Partial Fractions

Notes, Examples, & Practice Questions (with Answers)

Topics include decomposition, linear systems, factoring, long division, and more.

When a rational expression is composed of polynomials, it can be helpful to “break them apart” into separate, smaller fractions. (This is especially useful in Integration, in Calculus). Here is an introduction to partial fraction decomposition.

Mathplane.com

Partial Fractions

What are they? "Simpler Proper Fractions" or "Reduced Rational Expression"

Suppose you want to add two rational expressions...

Example: $\frac{2x}{x^2+5} + \frac{3}{x+4}$

Find the common denominator...

$$\frac{2x}{x^2+5}\cdot\frac{x+4}{x+4} + \frac{3}{x+4}\cdot\frac{x^2+5}{x^2+5} = \frac{2x^2+8x}{(x^2+5)(x+4)} + \frac{3x^2+15}{(x^2+5)(x+4)} =$$

Combine like terms and expand denominator...

$$\frac{5x^2+8x+15}{(x^2+5)(x+4)} = \boxed{\frac{5x^2+8x+15}{x^3+4x^2+5x+20}}$$

Now, suppose you want to reverse the above example.
How do you decompose the 'large rational expression' back into 'partial fractions'?

Example: Decompose $\frac{5x^2+8x+15}{x^3+4x^2+5x+20}$ into partial fractions...

Step 1: Factor the denominator

$$\frac{5x^2+8x+15}{x^3+4x^2+5x+20} = \frac{5x^2+8x+15}{(x^2+5)(x+4)}$$

$x^3+4x^2+5x+20$

factor by grouping...

$x^2(x+4) + 5(x+4)$

$(x^2+5)(x+4)$

Step 2: Split the denominator factors,
and place variables in the numerators...

$$\frac{5x^2+8x+15}{(x^2+5)(x+4)} = \frac{Ax+B}{(x^2+5)} + \frac{C}{(x+4)}$$

Note: the numerator degrees are 1 less than the denominator degrees!

Step 3: Combine the split expressions...

$$\frac{5x^2+8x+15}{(x^2+5)(x+4)} = \frac{(Ax+B)(x+4) + C(x^2+5)}{(x^2+5)(x+4)}$$

Step 4: Drop the denominators, and solve...

$$5x^2+8x+15 = (Ax+B)(x+4) + C(x^2+5)$$

Let $x = -4$: $5(-4)^2 + 8(-4) + 15 = 0 + C((-4)^2 + 5)$

this cancels A and B, helping us find C...

$63 = 21C$ $\boxed{C = 3}$

$$5x^2+8x+15 = (Ax+B)(x+4) + 3x^2+15$$

$$= Ax^2+4Ax+Bx+4B + 3x^2+15$$

Combine like terms to find A and B...

$$= (A+3)x^2 + (4A+B)x + (4B+15)$$

$$5x^2+8x+15$$

$4B+15 = 15$ $\boxed{B = 0}$

$A+3 = 5$ $\boxed{A = 2}$

Step 5: Replace the variables!

$$\frac{5x^2+8x+15}{(x^2+5)(x+4)} = \boxed{\frac{2x+0}{(x^2+5)} + \frac{3}{(x+4)}}$$

Partial Fractions

Example: $\dfrac{13x + 46}{12x^2 - 11x - 15}$

Step 1: Factor the denominator $\dfrac{13x + 46}{(4x + 3)(3x - 5)} =$

Step 2: Separate the terms $\dfrac{13x + 46}{(4x + 3)(3x - 5)} = \dfrac{A}{(4x + 3)} + \dfrac{B}{(3x - 5)}$

multiply both sides by $(4x + 3)(3x - 5)$

Step 3: Solve for A and B $13x + 46 = A(3x - 5) + B(4x + 3)$

To find A,

Let $x = -3/4$: $13(-3/4) + 46 = A(-9/4 - 5) + B(-3 + 3)$

("gets rid of the B term")

$\dfrac{-39}{4} + \dfrac{184}{4} = \dfrac{-29}{4} A$ $A = -5$

To find B,

Let $x = 5/3$: $13(5/3) + 46 = A(5 - 5) + B(20/3 + 3)$

("gets rid of the A term")

