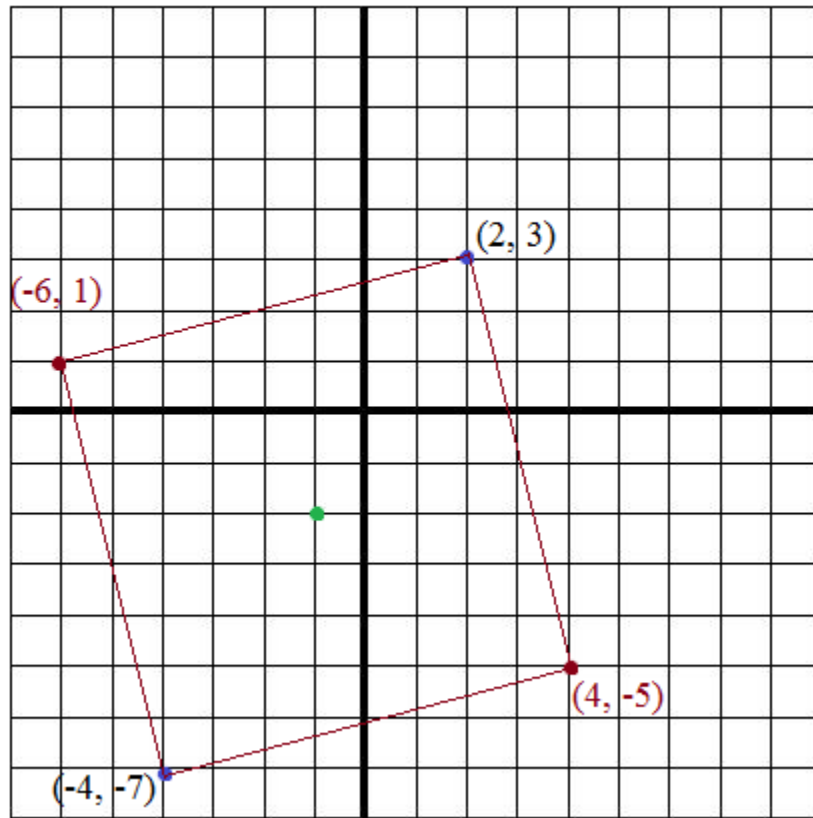


Coordinate Geometry 4 (Advanced)

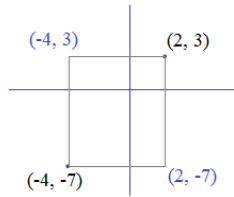


Topics include slope, reflection, centroid, circumcenter, altitudes, area, quadrilaterals, and more.

Example: $(-4, -7)$ and $(2, 3)$ are opposite vertices of a square.

What are the other two vertices?

If this were a vertical square, this would be easy...



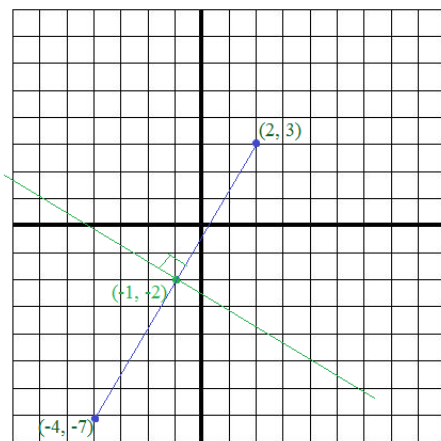
Unfortunately, that creates a rectangle...

This square will be 'tilted'...

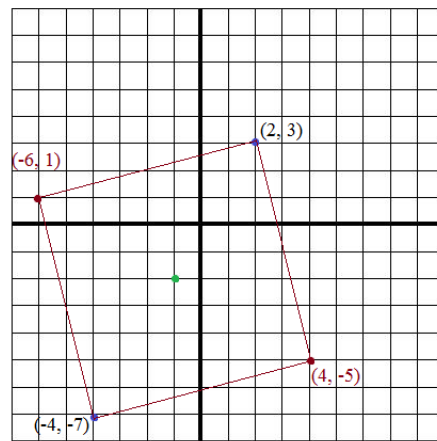
So, consider a square: the diagonals are perpendicular, congruent, and bisect each other...

slope of given diagonal: $(2, 3)$ and $(-4, -7)$ $\frac{-7-3}{-4-2} = \frac{5}{3}$

midpoint of given diagonal: $(2, 3)$ and $(-4, -7)$ $\left(\frac{2+(-4)}{2}, \frac{3+(-7)}{2}\right) = (-1, -2)$



slope of perpendicular line (the other diagonal) is the opposite reciprocal: $-\frac{3}{5}$



From the center of the square $(-1, -2)$, the given vertices are "up 5, right 3" and "down 5, left 3"... Therefore, the other vertices will be the opposite!

("up 3, left 5" and down 3, right 5")

To check: length of each side $\sqrt{68}$ ✓

sides are perpendicular (slopes are opposite reciprocals) ✓

Example: What is the distance between the lines $y = 3x + 4$ and $y = 3x - 7$?

The distance between two (parallel) lines is a segment that is perpendicular to both lines...

Step 1: find the equation of the perpendicular line.

Slope: Since the slope of each parallel line is 3, the slope of the perpendicular segment is $-\frac{1}{3}$ (opposite reciprocal)

Point: We know that the y-intercept will be $(0, 4)$
equation is $y = -\frac{1}{3}(x) + 4$

Step 2: find the intersection of the segment and other line.

$$\begin{aligned} y &= -\frac{1}{3}(x) + 4 \\ y &= 3(x) - 7 \end{aligned} \implies -\frac{1}{3}(x) + 4 = 3(x) - 7$$

$$0 = -10/3(x) + 11$$

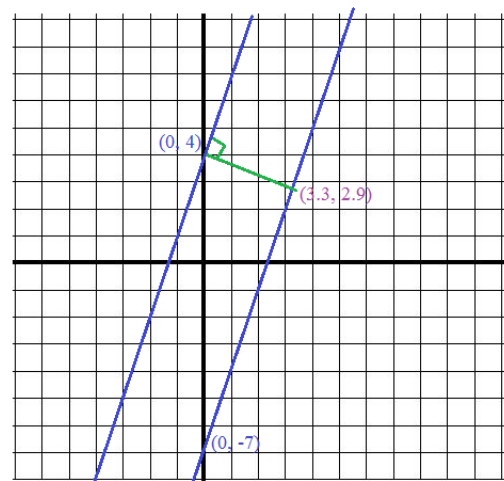
$$x = 3.3 \quad \text{and} \quad y = 2.9$$

Step 3: Find the distance between 2 points

$(0, 4)$ and $(3.3, 2.9)$

$$\text{distance} = \sqrt{(3.3 - 0)^2 + (2.9 - 4)^2}$$

$$= \sqrt{12.1} = 3.48 \text{ (approx.)}$$



Example: (0, 4) (-8, 10) and (-4, 2) are points on a circle.
Find the equation of the circle.

