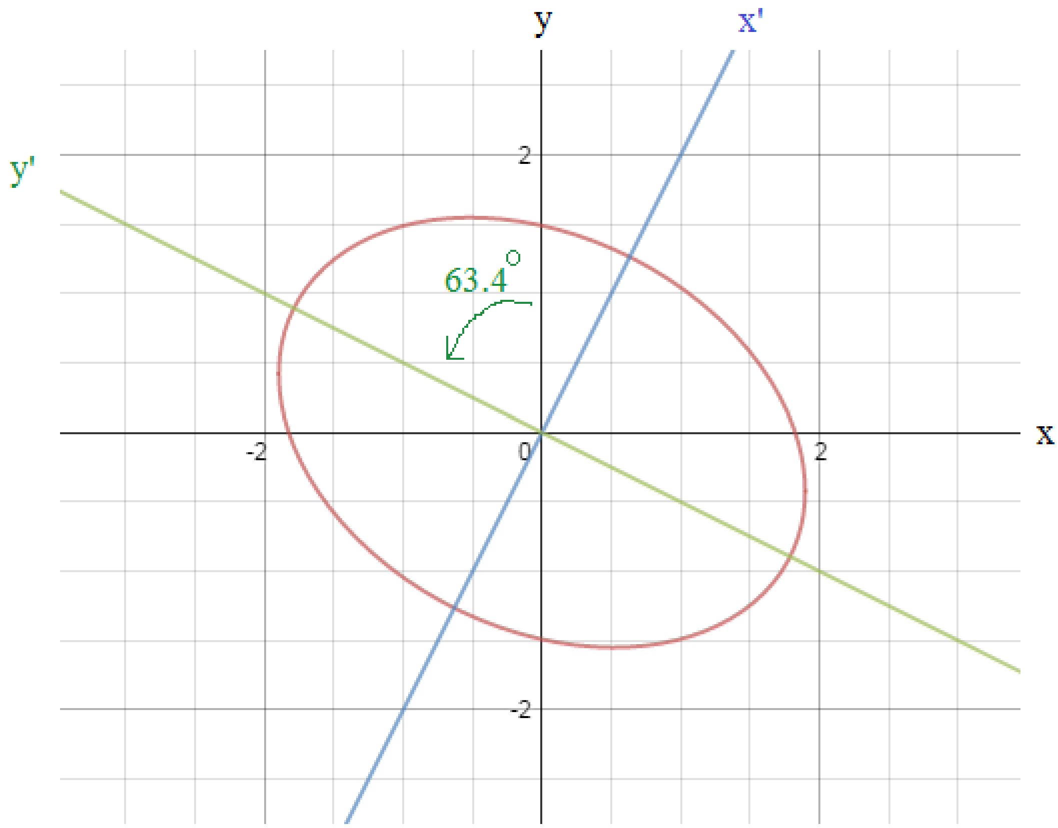
Rotation of Axes: Conics

Formulas, Examples, and practice test (with solutions)

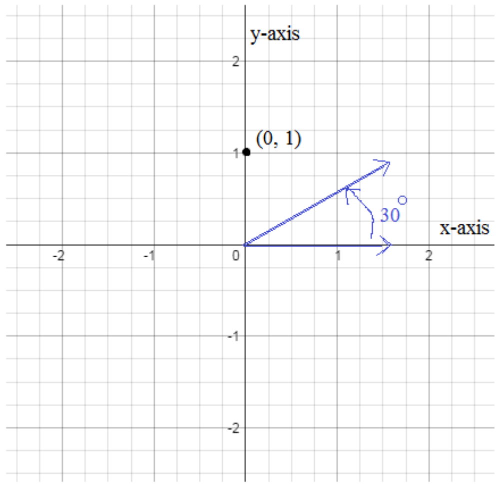


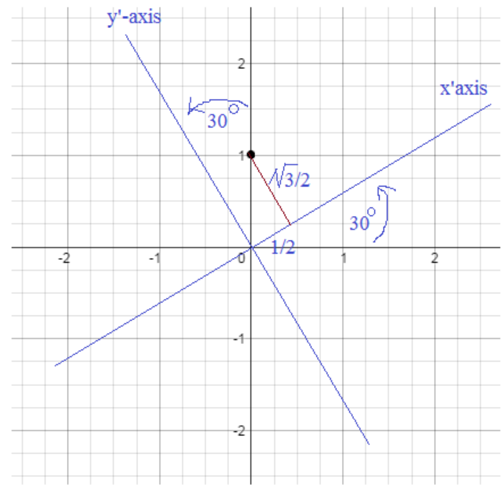
Rotation of Axes

Determine the x'y' coordinates of a given point if the coordinate axes are rotated through a given angle.

