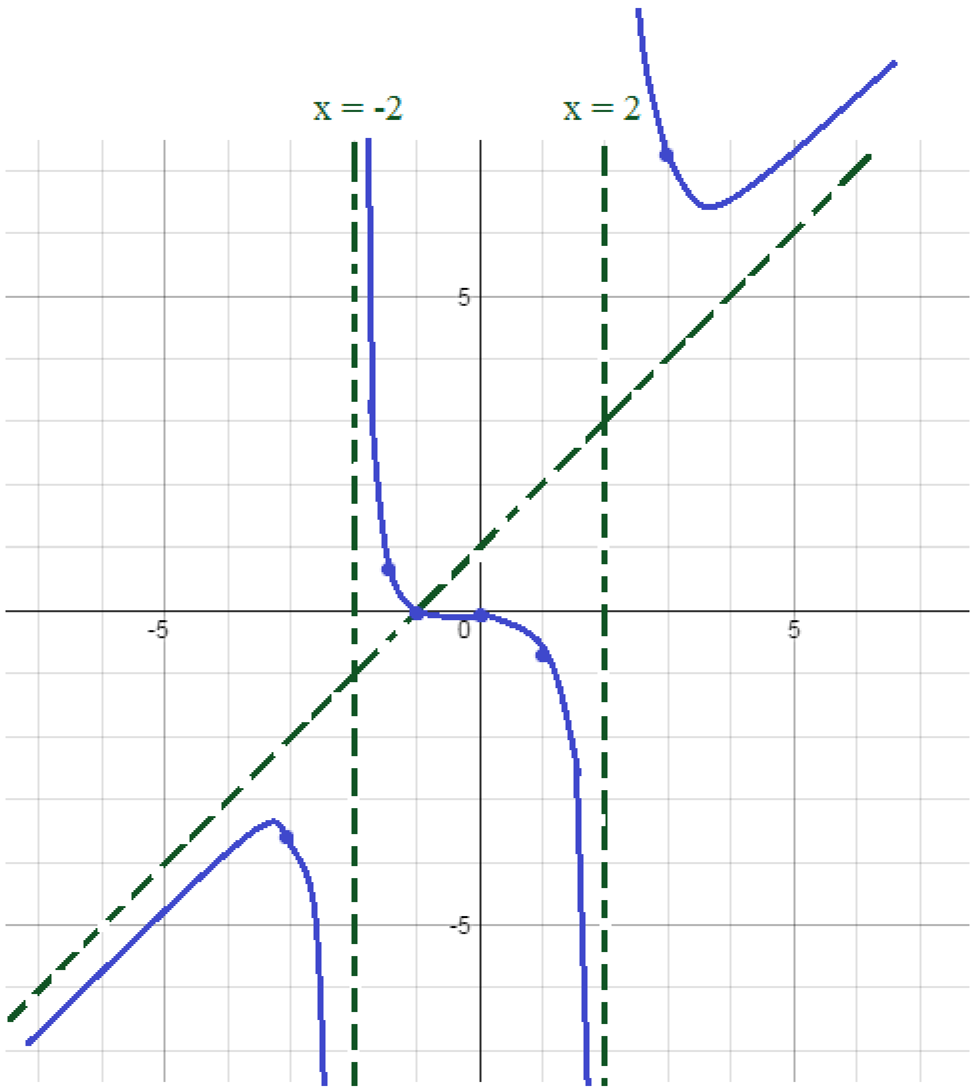
Sketching Rational Expressions II

Notes, Examples, and Practice Exercises (with Solutions)



Topics include end behavior asymptotes, intercepts, factoring, limits, graphing, and more.

Example: Graph the following function.
Identify the asymptotes and intercepts.

$$f(x) = \frac{x^3 + x^2}{x^2 - 4} \qquad f(x) = \frac{x^3 + x^2}{(x+2)(x-2)}$$

y-intercept: (0, 0)

The function crosses the y-axis at this point...

x-intercept: (0, 0) and (-1, 0)

The function lies on the x-axis.
(this occurs when $f(x) = 0$)

Vertical Asymptotes: $x = 2$ and $x = -2$

The function is undefined at these x values...

