Algebra 2: Parametric Topics

Topics include conics, lines, trig identities, graphing, projectile, and more.

Brief introduction to Parametric Equations

Equations are often expressed with 2 variables: a dependent variable and an independent variable..

For example, $y = 3x + 5$ For every change in x (the independent variable), y (the dependent variable) increases by 3 units...

The movement of x and y are related directly to each other..

Suppose we introduce a 3rd component, t.. As t moves, x moves in one way and y moves in another way.

Note: x and y are *directly* related to t.. And, they are *indirectly* related to each other..

Application: A plane is flying east at 500 miles per hour. And, the wind is blowing north at 50 miles per hour. Use parametric equations to express the direction of the plane relative to time (t).

$$x = 500t$$
$$y = 50t$$
$$t \geq 0$$

Every 1 hour (t increases by 1), the plane moves 500 miles in (x) direction and 50 miles in (y) direction.

Change parametric equation of line to symmetric equation:

Example:

$x = 2t + 4$ $y = 3t$ $z = 2t - 7$

solve each equation for t

$$t = \frac{x-4}{2} \qquad t = \frac{y}{3} \qquad t = \frac{z+7}{2}$$

Substitution

$$\frac{x-4}{2} = \frac{y}{3} = \frac{z+7}{2}$$

Change $y = f(x)$ into a parametric ('parameterize' the function)

let $x = t$

then, $y = f(t)$

Example: $y = x^2$

$$x = t$$
$$y = t^2$$

reminder: in some instances, the domain of t must be defined

Example: Write the equation of a line that passes through (2, 5) and (-3, 4) in parametric form.

$$\text{slope} = \frac{4-5}{-3-2} = \frac{1}{5}$$

During every increment (t), the y component moves +1, and the x component moves +5 units.

$$x = \quad + 5t$$
$$y = \quad + 1t$$

Then, we can choose either point to finish the parametric equations

$$x = 2 + 5t$$
$$y = 5 + t$$

Quick check: at $t = 0$, $(x, y) = (2, 5)$ ✓

at $t = -1$, $(x, y) = (-3, 4)$ ✓

Parametric Notes : Shortcuts and formulas

Circle: center: (h, k)

$x = h + r\cos\theta$

$y = k + r\sin\theta$

Example: A circle with center (3, −9) and radius 4

$x = 3 + 4\cos\theta$

$y = -9 + 4\sin\theta$

Line: slope: $\dfrac{\Delta y}{\Delta x}$

Example: (3, 1) (−2, 11)

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{10}{-5} = \frac{2}{-1}$$

$x = -1t$

$y = 2t$

then, add a point on the line...

$x = -1t + 3$ or $x = -1t - 2$

$y = 2t + 1$ $\quad$ $y = 2t + 11$

Parametric Notes: Conics

Example: A circle with center $(3, -7)$ is tangent to the x-axis. Write a parametric equation of the circle.

radius: 7 units

center: (3, -7)

$$(x-3)^2 + (y+7)^2 = 49$$

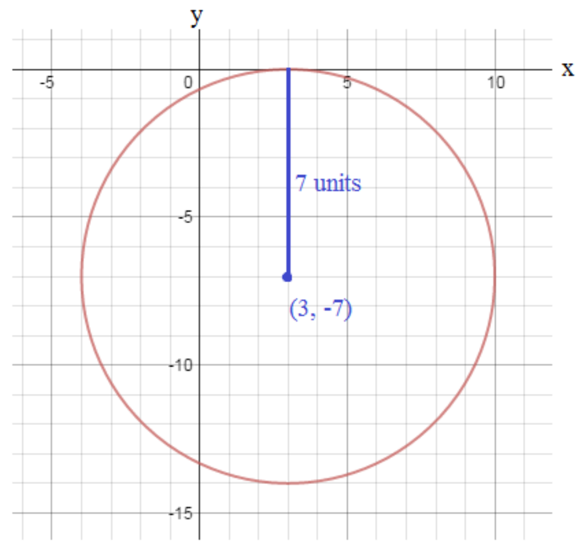
$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{(x-3)^2}{49} + \frac{(y+7)^2}{49} = 1$$

$$\cos^2\theta = \frac{(x-3)^2}{49} \qquad \sin^2\theta = \frac{(y+7)^2}{49}$$

$$\cos\theta = \frac{(x-3)}{7} \qquad \sin\theta = \frac{(y+7)}{7}$$

$$\boxed{\begin{aligned} x &= 7\cos\theta + 3 \\ y &= 7\sin\theta - 7 \end{aligned}}$$



Quick check:

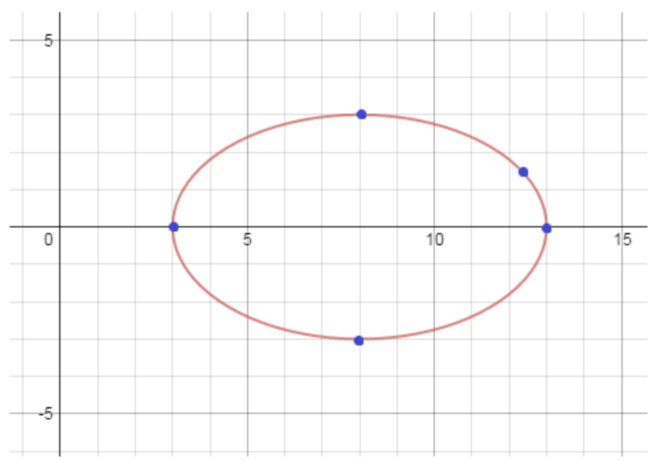
θ	x	y
0	10	-7
90	3	0
180	-4	-7
270	3	-14

Example: $x = 5\cos\theta + 8$

$y = 3\sin\theta$

a) Plot 5 points on an xy-coordinate plane

θ	x	y	
0	13	0	
30	$5\sqrt{3}/2 + 8$	3/2	(12.33, 1.5)
90	8	3	
180	3	0	
270	8	-3	



b) Convert to a rectangular equation

$$\cos\theta = \frac{(x-8)}{5} \qquad \sin\theta = \frac{y}{3}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{(x-8)}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{(x-8)^2}{25} + \frac{y^2}{9} = 1$$

Example: The equations of two intersecting planes are

$$2x + 3y - z = 8$$
$$x - 5y + 2z = -1$$

Use a parametric expression to describe the line of intersection...

