Trigonometry: Polar and Rectangular Equations

Notes, Examples, and Quiz (with solutions)

Topics include converting from polar to rectangular forms, graphing conics, eccentricity, directrix, trig functions, and more.

Mathplane.com

Polar Equations

Example: Convert $r^2 = \sin 2\theta$ into rectangular coordinates

$$x^2 + y^2 = 2\sin\theta\cos\theta$$

$$x^2 + y^2 = 2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right)$$

$$x^2 + y^2 = \frac{2xy}{x^2 + y^2}$$

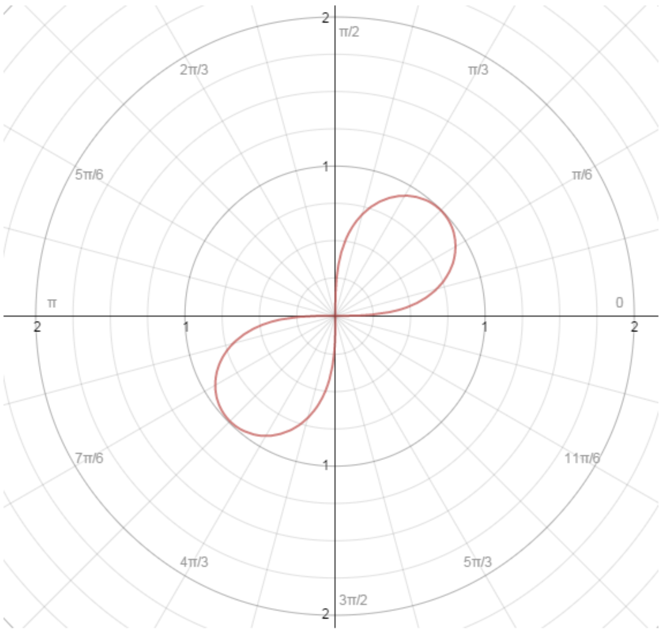
cross-multiply

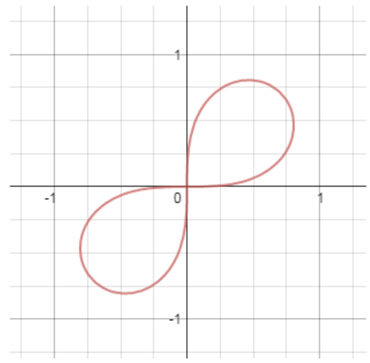
$$x^4 + 2x^2y^2 + y^4 = 2xy$$

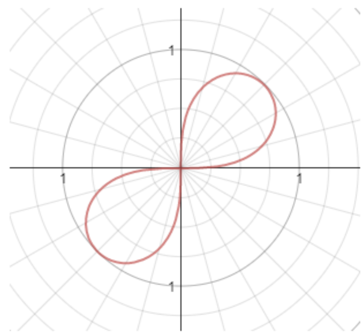
$$\boxed{x^4 + 2x^2y^2 - 2xy + y^4 = 0}$$

$$x = r\cos\theta \qquad \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$y = r\sin\theta \qquad \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$







Example: Convert $4x^2 - 9y^2 = 1$ into polar coordinates

(hyperbola)

$x = r\cos\theta \qquad y = r\sin\theta$

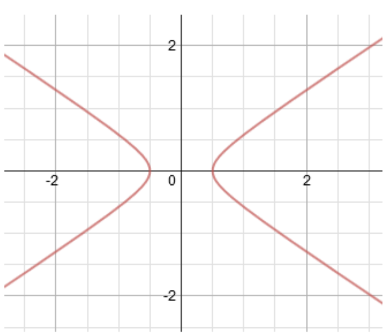
$$4r^2\cos^2\theta - 9r^2\sin^2\theta = 1$$

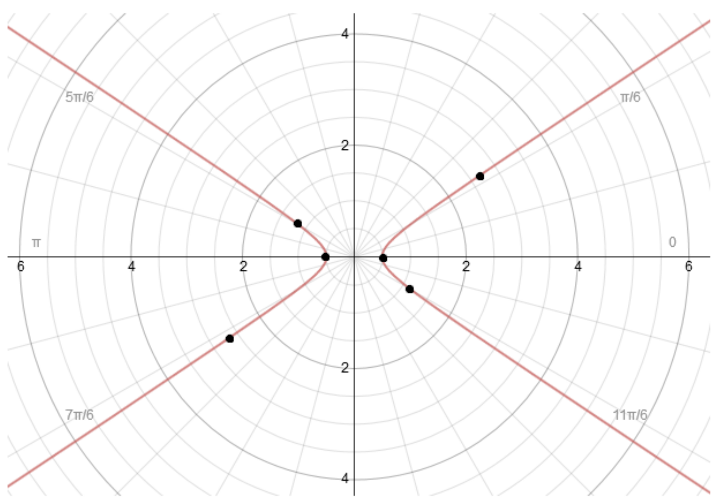
$$r^2(4\cos^2\theta - 9\sin^2\theta) = 1$$

$$r^2 = \frac{1}{(4\cos^2\theta - 9\sin^2\theta)}$$

$$r = \pm\sqrt{\frac{1}{(4\cos^2\theta - 9\sin^2\theta)}}$$

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	33°
r	1/2	DNE	DNE	1.15	2.65
	-1/2	DNE	DNE	-1.15	-2.65





Sketch the following polar conic $r = \dfrac{8}{4 + 2\cos\theta}$

Using POLAR form

rewrite in standard form (by dividing by 4)

$$r = \frac{2}{1 + (1/2)\cos\theta}$$

$e = 1/2$

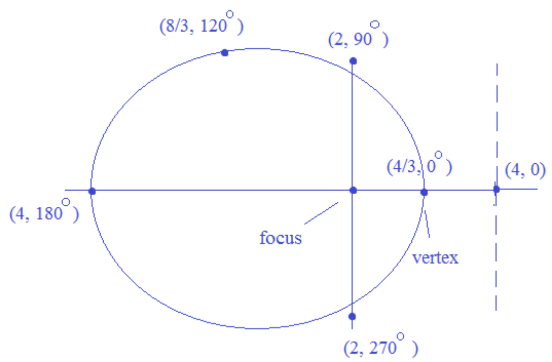
since $ep = 2$, $p = 4$

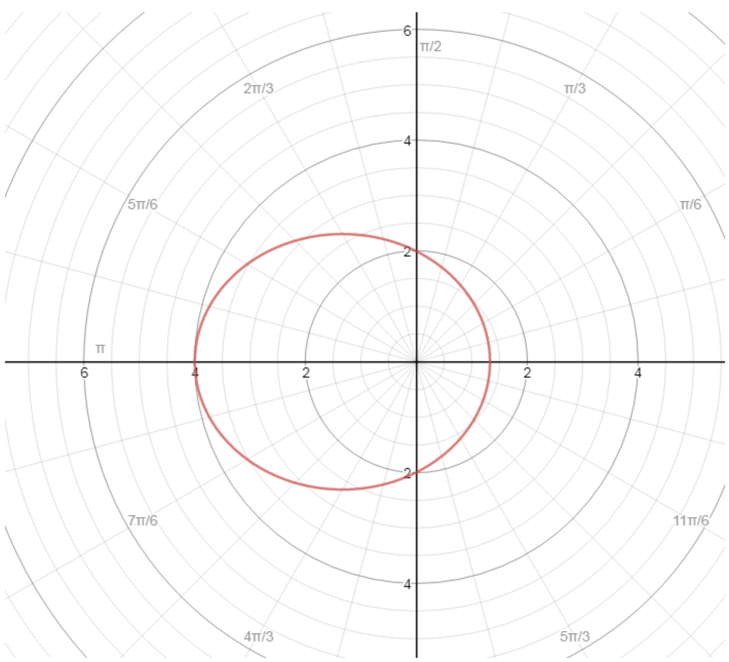
$$r = \frac{ep}{1 + e\cos\theta}$$

e = eccentricity

p is distance between focus and directrix
(and, focus is on the pole)

- since $e < 1$, it is an ellipse
- since the trig function is $\cos\theta$, it is a horizontal ellipse
- and, because it is positive, the right focus is on the pole...