Horizontal Asymptote

None. Since the degree of the numerator's lead term is 3 and the degree of the denominators lead term is 2, this expression is "top heavy"...

Slant Asymptote

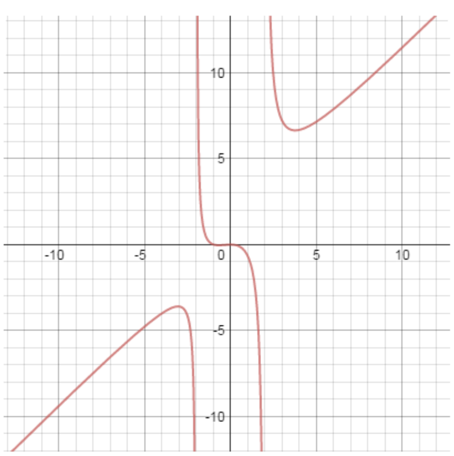
Since the degree of the numerator is 1 more than the degree of the denomiantor, there is a slant asymptote. (Use long division to find it)

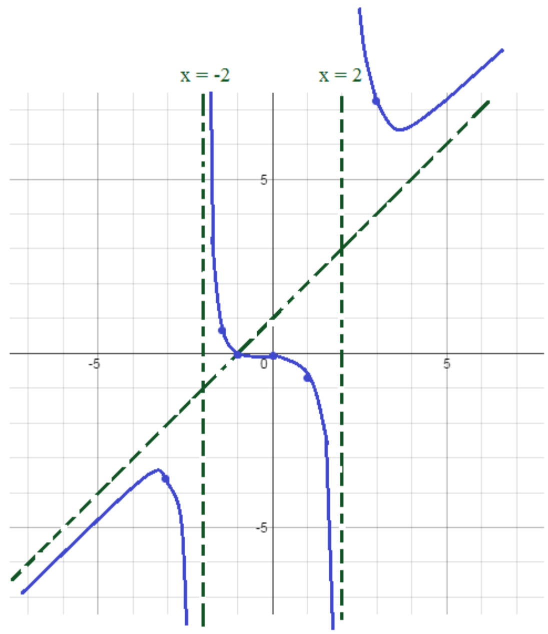
$$x + 1 + \frac{4x + 4}{x^2 - 4}$$

$$\begin{array}{r|l} x^2 - 4 & x^3 + x^2 + 0x + 0 \\ & -\;(x^3 \qquad - 4x) \\ \hline & 0 + x^2 + 4x \\ & -\;(x^2 \qquad - 4) \\ \hline & 0 + 4x + 4 \end{array}$$

As the function goes to infinity or negative infinity, the remainder approaches 0 ("bottom heavy"). therefore, we ignore the remainder and see the slant asymptote is

$$\boxed{x + 1} + \frac{4x + 4}{x^2 - 4}$$





To sketch the graph,

1) draw the asymptotes.

Note: The graph may not cross the vertical asymptote(s).
However, it is possible that the graph can cross through a horizontal or slant asymptote (i.e. "end behavior asymptote")

2) Plot the itntercepts. (0, 0) and (-1, 0)

3) Pick points in each region or make a larger table of values

Left of the asymptote, let's pick x = -3... $f(-3) = -18/5$

Middle of asymptotes, let's try x = -1.5... $f(-1.5) = .64$

x = 1... $f(1) = -2/3$

Right of asymptotes, use x = 3 $f(3) = 36/5$

4) Does the graph cross the slant asymptote ('end behavior asymptote')?

$$x + 1 = \frac{x^3 + x^2}{(x+2)(x-2)}$$

$$(x+1)(x+2)(x-2) = x^3 + x^2$$

$$x^3 + x^2 - 4x - 4 = x^3 + x^2$$

$$x = -1$$

Yes, at (-1, 0)

Quadratic End Behavior

Example: $f(x) = \dfrac{-x^4 + 2x^3 - 2x^2}{(x-1)^2}$ $\qquad \dfrac{-x^2(x^2 - 2x + 2)}{(x-1)(x-1)}$

vertical asymptote: $x = 1$

horizontal asymptote: None
("top-heavy": degree of numerator > degrees of denominator)

end behavior asymptote: Parabolic asymptote (because degree of numerator is 2 more than degree of denominator)