Step 1: Combine the equations to simplify

$$2x + 3y - z = 8$$
$$x - 5y + 2z = -1$$
$$-2x + 10y - 4z = 2$$

$$13y - 5z = 10$$

$$\boxed{y = \frac{5z + 10}{13}}$$

Step 2: "Parameterize" one of the variables and substitute

$$2x + 3y - z = 8$$

Let $2x = t$

$$\boxed{x = \frac{t}{2}}$$

$$t + 3y - z = 8$$

$$t + 3\left(\frac{5z + 10}{13}\right) - z = 8$$

$$t + \frac{15z}{13} + \frac{30}{13} - \frac{13z}{13} = 8$$

$$\frac{2z}{13} = 8 - \frac{30}{13} - t$$

$$2z = 104 - 30 - 13t$$

$$\boxed{z = 37 - \frac{13}{2}t}$$

$$y = \frac{5z + 10}{13}$$

$$y = \frac{5(37 - \frac{13}{2}t) + 10}{13}$$

$$y = \frac{195 - \frac{65}{2}t}{13}$$

$$\boxed{y = 15 - \frac{5}{2}t}$$

Step 3: Check your solutions

$$2x + 3y - z = 8$$
$$x - 5y + 2z = -1$$

Let $t = 0$

$x = 0$

$y = 15$

$z = 37$

$$2(0) + 3(15) - (37) = 8$$
$$8 = 8 \checkmark$$
$$(0) - 5(15) + 2(37) = -1$$
$$-1 = -1 \checkmark$$

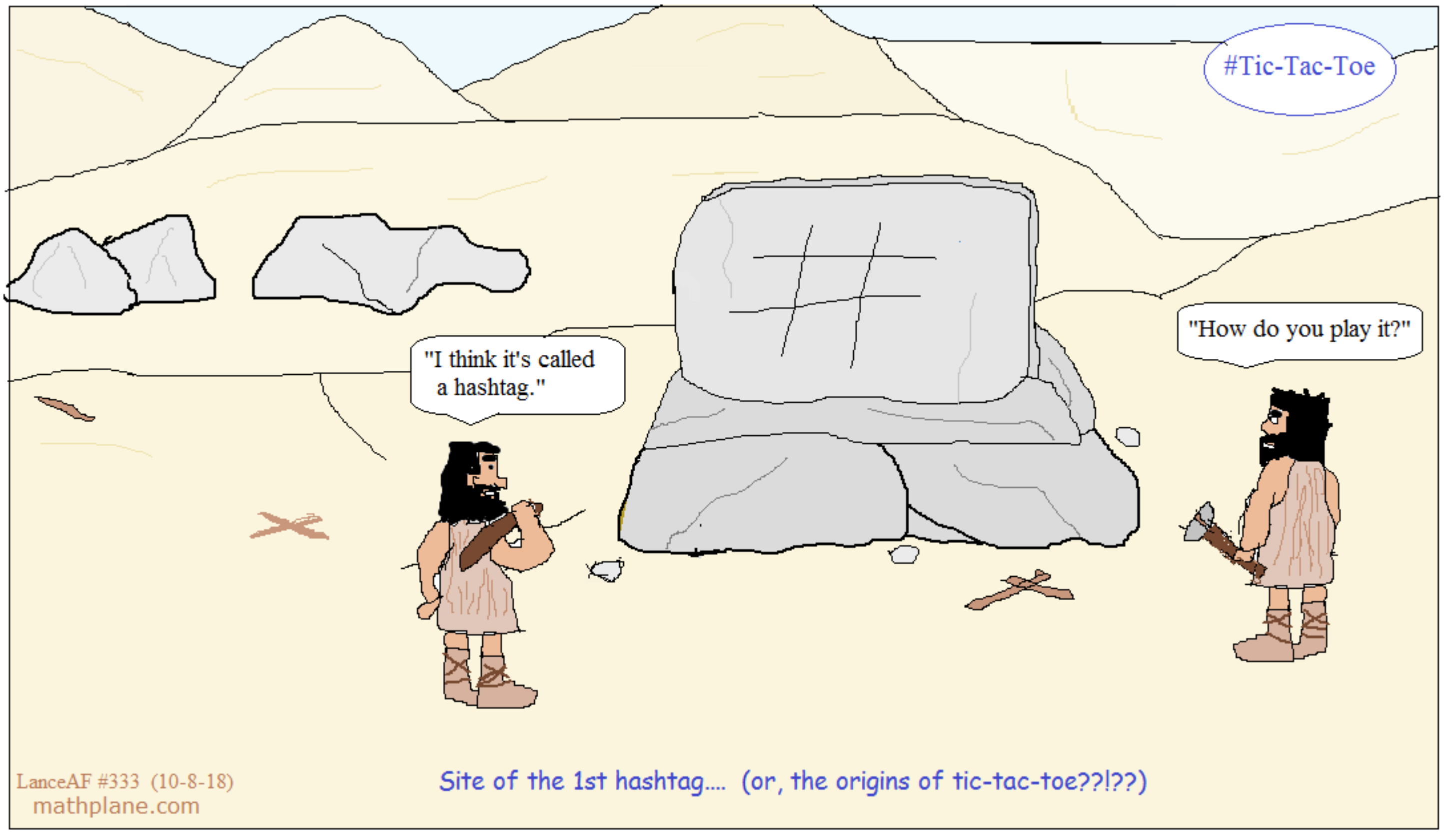
then, let $t = 1$

$x = \frac{1}{2}$

$y = \frac{25}{2}$

$z = \frac{61}{2}$

$$2(\frac{1}{2}) + 3(\frac{25}{2}) - (\frac{61}{2}) = 8$$
$$1 + 37.5 - 30.5 = 8 \checkmark$$
$$(\frac{1}{2}) - 5(\frac{25}{2}) + 2(\frac{61}{2}) = -1$$
$$.5 - 62.5 + 61 = -1 \checkmark$$