$\dfrac{65}{3} + \dfrac{138}{3} = \dfrac{29}{3} B$ $B = 7$

Step 4: Summarize (and check)

$$\frac{13x + 46}{12x^2 - 11x - 15} = \frac{-5}{(4x + 3)} + \frac{7}{(3x - 5)}$$

To check, rewrite with common denominators and add...

$$\frac{-5}{(4x + 3)} + \frac{7}{(3x - 5)} = \frac{-5(3x - 5)}{(4x + 3)(3x - 5)} + \frac{7(4x + 3)}{(3x - 5)(4x + 3)} = \frac{-15x + 25 + 28x + 21}{(3x - 5)(4x + 3)}$$

$$= \frac{13x + 46}{12x^2 - 11x - 15}$$ ✓

Example: Separate the following rational expression into partial fractions:

Partial Fractions

$$\frac{-4x^2 - 2x + 10}{(3x+5)(x+1)^2}$$

Step 1: Separate the denominator..

NOTE: since (x + 1) has a multiplicity of 2, it is represented twice

$$\frac{-4x^2 - 2x + 10}{(3x+5)(x+1)^2} = \frac{A}{(3x+5)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

Step 2: Solve for the variables in the numerators

$$-4x^2 - 2x + 10 = A(x+1)^2 + B(3x+5)(x+1) + C(3x+5)$$

(To find C, let x = -1, eliminating A and B)

$$\text{If } x = -1, \quad -4(-1)^2 - 2(-1) + 10 = A(0) + B(2)(0) + C(2)$$

$$8 = 2C \qquad C = 4$$

(To find A, let x = -5/3, eliminating B and C)

$$\text{If } x = -5/3, \quad -4(25/9) + 10/3 + 10 = A(4/9) + B(0)(-2/3) + C(0)$$

$$\frac{20}{9} = \frac{4}{9}A \qquad A = 5$$

Since we know A = 5 and C = 4, let x = 0 and substitute!

$$\text{If } x = 0, \quad -4(0)^2 - 2(0) + 10 = 5(0+1) + B(0+5)(0+1) + 4(0+5)$$

$$10 = 5 + 5B + 20$$

$$B = -3$$

Step 3: Summarize (and check)

$$\frac{-4x^2 - 2x + 10}{(3x+5)(x+1)^2} = \frac{5}{(3x+5)} - \frac{3}{(x+1)} + \frac{4}{(x+1)^2}$$

$$\frac{5(x+1)^2}{(3x+5)(x+1)^2} - \frac{3(3x+5)(x+1)}{(x+1)(3x+5)(x+1)} + \frac{4(3x+5)}{(x+1)^2(3x+5)}$$

$$\frac{5x^2 + 10x + 5 \quad - 9x^2 - 24x - 15 \quad + \quad 12x + 20}{(3x+5)(x+1)^2} = \frac{-4x^2 - 2x + 10}{(3x+5)(x+1)^2} \checkmark$$

Partial Fraction Decomposition: Using Linear System

Example: $\dfrac{-6x^2+3x+5}{x^3-x}$

step 1: Factor the denominator

$$x^3-x = x(x^2-1) = x(x+1)(x-1)$$

step 2: 'decompose into partial fractions'

$$\frac{-6x^2+3x+5}{x^3-x} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x-1)}$$

step 3: common denominators and simplify

$$= \frac{(x+1)(x-1)A}{x(x+1)(x-1)} + \frac{x(x-1)B}{x(x+1)(x-1)} + \frac{x(x+1)C}{x(x+1)(x-1)}$$

$$-6x^2+3x+5 = (x+1)(x-1)A + x(x-1)B + x(x+1)C$$

$$Ax^2 - 1A \quad + \quad Bx^2 - Bx \quad + \quad Cx^2 + Cx$$

(regroup the terms)

$$(A+B+C)x^2 + (-B+C)x + A(-1)$$

$$-6x^2 \quad + \quad 3x \quad + 5$$

"Express method"

Let x = 0 (to eliminate B and C)

$0+0+5 = -1A + 0B + 0C$

$A = -5$

Let x = -1 (to eliminate A and C)

$-6 + (-3) + 5 = 0A + 2B + 0C$

$B = -2$

Let x = 1 (to eliminate A and B)