(Note: If we KNOW 2 of the points are endpoints of a diameter, then this is a rather straight-forward question. But, we cannot assume.)

(the perpendicular bisector of a chord will go through the center of the circle)

Approach 1: Using Chords and bisectors.

Using chord 1: endpoints: (-4, 2) (-8, 10)
slope: -2
midpoint: (-6, 6)

therefore, the equation of the perpendicular bisector is

slope: 1/2 point: (-6, 6)
 $y - 6 = 1/2(x + 6)$

Using chord 2: endpoints: (-4, 2) (0, 4)

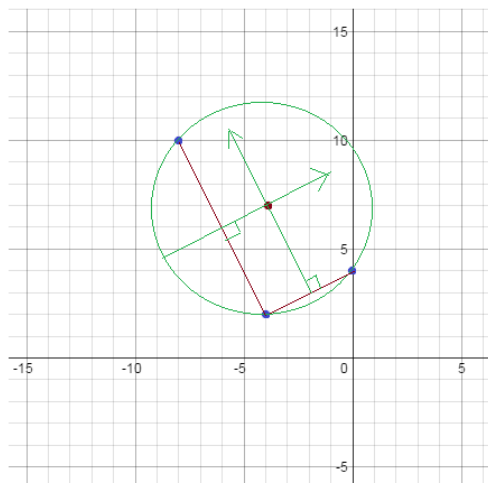
slope: 1/2
midpoint: (-2, 3)

therefore, the equation of the perpendicular bisector is

slope: -2 point: (-2, 3)
 $y - 3 = -2(x + 2)$

Note: Since the chords are perpendicular, they form an inscribed right angle... Therefore, that right angle is inscribed in a semicircle (where the hypotenuse is the diameter of the circle)

In the diagram is a sketch of the circle we're seeking..



the intersection of chord 1 and chord 2 will occur at the center...

$$\begin{aligned} y - 6 &= 1/2(x + 6) & \Rightarrow & y = 1/2(x) + 9 \\ y - 3 &= -2(x + 2) & \Rightarrow & y = -2x - 1 \end{aligned}$$

$$\begin{aligned} 1/2(x) + 9 &= -2x - 1 & x &= -4 \\ 5/2(x) &= -10 & y &= 7 \end{aligned}$$

Then, we see the radius is 5
(the distance from (-4, 7) to each of the 3 points)

$$(x + 4)^2 + (y - 7)^2 = 25$$

Approach 2: Using a system to solve algebraically

$$(x - h)^2 + (y - k)^2 = r^2 \begin{cases} (0, 4) & (0 - h)^2 + (4 - k)^2 = r^2 & \Rightarrow & h^2 + 16 - 8k + k^2 = r^2 & (1) \\ (-8, 10) & (-8 - h)^2 + (10 - k)^2 = r^2 & \Rightarrow & 64 + 16h + h^2 + 100 - 20k + k^2 = r^2 & (2) \\ (-4, 2) & (-4 - h)^2 + (2 - k)^2 = r^2 & \Rightarrow & 16 + 8h + h^2 + 4 - 4k + k^2 = r^2 & (3) \end{cases}$$

set (1) and (2) equal to each other...

$$\cancel{h^2} + 16 - 8k + \cancel{k^2} = 64 + 16h + \cancel{h^2} + 100 - 20k + \cancel{k^2}$$

$$-148 = 16h - 12k \quad (4)$$

set (1) and (3) equal to each other..

$$\cancel{h^2} + 16 - 8k + \cancel{k^2} = \cancel{h^2} + 8h + \cancel{h^2} + 4 - 4k + \cancel{k^2}$$

$$-4 = 8h + 4k \quad (5)$$

solve the resulting system (4) and (5)...

$$\begin{aligned} -148 &= 16h - 12k & \Rightarrow & -37 = 4h - 3k \\ -4 &= 8h + 4k & \Rightarrow & 2 = -4h - 2k \end{aligned}$$

$$k = 7 \quad \text{then, } h = -4$$

$$-35 = -5k$$

Example: On the coordinate plane, there is a triangle $\triangle ABC$, where the vertices are
 A: (0, -15)
 B: (4, -3)
 C: (12, 1)

What is the length of the altitude extended from C?

A quick sketch, and we see this is an obtuse triangle...
 So, we extend side \overline{AB} and draw the altitude.

To find the length \overline{MC} , we need to find the location of point M.

Equation of line \overline{AB} : slope: $\frac{-15 - (-3)}{0 - 4} = 3$

$$y = 3x - 15$$

y-intercept: (0, -15)

Equation of line \overline{MC} : slope: (perpendicular to AB)
 opposite reciprocal $\rightarrow -1/3$
 point: (12, 1)

$$y = -1/3(x) + b$$

$$y = -1/3(x) + 5$$

$$1 = -1/3(12) + b$$

$$1 = -4 + b$$

$$b = 5$$

Intersection of line \overline{AB} and \overline{MC} :
 $y = 3x - 15$
 $y = -1/3(x) + 5$

$$3x - 15 = -1/3(x) + 5$$

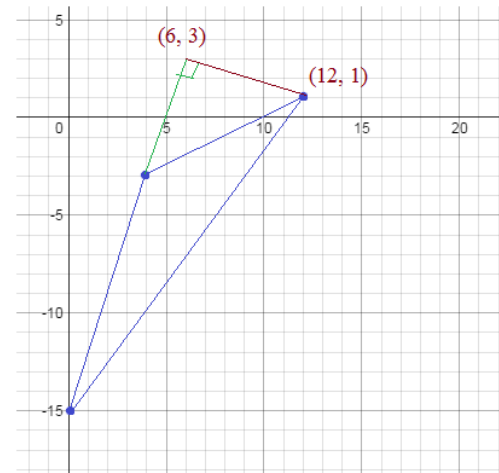
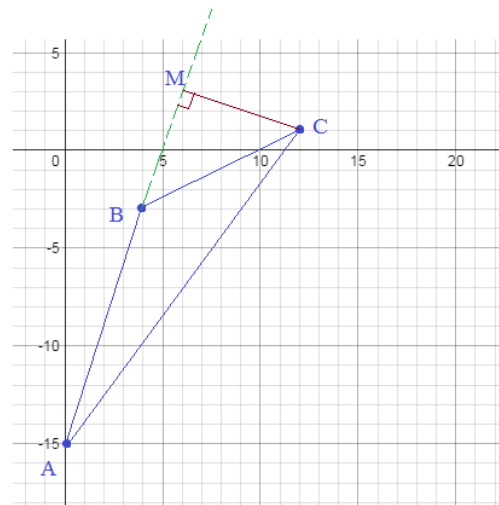
$$10/3(x) = 20$$

$$x = 6 \quad \text{then, } y = 3(6) - 15$$

$$y = 3$$

Therefore, the altitude extends from (12, 1) to (6, 3)