Using RECTANGULAR form

$$r = \frac{8}{4 + \frac{2x}{r}}$$

$x = r\cos\theta$

$x^2 + y^2 = r^2$

$\cos\theta = \dfrac{x}{r}$

cross multiply

$8 = 4r + 2x$

$8 - 2x = 4r$

$64 - 32x + 4x^2 = 16r^2$

$64 - 32x + 4x^2 = 16x^2 + 16y^2$

$12x^2 + 16y^2 + 32x = 64$

$3x^2 + 4y^2 + 8x = 16$

ellipse..... Then, complete the square to put into standard form...

$$3\left(x^2 + \frac{8}{3}x + \frac{16}{9}\right) + 4y^2 = 16 + \frac{16}{3}$$

$$3\left(x + \frac{4}{3}\right)^2 + 4y^2 = \frac{64}{3}$$

$$\frac{9\left(x + \frac{4}{3}\right)^2}{64} + \frac{3y^2}{16} = 1$$

center: $\left(-\frac{4}{3}, 0\right)$

$a^2 = \dfrac{64}{9}$ $\quad b^2 = \dfrac{16}{3}$

$c^2 = a^2 - b^2 = \dfrac{16}{9}$

$a = \dfrac{8}{3}$ vertices: $\left(\frac{4}{3}, 0\right)$ $\left(-\frac{12}{3}, 0\right)$

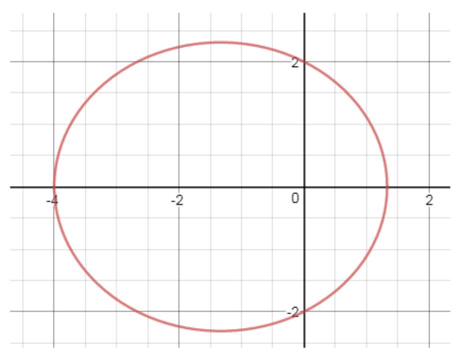
$b = \dfrac{4}{\sqrt{3}}$ co-vertices: $\left(-\frac{4}{3}, \frac{4}{\sqrt{3}}\right)$ $\left(-\frac{4}{3}, \frac{-4}{\sqrt{3}}\right)$

(-1.33, 2.31) (-1.33, -2.31)

$c = \dfrac{4}{3}$ foci: $(0, 0)$ $\left(-\frac{8}{3}, 0\right)$

directrix $= \dfrac{a^2}{c} = \dfrac{64/9}{4/3} = \dfrac{16}{3}$

so, 5.33 to the right of the center $x = 4$
and, 5.33 to the left of the center... $x = -20/3$



Sketch the following polar conic $r = \dfrac{12}{4 + 8\cos\theta}$

Using POLAR form

rewrite in standard polar form (by dividing by 4)

$$r = \frac{3}{1 + 2\cos\theta}$$

$$r = \frac{ep}{1 + e\cos\theta}$$

e = eccentricity

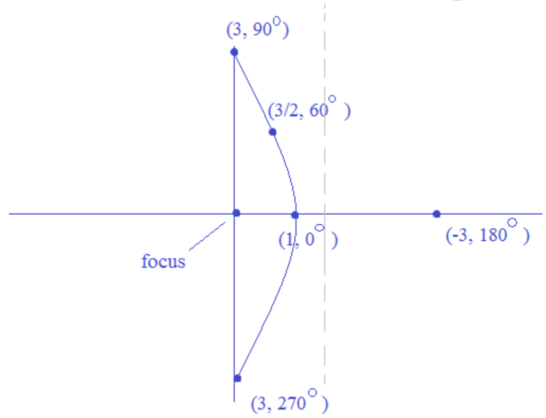
p is distance between focus and directrix
(and, focus is on the pole)

$e = 2$

since $ep = 3$, $p = \frac{3}{2}$

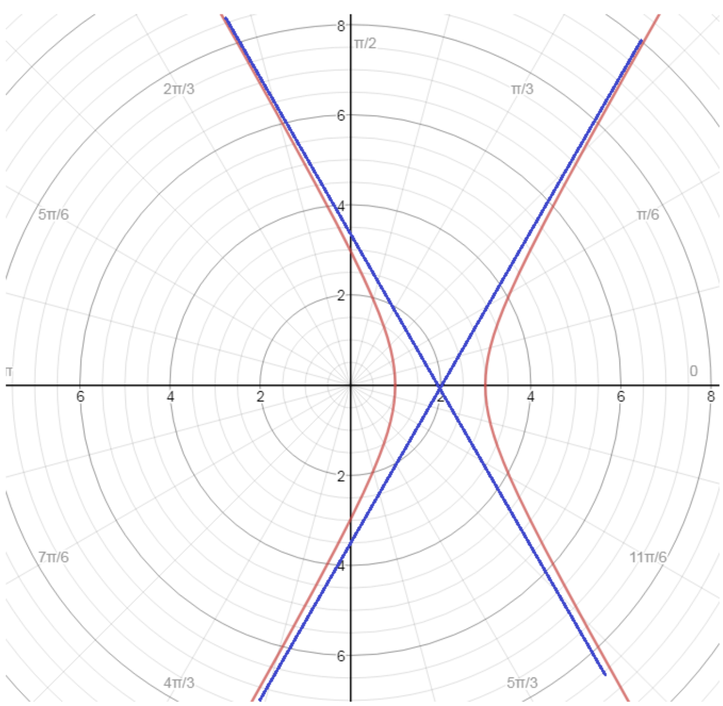
- since $e > 1$, it is a hyperbola
- since the trig function is $\cos\theta$, it is a horizontal hyperbola
- and, because it is positive, the left focus is on the pole

and, the left (vertical) directrix is $\frac{3}{2}$ units from focus...



***When $\theta = 120^\circ$ or 240° , the function is undefined!
(those are the asymptotes...)

and, the asymptotes will cross at $(2, 0^\circ)$ because 2 is midpoint of 1 and 3...



Using RECTANGULAR form

$x^2 + y^2 = r^2$

$x = r\cos\theta$

$$r = \frac{12}{4 + 8\frac{x}{r}}$$

cross multiply

$$12 = 4r + 8x$$

divide by 4 and rearrange

$$r = 3 - 2x$$

square both sides

$$r^2 = 9 - 12x + 4x^2$$

$$x^2 + y^2 = 9 - 12x + 4x^2$$

$$3x^2 - 12x - y^2 = -9$$

$$3(x^2 - 4x + 4) - y^2 = -9 + 12$$

$$3(x - 2)^2 - y^2 = 3$$

$$\frac{(x-2)^2}{1} - \frac{y^2}{3} = 1$$

$a^2 = 1$ $b^2 = 3$ $c^2 = 4$

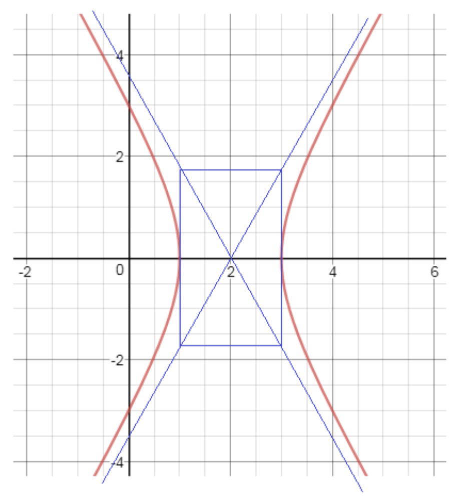
Horizontal hyperbola

center: (2, 0)