$-6 + 3 + 5 = 0A + 0B + 2C$

$C = 1$

step 4: solve the system

$$A + B + C = -6$$
$$-B + C = 3$$
$$-A = 5$$

$A = -5$

$-5 + B + C = -6$
$-B + C = 3$
$2C = 2$
$C = 1$

$-B + C = 3$
$-B + 1 = 3$
$B = -2$

Step 5: Express the Partial Fraction!

$$\frac{-6x^2+3x+5}{x^3-x} = \boxed{\frac{-5}{x} + \frac{-2}{(x+1)} + \frac{1}{(x-1)}}$$

Partial Fractions

Example: $\dfrac{x^2 - 2x - 2}{x^3 - 1}$

Factor the denominator, then decompose into partial fractions

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2+x+1)}$$

Solve the rational equation with common denominators...

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A\,(x^2+x+1)}{(x-1)\,(x^2+x+1)} + \frac{(Bx+C)\,(x-1)}{(x^2+x+1)\,(x-1)}$$

$$x^2 - 2x - 2 = A\,(x^2+x+1) + (Bx+C)(x-1)$$

$$x^2 - 2x - 2 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

"Collect Like Terms"

x^2	x^2	$= (A+B)x^2$
x	$-2x$	$= (A-B+C)x$
constant	-2	$= (A - C)$

Solve the system:

$A + B = 1$

$A - B + C = -2$

$A - C = -2$

$2A - B = -4$

$3A = -3$

$A = -1$

$B = 2$

$C = 1$

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2+x+1)}$$

$$= \boxed{\frac{-1}{(x-1)} + \frac{2x+1}{(x^2+x+1)}}$$

"Telescoping Series" and Partial Fractions

Example: Find the sum of the series:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

Change to a partial fraction:

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} \longrightarrow \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$$

$$\sum_{n=2}^{\infty} \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$$

$$\boxed{\frac{1}{2}} - \frac{1}{6} + \boxed{\frac{1}{4}} - \frac{1}{8} + \frac{1}{6} - \frac{1}{10} + \frac{1}{8} - \frac{1}{12} + \ldots$$

The sequence approaches zero as n goes to ∞

Note: this is a telescoping series -- its partial sums eventually have a fixed number of terms (after cancellations)

so, the series = $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

This technique of finding a partial sum is also known as using a 'method of differences'

Changing the equation to partial fractions:

