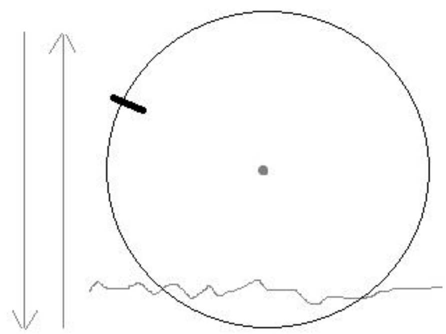
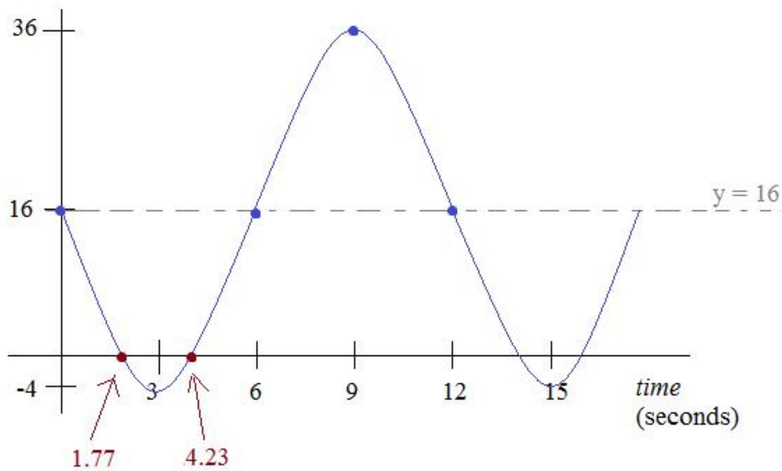
Periodic Trig Function Models - Word Problems





A ferris wheel is 4 feet off the ground. It has a diameter of 26 feet, and rotates once every 32 seconds. If you begin the ride sitting in a chair that is 6 feet above the ground, how high will you be 10 seconds into the ride? During the first minute, when will you be 20 feet high?

To solve, let's model your position. (write an equation that shows your *height as a function of time*)

$y = A\sin B(x - C) + D$

Step 1: Find the vertical shift

The bottom of the ferris wheel is 4 feet off the ground.
Since the diameter is 26 feet, the top of the ferris wheel is 30 feet off the ground.

The 'wave center' of the sine function is $\frac{(4 + 30)}{2}$ (halfway between the maximum and minimum)

vertical shift $D = +17$

Step 2: Find the amplitude

The diameter of the ferris wheel is 26 feet. The amplitude is 1/2 the distance from top to bottom.

amplitude $A = 13$

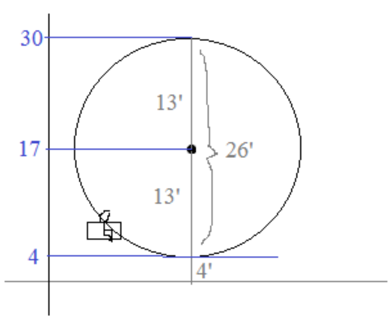
Step 3: Find the period

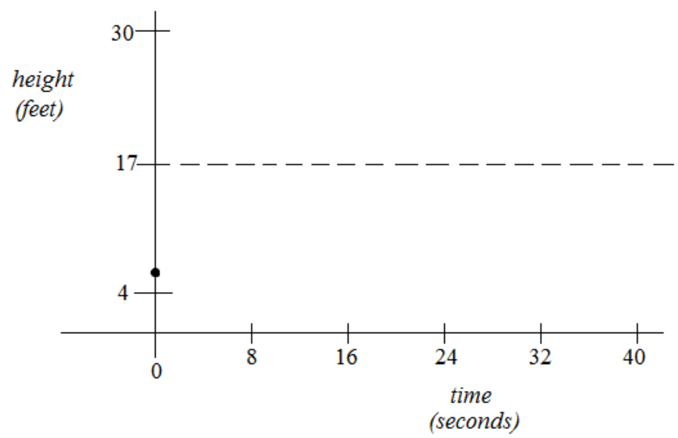
The ferris wheel rotates once every 32 seconds. The period is 32 seconds.