Long division:

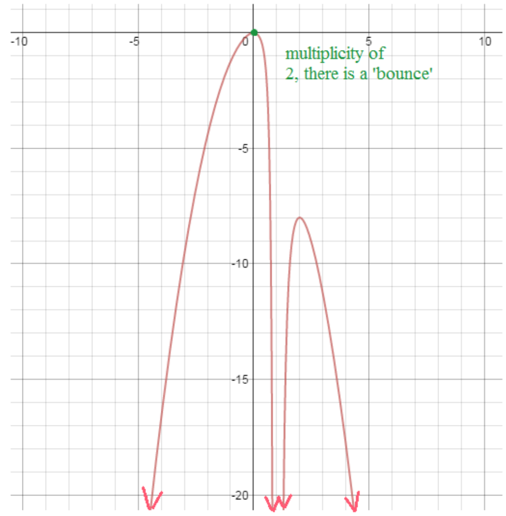
$$
\begin{array}{r}
-x^2 + 0 - 1 \\
x^2 - 2x + 1 \,\big)\, -x^4 + 2x^3 - 2x^2 \\
-\;\; (-x^4 + 2x^3 - x^2) \\
\hline
0 + 0 - x^2 \\
-\;\; (-x^2 + 2x - 1) \\
\hline
0 - 2x + 1
\end{array}
$$

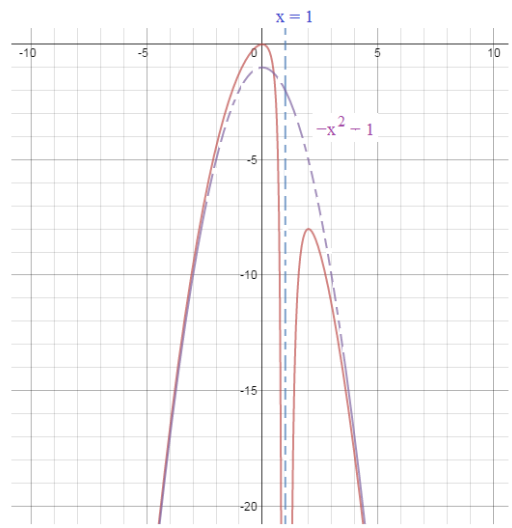
end behavior is $\boxed{-x^2 - 1} + \dfrac{-2x + 1}{(x^2 - 2x + 1)}$

y-intercept: (0, 0)

x-intercept: (0, 0) $\qquad -x^2(x^2 - 2x + 2) = 0$

roots are 0 with multiplicity of 2
and, the other two roots are
imaginary (because the discriminate < 0)





Where does the graph cross the end behavior asymptote?

$$-x^2 - 1 = \frac{-x^4 + 2x^3 - 2x^2}{(x-1)^2}$$

$$(-x^2 - 1)(x-1)^2 = -x^4 + 2x^3 - 2x^2$$

$$(-x^2 - 1)(x^2 - 2x + 1) = -x^4 + 2x^3 - 2x^2$$

$$-x^4 + 2x^3 - 2x^2 + 2x - 1 = -x^4 + 2x^3 - 2x^2$$

$$2x - 1 = 0$$

$$x = 1/2 \quad \text{YES!}$$

Cubic End Behavior

Example: Graph $y = \dfrac{x^4 - 3x^3 + 6}{x - 3}$

The degree of the numerator is 3 more than the degree of the denominator.
So, there is a "cubic asymptote"..