$c = 2$ foci: (0, 0) and (4, 0)

$a = 1$ vertices: (1, 0) and (3, 0)

$b = \sqrt{3}$ co-vertices: $(2, \sqrt{3})$ and $(2, -\sqrt{3})$



Note: the asymptotes have a slope of $\frac{\sqrt{3}}{1}$ and $-\frac{\sqrt{3}}{1}$

$\tan^{-1}(\sqrt{3}) = 60^\circ$, $\boxed{240^\circ}$ $\tan^{-1}(-\sqrt{3}) = \boxed{120^\circ}$

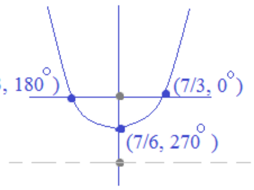
sketch $r = \dfrac{7}{3 - 3\sin\theta}$ identify the conic and main features (eccentricity, focus, vertices, etc..)

Using POLAR form

$$r = \frac{7/3}{1 - \sin\theta}$$

plotting 3 easy points...

(undefined at 90°)



$$r = \frac{ep}{1 - e\sin\theta}$$

eccentricity (e) = 1
distance between focus and directrix (p) = 7/3

- Since the coefficient of the trig function is 1, it is a parabola...
- Since it is $\sin\theta$, it is a vertical parabola...
- Since it is Negative sine, it opens upward (directrix is below the focus)

Note: there is a slight difference between "p" in polar form and "p" in rectangular form!

polar "p" is distance from directrix to focus
rectangular "p" is distance from vertex to focus

Using RECTANGULAR form

$$r = \frac{7}{3 - 3\left(\frac{y}{r}\right)}$$

$\boxed{x^2 + y^2 = r^2 \qquad y = r\sin\theta}$

cross multiply

$$3r - 3y = 7$$

$$3\left(\sqrt{x^2 + y^2}\right) - 3y = 7$$

$$3\left(\sqrt{x^2 + y^2}\right) = 3y + 7$$

$$9x^2 + 9y^2 = 9y^2 + 42y + 49$$

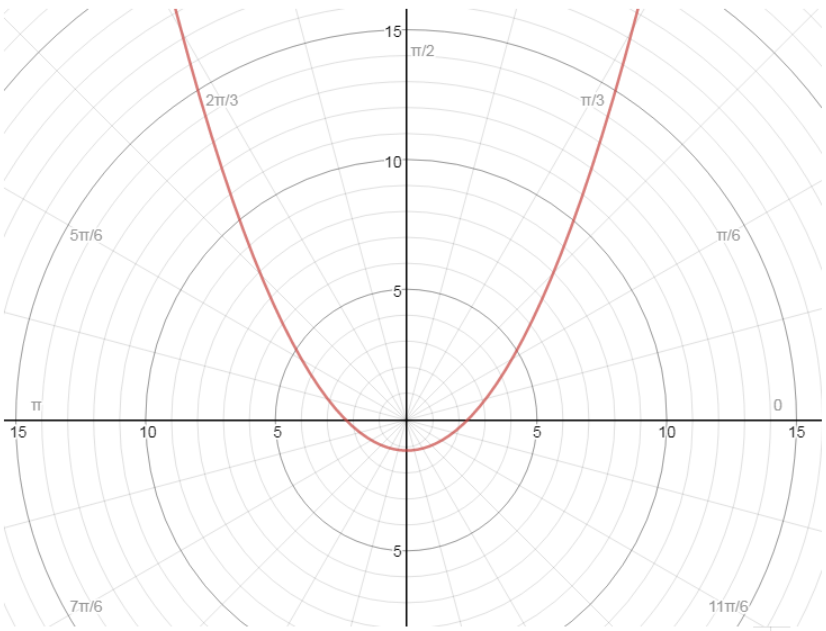
$$9x^2 = 42y + 49$$

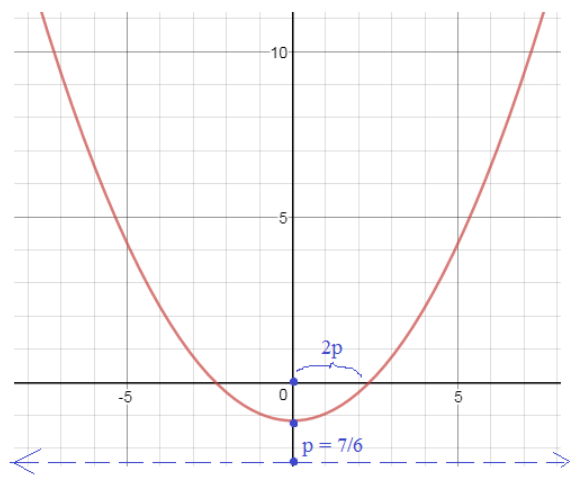
$$\boxed{y = \frac{9}{42}x^2 - \frac{7}{6}}$$

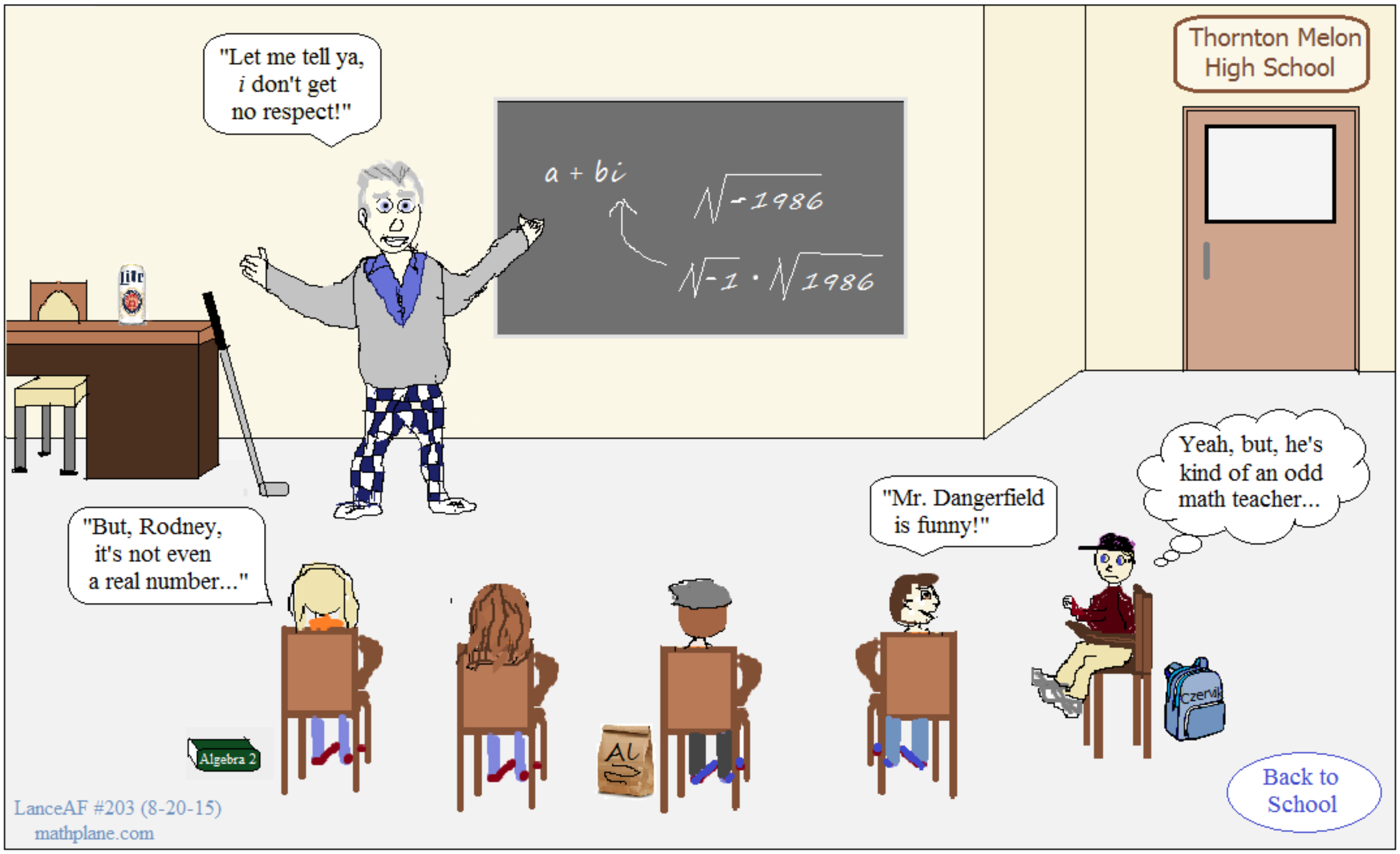
$$p = \frac{1}{4a} = \frac{1}{4(9/42)} = \frac{7}{6}$$