#Tic-Tac-Toe
"I think it's called a hashtag."
"How do you play it?"
LanceAF #333 (10-8-18)
mathplane.com
Site of the 1st hashtag.... (or, the origins of tic-tac-toe??!??)

Example: A baseball batter hits a fastball at 100 miles per hour.
The contact occurs 3 feet above home plate at a 15 degree angle.
The fence is 400 feet away and 10 feet high.

Parametric Equations: Projectile Motion

$x = v_0 \cos\theta\, t$

$y = h_0 + v_0 \sin\theta\, t - 16t^2$

where v_0 is initial velocity (feet/second)

θ is angle of contact

t is time (seconds)

h_0 is initial height

a) When is the ball 20 feet above the ground?

This occurs when $y = 20$...

100 miles/1 hour = 528000 feet/3600 seconds

= 146.67 feet/second

$$y = h_0 + v_0 \sin\theta\, t - 16t^2$$

$$20\text{ft} = 3\text{ft} + 146.67 \text{ ft/sec} \cdot \sin(15^\circ)t - 16t^2$$

$$16t^2 - 37.96t + 17 = 0$$

when $t = .60$ and 1.77

b) How high does the ball reach?

We need to find the vertex....
Since the initial height is 3 feet (i.e $y = 3$ when $t = 0$), let's find another point where $y = 3$

$$3 = 3 + 146.67 \cdot \sin(15^\circ)t - 16t^2$$

$$16t^2 - 37.96t = 0$$

$$t = 0 \text{ and } 2.37$$

The midpoint, $t = 1.18$ is when the max height is reached (i.e. vertex)

$$y = h_0 + v_0 \sin\theta\, t - 16t^2$$

$$y = 3 + (146.67)(.2588)(1.18) - 16(1.18)^2$$

$$= 3 + 44.79 - 22.28 = 25.5 \text{ approx.}$$

c) Does the ball clear the fence?

The ball reaches the fence when $x = 400$

$$x = v_0 \cos\theta\, t$$

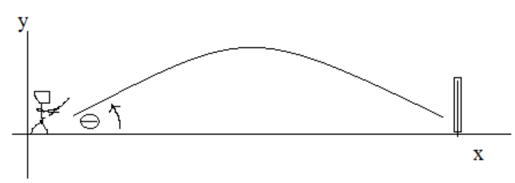
$$400 = 146.67 \cdot \cos(15^\circ)t$$

$$t = \frac{400}{146.67 \cdot .966} = 2.82 \text{ seconds}$$

Then, at $t = 2.82$ seconds,

$$y = 3 + 146.67 \cdot \sin(15^\circ)(2.82) - 16(2.82)^2$$

$$= 3 + 107.0 - 127.2 = -.107 < 10 \text{ (foot fence)}$$ NO



d) Suppose the bat's contact occurs at a 25 degree angle.
Would that be a home run (i.e. clear the 10 foot fence) ?

First, find when the ball reaches the fence....

$$x = v_0 \cos\theta\, t$$

$$400 = 146.67 \cdot \cos(25^\circ)t$$

$$t = \frac{400}{146.67(.906)} = 3.01 \text{ seconds}$$

and, the height (y) at $t = 3$ seconds...

$$y = h_0 + v_0 \sin\theta\, t - 16t^2$$

$$y = 3 + 146.67 \cdot \sin(25^\circ)(3) - 16(3)^2$$

$$= 44.96 \text{ feet}$$ YES, it clears the fence!

e) What angle is necessary to clear the fence?

So, 15 degrees is not enough of an angle.. And,
25 degrees easily clears the fence...

To find the height necessary, we must solve for θ

$$x = v_0 \cos\theta\, t \qquad y = h_0 + v_0 \sin\theta\, t - 16t^2$$

$$400 = 146.67 \cdot \cos\theta\, t \qquad 10 = 3 + 146.67 \cdot \sin\theta\, t - 16t^2$$

$$t = \frac{400}{146.67 \cdot \cos\theta}$$

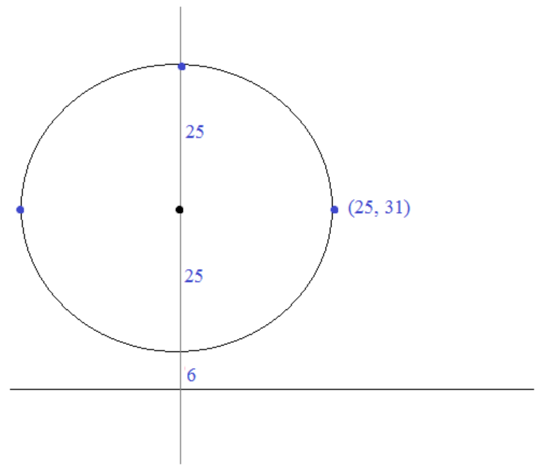
$$7 = 146.67 \cdot \sin\theta \frac{400}{146.67 \cdot \cos\theta} - 16\left(\frac{400}{146.67 \cdot \cos\theta}\right)^2$$