Example: (0, 1) 30°

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$





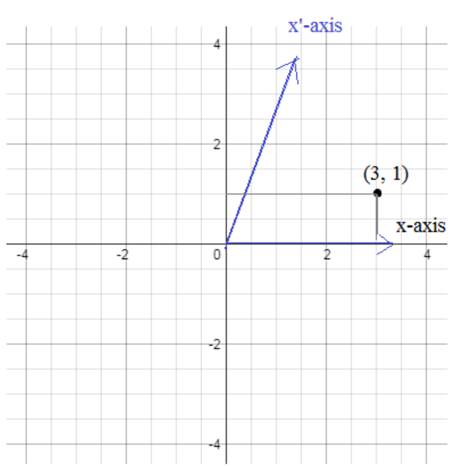
$x' = 0\cos(30) + 1\sin(30)$ $\quad x' = 1/2$

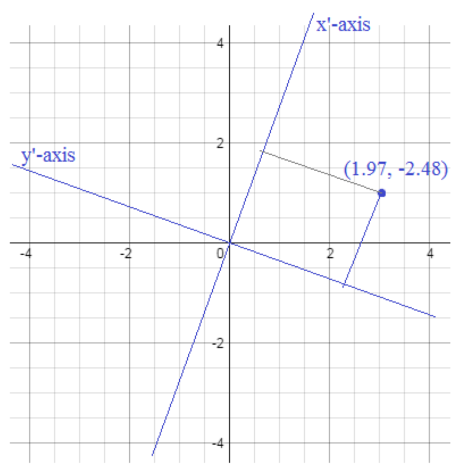
$y' = -0\sin(30) + 1\cos(30)$ $\quad y' = \sqrt{3}/2$

The coordinates of the point *related to the xy-axes* (0, 1)

The coordinates of the point *related to the rotated x'y'-axis* $(1/2, \sqrt{3}/2)$

Example: (3, 1) 70°





$x' = 3\cos(70) + 1\sin(70)$ $\quad x' = 1.97$

$y' = -3\sin(70) + 1(\cos 70)$ $\quad y' = -2.48$

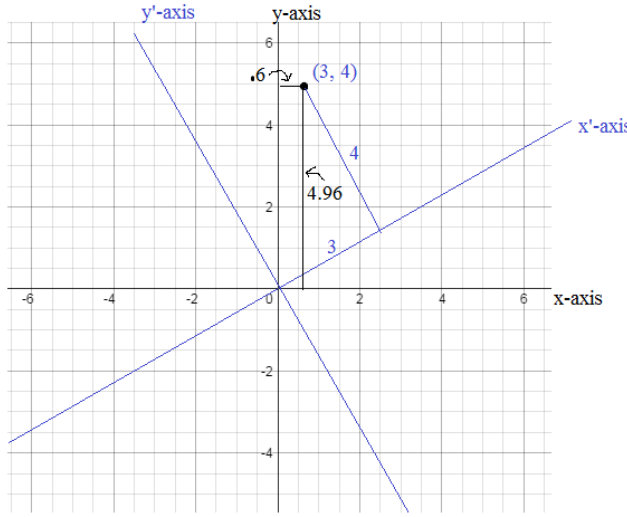
The coordinates of the point *related to the xy-axes* (3, 1)

The coordinates of the point *related to the rotated x'y'-axis* (1.97, −2.48)

Rotation of Axes

Determine the original xy-coordinates from a given point in a rotated x'y'-coordinate axes.

Example: (3, 4) inside a 30 degree rotated xy-axes



$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

$x = 3\cos(30) - 4\sin(30) \qquad \frac{3\sqrt{3}}{2} - 2 \qquad = .60$

$y = 3\sin(30) + 4\cos(30) \qquad \frac{3}{2} + 2\sqrt{3} \qquad = 4.96$

Application/Example: Show that xy = 4 is a conic rotated though an angle of 45 degrees.

$x = x'\cos(45) - y'\sin(45) \qquad x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'$

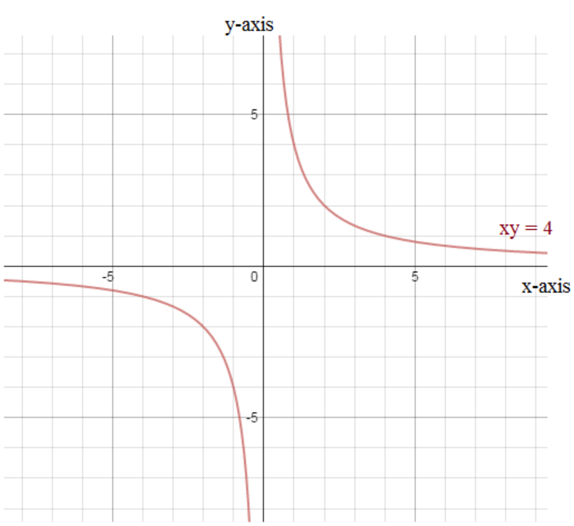
$y = x'\sin(45) + y'\cos(45) \qquad y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$