$$\frac{A}{(n+1)} + \frac{B}{(n-1)} = \frac{1}{n^2-1}$$

$$\frac{A(n-1) + B(n+1)}{n^2-1 \quad\quad n^2-1} = \frac{1}{n^2-1}$$

$$A(n-1) + B(n+1) = 1$$

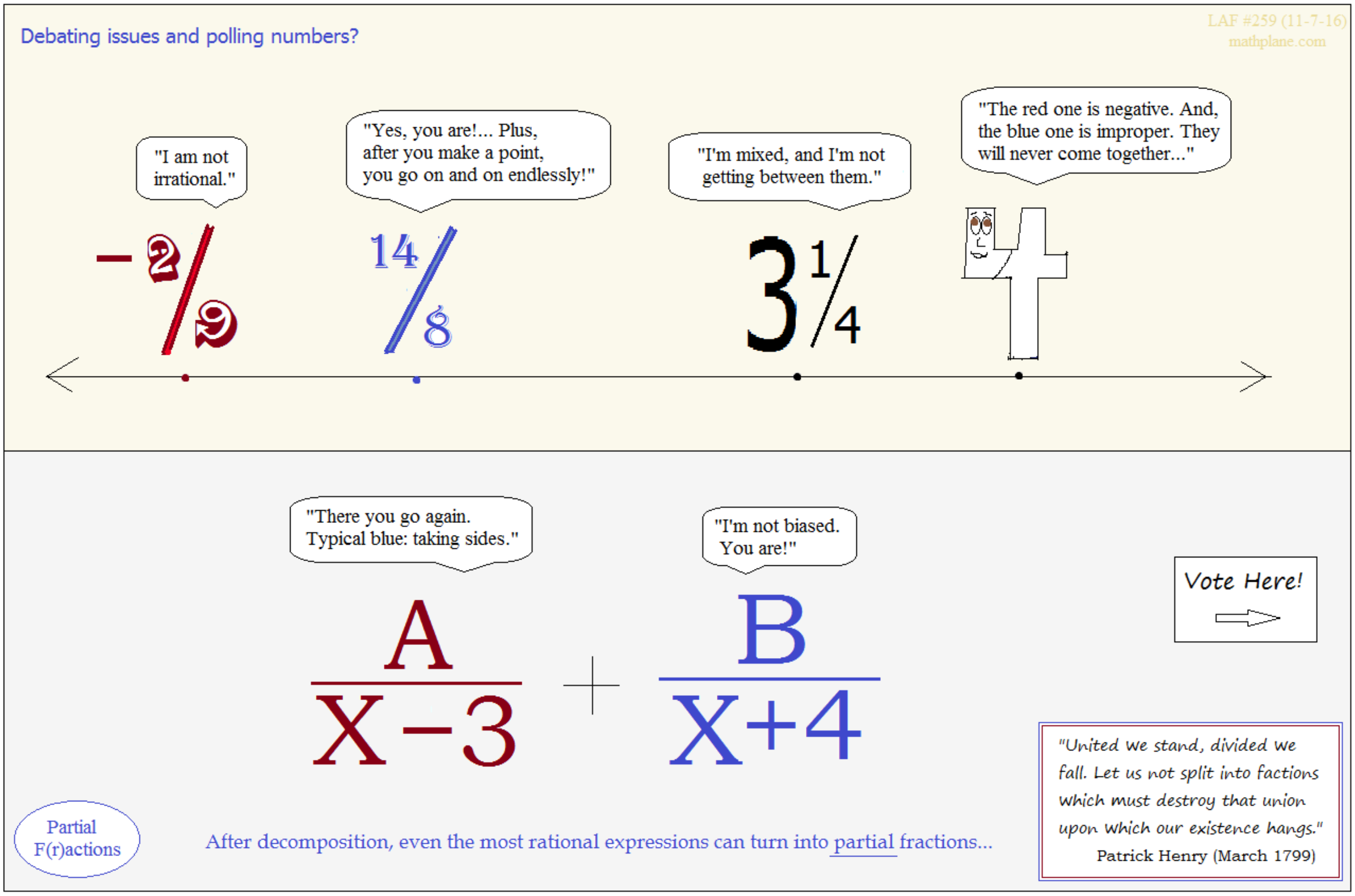
If $n = 1$, then $A(1-1) + B(1+1) = 1$

$$2B = 1$$
$$B = 1/2$$

If $n = -1$, then $A(-1-1) + B(-1+1) = 1$

$$-2A = 1$$
$$A = -1/2$$

$$\boxed{\frac{-1/2}{(n+1)} + \frac{1/2}{(n-1)} = \frac{1}{n^2-1}}$$



PRACTICE QUESTIONS-→

Find the partial fractions (using decomposition):