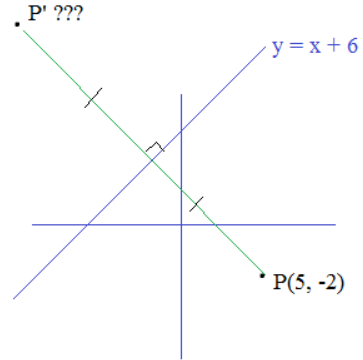
distance formula: $d = \sqrt{(12 - 6)^2 + (1 - 3)^2} = \sqrt{40} = 2\sqrt{10}$



Example: If the point $P(5, -2)$ is reflected over the line $y = x + 6$, what is the coordinate of P' ?

Step 1: Draw a quick sketch to estimate the result.

Notice, the distance from P to $y = x + 6$ is a straight (perpendicular) line segment
And, the identical distance from P' to $y = x + 6$ is a congruent line segment.



Step 2: Find the equation of the perpendicular line

To find the equation of a line, we need the slope and a point...

The point we'll use is $(5, -2)$.

The slope of a perpendicular line is the opposite reciprocal.
(slope of $y = x + 6$ is 1)

The slope of line is -1

Equation of line: $y + 2 = -1(x - 5)$

$$y = -x + 3$$

Step 3: Find the intersection of the 2 lines

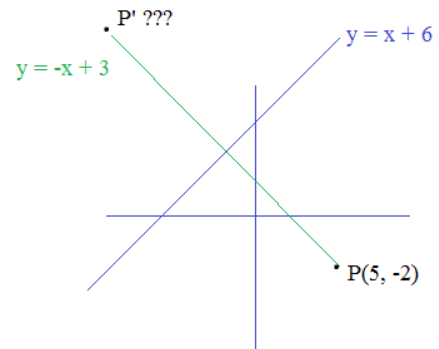
$$y = x + 6$$

using substitution: $x + 6 = -x + 3$

$$y = -x + 3$$

$$2x = -3$$

$$x = -3/2 \quad y = 9/2$$



Step 4: Use the use the midpoint to determine the reflected point

$$P(5, -2) \quad M(-3/2, 9/2) \quad P'(x, y)$$

The distance from 5 to $-3/2$ is $-13/2$... So, the distance from $-3/2$ to x is $-13/2$: -8

The distance from -2 to $9/2$ is $13/2$... So, the distance from $9/2$ to y is $13/2$: 11

P' is $(-8, 11)$

Example: Is this a right triangle?

It looks like a right angle, but look at the coordinates!

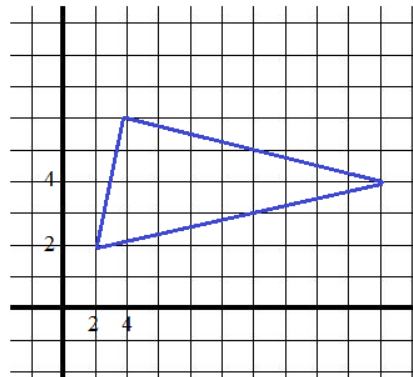
$(2, 2)$ to $(4, 6)$

slope is 2

$(4, 6)$ to $(20, 4)$

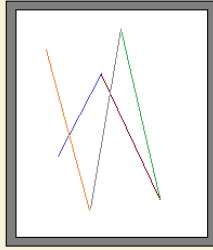
slope is $-1/8$

Since the slopes are not opposite reciprocals, this is NOT a right triangle!



Gallery Exhibit...

"I don't get it..."



Title: "Lines in Space"

"Me, neither"



Abstract Art
and Math

(Sometimes, it takes multiple views to
understand and appreciate...)

Geometry Lesson....

Lines in space



$y = 4$ is
parallel to
the xz -plane

"I don't get it..."



"Me, neither."



LanceAF #244 (7-4-16)
Mathplane.com

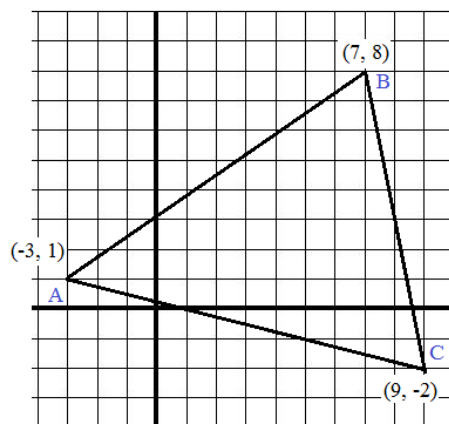
Practice Quiz-→

A) Find lines that include the following (from triangle ABC):

1) The median from A to \overline{BC}
(write as a linear equation in *point-slope form*)

2) The Altitude from B to \overline{AC}
(write as a linear equation in *standard form*)

3) The Perpendicular Bisector of \overline{BC}
(write the linear equation *slope intercept form*)



B) Given: Right triangle SMR with altitude \overline{MP}
and horizontal hypotenuse \overline{SR}

M: (3, 4) S: (-5, -1)

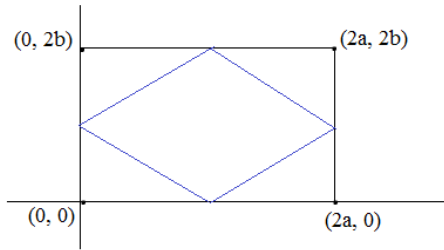
Find: Coordinate R

C) Find the coordinate of the circumcenter from the triangle TRI $\begin{matrix} T & R & I \\ (0, 2) & (6, -4) & (8, 4) \end{matrix}$

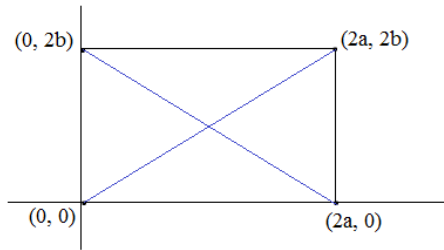
D) Find the coordinate of the centroid in the triangle XYZ $\begin{matrix} X & Y & Z \\ (1, 0) & (7, 15) & (13, 0) \end{matrix}$

E) Find the coordinate of the orthocenter of the triangle ABC $\begin{matrix} A & B & C \\ (-1, 4) & (6, 7) & (9, 0) \end{matrix}$

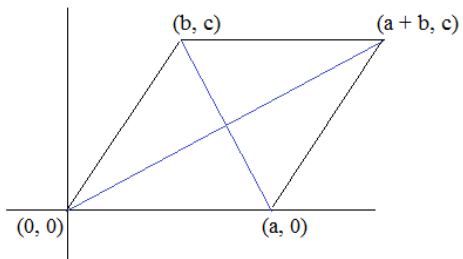
Prove: The connected midpoints of a rectangle form a parallelogram.