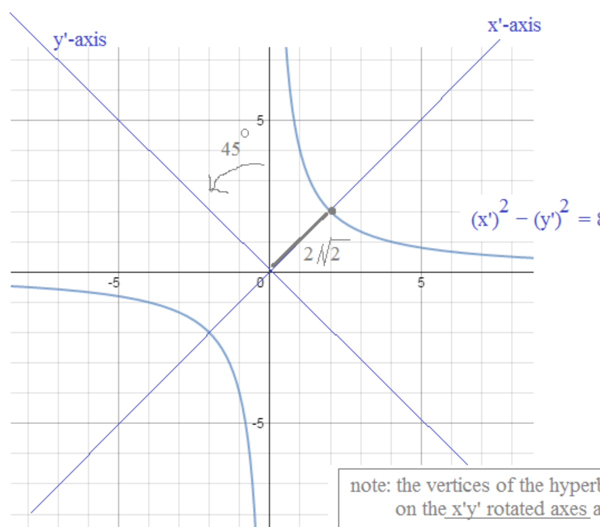
$$x = \frac{\sqrt{2}}{2}(x' - y')$$
$$y = \frac{\sqrt{2}}{2}(x' + y')$$

Then, substitute:

$$xy = 4$$
$$\frac{\sqrt{2}}{2}(x' - y') \cdot \frac{\sqrt{2}}{2}(x' + y') = 4$$
$$\frac{2}{4}(x' - y') \cdot (x' + y') = 4$$
$$(x'^2 - y'^2) = 8$$

Hyperbola!





note: the vertices of the hyperbola on the x'y' rotated axes are $(2\sqrt{2}, 0)$ and $(-2\sqrt{2}, 0)$

General Form: $A^2 + Bxy + C^2 + Dx + Ey + F = 0$

$$B^2 - 4AC < 0 \Rightarrow A'C' > 0 \Rightarrow A' \text{ and } C' \text{ are the same sign} \Rightarrow \text{is an } \underline{\text{ellipse}} \; ;$$

$$B^2 - 4AC > 0 \Rightarrow A'C' < 0 \Rightarrow A' \text{ and } C' \text{ are of different sign} \Rightarrow \text{is a } \underline{\text{hyperbola}} \; ;$$

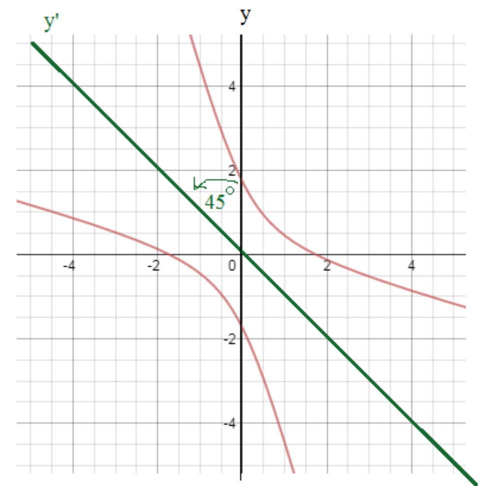
$$B^2 - 4AC = 0 \Rightarrow A'C' = 0 \Rightarrow A' \text{ or } C' \text{ is zero} \Rightarrow \text{is a } \underline{\text{parabola}} \; .$$

Example: $x^2 + 4xy + y^2 - 3 = 0$

What type of conic is it?

It appears to be a circle, because the A and C terms are the same..
But, there is a B term...

$\boxed{B^2 - 4AC = 12 > 0 \quad \text{therefore, it is a hyperbola!}}$



Rotate the axes so that the new expression contains no "xy" term.

$$\cot(2\theta) = \frac{A - C}{B}$$

$$\cot(2\theta) = \frac{1 - 1}{4} = 0$$

$$2\theta = 90^\circ$$

$$\theta = 45^\circ$$

Convert the x and y coordinates into x' and y' terms...

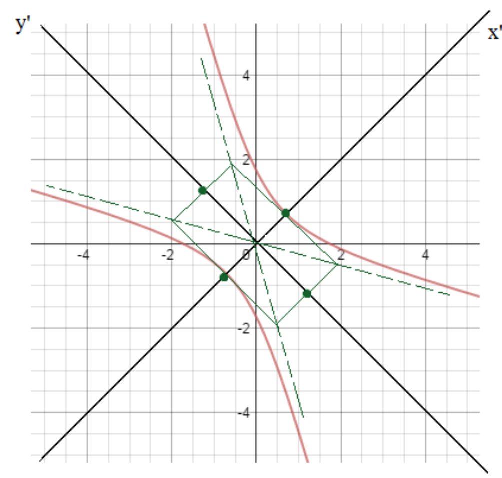
$\boxed{\begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{array}}$

$$x = x'\cos(45) - y'\sin(45)$$

$$x = \frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'$$

$$y = x'\sin(45) + y'\cos(45)$$

$$y = \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$$



Substitute and simplify...