1) $\dfrac{4 - x}{(x + 2)(x + 4)}$

2) $\dfrac{10x^{3} - 15x^{2} - 35x}{x^{2} - x - 6}$

3) $\dfrac{8x + 5}{x^2 + 3x - 10}$

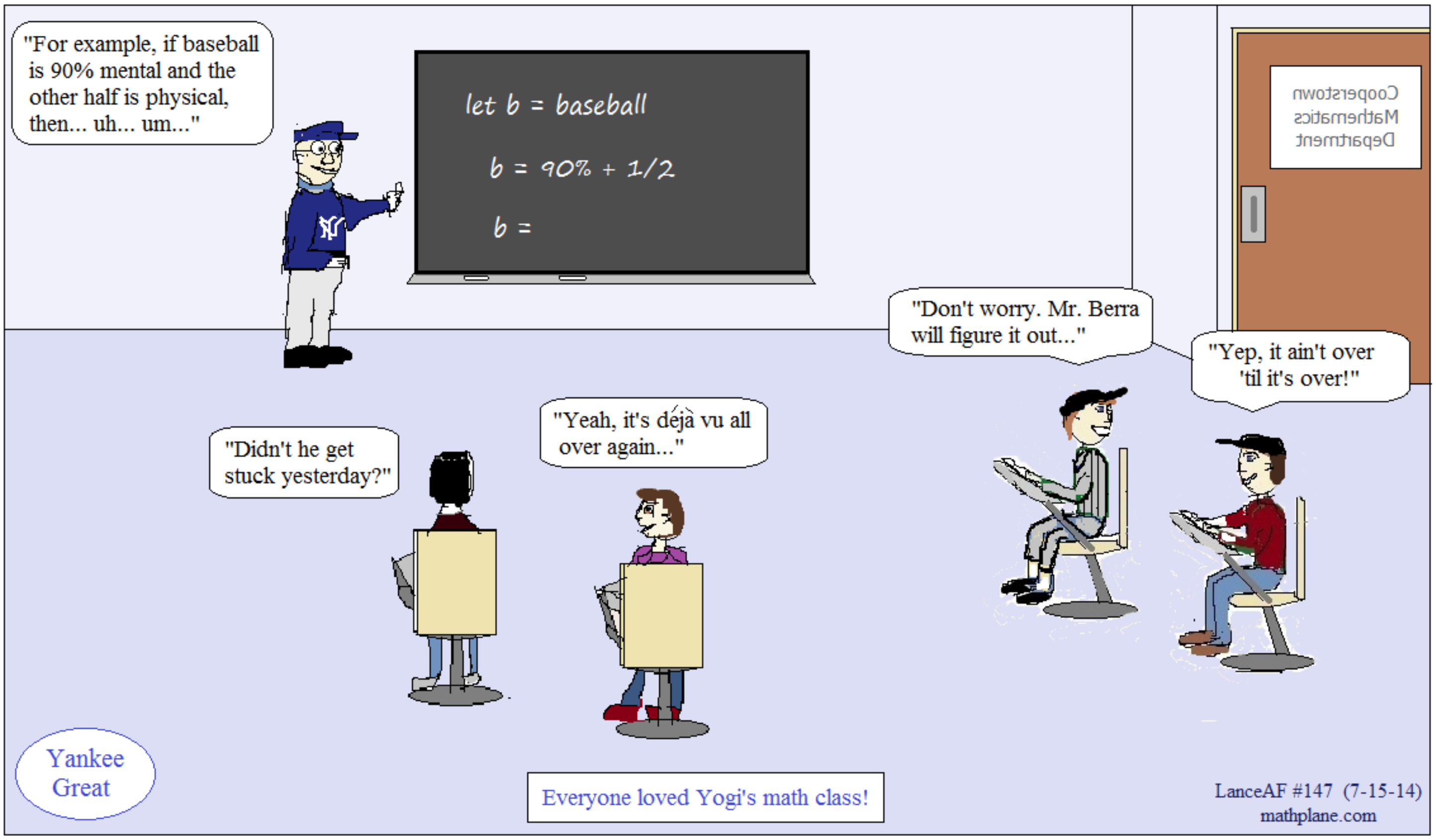
4) $\dfrac{x^2 + 6}{3x^3 + 4x^2 - 4x}$

5) $\frac{3x^2 - 2x + 12}{(x^2 + 4)^2}$

6) $\frac{5}{(x + 2)^2 (x + 1)}$

7) $\dfrac{13x^2 - 8x - 25}{2x^3 - 6x^2 - x + 3}$

8) $\dfrac{x^5 - 5x^4 + 7x^3 - x^2 - 4x + 12}{x^3 - 3x^2}$



Everyone loved Yogi's math class!

ANSWERS-→

Find the partial fractions:

Solutions

1) $\frac{4 - x}{(x+2)(x+4)}$

$$\frac{A}{(x+2)} + \frac{B}{(x+4)} = \frac{4 - x}{(x+2)(x+4)}$$

$$A(x+4) + B(x+2) = 4 - x$$

To find A, let x = -2 (eliminating B)

$$A(-2+4) + B(-2+2) = 4 - (-2)$$

$$2A = 6$$

$$A = 3$$

To find B, let x = -4 (eliminating A)

$$A(-4+4) + B(-4+2) = 4 - (-4)$$

$$-2B = 8$$

$$B = -4$$

$$\boxed{\frac{3}{(x+2)} - \frac{4}{(x+4)}}$$

2) $\frac{10x^3 - 15x^2 - 35x}{x^2 - x - 6}$

Step 1: Use long division

$$\begin{array}{r|l} & 10x - 5 \\ x^2 - x - 6 & 10x^3 - 15x^2 - 35x \\ & -\ 10x^3 - 10x^2 - 60x \\ & 0 - 5x^2 + 25x \\ & -\ \ -5x^2 + 5x + 30 \\ & 0 + 20x - 30 \end{array}$$