Prove: The diagonals of a rectangle bisect each other.



Prove: The diagonals of a rhombus are perpendicular to each other.



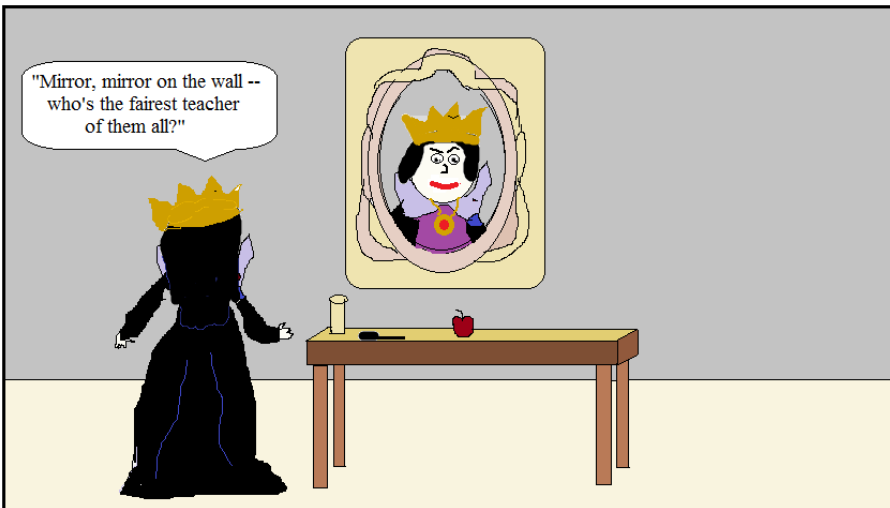
- 1) If point P (8, 12) is reflected over the line $y = 3x + 8$,
then what is the coordinate of point P' ?

- 2) If point Q (-7, -1) is reflected over the line $2x + 5y = 10$,
then what is the coordinate of point Q'?

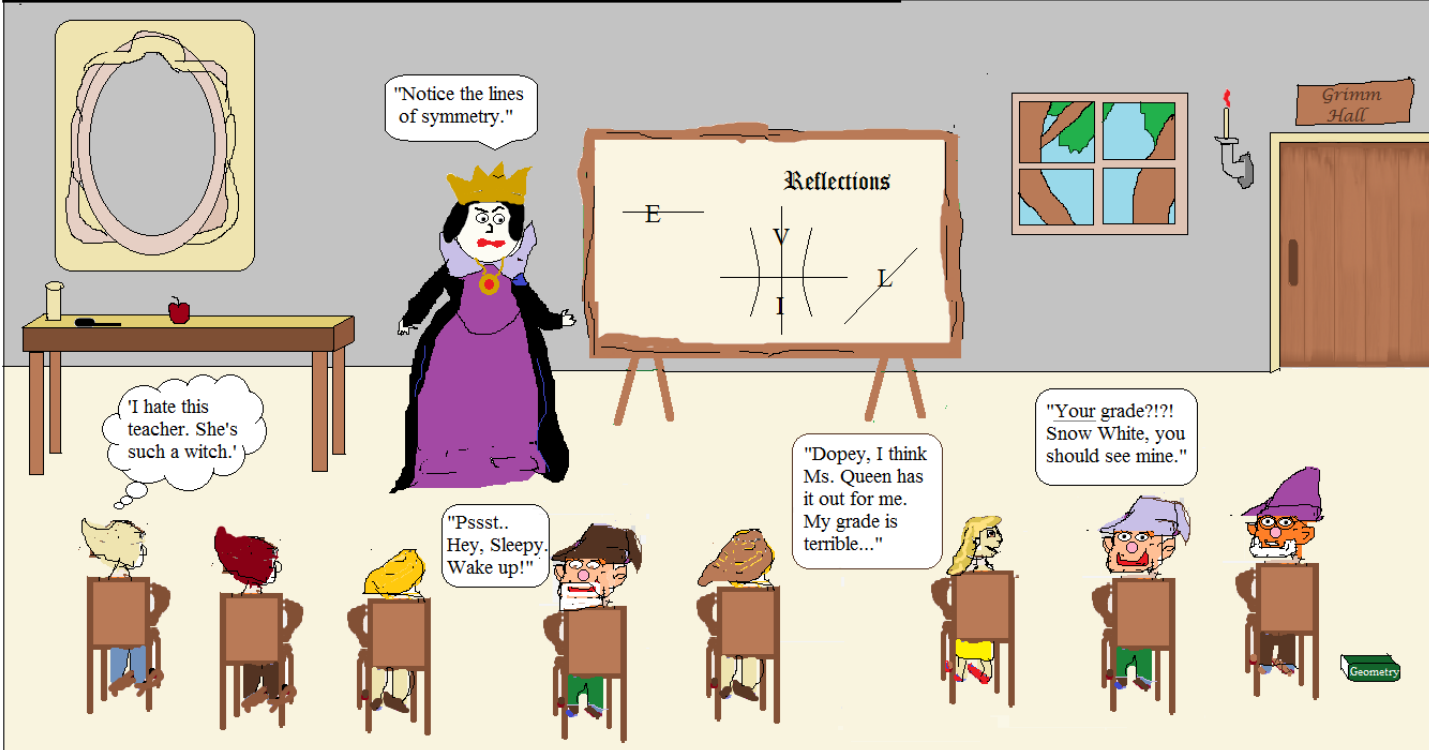
- 3) If Point B (-8, 4) is reflected over the line $y = -1$,
then what is the coordinate of point B' ?

- 4) ***Challenge

If point A (-3, 5) is reflected over the y-axis and
THEN reflected over the line $y = -4x - 6$,
where does the point land?



Doc excelled...
And, Happy and Bashful didn't care...
But, the rest suffered in this math class...