Since $B = \frac{2\pi}{\text{period}} = \frac{2\pi}{32 \text{ sec.}}$

(B is the cycles per 2π)

The value $B = \frac{\pi}{16}$





Step 4: Find the horizontal shift

To find the horizontal shift, we'll plug in the initial position:

$y = A\sin B(x - C) + D$

$$6 = 13\sin\frac{\pi}{16}(0 - C) + 17$$

$$-11 = 13\sin\frac{\pi}{16}(0 - C)$$

$$\frac{-11}{13} = \sin\frac{\pi}{16}(0 - C)$$

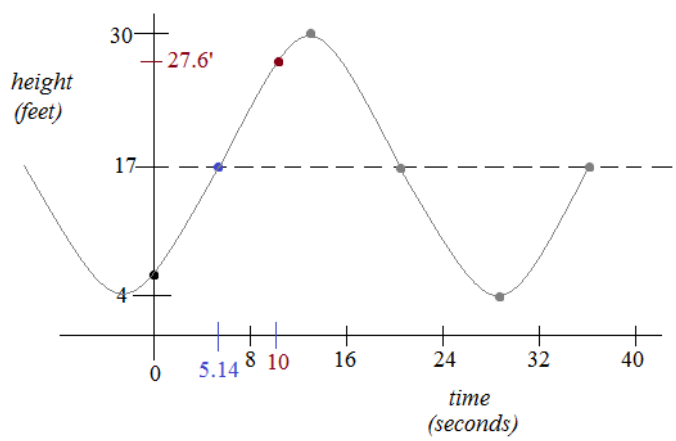
(inverse sine of both sides)

$$-1.009 \text{ radians} = \frac{\pi}{16}(0 - C)$$

$$-5.14 = (0 - C)$$

$$C = 5.14$$

The horizontal shift is $C = 5.14$
(5.14 seconds to the right!)



Step 5: Answer the questions using the function!

$$y = 13\sin\frac{\pi}{16}(x - 5.14) + 17$$

Ten seconds into the ride: $y = 13\sin\frac{\pi}{16}(10 - 5.14) + 17$

$x = 10$

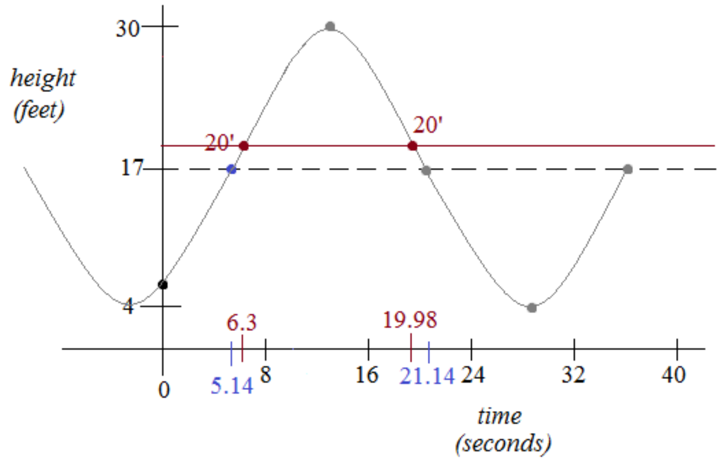
$y = 13\sin(.954) + 17 \qquad = 27.6$ feet

$y = 13\sin\frac{\pi}{16}(x - 5.14) + 17$

When will you be 20 feet high? $y = 20$

$$20 = 13\sin\frac{\pi}{16}(x - 5.14) + 17$$

$$\frac{3}{13} = \sin\frac{\pi}{16}(x - 5.14) \qquad x = 6.3 \text{ seconds...}$$



Since you're 20 feet high at 6.3 seconds, you'll be the same height every 32 seconds afterwards....

6.3 seconds, 38.3 seconds, 70.3 seconds, etc...

That accounts for the chair *going up*..

The difference between 5.14 and 6.3 is 1.16 seconds...
Considering the symmetry of the sine function, subtract 1.16 seconds from 21.14 --- 19.98 seconds

Since you're 20 feet high at 19.98 seconds, you'll be the same height every 32 seconds afterwards....

19.98 seconds, 51.98 seconds, 83.98 seconds, etc....

That accounts for the chair *going down*...

During the first minute, the chair will be 20 feet above ground at 6.3, 19.98, 38.3, and 51.98 seconds....

A car's tire has a diameter of 32 inches. It runs over a nail, but it is able to continue moving. Write the cosine function that describes the *height of the nail above the ground* as a function of the wheel's *angular distance*.

For discussion: Assume the tire rims cover the top 1/2 of each tire. When will the nail be visible?