$$10x - 5 + \frac{20x - 30}{x^2 - x - 6}$$

Step 2: Separate rational expression into partial fractions

$$\frac{20x - 30}{x^2 - x - 6} = \frac{A}{(x-3)} + \frac{B}{(x+2)}$$

$$20x - 30 = A(x+2) + B(x-3)$$

Let x = -2: $-40 - 30 = A(0) + B(-5)$

B = 14

Let x = 3: $60 - 30 = A(5) + B(0)$

A = 6

Step 3: Summarize and Check

$$\boxed{\frac{10x^3 - 15x^2 - 35x}{x^2 - x - 6} = 10x - 5 + \frac{6}{(x-3)} + \frac{14}{(x+2)}}$$

3) $\dfrac{8x+5}{x^2+3x-10}$ $\qquad \dfrac{A}{(x+5)} + \dfrac{B}{(x-2)} = \dfrac{8x+5}{x^2+3x-10}$

$(x+5)(x-2)$ $\qquad \dfrac{A(x-2)}{(x+5)(x-2)} + \dfrac{B(x+5)}{(x+5)(x-2)} = \dfrac{8x+5}{x^2+3x-10}$

$$A(x-2) + B(x+5) = 8x+5$$

$$Ax - 2A + Bx + 5B = 8x + 5$$

$$(A+B)(x) - 2A + 5B = 8x + 5$$

Then, we know $A + B = 8$

$-2A + 5B = 5$

$2A + 2B = 16$

$-2A + 5B = 5$

$7B = 21$

$B = 3$

then, $A = 5$

$A = 5$

$B = 3$

$$\boxed{\dfrac{5}{(x+5)} + \dfrac{3}{(x-2)}}$$

4) $\dfrac{x^2+6}{3x^3+4x^2-4x}$ $\qquad \dfrac{A}{x} + \dfrac{B}{(3x-2)} + \dfrac{C}{(x+2)} = \dfrac{x^2+6}{3x^3+4x^2-4x}$

$x(3x^2+4x-4)$

$x(3x-2)(x+2)$

Creating common denominators....

$$\dfrac{(3x-2)(x+2)A}{x(3x-2)(x+2)} + \dfrac{x(x+2)B}{x(3x-2)(x+2)} + \dfrac{x(3x-2)C}{x(3x-2)(x+2)} = \dfrac{x^2+6}{3x^3+4x^2-4x}$$

$$(3x-2)(x+2)A + x(x+2)B + x(3x-2)C = x^2+6$$

Pick values to find A, B, and C...

If we let x = -2, that will eliminate A and B....

$$(-8)(0)A + -8(0)B + -2(-8)C = (-2)^2 + 6$$

$16C = 10 \qquad C = 5/8$

If we let x = 0, then we eliminate B and C...

$$(-2)(2)A + 0(2)B + 0(-2)C = (0)^2 + 6$$

$-4A = 6 \qquad A = -3/2$

If x = 2/3, then this eliminates A and C...

$$(0)(8/3)A + (2/3)(8/3)B + (2/3)(0)C = 4/9 + 6$$

$\dfrac{16}{9}B = \dfrac{58}{9} \qquad B = \dfrac{29}{8}$

$$\boxed{\dfrac{-3/2}{x} + \dfrac{29/8}{(3x-2)} + \dfrac{5/8}{(x+2)}}$$

SOLUTIONS

5) $\dfrac{3x^2 - 2x + 12}{(x^2 + 4)^2}$

Decompose using partial fractions...

$$\frac{Ax + B}{(x^2 + 4)} + \frac{Cx + D}{(x^2 + 4)^2} = \frac{3x^2 - 2x + 12}{(x^2 + 4)^2}$$

We want common denominators...

$$\frac{(Ax + B)(x^2 + 4)}{(x^2 + 4)^2} + \frac{Cx + D}{(x^2 + 4)^2} = \frac{3x^2 - 2x + 12}{(x^2 + 4)^2}$$

$$(Ax + B)(x^2 + 4) + Cx + D = 3x^2 - 2x + 12$$

Expand and regroup...