Using Synthetic Division:

3	1	-3	0	0	6
		3	0	0	0
	1	0	0	0	6

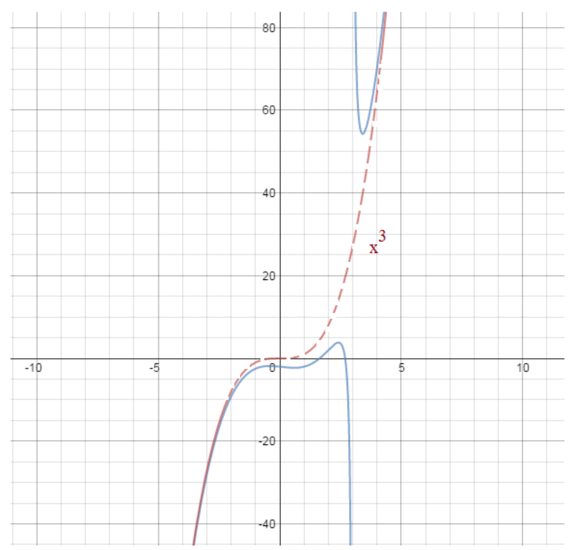
End behavior is x^3

$$x^3 + \frac{6}{x^4 - 3x^3 + 6}$$

x-intercepts: (1.6, 0) (2.7, 0)

y-intercept: (0, -2)

The cubic asymptote and vertical
asymptote intersect at (3, 27)