latus rectum = $4p = \dfrac{14}{3}$

vertex: (0, -7/6) directrix: y = -7/3 focus: (0, 0)







Thornton Melon
High School
"Let me tell ya, i don't get no respect!"
$a + bi$
$\sqrt{-1986}$
$\sqrt{-1} \cdot \sqrt{1986}$
"But, Rodney, it's not even a real number..."
"Mr. Dangerfield is funny!"
Yeah, but, he's kind of an odd math teacher...
Algebra 2
Al
Back to School
LanceAF #203 (8-20-15)
mathplane.com

Quick Quiz-→

A) Express $(3, \frac{5\pi}{6})$ where $r < 0$ and $-2\pi < \theta < 0$

where $r > 0$ and $-2\pi < \theta < 0$

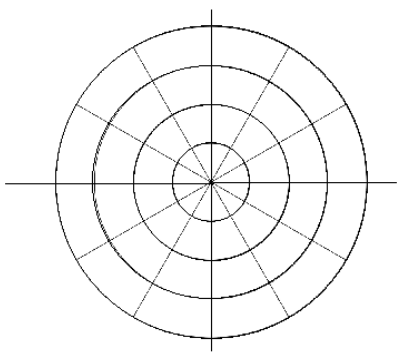
where $r < 0$ and $0 < \theta < 2\pi$

Express $(6, 225^\circ)$ where $r > 0$ and $-360^\circ < \theta < 0^\circ$

where $r < 0$ and $-360^\circ < \theta < 0^\circ$

where $r < 0$ and $0^\circ < \theta < 360^\circ$

B) On the polar graph, label the following coordinates:



$A = (2, \frac{2\pi}{3})$

$B = (-4, \frac{\pi}{4})$

$C = (0, \frac{\pi}{2})$

$D = (3, 0)$

$E = (-1, \frac{-5\pi}{3})$

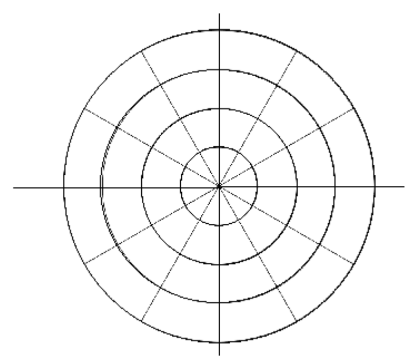
$G = (3, 90^\circ)$

$H = (-2, -150^\circ)$

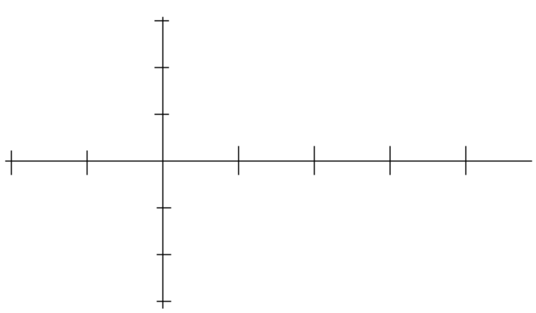
$I = (4, 390^\circ)$

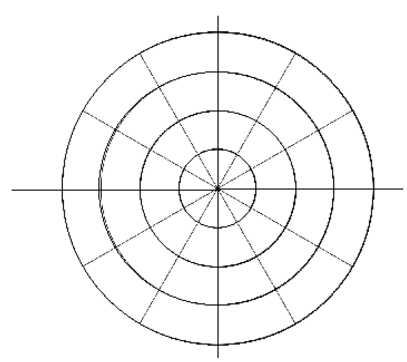
$J = (-1, 405^\circ)$

$K = (0, 60^\circ)$



C) Sketch $y = 3\sin x$ on the xy axis. then, sketch $r = 3\sin\theta$ on the polar graph