$$Ax^3 + 4Ax + Bx^2 + 4B + Cx + D = 3x^2 - 2x + 12$$

$Ax^3 = 0x^3$ $\quad$ $A = 0$

$Bx^2 = 3x^2$ $\quad$ $B = 3$

$4Ax + Cx = -2x$ $\quad$ Since $A = 0$, then $C = -2$

$4B + D = 12$ $\quad$ Since $B = 3$, then $D = 0$

$$\boxed{\frac{3}{(x^2 + 4)} + \frac{-2x}{(x^2 + 4)^2}} = \frac{3x^2 - 2x + 12}{(x^2 + 4)^2}$$

6) $\dfrac{5}{(x + 2)^2 (x + 1)}$

$$\frac{A}{(x + 1)} + \frac{B}{(x + 2)} + \frac{C}{(x + 2)^2} = \frac{5}{(x + 2)^2 (x + 1)}$$

Common Denominators...

$$\frac{A(x + 2)^2}{(x + 1)(x + 2)^2} + \frac{B(x + 1)(x + 2)}{(x + 1)(x + 2)^2} + \frac{C(x + 1)}{(x + 1)(x + 2)^2} = \frac{5}{(x + 2)^2 (x + 1)}$$

"Drop the denominators"...

$$A(x + 2)^2 + B(x + 1)(x + 2) + C(x + 1) = 5$$

Using Elimination, let $x = -1$ $\quad$ $A(-1 + 2)^2 + B(0) + C(0) = 5$

$A = 5$

let $x = -2$ $\quad$ $A(0) + B(0) + C(-2 + 1) = 5$

$C = -5$

Now, we know $A = 5$ and $C = -5$....

To find B, we'll let $x = 0$,

$5(2)^2 + B(1)(2) + -5(1) = 5$

$20 + 2B - 5 = 5$

$B = -5$

$$\boxed{\frac{5}{(x + 1)} + \frac{-5}{(x + 2)} + \frac{-5}{(x + 2)^2}} = \frac{5}{(x + 2)^2 (x + 1)}$$

SOLUTIONS

7) $\dfrac{13x^2 - 8x - 25}{2x^3 - 6x^2 - x + 3}$

Step 1: Factor the denominator

Using grouping...

$2x^2(x - 3) - 1(x - 3)$

$(2x^2 - 1)(x - 3)$

$$\frac{13x^2 - 8x - 25}{(2x^2 - 1)(x - 3)}$$

Step 2: Split the factors

$$\frac{13x^2 - 8x - 25}{(2x^2 - 1)(x - 3)} = \frac{Ax + B}{(2x^2 - 1)} + \frac{C}{(x - 3)}$$

(the numerators are 1 degree less than the denominators)

Step 3: Get common denominators

$$\frac{13x^2 - 8x - 25}{(2x^2 - 1)(x - 3)} = \frac{Ax + B}{(2x^2 - 1)}\frac{(x - 3)}{(x - 3)} + \frac{C}{(x - 3)}\frac{(2x^2 - 1)}{(2x^2 - 1)}$$

$$\frac{13x^2 - 8x - 25}{(2x^2 - 1)(x - 3)} = \frac{Ax^2 - 3Ax + Bx - 3B + 2Cx^2 - C}{(2x^2 - 1)(x - 3)}$$

Step 4: Solve for A, B, and C

$$13x^2 - 8x - 25 = Ax^2 - 3Ax + Bx - 3B + 2Cx^2 - C$$

$$13x^2 - 8x - 25 = (A + 2C)x^2 + (-3A + B)x + (-3B - C)$$

$A + 2C = 13 \Rightarrow 3A + 6C = 39$

$-3A + B = -8 \Rightarrow -3A + B = -8$

$B + 6C = 31 \Rightarrow 3B + 18C = 93$

$-3B - C = -25$

$17C = 68$

$C = 4$

$B = 7$

$A = 5$

$$\boxed{\frac{5x + 7}{(2x^2 - 1)} + \frac{4}{(x - 3)}}$$

8) $\dfrac{x^5 - 5x^4 + 7x^3 - x^2 - 4x + 12}{x^3 - 3x^2}$

Step 1: Long division..

$$\begin{array}{r|l} & x^2 - 2x + 1 \\ x^3 - 3x^2 & x^5 - 5x^4 + 7x^3 - x^2 - 4x + 12 \\ & -(x^5 - 3x^4) \\ & \quad -2x^4 + 7x^3 \\ & \quad -(-2x^4 + 6x^3) \\ & \qquad x^3 - x^2 \\ & \qquad -(x^3 - 3x^2) \\ & \qquad\quad 2x^2 - 4x + 12 \end{array}$$

$$\boxed{x^2 - 2x + 1 + \frac{-4}{x^2} + \frac{2}{x - 3}}$$

Step 2: Partial Fractions of remainder...