Step 1: Identify the vertical shift

$y = A\cos B(x - C) + D$

The tire is on the ground. (the minimum is 0 feet above ground)
The top of the tire is the highest the nail will reach. (the maximum is 32 inches above ground)

Vertical shift $D = +16$

The vertical shift is the halfway point between min and max. 16 inches

Step 2: Identify the amplitude

Amplitude $A = \cancel{16}$

The radius of the tire is 16 inches.

$A = -16$

Since the nail begins at the *bottom of the tire* (i.e. the minimum), the cosine function is *negative*!

Step 3: Find the horizontal shift

At the instance of the nail sticking into the tire, we begin the cosine function.
Therefore, there is no horizontal shift.

horizontal shift $C = 0$

Step 4: Find the cycle (distance of each rotation) and period

Since the function will consist of angular distance, we'll use 360 degrees for each cycle.

$B = 1$

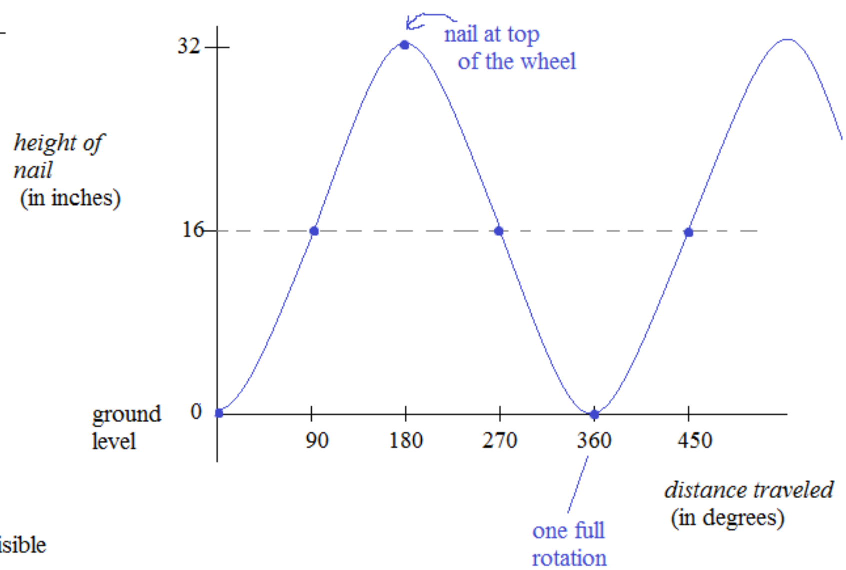
The B value = $\dfrac{2\cancel{\pi}}{360 \text{ degrees}}$

The cosine function is

$y = -16\cos 1(\theta - 0) + 16$

where y = height of nail above ground
θ is angular distance traveled

$y = -16\cos\theta + 16$



If the tire's rim covers the top half, then the nail will be visible in the following intervals:

$0 < \theta < 90 \qquad 270 < \theta < 450$ etc...

AND $\theta \neq 360n$

because the nail is under the tire!

Study Break:
Math Snacks

LanceAF #35 6-3-12
www.mathplane.com

Preferable to ordinary computer cookies...

Essential part of a well-rounded, academic diet.

Try with (t), or any beverage...

Also, look for Honey Graham Squares
in the geometry section of your local store...

Each day, the tide continuously goes in and out, raising and lowering a boat (sinusoidally) in the harbor.
At low tide, the boat is only 2 feet above the ocean floor.
And, 6 hours later, at peak high tide, the boat is 40 feet above the ocean floor.
Write a sine function that describes the boat's *distance above the ocean floor as it relates to time.*

For safety, the boat needs 14 feet of depth to sail.
If high tide *occurs at noon*, between what times can the boat go out to sea?

$y = A\sin B(x - C) + D$

Step 1: Find the vertical shift

The low point is 2 feet. (function minimum)
The high point is 40 feet. (function maximum)
The median (halfway point) is 21 feet.

Vertical shift D: $+21$

Step 2: Find the amplitude

The low point is 2 feet. The high point is 40 feet.
Amplitude = 1/2 the range 1/2(40 - 2) = 19

Amplitude A: 19

Step 3: Find the period

The tide goes from low to high to low in 12 hours. (i.e. one cycle is 12 hours)

$$B = \frac{2\pi}{\text{period}} = \frac{2\pi}{12 \text{ hours}}$$

B: $\frac{\pi}{6}$

Step 4: Determine a horizontal shift

High tide occurs at noon. (max)
Low tide occurs at 6:00am (min) or 6:00pm.
Then, the middle occurs at 9:00am or 3:00pm.
In this case, we'll use a 3 hour horizontal shift to the left.