$$x^2 + 4xy + y^2 - 3 = 0$$

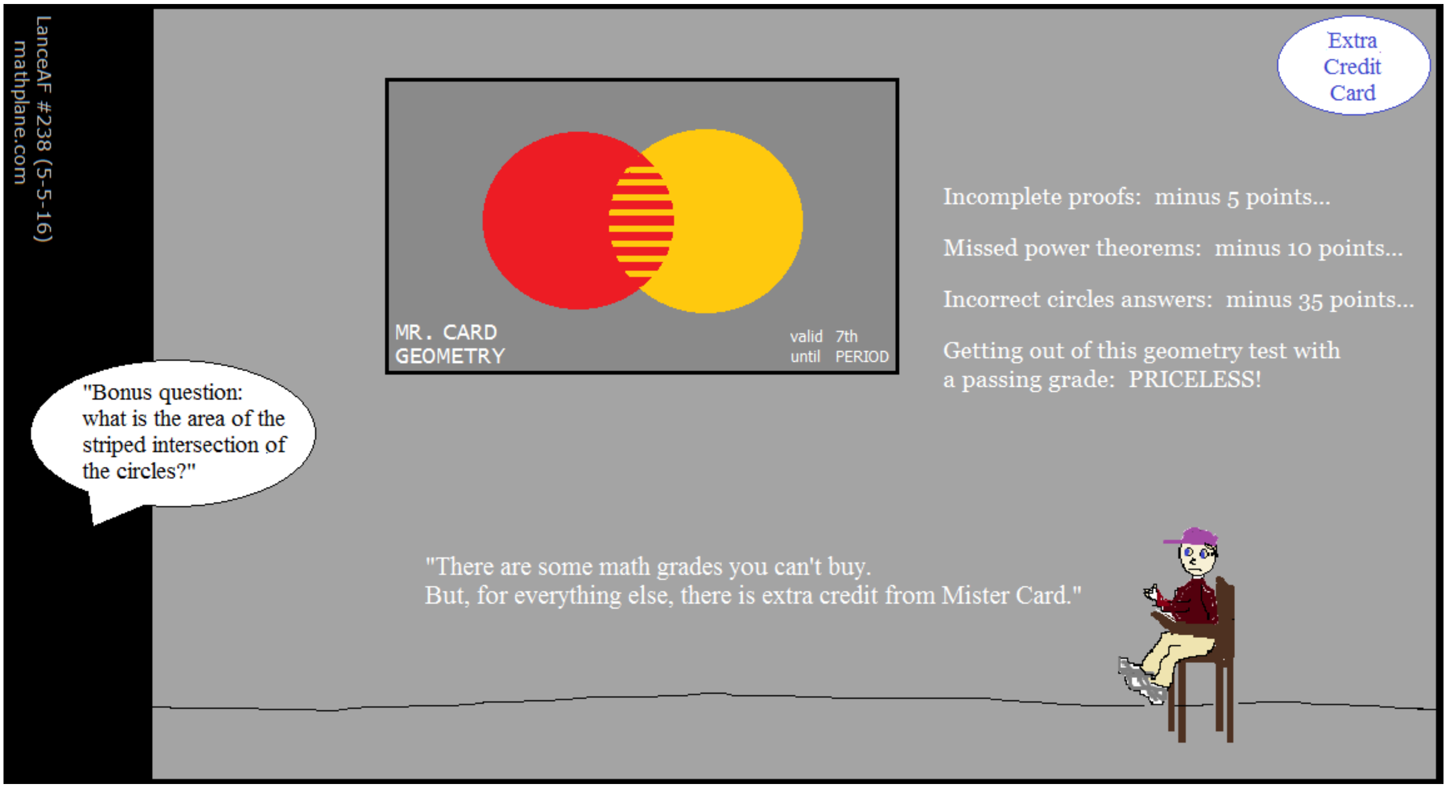
$$\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 + 4\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + \left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 = 3$$

$$\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2 + 4\left(\frac{1}{2}x'^2 - \frac{1}{2}y'^2\right) + \frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2 = 3$$

$$3x'^2 - y'^2 = 3$$

$$\frac{x'}{1} - \frac{y'}{3} = 1$$

center: (0, 0)
vertex: (1, 0) and (-1, 0) on the x'y'-coordinate plane..
foci: (2, 0) and (-2, 0) on the x'y'-coordiante plane..
asymptotes: $y' = \sqrt{3}\,x'$ and $y' = -\sqrt{3}\,x'$