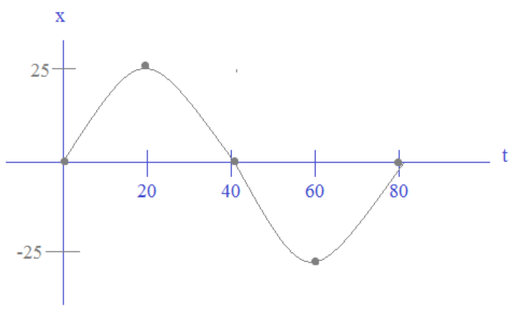
$$7 = 400\tan\theta - \frac{119}{\cos^2\theta} \qquad \theta = 19.38^\circ \text{ (approximately)}$$

Example: Write a parametric model describing a ferris wheel that is 6 feet off the ground, has a span (diameter) of 50 feet, and takes 80 seconds per revolution...

First, let's break up the components:

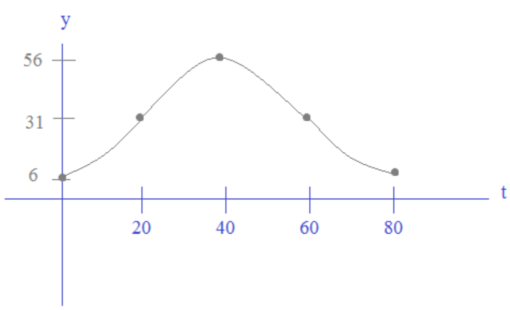
t	(x, y)
t = 0	(0, 6)
t = 20	(25, 31)
t = 40	(0, 56)
t = 60	(-25, 31)
t = 80	(0, 6)





the x-component (oscillates from left to right)

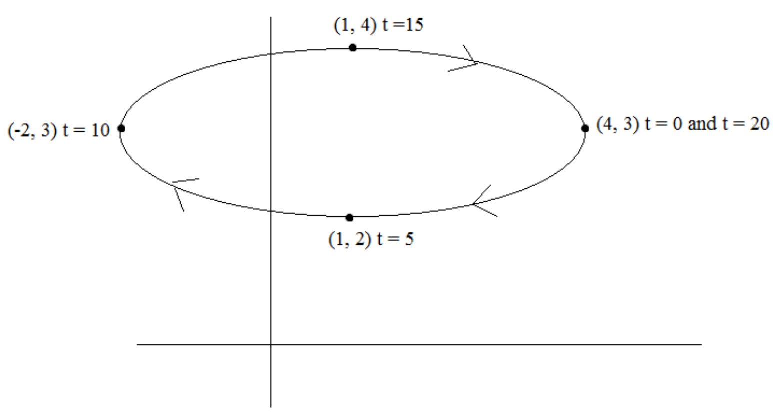
$$x = 25\sin(\frac{\pi}{40})t$$



the y-component (oscillates up and down)

$$y = -25\cos(\frac{\pi}{40})t$$

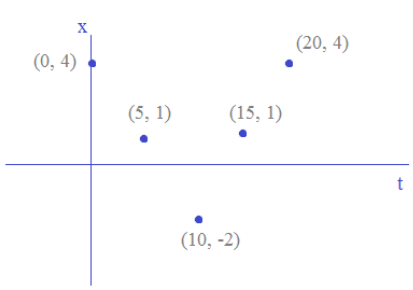
Example: Find the parametric equation:



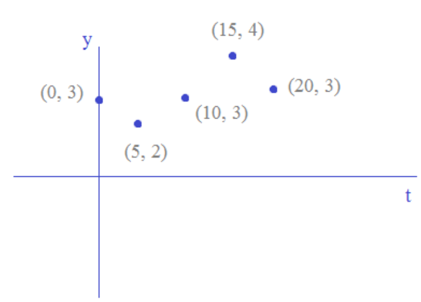
Step 1: Create a table

t	(x, y)
0	(4, 3)
5	(1, 2)
10	(-2, 3)
15	(1, 4)
20	(4. 3)

Step 2: Graph the components



Step 3: Write the equations



$$x = 3\cos(\frac{\pi}{10})t + 1$$

$$y = -\sin(\frac{\pi}{10})t + 3$$

Parametric Notes

Example: A car's location on a map is modeled by the parametric equation $x = 3 + t$
$y = 4t - 7$

where x is the east/west distance from the town center
y is the north/south distance from the town center

t is time (minutes)

a) After 5 minutes, how far has the car traveled?

Every minute (t), the car travels 1 mile east and 4 miles north...

(using distance formula), the car travels $\sqrt{17}$ miles per minute

after 5 minutes, the car traveled $\boxed{5\sqrt{17}}$

$x = 3 + 1t$
$y = 4t - 7$

rate of change (related to *t*)
1 east and 4 north

b) When the car is 21 miles north of the town center, how far east is it?

The car is 21 miles north when $y = 21$..
If $y = 21$, then
$21 = 4t - 7$ and $t = 7$ minutes

When $t = 7$, the car's position east is $x = 3 + (7) = \boxed{10 \text{ miles}}$

NOTE: if we remove the parameter,

$y = 4(x - 3) - 7$

$y = 4x - 19$

(10, 21) is a point on that line

Example: Convert the parametric equations into cartesian coordinate system:

$x = 2 + 3\sec t$

$y = 1 + 4\tan t$

then, graph...