Horizontal shift C: 3

Step 5: Write the equation (and check)

$$y = 19\sin\frac{\pi}{6}(x + 3) + 21$$

We can sketch the graph and check points to confirm our model:

let x = -6: $y = 19\sin\frac{\pi}{6}(-6 + 3) + 21$

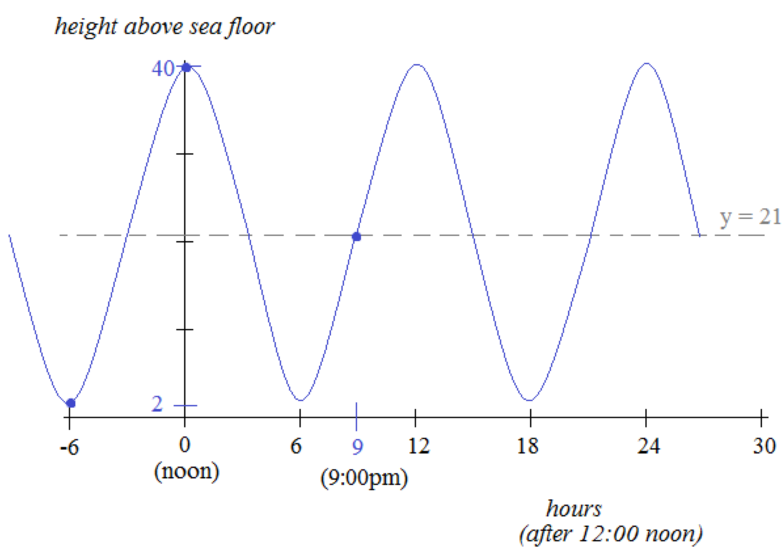
$y = 19\sin\left(-\frac{\pi}{2}\right) + 21 = 2$ ✓