Polar/Rectangular Coordinates

1) Convert to polar coordinates. Give 2 answers where $0 \le \theta < 2\pi$

a) $(-5, 5\sqrt{3})$

b) $(0, 7)$

c) $(10, -24)$

2) Convert to rectangular coordinates

a) $\theta = \frac{5\pi}{6}$

b) $r = \frac{1}{3\cos\theta + 8\sin\theta}$

3) Convert $4x^2 + y^2 = 1$ into polar coordinates

4) Convert $r^2 = \cos 2\theta$ into rectangular coordinates

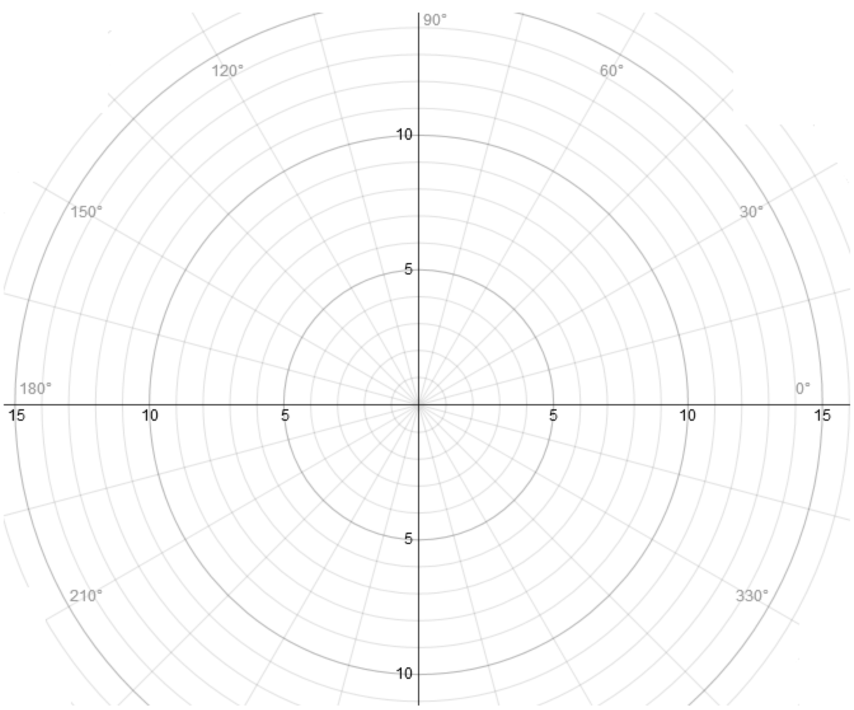
Polar/Rectangular Coordinates

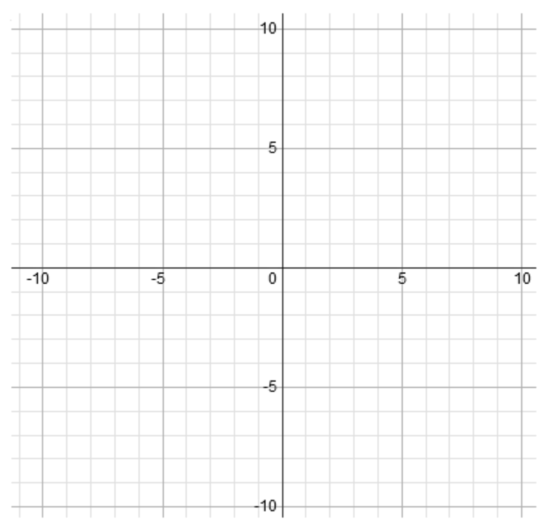
5) Convert the point (-4, 5) into polar coordinates.

Then, write the equation of a circle that passes through that point (in polar form)

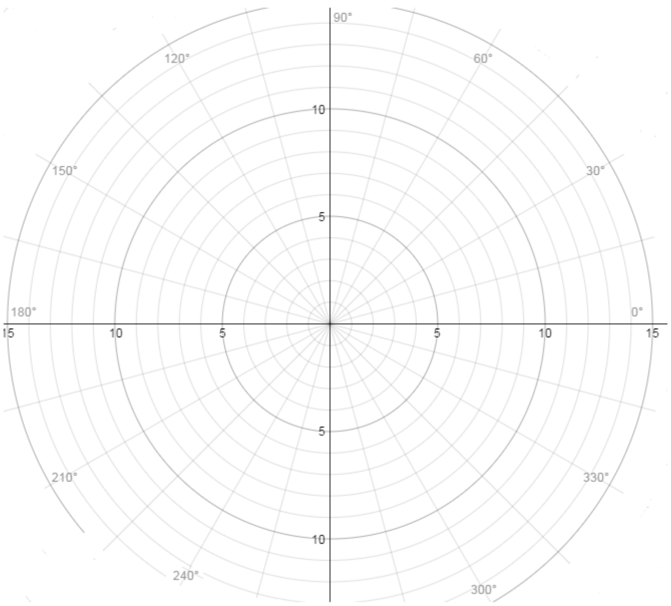
6) Convert $xy = 5$ into polar coordinates

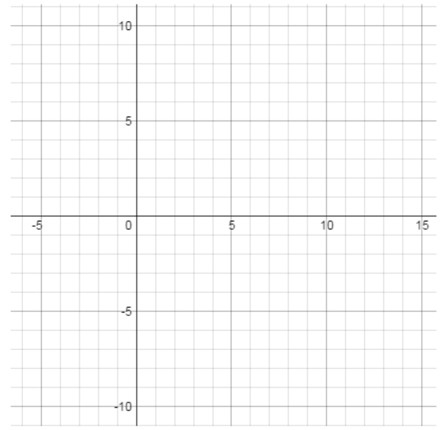
Sketch the graphs and compare...





7) $r = \dfrac{2}{1 - \cos\theta}$ Convert to rectangular coordinates.
Then, graph each equation to confirm.