Practice Quiz-→

Rotation of Conics Exercise

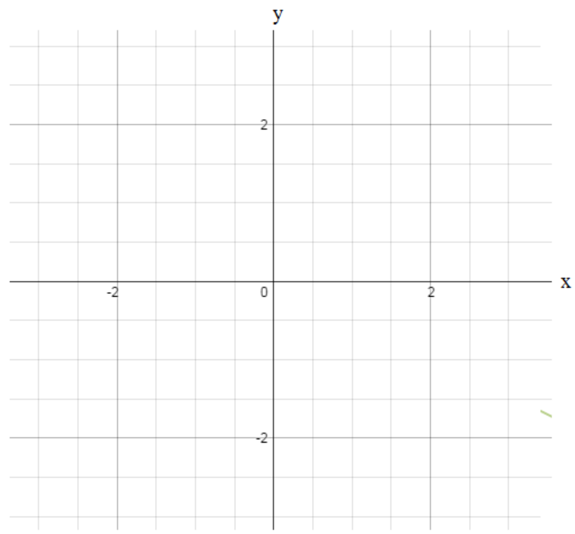
In the following general equations,

a) Identify the conic

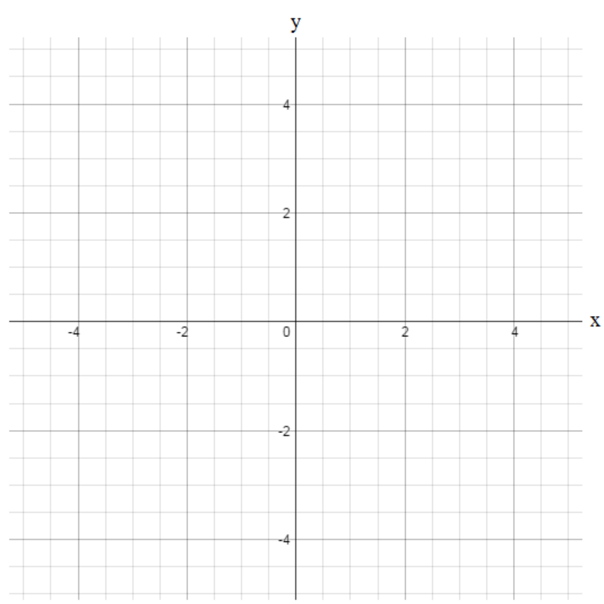
b) Rotate the axes, and write the new expression containing no 'xy' term