$\sec t = \dfrac{x-2}{3}$ $\tan t = \dfrac{y-1}{4}$

recognize the tangent and secant:

trig identity: $1 + \tan^2 x = \sec^2 x$

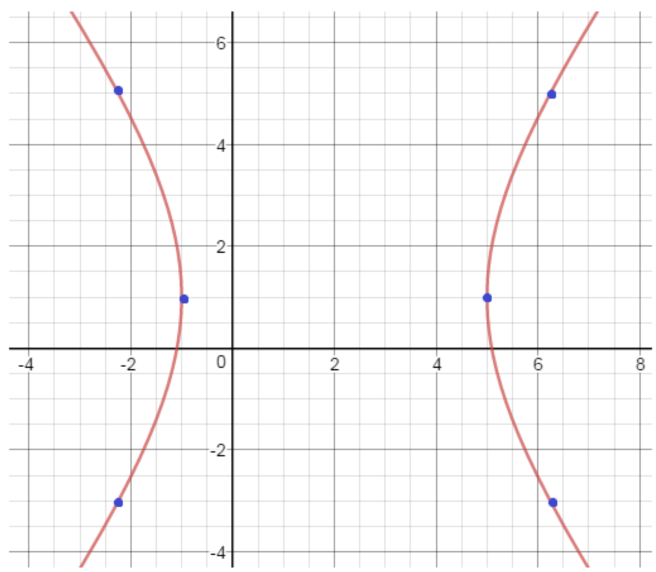
$$1 + (\tan t)^2 = (\sec t)^2$$

$$1 + \left(\frac{y-1}{4}\right)^2 = \left(\frac{x-2}{3}\right)^2$$

$$\left(\frac{x-2}{3}\right)^2 - \left(\frac{y-1}{4}\right)^2 = 1$$

Hyperbola!

$$\frac{(x-2)^2}{9} - \frac{(y-1)^2}{16} = 1$$



Quick check:

$\ominus$	x	y	$\ominus$	x	y
0	5	1	45	6.24	5
90	undef	undef	135	-2.24	-3
180	-1	1	225	-2.24	5
270	undef	undef	315	6.24	-3

Example: The following are 2 parametric equations, describing points on a plane as a function of time (seconds). Where and when do the equations intersect?

$$\begin{cases} x_1 = 2t - 5 \\ y_1 = -t + 1 \end{cases} \qquad \begin{cases} x_2 = t + 3 \\ y_2 = t - 15 \end{cases}$$

parametric graph for $0 < t < 10$

If the curves/lines meet, then $x_1 = x_2$

$$2t - 5 = t + 3$$

$$t = 8$$

So, if they meet, it would occur at $t = 8$

and, at $t = 8$

$$x_1 = x_2 = 11$$

$$y_1 = y_2 = -7$$

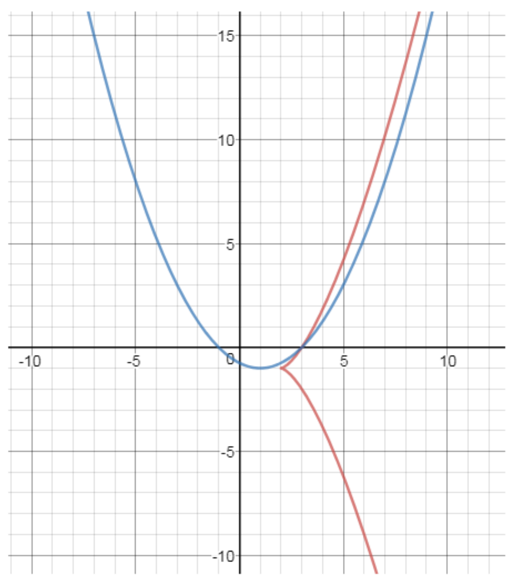
the equations intersect

at $t = 8$ seconds

at (11, -7)

Example: The following are 2 parametric equations, describing points on a plane as a function of time (seconds). Where and when do the equations intersect?

$$\begin{cases} x_1 = t^2 + 2 \\ y_1 = t^3 - 1 \end{cases} \qquad \begin{cases} x_2 = -2t + 1 \\ y_2 = t^2 - 1 \end{cases}$$



If the curves meet, then $x_1 = x_2$ at some time (t)...

$$t^2 + 2 = -2t + 1$$

$$t^2 + 2t + 1 = 0$$

$$(t + 1)(t + 1) = 0$$

$$t = -1$$

If $t = -1$, then $x_1 = x_2 = 3$

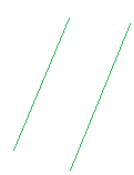
If $t = -1$, then $y_1 = -2$ and $y_2 = 0$

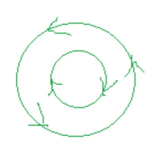
THERE IS NO SOLUTION!!

Note: 2 equations may not intersect if they don't overlap....

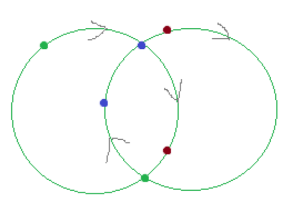
EX: parallel lines

EX: 2 concentric circles





Also, 2 parametric equations may not intersect, if the points are occupied at different times!



Parametric Conics System

Example: $x_1 = 3\cos t$ $\quad x_2 = -3 + \cos t$

$y_1 = 2\sin t$ $\quad y_2 = 1 + \sin t$

a) Graph the conics, and determine their intersection...

b) If the path of a particle (x_1, y_1) is the first equation,

and the path of a particle (x_2, y_2) is the second equation,

do the particles collide?

$x_1 = 3\cos t$ $\quad x_2 = -3 + \cos t$

$y_1 = 2\sin t$ $\quad y_2 = 1 + \sin t$

Set the x's and y's equal to each other

$3\cos t = -3 + \cos t$ $\qquad 2\sin t = 1 + \sin t$

$2\cos t = -3$ $\qquad \sin t = 1$

$\cos t = -3/2$ $\qquad t = 90^\circ$

no solution $\qquad$ at $t = 90^0$

$x_1 = 3(0)$ $\quad x_2 = -3 + (0)$

$y_1 = 2(1)$ $\quad y_2 = 1 + (1)$

(0, 2) $\qquad$ (-3, 2)

particles will never collide...