let x = 0: (noon) $y = 19\sin\frac{\pi}{6}(0 + 3) + 21$

$y = 19\sin\frac{\pi}{2} + 21 = 40$ ✓

let x = 9: (9:00pm) $y = 19\sin\frac{\pi}{6}(9 + 3) + 21$

$y = 19\sin 2\pi + 21 = 21$ ✓



For safety, the boat may go out to sea whenever the graph is above y = 14

$$y = 19\sin\frac{\pi}{6}(x + 3) + 21$$

To find critical values, let y = 14

$$14 = 19\sin\frac{\pi}{6}(x + 3) + 21$$

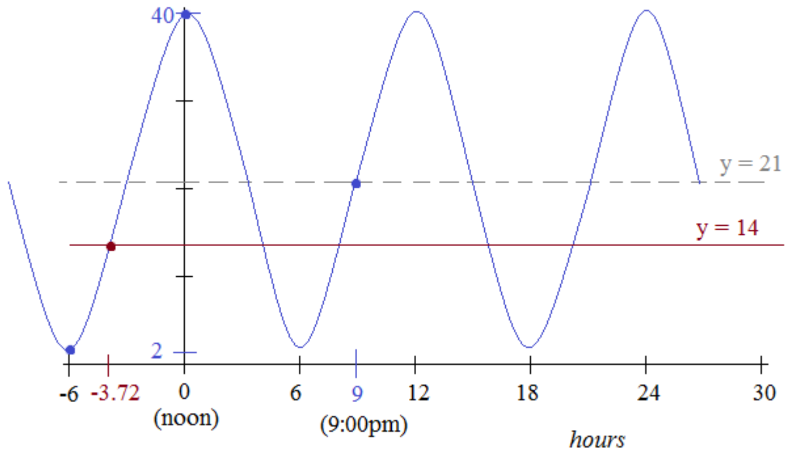
$$-7 = 19\sin\frac{\pi}{6}(x + 3)$$

$$\frac{-7}{19} = \sin\frac{\pi}{6}(x + 3)$$

$$\sin^{-1}\left(\frac{-7}{19}\right) = \frac{\pi}{6}(x + 3)$$

$$-.72 = (x + 3)$$

x = -3.72 approximately 8:17am

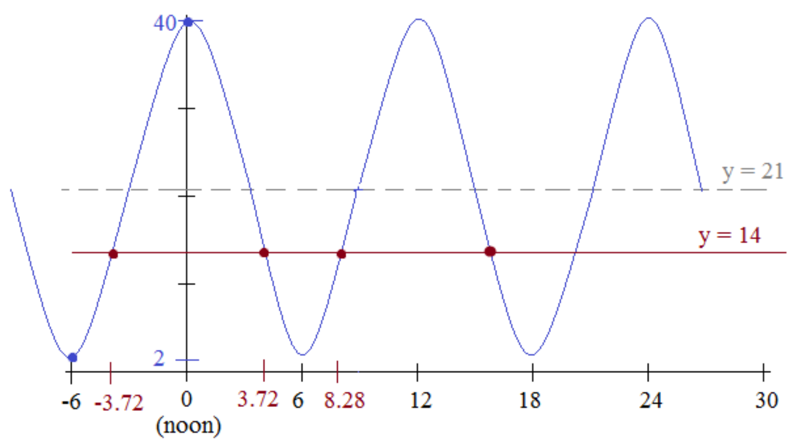


Then, to find other critical values, observe the graph...

-3.72 is 2.28 from -6 (the minimum)

And, 2.28 from 6 is 3.72...

x = 3.72 approximately 3:43pm



The boat is safe to sail out to sea between 8:17am and 3:43pm.. And, between 8:17pm and 3:43am...

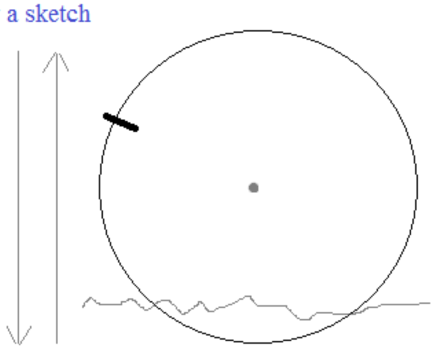
The following trig function models the position of a rung on a waterwheel:

$$y = -20\sin\left(\frac{\pi}{6}t\right) + 16$$

where t = seconds
y = number of feet *above* water level

a) What is the diameter of the wheel?
b) At the top of the wheel, how high is the rung above water level?
c) How many rotations per minute does the wheel make?
d) What percentage of time does a rung spend under water?

Step 1: Draw a sketch



Step 2: Identify the measurements

amplitude: 20 feet (distance from middle to peak)

vertical shift: up 16 feet (position of sine *wave center*)

horizontal shift: None

period: $\frac{2\pi}{B} = \frac{2\pi}{\pi/6} = 12$ seconds

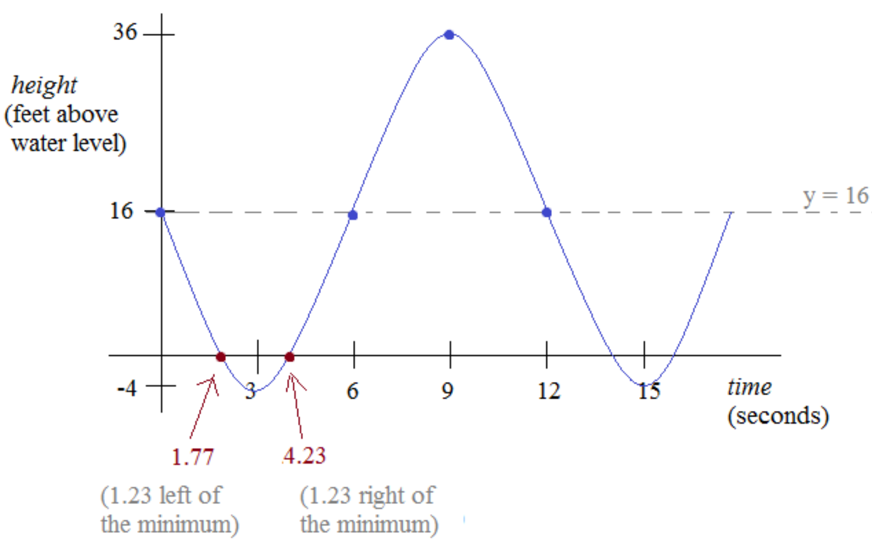
Step 3: Answer the questions

a) Since the amplitude is 20 feet, the $\boxed{\text{diameter of the wheel is 40 feet}}$

b) Since the vertical shift is up 16 feet, the new 'wave center' is y = 16.
Therefore, the top of the waterwheel is 16 + 20 = $\boxed{\text{36 feet above the water level.}}$
(and, the bottom is -4 feet or, 4 feet under water.)

c) The period (one rotation) is 12 seconds. Therefore, the $\boxed{\text{wheel rotates 5 times per minute.}}$

d) To determine when the rung is under water, let's sketch the graph:



To find when the rung is at water level,
let y = 0

$$0 = -20\sin\left(\frac{\pi}{6}t\right) + 16$$

$$-16 = -20\sin\left(\frac{\pi}{6}t\right)$$

$$\frac{4}{5} = \sin\left(\frac{\pi}{6}t\right)$$

(radian mode)

$$\sin^{-1}(.80) = \left(\frac{\pi}{6}t\right)$$

$$.927 \approx .523t$$

$$t \approx 1.77$$

Then, the rung is under water level from 1.77 seconds to 4.23 seconds.