SOLUTIONS-→

A) Find lines that include the following (from triangle ABC):

- 1) The median from A to \overline{BC}
(write as a linear equation in *point-slope form*)

To express the equation of a line, we need the slope and a point:

Point: A -- (-3, 1)

Slope: the slope going through A and the midpoint of \overline{BC}

$$\text{Midpoint } M = \left(\frac{7+9}{2}, \frac{8+(-2)}{2} \right) = (8, 3)$$

$$\text{Slope of line going through A and M: } \frac{3-1}{8-(-3)} = \frac{2}{11}$$

$$y - 1 = \frac{2}{11}(x + 3)$$

or

$$y - 3 = \frac{2}{11}(x - 8)$$

- 2) The Altitude from B to \overline{AC}
(write as a linear equation in *standard form*)

We need a point and the slope...

Point: B (7, 8)

Slope: *perpendicular* to \overline{AC}

$$\text{slope of } \overline{AC} \text{ is } \frac{1 - (-2)}{-3 - 9} = \frac{3}{-12}$$

slope of line perpendicular to \overline{AC} is 4 (opposite reciprocal)

Altitude line:

$$y - 8 = 4(x - 7)$$

$$y - 8 = 4x - 28$$

$$4x - y = 20$$

- 3) The Perpendicular Bisector of \overline{BC}
(write the linear equation *slope intercept form*)

Need the midpoint of \overline{BC}
and the slope of a line perpendicular to \overline{BC}

$$\text{Midpoint of } \overline{BC} = (8, 3) \text{ (found in question 1)}$$

$$\text{slope of } \overline{BC} = \frac{8 - (-2)}{7 - 9} = -5$$

slope of line perpendicular to $\overline{BC} = 1/5$

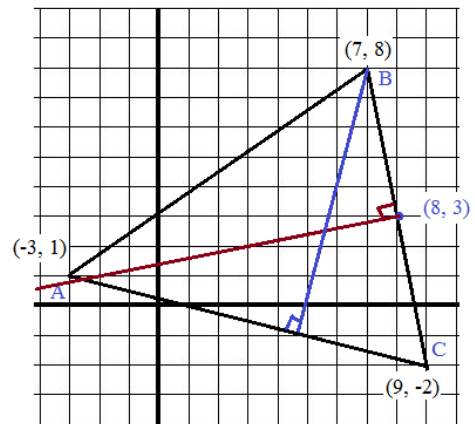
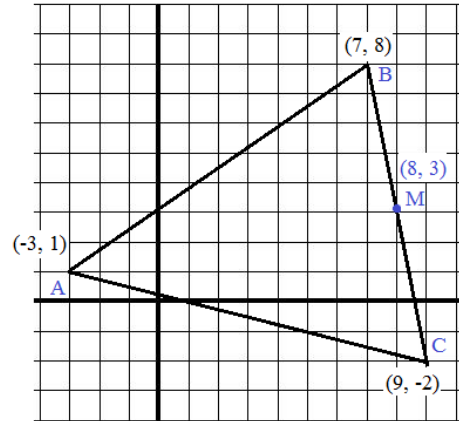
linear equation of
perpendicular bisector:

$$y - 3 = 1/5(x - 8)$$

$$y - 3 = 1/5x - 8/5$$

$$y = 1/5x + 7/5$$

SOLUTIONS



B) Given: Right triangle SMR with altitude \overline{MP}
and horizontal hypotenuse \overline{SR}

M: (3, 4) S: (-5, -1)

Find: Coordinate R

Since MP is an altitude, it is perpendicular to SR...
If SR is horizontal, then MP is vertical and P is (3, -1)

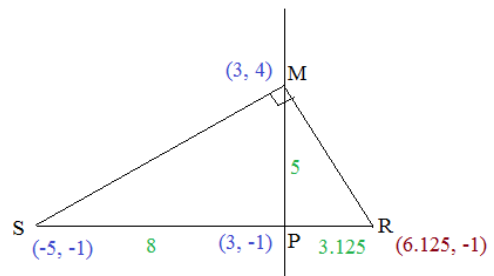
Length of MP is 5 and SP is 8

"Altitude to Hypotenuse": MP is the geometric mean of PR and SP

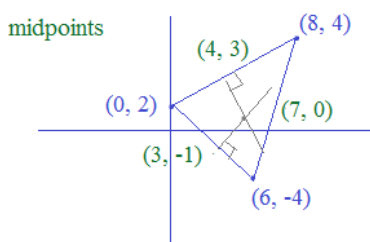
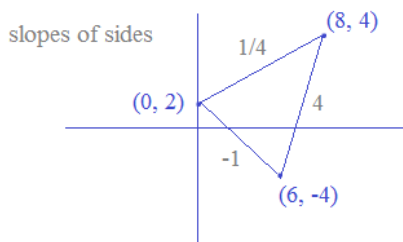
$$\frac{8}{5} = \frac{5}{PR}$$

$$8(PR) = 25 \quad PR = 3.125$$

Therefore, R is (6.125, -1)



C) Find the coordinate of the circumcenter from the triangle TRI (0, 2) (6, -4) (8, 4)



find 2 perpendicular bisectors...

\overline{TR} perp. bisector: slope: 1 point: (3, -1)
 $y + 1 = 1(x - 3)$

\overline{TI} perp. bisector: slope: -4 point: (4, 3)
 $y - 3 = -4(x - 4)$

and, their intersection...

$y = x - 4$ $y = -4x + 19$

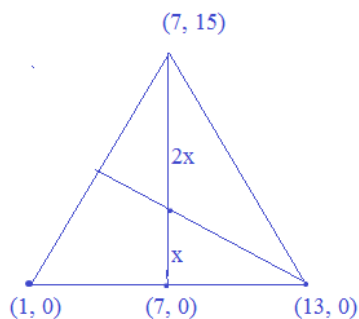
$x - 4 = -4x + 19$

$5x = 23$ $x = 4.6$
 $y = .6$

D) Find the coordinate of the centroid in the triangle XYZ (1, 0) (7, 15) (13, 0)

Isosceles triangle

Centroid is (7, 5)



Centroid is 2/3 distance from each vertex!

distance from (7, 0) to (7, 15) is 15

$x + 2x = 15$

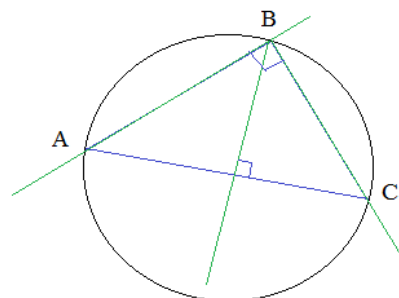
$x = 5$

E) Find the coordinate of the orthocenter of the triangle ABC (-1, 4) (6, 7) (9, 0)