$x_1 = 3\cos t$

$y_1 = 2\sin t$

at (-3, 0) ----> $3\cos t = -3$ $\qquad t = 180^\circ$ (or π)

$2\sin t = 0$

$x_2 = -3 + \cos t$

$y_2 = 1 + \sin t$

if $t = 180^\circ$ (or π) then $-3 + \cos(180^\circ) = -4$

$1 + \sin(180^\circ) = 1$

(-4, 1)

Removing the Parameter to graph

$x_1 = 3\cos t$ $\qquad y_1 = 2\sin t$

$\cos t = \frac{x}{3}$ $\qquad \sin t = \frac{y}{2}$

$\sin^2 + \cos^2 = 1$

$$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1 \qquad \frac{x^2}{9} + \frac{y^2}{4} = 1$$

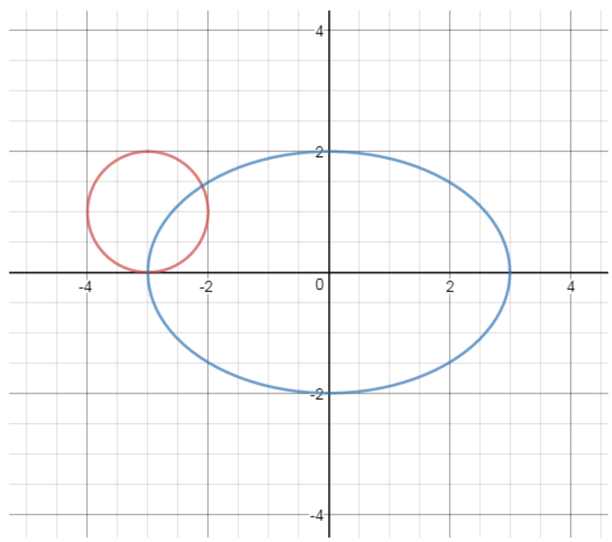
$x_2 = -3 + \cos t$ $\qquad y_2 = 1 + \sin t$

$\cos t = x + 3$ $\qquad \sin t = y - 1$

$\sin^2 + \cos^2 = 1$ $\qquad (y-1)^2 + (x+3)^2 = 1$

intersection:

(-3, 0) and (-2.1, 1.4)



Parametric Equations Exploration: Domain and Orientation

Compare and contrast the following Parametric Equations:

a) $x = t$, $y = t^2$

b) $x = \sqrt{t}$, $y = t$

c) $x = \sin(t)$, $y = (\sin(t))^2$

d) $x = 3^t$, $y = 3^{2t}$

***NOTE: When you convert the 4 into rectangular coordinate equations,

the result is $y = x^2$

But, what about the graphs?!?!?

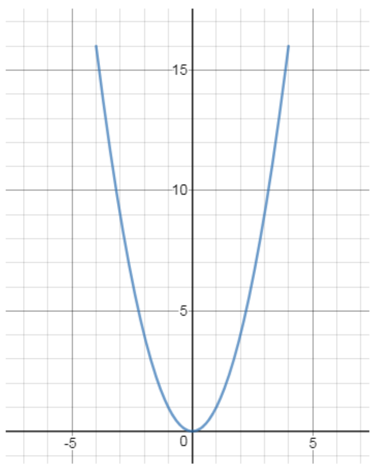
a)

t	x	y
-2	-2	4
-1	-1	1
0	0	0
1	1	1
2	2	4

$x = t$

$y = t^2$

Graph is for
$-4 < t < 4$



Domain: all real numbers

Range: $y \geq 0$

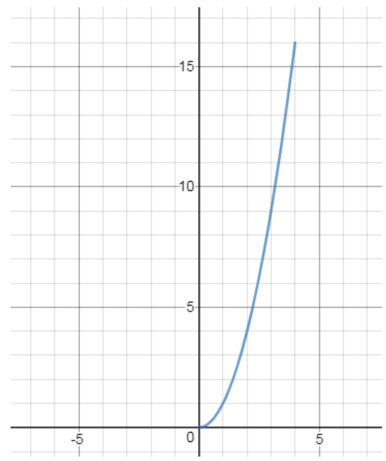
b)

t	x	y
-2	Undefined	-2
-1	Undefined	-1
0	0	0
1	1	1
2	$\sqrt{2}$	2
4	2	4

$x = \sqrt{t}$

$y = t$

Graph is for
$0 < t < 16$



Domain: $x \geq 0$

$y \geq 0$

c)

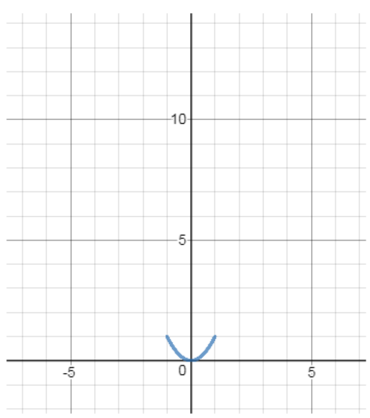
t	x	y
0	0	0
$\frac{\pi}{6}$	1/2	1/4
$\frac{\pi}{2}$	1	1
$\frac{7\pi}{6}$	-1/2	1/4
$\frac{3\pi}{2}$	-1	1