Therefore, every 12 seconds, the rung is under water approximately 2.46 seconds.

$\frac{2.46}{12} = .205$ or $\boxed{\text{20.5\% of the time}}$

The motion of a swing hanging from a tree next to a lake can be modeled by a sinusoid.
The tree is 10 feet from the water, and the swing can extend 20 feet from the tree in each direction.
If it takes 2 seconds to swing from one side to the other side,

a) write an equation that models the position of the swing as a function of time.
b) determine the interval of time that the swing is above the water?
c) what percentage of time is the swing above the water?

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a) write an equation that models the position of the swing as a function of time.

$y = A\cos B(x - C) + D$

When the swing is at rest, its position is 0.
So, its maximum is 20 and its minimum is -20

Since the motion of the swing starts at one extreme, we'll use a cosine function...

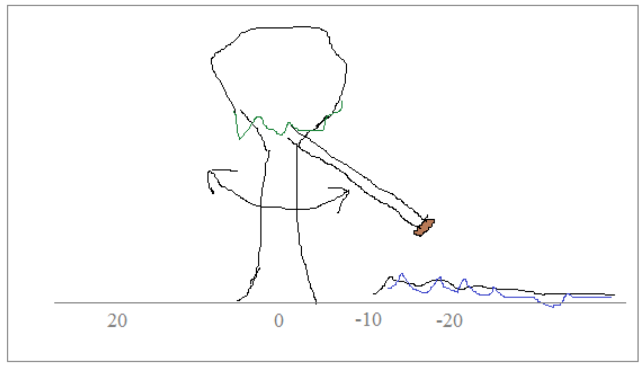
A: (Amplitude): 20

B: Since it takes 2 seconds to swing from one side to the other, the period for one full cycle is 4 seconds

$$B = \frac{2\pi}{4} = \frac{\pi}{2}$$

C: (Horizontal Shift): None for a cosine model

D: (Vertical Shift): None, because the axis of wave (middle) occurs when the swing is even with the tree (0)



$d(t) = 20\cos(\frac{\pi}{2}t)$ where t is the time in seconds
and $d(t)$ represents the distance from the tree
where positive values are feet left of the tree
and negative values are feet right of the tree (toward the water)

b) what interval of time is the swing above the water?

The swing is above the edge of the water when $d(t) = -10$

So, when does that occur? $-10 = 20\cos(\frac{\pi}{2}t)$

$$\frac{-1}{2} = \cos(\frac{\pi}{2}t)$$

$$\cos^{-1}\left(\frac{-1}{2}\right) = \frac{\pi}{2}t$$

$$\frac{2\pi}{3} = \frac{\pi}{2}t \implies t = \frac{4}{3}$$

$$\frac{4\pi}{3} = \frac{\pi}{2}t \implies t = \frac{8}{3}$$

[1.33, 2.67]

[5.33, 6.67]

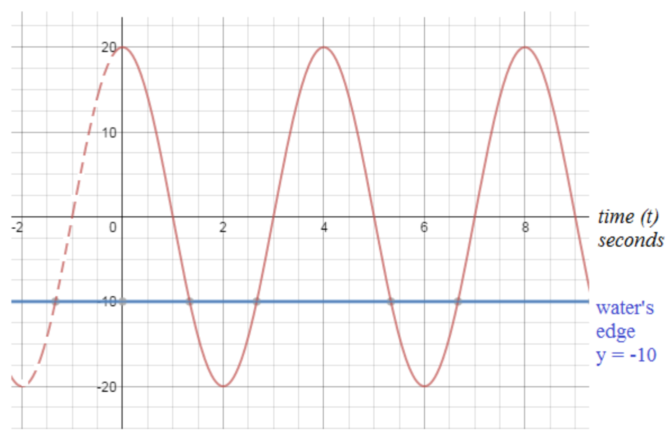
[9.33, 10.67]

etc...