\overline{AB} --- slope is 3/7
 \overline{BC} --- slope of -7/3

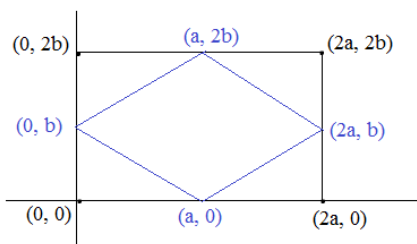
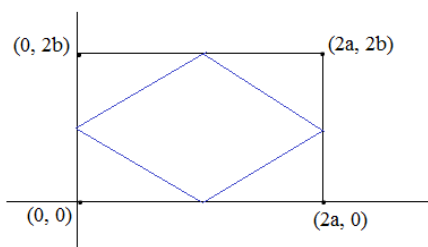
Since $\overline{AB} \perp \overline{BC}$, it is a right triangle.
 Therefore, orthocenter is at the vertex B

(6, 7)



Inscribed right triangle in a semicircle...

Prove: The connected midpoints of a rectangle form a parallelogram.



Using the midpoint formula, we can identify all the midpoints...

Then, find all the slopes...

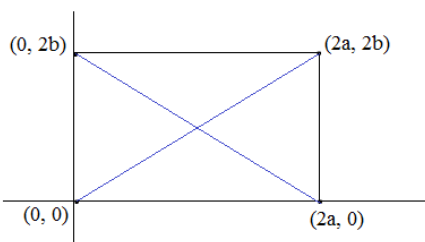
$$(0, b) \text{ and } (a, 2b): \frac{2b - b}{a - 0} = b/a \quad (a, 2b) \text{ and } (2a, b): \frac{b - 2b}{2a - a} = -b/a$$

Since the opposite sides have the same slopes, they are parallel lines...

$$(a, 0) \text{ and } (2a, b): \frac{b - 0}{2a - a} = b/a \quad (0, b) \text{ and } (a, 0): \frac{b - 0}{0 - a} = -b/a$$

Since opposite sides are parallel, then it's a parallelogram...

Prove: The diagonals of a rectangle bisect each other.

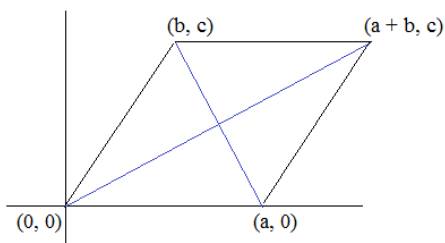


The midpoint of $(0, 2b)$ and $(2a, 0)$ is (a, b)

The midpoint of $(0, 0)$ and $(2a, 2b)$ is (a, b)

Since each diagonal has the same midpoint, then they bisect each other.

Prove: The diagonals of a rhombus are perpendicular to each other.



The slope of the diagonals:

$$(0, 0) \text{ and } (a+b, c): \frac{c-0}{a+b-0} = \frac{c}{a+b} \quad (1)$$

$$(a, 0) \text{ and } (b, c): \frac{c-0}{b-a} = \frac{c}{b-a} \quad (2)$$

$$b^2 + c^2 = a^2 \quad (\text{because all sides of rhombus are congruent and equal the length } a)$$

$$c^2 = a^2 - b^2$$

$$c^2 = (a+b)(a-b)$$

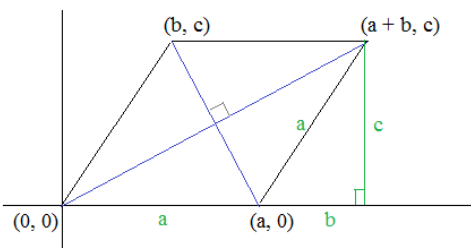
$$(a+b) = \frac{c^2}{(a-b)} \quad \text{or} \quad (a-b) = \frac{c^2}{(a+b)}$$

$$(1) \frac{c}{a+b} = \frac{c}{\frac{c^2}{(a-b)}} = \frac{(a-b)}{c}$$

$$(2) \frac{c}{b-a} = \frac{-c}{a-b}$$

Opposite reciprocals!

Since the slopes of the diagonals are opposite reciprocals, the diagonals are perpendicular.



1) If point P (8, 12) is reflected over the line $y = 3x + 8$,

then what is the coordinate of point P' ?

ANSWER:
(-4, 16)

Step 1: Draw quick sketch to get estimate

Step 2: Find equation of "reflection line segment"

We know the slope of the line $y = 3x + 8$ is 3

Therefore, the slope of the reflection line (perpendicular)

is $-1/3$

Since it goes through (8, 12), the equation of the line

is $y - 12 = \frac{-1}{3}(x - 8)$

Step 3: Find the intersection of the "reflection line" and original line

$$y = 3x + 8$$

$$y - 12 = \frac{-1}{3}(x - 8)$$

Using substitution:

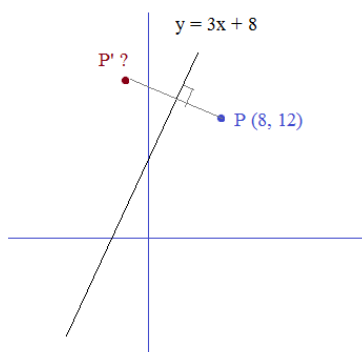
$$(3x + 8) - 12 = \frac{-1}{3}(x - 8)$$

$$3x - 4 = \frac{-1}{3}(x - 8)$$

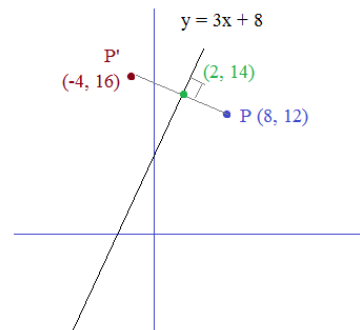
$$9x - 12 = -1(x - 8)$$

$$10x = 20$$

$$x = 2 \quad \text{then, } y = 14$$



SOLUTIONS



Step 4: Utilize the midpoint formula

Recognizing that the intersection (2, 14) is the midpoint of P and P',

$$\begin{matrix} (x - 6, y + 2) & (x - 6, y + 2) \\ (8, 12) & \Rightarrow & (2, 14) & \Rightarrow & (-4, 16) \\ P & & \text{Midpoint} & & P' \end{matrix}$$

Midpoint of P and P' is M

$$\left(\frac{8 + x}{2}, \frac{y + 12}{2} \right) = (2, 14)$$

$$\frac{8 + x}{2} = 2 \quad x = -4$$

$$\frac{y + 12}{2} = 14 \quad y = 16$$

2) If point Q (-7, -1) is reflected over the line $2x + 5y = 10$,

then what is the coordinate of point Q' ?

