(n)n^x$$

find dy/dx

power rule logarithm/exponent rule

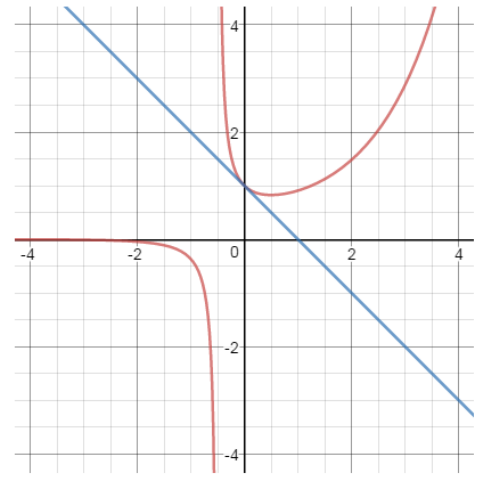
f) Find the equation of the line tangent to $y = \frac{e^x}{2x+1}$ @ $x=0$ SOLUTIONS

The coordinate of the point is $y = \frac{e^0}{2(0)+1} = \frac{1}{1} = 1$

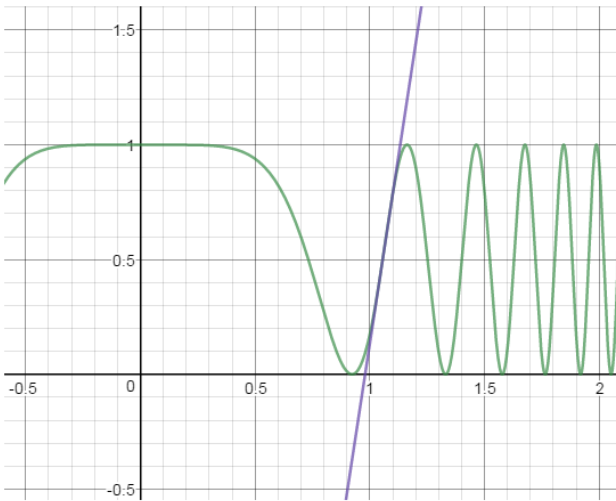
The slope at that point is

$$y' = \frac{e^x(2x+1) - 2(e^x)}{(2x+1)^2} \quad @ x=0 \quad \Rightarrow \quad \frac{1-2}{1} = -1$$

$$y - 1 = -1(x - 0) \quad \Rightarrow \quad \boxed{y = -x + 1}$$



g) Find the equation of the line tangent to $f(x) = \cos^2(2x^3)$ @ $x = \frac{\pi}{3}$



$$[\cos(2x^3)]^2$$

$$2[\cos(2x^3)]^1 \cdot -\sin(2x^3) \cdot 6x^2$$

at $x = \frac{\pi}{3}$ $f'(x) = 6.533$

and, at $\frac{\pi}{3}$ $f(x) = .441$

slope is 6.533

the point is $(\frac{\pi}{3}, .441)$

$$\boxed{y - .441 = 6.533(x - \frac{\pi}{3})}$$

h) $y = \sin(x^2)\cos(y)$ $dy/dx =$

$$(1) \frac{dy}{dx} = \cos(x^2) \cdot 2x \cdot \cos(y) + (-\sin(y) \frac{dy}{dx}) \cdot \sin(x^2)$$

product rule
chain rule
implicit differentiation
trig derivatives

$$\frac{dy}{dx} + \sin(x^2)\sin(y) \frac{dy}{dx} = \cos(x^2) \cdot 2x \cdot \cos(y)$$

$$\frac{dy}{dx} = \frac{\cos(x^2) \cdot 2x \cdot \cos(y)}{1 + \sin(x^2)\sin(y)}$$

j) $y = 5^{(2x-3)}$ $dy/dx =$

$$\boxed{\frac{dy}{dx} = \ln 5 \cdot 5^{(2x-3)} \cdot 2}$$

"ln of the base" "itself" "derivative of the exponent"

i) $y = \sec^2(5x)$ $y' =$

$$2\sec(5x) \cdot \sec(5x)\tan(5x) \cdot 5$$

power rule
chain rule
trig derivatives

$$\boxed{10\sec^2(5x)\tan(5x)}$$

k) $y = \ln(\cos 2x)$ find the first derivative.

$$y' = \frac{\text{"derivative"} \cdot \text{"itself"}}{\text{"itself"}} = \frac{-\sin(2x) \cdot 2}{\cos(2x)} = \boxed{-2\tan(2x)}$$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers

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