So, during a 4 second ride, the swing is over the water between

1.33 and 2.67 seconds

Since the graph's horizontal axis is time (t), the domain is $t \geq 0$



The intervals where the red cosine curve is below the horizontal line y =-10 represent the time when the swing is over the water.

c) what percentage of time is the swing above the water?

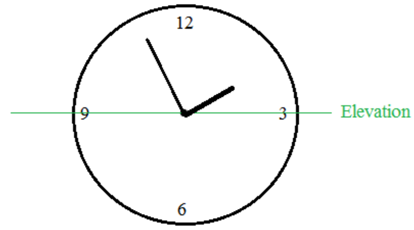
during a 4 second swing back and forth, you're over the water for 1.33 seconds...

$$\frac{1.33}{4} = .33 \quad \text{(33\% of the time)}$$

A clock with a 14 inch diameter has a minute hand that is 6" long and an hour hand that is 4" in length.
Assume a line drawn from the 9 to the 3 represents an elevation of zero.

a) Write a sine model showing the position (elevation) of the minute hand tip as it relates to time.
   What is the elevation of the minute hand tip at 53 minutes past the hour?
   When is the elevation of the minute hand tip 2 inches below the elevation line?

b) Write a cosine model that describes the position of the hour hand tip as it relates to time.
   What is the elevation of the hour hand tip at 5:00?
   When is the elevation 3 inches above the elevation line?



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a) Write a sine model showing the position (elevation) of the minute hand tip as it relates to time.

$y = A\sin B(x - C) + D$

A: Amplitude -- since length of minute hand is 6"   6

B: Period -- one cycle is 60 minutes   The "B" value is $\frac{2\pi}{60} = \frac{\pi}{30}$

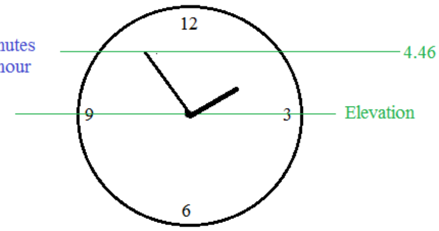
C: Horizontal shift -- since it is a sine function, the cycle starts at "9"... C = 45 (minutes)

D: Vertical shift -- none

$y = 6\sin\frac{\pi}{30}(t - 45)$ where t is minutes past the hour...

Minute "snapshot"
1:53

at :53 minutes past the hour



> Note: If you use a cosine model, amplitude, period, and vertical shift are the same...
>
> $y = 6\cos(\frac{\pi}{30})t$ where t is minutes past the hour (i.e. movement past 12)

(radians)

at 53 minutes past the hour, $y = 6\sin\frac{\pi}{30}(53 - 45) = \boxed{4.46}$

2 inches below occurs at $-2 = 6\sin\frac{\pi}{30}(t - 45)$

$\sin^{-1}(-\frac{2}{6}) = \sin^{-1}(\sin\frac{\pi}{30}(t - 45))$

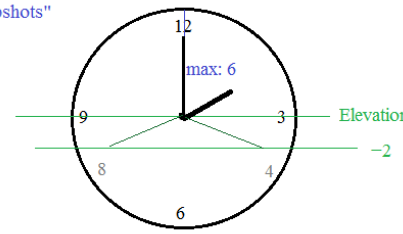
$-.34 = \frac{\pi}{30}(t - 45)$ $3.48 = \frac{\pi}{30}(t - 45)$

$-3.25 = (t - 45)$ $33.25 = (t - 45)$

$t = 41.75$ $\boxed{\text{41 minutes, 45 seconds}}$ $t = 78.25$ or 18.25 $\boxed{\text{18 minutes, 15 seconds}}$

Minute "snapshots"

y = 6
y = -2



b) Write a cosine model that describes the position of the hour hand tip as it relates to time.