$$\frac{2x^2 - 4x + 12}{x^3 - 3x^2} = \frac{Ax + B}{x^2} + \frac{C}{x - 3}$$

Step 3: common denominator and solve

$$2x^2 - 4x + 12 = (Ax + B)(x - 3) + C(x^2)$$

Let $x = 3$ (to eliminate the A and B terms)

$$2(3)^2 - 4(3) + 12 = (3A + B)(0) + 9C$$

$$18 = 9C$$

$$C = 2$$

$$2x^2 - 4x + 12 = (Ax + B)(x - 3) + 2x^2$$

$$-4x + 12 = Ax^2 - 3Ax + Bx - 3B$$

since there is no x^2 term, $A = 0$

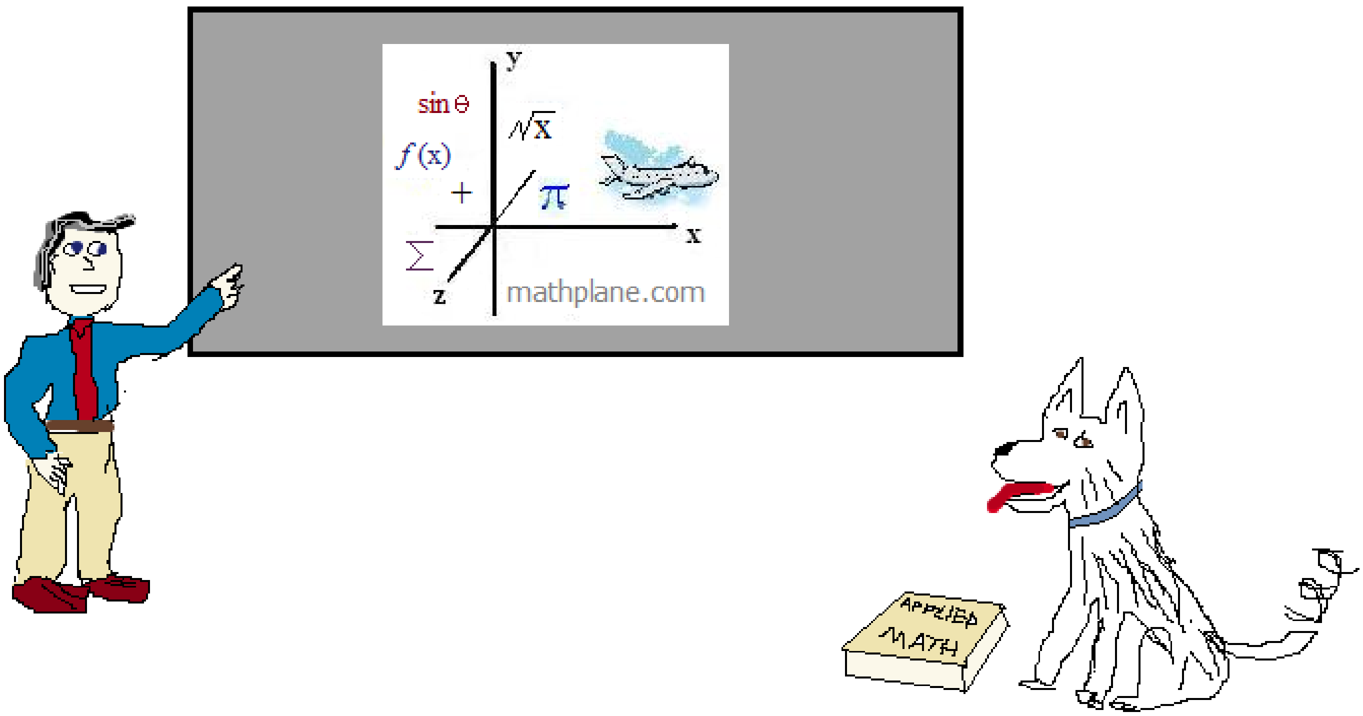
Since $A = 0$, $B = -4$

$$\frac{-4}{x^2} + \frac{2}{x - 3}$$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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