ANSWER:
(-3, 9)

Step 1: Draw quick sketch to get estimate

Step 2: Find equation of "reflection line segment"

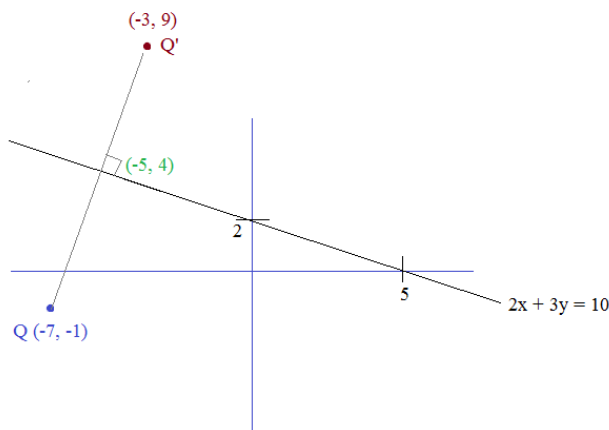
Find the slope of line: $2x + 5y = 10$

$$y = \frac{-2}{5}x + 2 \quad \text{slope is } -2/5$$

Therefore, the slope of the perpendicular reflection line segment is $5/2$

Since it must pass through (-7, -1), the equation of the line is

$$y + 1 = \frac{5}{2}(x + 7)$$



Step 3: Find the intersection of the "reflection line" and original line

$$\begin{matrix} y + 1 = \frac{5}{2}(x + 7) \\ \boxed{2x + 5y = 10} \\ \rightarrow \\ \boxed{2y + 2 = 5x + 35} \\ \boxed{-5x + 2y = 33} \end{matrix}$$

$$\begin{matrix} 10x + 25y = 50 \\ -10x + 4y = 66 \\ \hline 29y = 116 \\ y = 4 \quad \text{so, } x = -5 \end{matrix}$$

Step 4: Utilize the midpoint formula

Since endpoint Q (-7, -1) to midpoint (-5, 4)

$$\text{is } x + 2 \quad \text{and } y + 5 \dots$$

then, midpoint (-5, 4) to endpoint Q'

$$\text{is } -5 + 2 \quad \text{and } 4 + 5 \quad \text{-----} \rightarrow (-3, 9)$$

$$\begin{matrix} x & y \end{matrix}$$

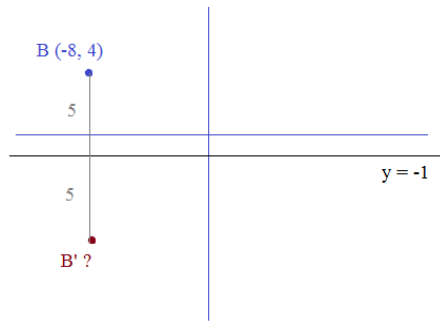
- 3) If Point B (-8, 4) is reflected over the line $y = -1$,
then what is the coordinate of point B' ?

SOLUTIONS

Step 1: Sketch the diagram

Step 2: Find distance from point to line..

***Since this is a horizontal line of reflection
(and the slope is 0), the direction of the point
will be directly down!
(i.e the slope from B to B' is undefined)



the distance from (-8, 4) to $y = -1$ is 5 units

Step 3: Duplicate the distance from line of reflection
to mirror point B'

Since distance from B (-8, 4) to $y = -1$ is 5 units,
we'll continue down another 5 units from (-8, -1) to

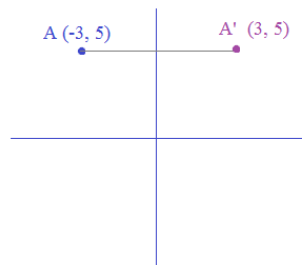
$$(-8, -6)$$

- 4) ***Challenge

If point A (-3, 5) is reflected over the y-axis and
THEN reflected over the line $y = -4x - 6$,
where does the point land?

Step 1: Sketch and identify the first reflection

The length from A to the y-axis is 3 units,
so, the coordinate of A' is (3, 5)



Step 2: Sketch the second reflection and identify the
line segment perpendicular to the line of reflection

Since the slope of $y = -4x - 6$ is -4,

the slope of the segment A'A'' is 1/4

then, using the point (3, 5), the equation of
the segment is

$$y - 5 = \frac{1}{4}(x - 3)$$

Step 3: find the intersection of segment and line of reflection..

$$\begin{aligned} y - 5 &= \frac{1}{4}(x - 3) \\ y &= -4x - 6 \end{aligned}$$

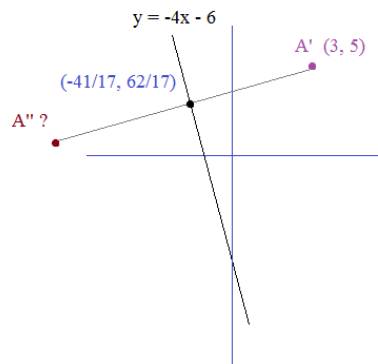
Using substitution, $(-4x - 6) - 5 = \frac{1}{4}(x - 3)$

$$\begin{aligned} -4x - 11 &= \frac{1}{4}x - \frac{3}{4} \\ \text{multiply by 4} & \\ \text{to get rid of} & \\ \text{fractions} & \\ -16x - 44 &= x - 3 \end{aligned}$$

$$\begin{aligned} -41 &= 17x \\ x &= -41/17 \end{aligned}$$

$$y = -4(-41/17) - 6$$

$$y = \frac{164}{17} - \frac{102}{17} = \frac{62}{17}$$



$$A'' (-133/17, 39/17)$$

Step 4: Using midpoint formula, find the reflection point..