c) Graph

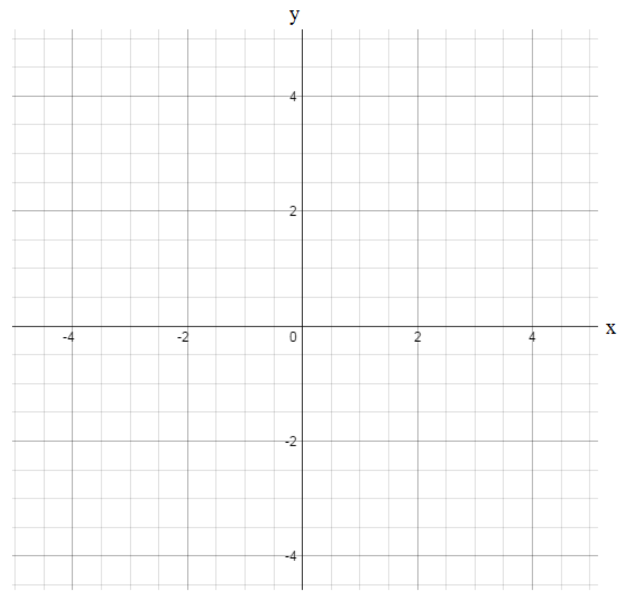
1) $6x^2 + 4xy + 9y^2 - 20 = 0$



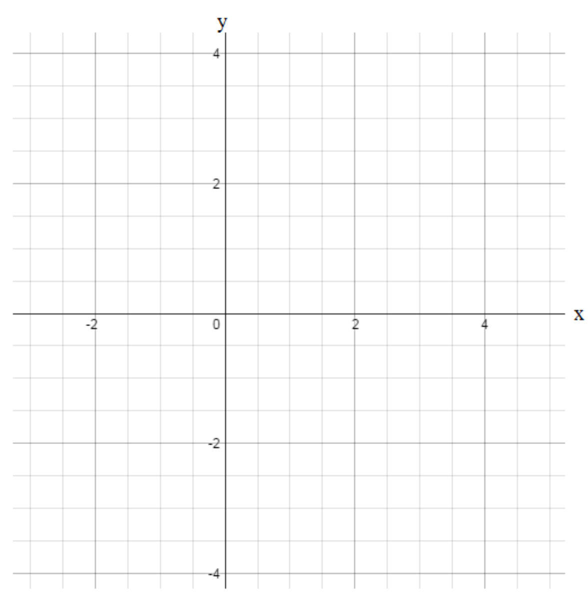
2) $4x^2 - 12xy + 9y^2 + 12x + 8y = 0$



3) $2x^2 - 8xy + 2y^2 - 6 = 0$



4) $4x^2 - 6xy + 4y^2 - 6y - 2 = 0$



In the following general equations,

a) Identify the conic

b) Rotate the axes, and write the new expression containing no 'xy' term

c) Graph

Rotation of Conics Exercise

SOLUTIONS

mathplane.com

1) $6x^2 + 4xy + 9y^2 - 20 = 0$

a) $B^2 - 4AC$

$(4)^2 - 4(6)(9) = -200 < 0$

Since less than zero, it's a $\boxed{\text{rotated ellipse...}}$

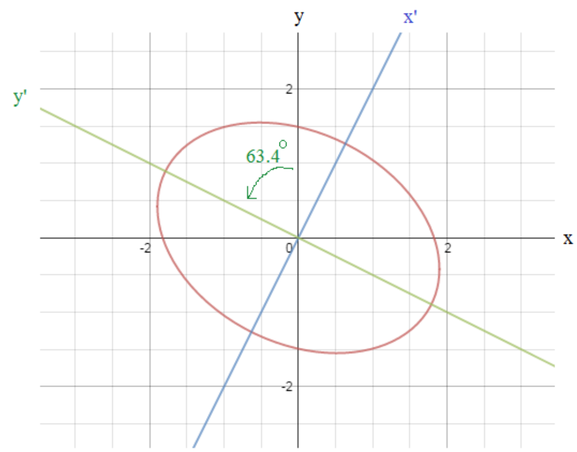
b) $\cot(2\theta) = \dfrac{A - C}{B}$

$\cot(2\theta) = \dfrac{6 - 9}{4} = -3/4$

$\text{arccot}(-3/4) = 2\theta$

$126.87 = 2\theta$

$\boxed{\theta \approx 63.4^\circ}$



c) $x = x'\cos(63.4) - y'\sin(63.4)$

$x = .45x' - .89y'$

$y = x'\sin(63.4) + y'\cos(63.4)$

$y = .89x' + .45y'$

$\boxed{\begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{array}}$

$\boxed{\begin{array}{l} \tan(63.4) = 2 \text{ (slope of x'-axis)} \\ \quad \text{then, } -1/2 \text{ (slope of y'-axis)} \end{array}}$

then, substitute:

$6x^2 + 4xy + 9y^2 - 20 = 0 \Rightarrow 6(.45x' - .89y')^2 + 4(.45x' - .89y')(.89x' + .45y') + 9(.89x' + .45y')^2 = 20$

$6(.20x'^2 - .8x'y' + .79y'^2) + 4(.40x'^2 - .79x'y' + .20x'y' - .40y'^2) + 9(.79x'^2 + .8x'y' + .20y'^2) = 20$

$9.91x'^2 + 0x'y' + 4.94y'^2 = 20$ $\boxed{\dfrac{x'^2}{2} + \dfrac{y'^2}{4} = 1}$ center: $(0, 0)$

minor semi-axis: 1.4
major semi-axis: 2

2) $4x^2 - 12xy + 9y^2 + 12x + 8y = 0$

a) $B^2 - 4AC$

$(-12)^2 - 4(4)(9) = 0$

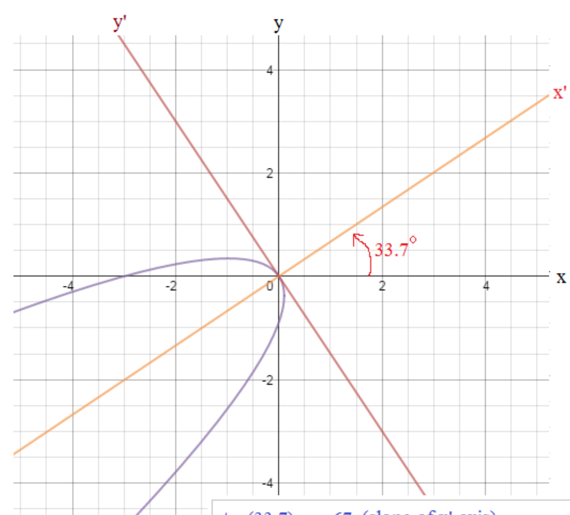
Since it equals 0, it's a $\boxed{\text{rotated parabola...}}$

b) $\cot(2\theta) = \dfrac{A - C}{B}$

$\cot(2\theta) = \dfrac{4 - 9}{-12} = 5/12$

$2\theta = 67.38$

$\boxed{\theta \approx 33.7^\circ}$



c) $x = x'\cos(33.7) - y'\sin(33.7)$

$x = .83x' - .55y'$

$y = x'\sin(33.7) + y'\cos(33.7)$

$y = .55x' + .83y'$

$\boxed{\begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{array}}$

then, substitute..