$x = \sin(t)$

$y = (\sin(t))^2$

Graph is for
$-20 < t < 20$

sine must be between
-1 and 1



Domain: $-1 \leq x \leq 1$

Range: $0 \leq x \leq 1$

d)

t	x	y
-3	1/27	1/729
-2	1/9	1/81
-1	1/3	1/9
0	1	1
1	3	9
2	9	81

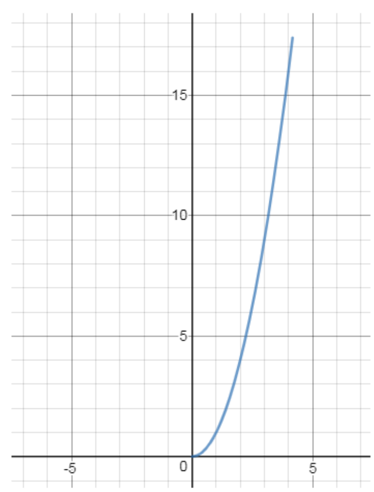
$x = 3^t$

$y = 3^{2t}$

Graph is for
$-20 < t < 1.3$

Note: when $t = 0$,
(x, y) is $(1, 1)$

$(0, 0)$ does not exist



Domain: $x > 0$

Range: $y > 0$

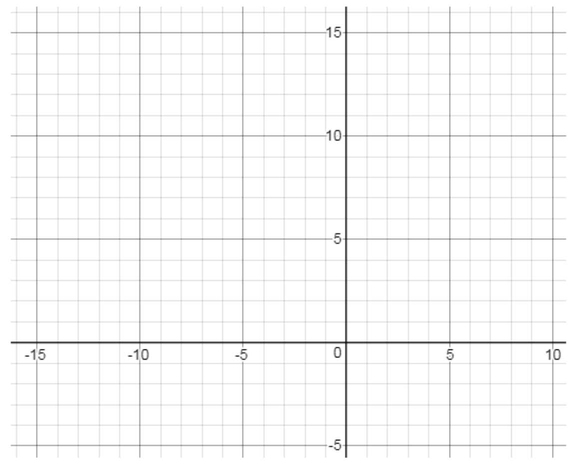
Parametrics Quiz

1) Write 2 different sets of parametric equations for $y = (x-2)^3 + 4$

2) Convert the parametric equations (into rectangular form) and graph:

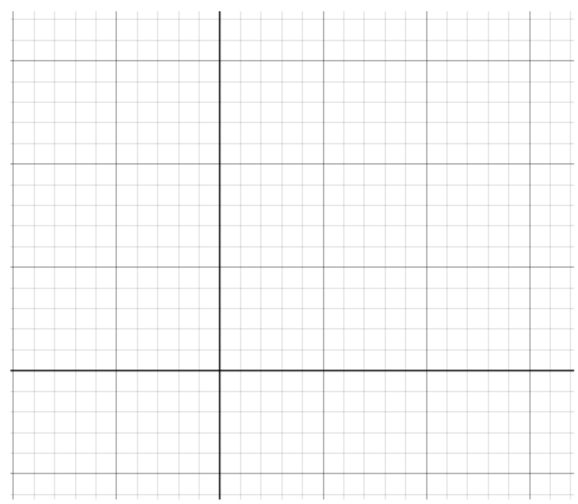
$x = -4 + 3\tan t$

$y = 7 - 2\sec t$



3) Convert and graph the following $x = t^2 + 2$

$y = 4t^2 - 3$



4) An ellipse has vertices at (5, 0) and (-5, 0)
and foci at (3, 0) and (-3, 0)...

What is the equation of the ellipse in parametric form?

Parametrics Quiz

mathplane.com

SOLUTIONS

1) Write 2 different sets of parametric equations for $y = (x-2)^3 + 4$

$\boxed{1)\quad y = t}$

$t = (x-2)^3 + 4$

$x - 2 = \sqrt[3]{t-4}$

$\boxed{x = \sqrt[3]{t-4} + 2}$

$\boxed{\begin{array}{l} 2)\quad x = t \\ y = (t-2)^3 + 4 \end{array}}$

2) Convert the parametric equations (into rectangular form) and graph:

$x = -4 + 3\tan t$

$y = 7 - 2\sec t$

$x + 4 = 3\tan t$ $\qquad$ $y - 7 = -2\sec t$

$\frac{(x+4)}{3} = \tan t$ $\qquad$ $\frac{(y-7)}{-2} = \sec t$

points on the graph

t	x	y
0	-4	5
$\pi/4$	-1	4.17
$\pi/2$	undef	undef
$3\pi/4$	-7	9.82
π	-4	9

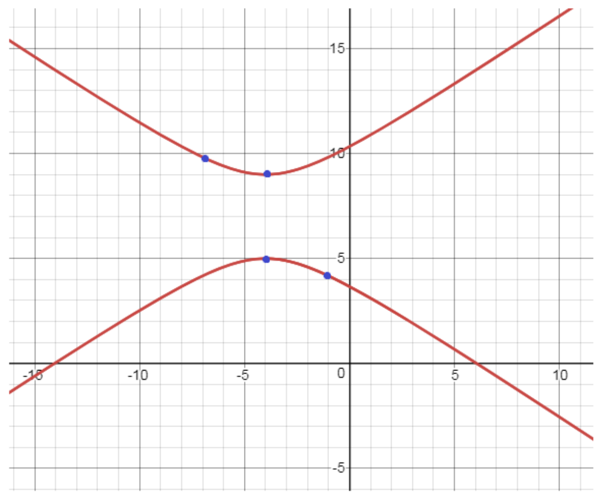
recognize the Pythagorean Trig Identity

$1 + \tan^2 x = \sec^2 x$

$$1 + \left(\frac{(x+4)}{3}\right)^2 = \left(\frac{(y-7)}{-2}\right)^2$$

$$\boxed{1 = \left(\frac{(y-7)}{-2}\right)^2 - \left(\frac{(x+4)}{3}\right)^2}$$

Hyperbola!!