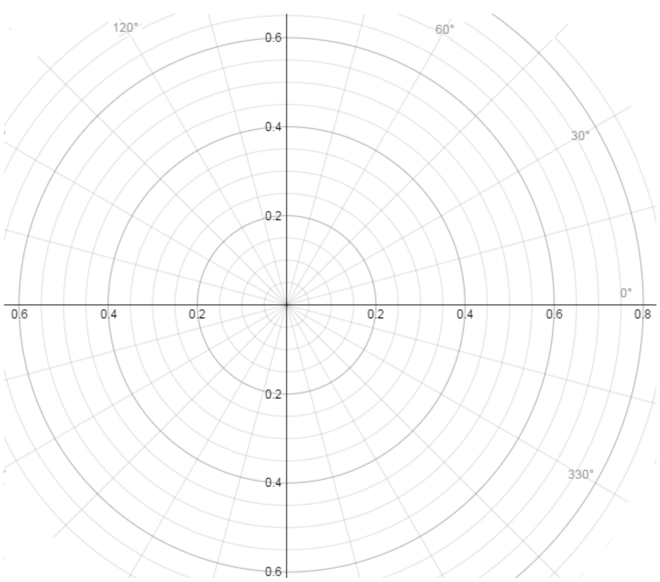


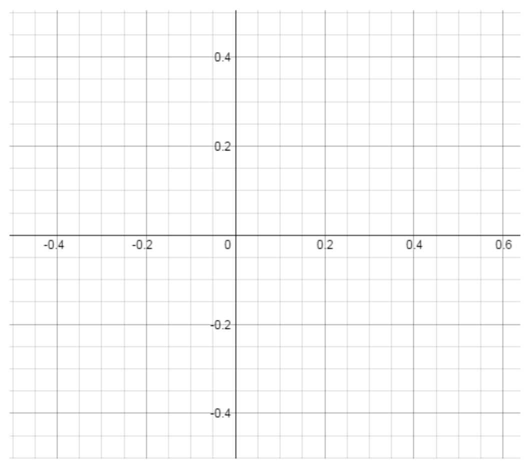


$r = \dfrac{2}{1 - \cos\theta}$

θ									
r									

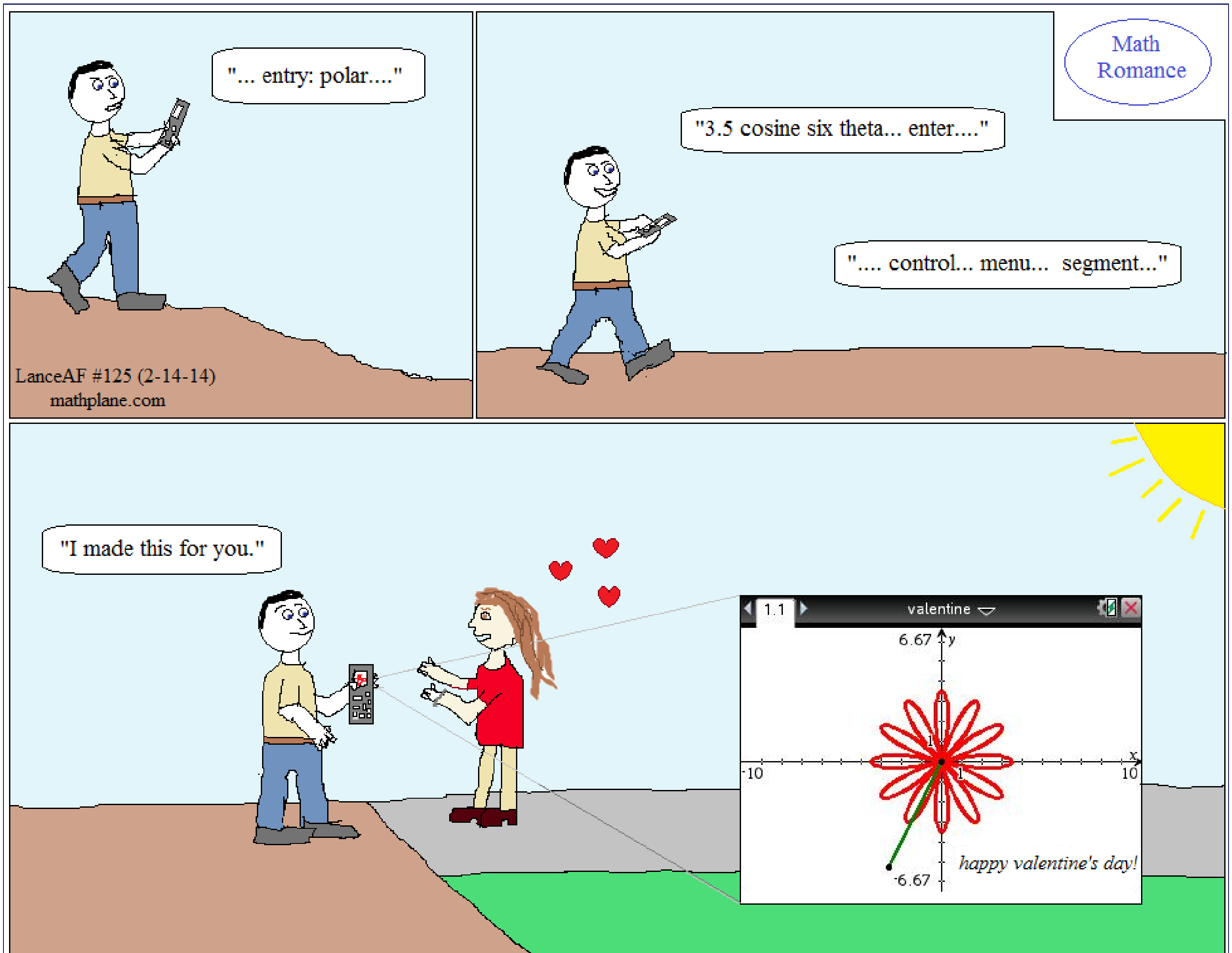
8) $r = \dfrac{1}{3 - \cos\theta}$ Convert to rectangular coordinates.
Then, graph each equation to confirm.





$r = \dfrac{1}{3 - \cos\theta}$

θ									
r									



A Valentine's Day Flower that lasts forever... (as long as you recharge the batteries!)

SOLUTIONS-→

Warm-Up

SOLUTIONS

Polar/Rectangular Coordinates

A) Express $(3, \frac{5\pi}{6})$ where $r < 0$ and $-2\pi < \theta < 0$ $\quad (-3, \frac{-\pi}{6})$

where $r > 0$ and $-2\pi < \theta < 0$ $\quad (3, -\frac{7\pi}{6})$

where $r < 0$ and $0 < \theta < 2\pi$ $\quad (-3, \frac{11\pi}{6})$

Express $(6, 225^\circ)$ where $r > 0$ and $-360^\circ < \theta < 0^\circ$

(find the coterminal angle..) $(6, -135^\circ)$

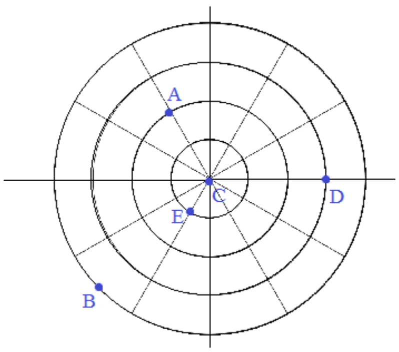
where $r < 0$ and $-360^\circ < \theta < 0^\circ$

$(-6, -315^\circ)$

where $r < 0$ and $0^\circ < \theta < 360^\circ$

$(-6, 45^\circ)$

B) On the polar graph, label the following coordinates:



$A = (2, \frac{2\pi}{3})$

$B = (-4, \frac{\pi}{4})$

$C = (0, \frac{\pi}{2})$

$D = (3, 0)$

$E = (-1, \frac{-5\pi}{3})$

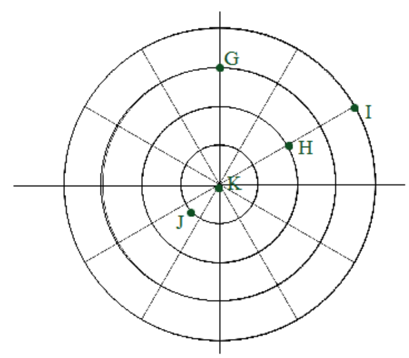
$G = (3, 90^\circ)$

$H = (-2, -150^\circ)$

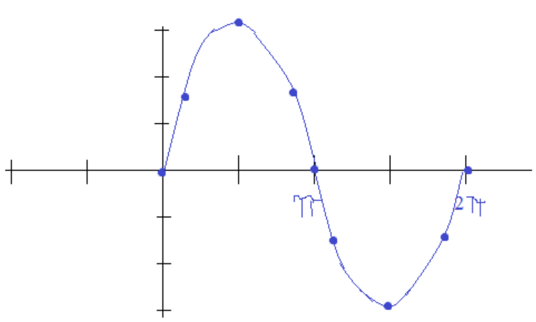
$I = (4, 390^\circ)$

$J = (-1, 405^\circ)$

$K = (0, 60^\circ)$



C) Sketch $y = 3\sin x$ on the xy axis. then, sketch $r = 3\sin\theta$ on the polar graph

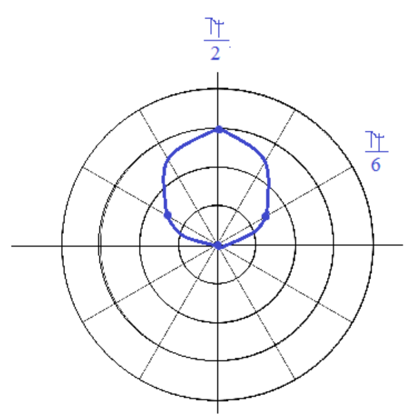


$3\sin(0) = 0$

$3\sin(\frac{\pi}{6}) = 1.5$

$3\sin(\frac{\pi}{2}) = 3$

etc...



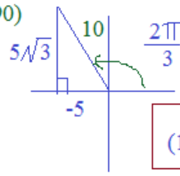
SOLUTIONS

Polar/Rectangular Coordinates

1) Convert to polar coordinates. Give 2 answers where $0 \le \theta < 2\pi$

a) $(-5, 5\sqrt{3})$

(30-60-90) right triangle



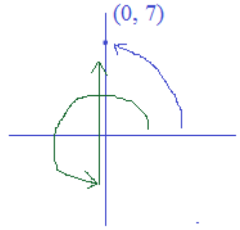
$\boxed{(10, \frac{2\pi}{3})}$

$\boxed{(-10, \frac{5\pi}{3})}$

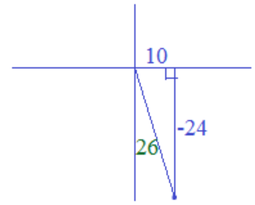
b) (0, 7)