$\boxed{\begin{array}{l} \tan(33.7) = .67 \text{ (slope of x'-axis)} \\ \quad \text{then, } -1.5 \text{ (slope of y'-axis)} \end{array}}$ perpendicular

$4(.83x' - .55y')^2 - 12(.83x' - .55y')(.55x' + .83y') + 9(.55x' + .83y')^2 + 12(.83x' - .55y') + 8(.55x' + .83y') = 0$

$4(.69x'^2 - .91x'y' + .30y'^2) - 12(.46x'^2 + .39x'y' - .46y'^2) + 9(.30x'^2 + .91x'y' + .69y'^2) + 9.96x' - 6.6y' + 4.4x' + 6.64y' = 0$

$0x'^2 + 0x'y' + 12.9y'^2 + 14.35x' + 0y' = 0 \Rightarrow 14.35x' = -12.9y'^2$ $\boxed{x' = -.9(y')^2}$

vertex: $(0, 0)$ Opens to the left...

Rotation of Conics Exercise

SOLUTIONS

3) $2x^2 - 8xy + 2y^2 - 6 = 0$

a) $B^2 - 4AC$

$(-8)^2 - 4(2)(2) = 48 > 0$

Since it is greater than 0, it's a $\boxed{\text{rotated hyperbola ...}}$

b) $\cot(2\theta) = \dfrac{A - C}{B}$

$\cot(2\theta) = \dfrac{2 - 2}{-8} = 0$

$2\theta = 90^\circ$

$\boxed{\theta = 45^\circ}$

$$\boxed{\begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{array}}$$

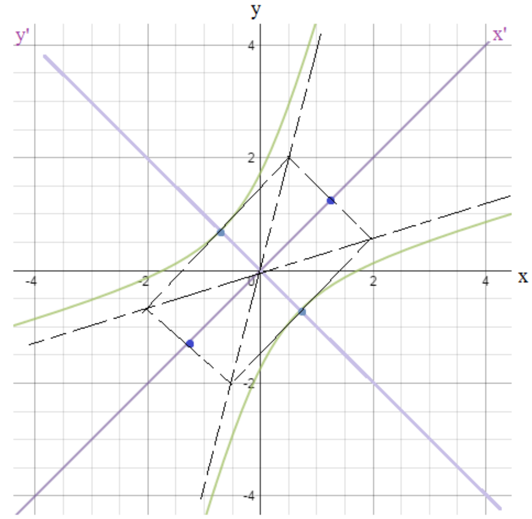
c) $x = x'\cos(45) - y'\sin(45)$

$x = \dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'$

$y = x'\sin(45) + y'\cos(45)$

$y = \dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'$

then, substitute..



$$2\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)^2 - 8\left(\frac{\sqrt{2}}{2}x' - \frac{\sqrt{2}}{2}y'\right)\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right) + 2\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 = 6$$

$$2\left(\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2\right) - 8\left(\frac{1}{2}x'^2 - \frac{1}{2}y'^2\right) + 2\left(\frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2\right) = 6$$

$$-2x'^2 + 0x'y' + 6y'^2 = 6$$

$$\boxed{\frac{y'^2}{1} - \frac{x'^2}{3} = 1}$$

tan(45) = 1 so, y = (1)x becomes the x'axis

and, y = (-1)x becomes the y'axis

'vertical hyperbola' center: (0, 0)

vertex (on the x'y'- coordinate plane): (0, 1) (0, -1)

co-vertex (on the x'y'-coordinate plane): $(\sqrt{3}, 0)$ $(-\sqrt{3}, 0)$

4) $4x^2 - 6xy + 4y^2 - 6y - 2 = 0$

$$\boxed{\begin{array}{l} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{array}}$$

a) $B^2 - 4AC$

$(-6)^2 - 4(4)(4) = -28 < 0$

$\boxed{\text{rotated AND shifted ellipse}}$

b) $\tan(2\theta) = \dfrac{B}{A - C}$

$\tan(2\theta) = \dfrac{-6}{4 - 4}$ undefined

$2\theta = 90^\circ$

$\boxed{\theta = 45^\circ}$

c) $x = x'\cos(45) - y'\sin(45)$

$x = \dfrac{\sqrt{2}}{2}x' - \dfrac{\sqrt{2}}{2}y'$

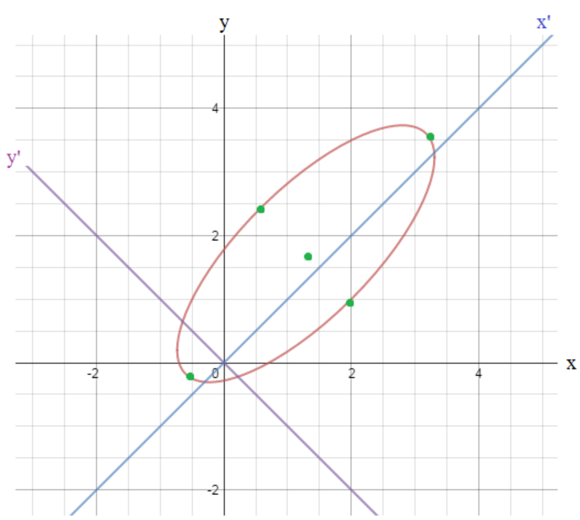
$x = \dfrac{\sqrt{2}}{2}(x' - y')$

$y = x'\sin(45) + y'\cos(45)$

$y = \dfrac{\sqrt{2}}{2}x' + \dfrac{\sqrt{2}}{2}y'$

$y = \dfrac{\sqrt{2}}{2}(x' + y')$

then, substitute..