3) Convert and graph the following $\quad x = t^2 + 2$

$y = 4t^2 - 3$

t	-2	-1	0	1	2	3
x	6	3	2	3	6	11
y	13	1	-3	1	13	33

Remove the parameter (t)

$t^2 = x - 2$

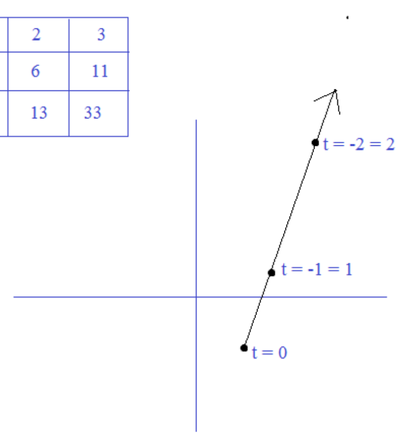
then, substitute into the 2nd equation...

$y = 4(x-2) - 3$

$\boxed{y = 4x - 11}$

***Now, we must see if there is a domain restriction!

$\boxed{x \geq 2}$



4) An ellipse has vertices at (5, 0) and (-5, 0)
and foci at (3, 0) and (-3, 0)...

What is the equation of the ellipse in parametric form?

Using properties of ellipses, we can conclude the standard form of equation is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

since $a^2 - b^2 = c^2$

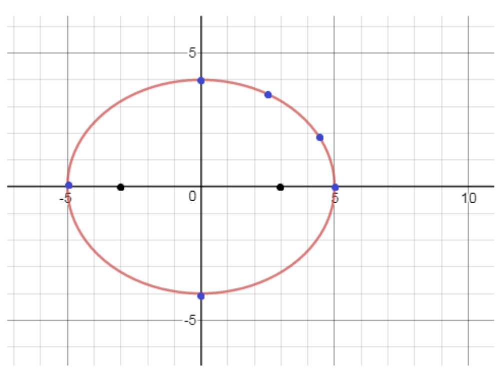
$5^2 - b^2 = 3^2$

$b = 4$

$\cos^2 t + \sin^2 t = 1$

$\cos t = \frac{x}{5} \qquad \sin t = \frac{y}{4}$

$\boxed{x = 5\cos t \qquad y = 4\sin t}$



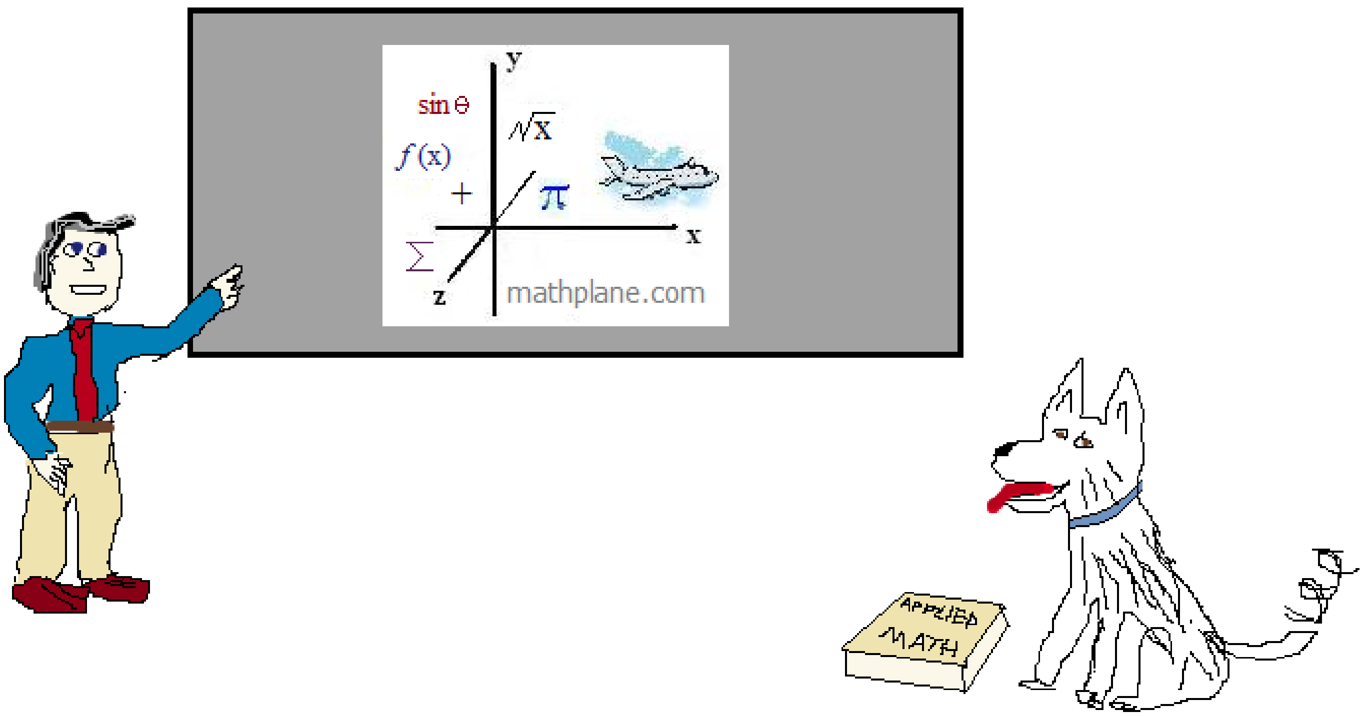
points on the graph

t	0	$\pi/6$	$\pi/3$	$\pi/2$	π	$3\pi/2$
x	5	4.33	2.5	0	-5	0
y	0	2	3.46	4	0	-4

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

Cheers



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And, the mathplane stores at TES and TeachersPayTeachers