$\boxed{(7, \frac{\pi}{2})}$

$\boxed{(-7, \frac{3\pi}{2})}$



c) $(10, -24)$



$\tan^{-1}(-24/10)$

$(26, -67.38^{\circ})$

$(26, -1.18)$

$+2\pi$

$\boxed{(26, 5.10)}$

$(-26, 112.62^{\circ})$

$\boxed{(-26, 1.96)}$

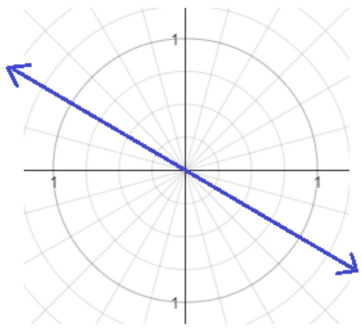
2) Convert to rectangular coordinates

a) $\theta = \frac{5\pi}{6}$

$\tan\theta = \tan\frac{5\pi}{6}$

$\frac{y}{x} = \frac{1}{-\sqrt{3}}$ $\quad x = -\sqrt{3}\,y$

$\boxed{y = \frac{1}{-\sqrt{3}}x}$



b) $r = \dfrac{1}{3\cos\theta + 8\sin\theta}$

$r(3\cos\theta + 8\sin\theta) = 1$

$3r\cos\theta + 8r\sin\theta = 1$

$\boxed{3x + 8y = 1}$

$x = r\cos\theta$

$y = r\sin\theta$

3) Convert $4x^2 + y^2 = 1$ into polar coordinates

(ellipse)

$x = r\cos\theta \quad y = r\sin\theta$

$4r^2\cos^2\theta + r^2\sin^2\theta = 1$

$r^2(4\cos^2\theta + \sin^2\theta) = 1$

$r^2 = \dfrac{1}{(4\cos^2\theta + \sin^2\theta)}$

$\boxed{r = \sqrt{\dfrac{1}{(4\cos^2\theta + \sin^2\theta)}}}$

4) Convert $r^2 = \cos 2\theta$ into rectangular coordinates

$x^2 + y^2 = \cos^2\theta - \sin^2\theta$

$x^2 + y^2 = \left(\dfrac{x}{\sqrt{x^2+y^2}}\right)^2 - \left(\dfrac{y}{\sqrt{x^2+y^2}}\right)^2$

$x = r\cos\theta \qquad \cos\theta = \dfrac{x}{r} = \dfrac{x}{\sqrt{x^2+y^2}}$

$y = r\sin\theta \qquad \sin\theta = \dfrac{y}{r} = \dfrac{y}{\sqrt{x^2+y^2}}$

$x^2 + y^2 = \dfrac{x^2 - y^2}{x^2 + y^2}$

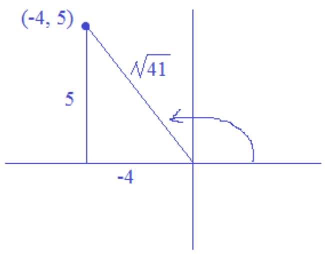
$(x^2 + y^2)^2 = x^2 - y^2$

$\boxed{x^4 + 2x^2y^2 + y^4 - x^2 + y^2 = 0}$

SOLUTIONS

5) Convert the point (-4, 5) into polar coordinates.

Then, write the equation of a circle that passes through that point (in polar form)



Use Pythagorean Theorem to get r
Or,

$x^2 + y^2 = r^2$

$\tan\theta = \frac{y}{x} = \frac{5}{-4} = -51.3$

$\boxed{(\sqrt{41}, 128.7^\circ) \text{ or } (\sqrt{41}, 2.246)}$

**since the point is in Quadrant II, add 180 degrees....

$-51.3 + 180 = 128.7^\circ$ or 2.246 radians

The radius is $\sqrt{41}$ so the circle is all points (in every direction) $\sqrt{41}$ from the origin..

$\boxed{r = \sqrt{41}}$

(Note: There are an infinite number of circles that can pass through (-4, 5)... We chose the one where the center is at the origin)

6) Convert xy = 5 into polar coordinates
Sketch the graphs and compare...

$r\cos\theta\,(r\sin\theta) = 5$

(y = 5/x reciprocal function)

$r^2 \sin\theta\cos\theta = 5$

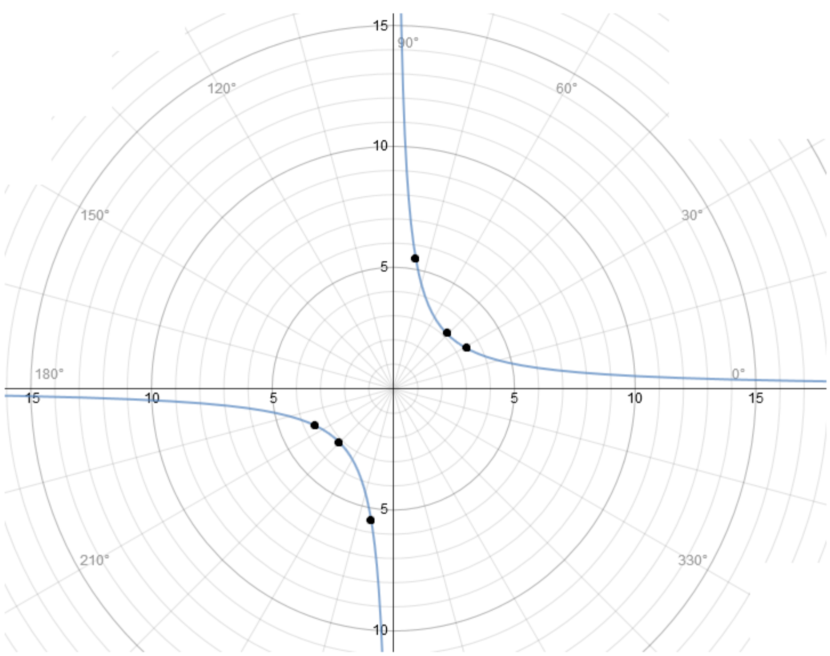
$\boxed{r = \sqrt{\dfrac{5}{\sin\theta\cos\theta}}}$

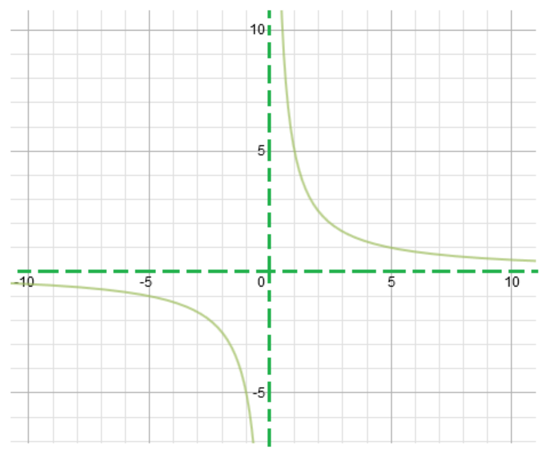
Note: the equation is undefined at 0, 90, 180, and 270!

Note: asymptotes are the x and y-axis!

(negative values)

θ	0	30	45	80	120	150	170
$\pm r$	DNE	3.4	3.16	5.4	DNE	DNE	DNE





7) $r = \dfrac{2}{1 - \cos\theta}$ Convert to rectangular coordinates. Then, graph each equation to confirm.