Looking at y values: distance from 5 to 62/17 is 23/17

therefore, 62/17 to A'' must be 23/17 ----> 39/17

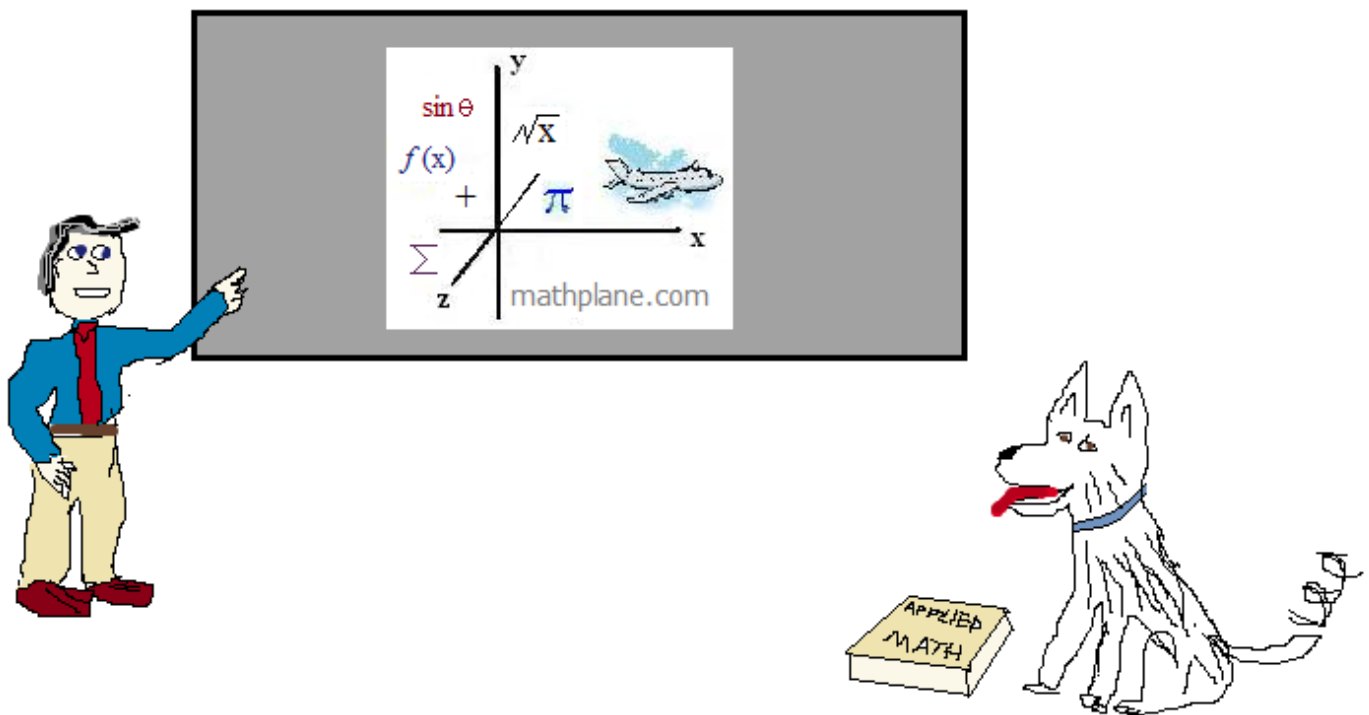
Looking at x values: distance from 3 to -41/17 is 92/17

therefore, -41/17 to A'' must be 92/17 -----> -133/17

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

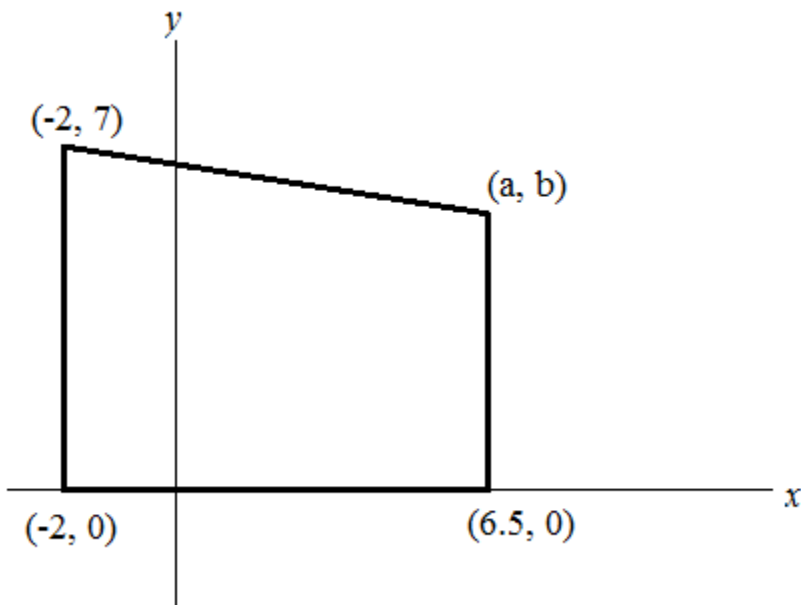
Cheers



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If the area of the trapezoid is 53,
Then what is the coordinate of (a, b) ?



Answer-→

If the area of the trapezoid is 53,
what is the coordinate (a, b)?

Area of trapezoid:

$$\frac{1}{2} (\text{base1} + \text{base2})(\text{height})$$

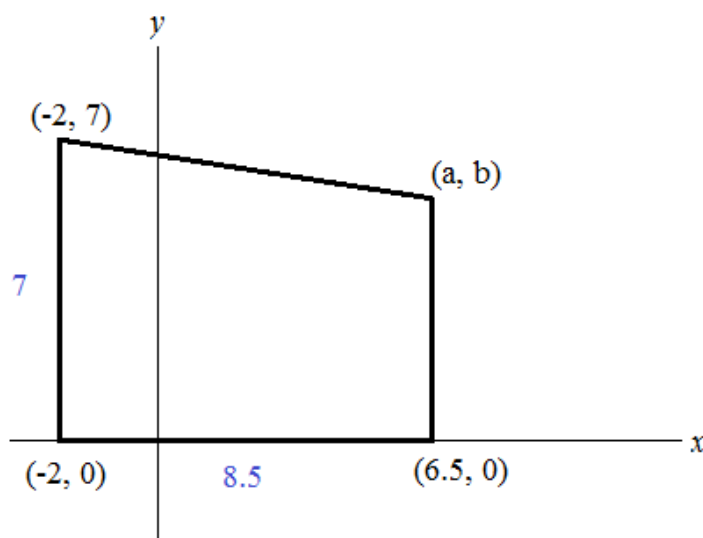
$$\frac{1}{2} (7 + \text{base2})(6.5 - (-2))$$

$$\frac{1}{2} (7 + \text{base2})(8.5) = 53$$

$$(7 + \text{base2})(8.5) = 106$$

$$(7 + \text{base2}) = 12.47$$

$$\text{base2} = 5.47$$



(6.5, 5.47)

Also, suppose the diagram isn't drawn to scale!

In other words, what if the horizontal lines
are parallel, and the vertical lines (which appear
parallel) are not?!?!?

$$\frac{1}{2} (8.5 + \text{base 2}') (7) = 53$$

$$(8.5 + \text{base 2}') (7) = 106$$

$$(8.5 + \text{base 2}') = 15.14$$

$$\text{base 2}' = 6.64$$

(4.64, 7)