$y = A\cos B(x - C) + D$

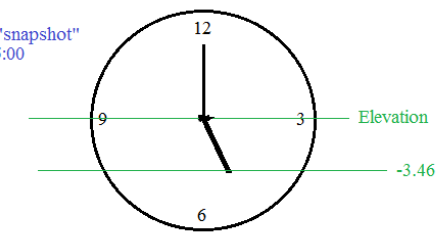
A: Amplitude -- since the length of the hour hand is 4"   4

B: Period -- once cycle is 12 hours   The "B" value is $\frac{2\pi}{12} = \frac{\pi}{6}$

C: Horizontal shift -- since it is a cosine function, the cycle starts at the max: "12"   so, there is NO shift

D: Vertical shift -- none $y = 4\cos\frac{\pi}{6}t$ where t is hours past midnight or noon

Hour "snapshot"
5:00



at 5:00, t = 5 $y = 4\cos\frac{\pi}{6}(5) = 4(\frac{-\sqrt{3}}{2}) = \boxed{-3.46}$

3 inches above occurs at $3 = 4\cos\frac{\pi}{6}(t)$

$\cos^{-1}(.75) = \cos^{-1}(\cos\frac{\pi}{6}(t))$

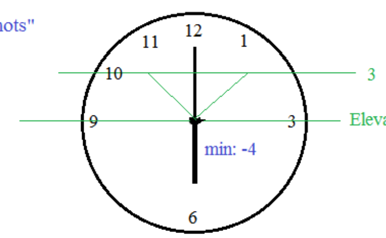
$.72 = \frac{\pi}{6}(t)$ $5.56 = \frac{\pi}{6}(t)$

$t = 1.38$ $t = 10.62$

$\boxed{1:22}$ $\boxed{10:37}$

Hour "snapshots"

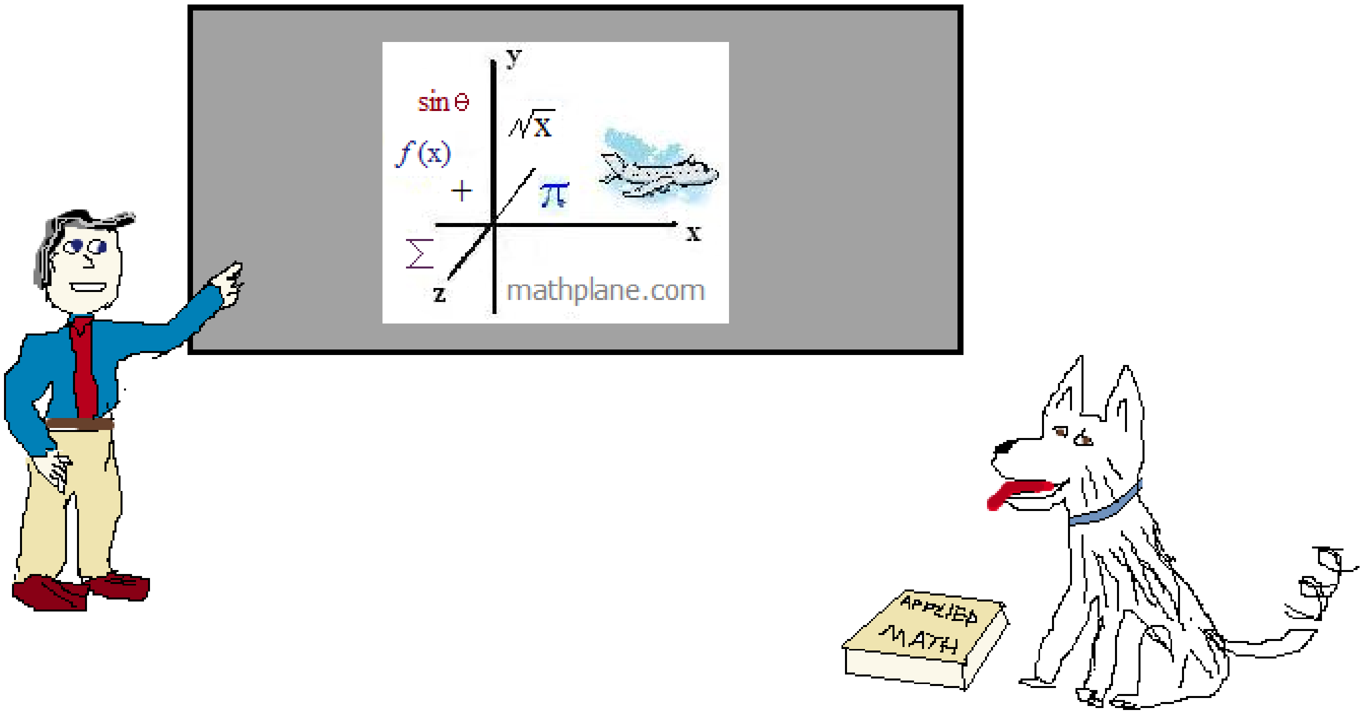
y = -4
y = 3



Thanks for visiting. Hope it helps!

If you have questions, suggestions, or requests, let us know.

Cheers



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