cross multiply: $r - r\cos\theta = 2$

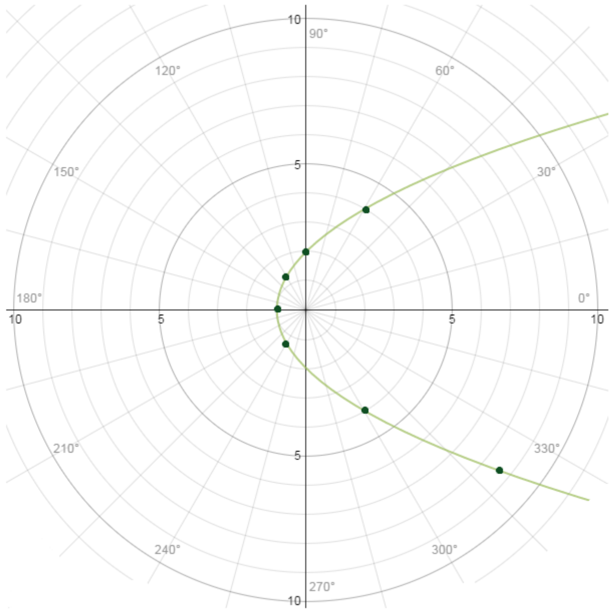
$$\sqrt{x^2 + y^2} - x = 2$$

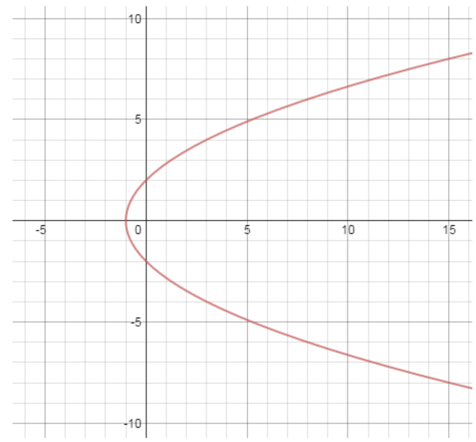
$$\sqrt{x^2 + y^2} = 2 + x$$

$$x^2 + y^2 = x^2 + 4x + 4$$

$$y^2 = 4(x + 1)$$

Parabola!





Vertex: (-1, 0)

Focus: (0, 0)

Directrix: x = -1

$r = \dfrac{2}{1 - \cos\theta}$

θ	0	60	90	120	180	240	300	320	350
r	DNE	4	2	4/3	1	4/3	4	8.55	131.6

8) $r = \dfrac{1}{3 - \cos\theta}$ Convert to rectangular coordinates. Then, graph each equation to confirm.

$$3r - r\cos\theta = 1$$

$$3\sqrt{x^2 + y^2} - x = 1$$

$$3\sqrt{x^2 + y^2} = x + 1$$

$$9x^2 + 9y^2 = x^2 + 2x + 1$$

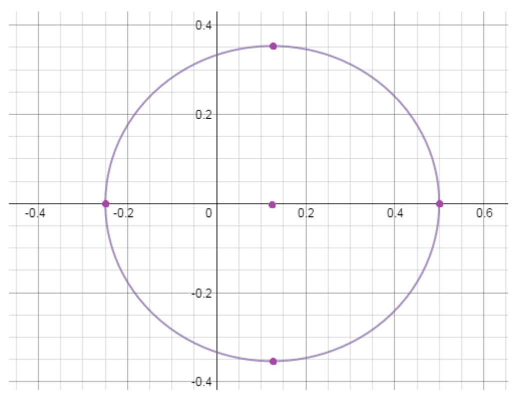
$$8x^2 + 9y^2 - 2x = 1$$

$$8(x^2 - \frac{1}{4}x + \frac{1}{64}) + 9y^2 = 1 + \frac{1}{8}$$

$$8(x - \frac{1}{8})^2 + 9y^2 = \frac{9}{8}$$

$$\frac{(x - \frac{1}{8})^2}{\frac{9}{64}} + \frac{y^2}{\frac{1}{8}} = 1$$

Ellipse!



center: (1/8, 0)

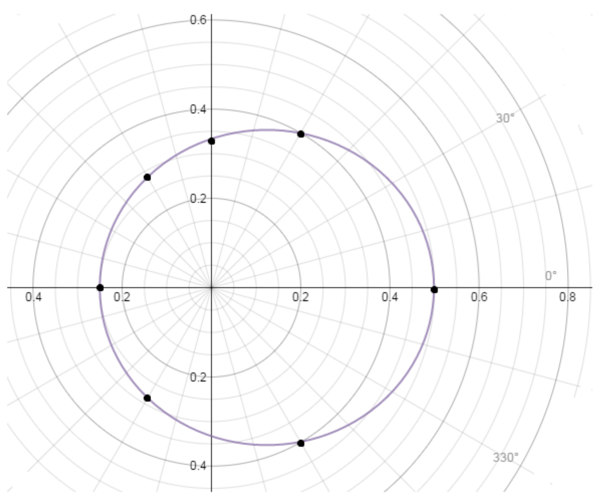
vertices: (1/2, 0) (-1/4, 0)

covertices: $(1/8, \sqrt{2}/4)$

$(1/8, -\sqrt{2}/4)$

$r = \dfrac{1}{3 - \cos\theta}$

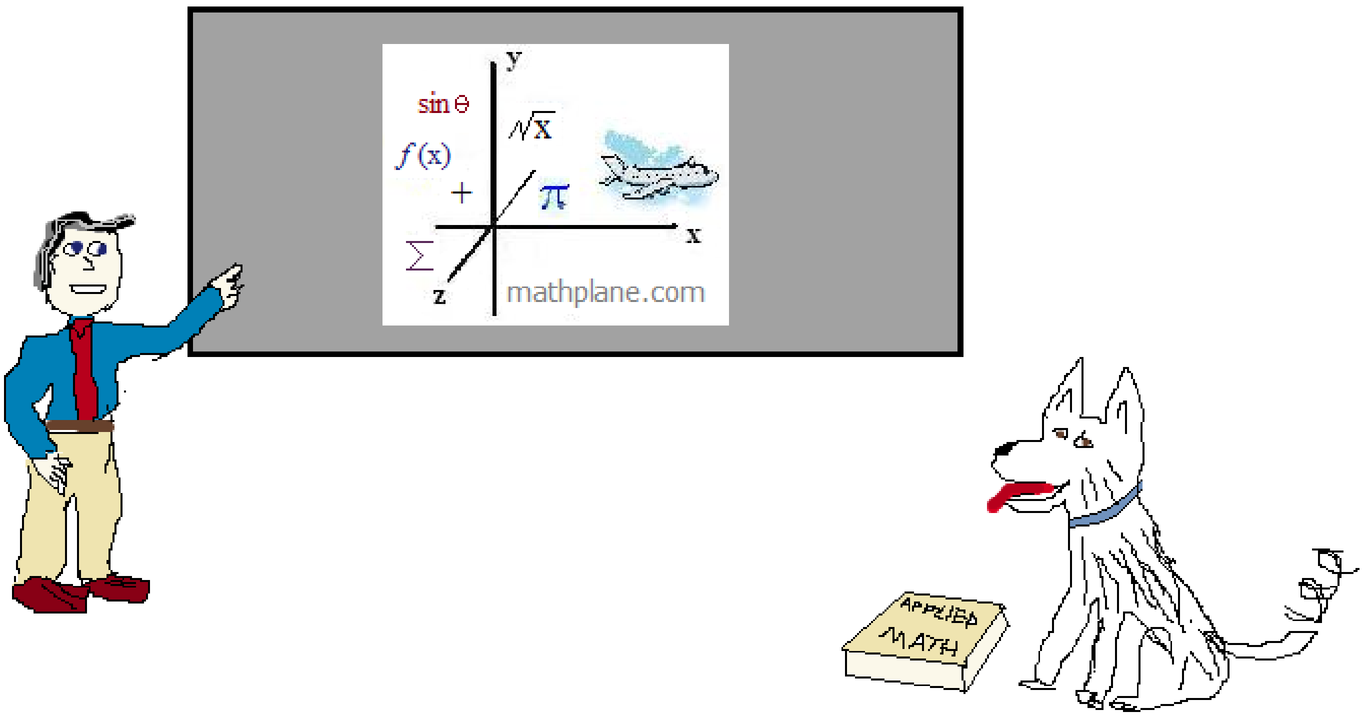
θ	0	60	90	120	180	240	300	360
r	1/2	2/5	1/3	2/7	1/4	2/7	2/5	1/2



Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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