$$2(x' - y')^2 - 3(x'^2 - y'^2) + 2(x' + y')^2 - 3\sqrt{2}(x' + y') = 2$$

$$x'^2 + 0x'y' + 7y'^2 - 3\sqrt{2}x' - 3\sqrt{2}y' = 2$$

(complete the square)

$$x'^2 - 3\sqrt{2}x' + \frac{9}{2} + 7\left(y'^2 - \frac{3\sqrt{2}y'}{7} + \frac{18}{196}\right) = 2 + \frac{9}{2} + \frac{18}{28}$$

$$\left(x' - \frac{3}{\sqrt{2}}\right)^2 + 7\left(y' - \frac{3}{\sqrt{98}}\right)^2 = \frac{50}{7} \qquad (x' - 2.12)^2 + 7(y' - .30)^2 = 7.14$$

center: (2.12, .30) on the x'y'-coordinate plane

vertices: (-.55, .30) and (4.79, .30)

co-vertices: (2.12, 1.30) and (2.12, -.70)

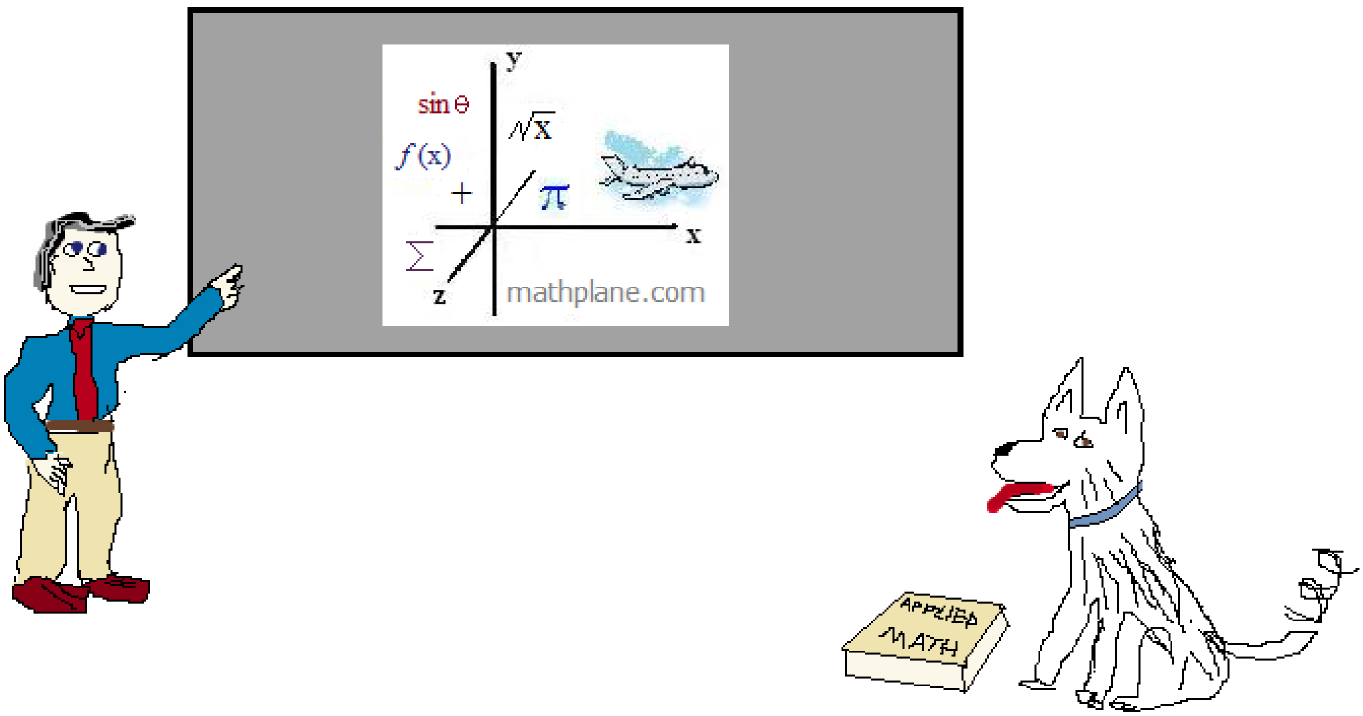
(approximate values)

$$\frac{(x' - 2.12)^2}{7.14} + \frac{(y' - .30)^2}{1.02} = 1$$

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know

Cheers



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