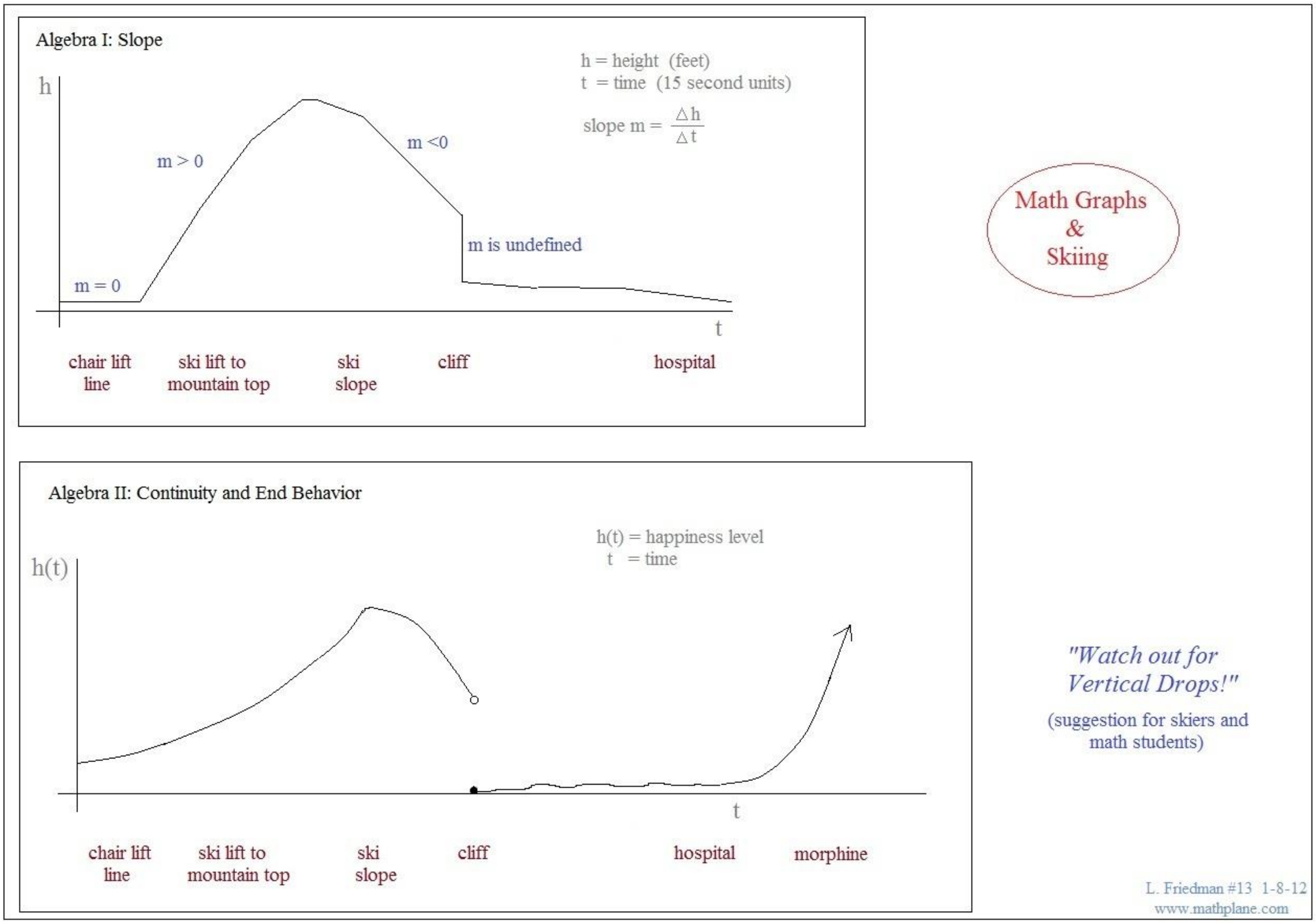


Observation: Note how the graph looks like a cubic function interrupted/divide by a vertical asymptote slicing through at x = 3

Does the end behavior asymptote intersect the function?

$$x^3 = \frac{x^4 - 3x^3 + 6}{x - 3} \implies x^4 - 3x^3 = x^4 - 3x^3 + 6$$

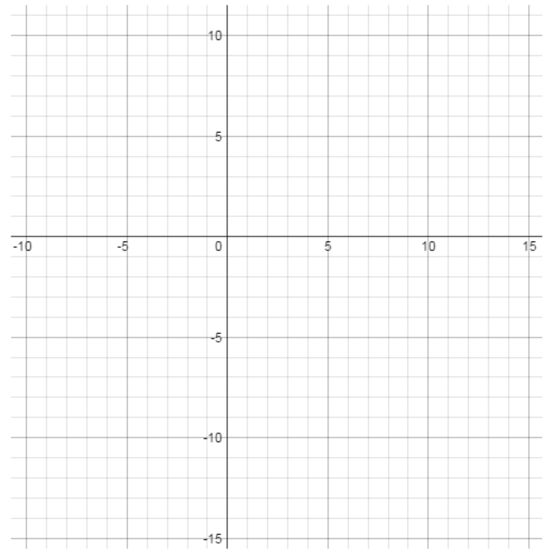
$0 = 6$ NO! they do not intersect....



Algebra I: Slope
h = height (feet)
t = time (15 second units)
slope m = △h / △t
h
m > 0
m <0
m is undefined
m = 0
t
chair lift line
ski lift to mountain top
ski slope
cliff
hospital
Math Graphs & Skiing
Algebra II: Continuity and End Behavior
h(t) = happiness level
t = time
h(t)
t
chair lift line
ski lift to mountain top
ski slope
cliff
hospital
morphine
"Watch out for Vertical Drops!"
(suggestion for skiers and math students)
L. Friedman #13 1-8-12
www.mathplane.com

Exercises-→

1) $y = \dfrac{x^2(x-7)}{(x-4)^2(x+1)}$



Vertical Asymptote(s)

Horizontal Asymptote
(or, End Behavior Asymptote)

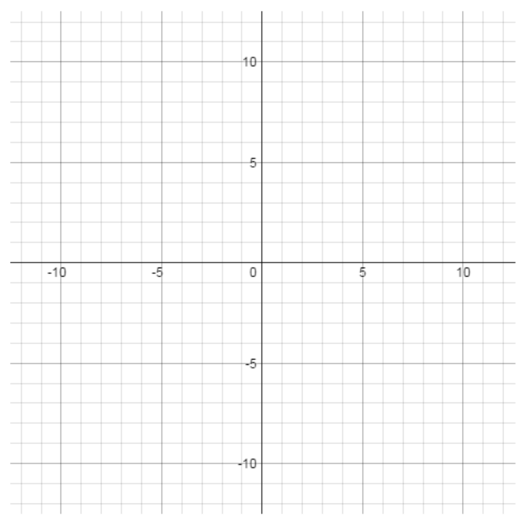
x-intercept(s)

y-intercept

Domain

Does the graph cross over the Horizontal/End Behavior Asymptote?

2) $f(x) = \dfrac{2x^2 - x - 1}{x^2 + 2x - 3}$



Vertical Asymptote(s)

Horizontal Asymptote
(or, End Behavior Asymptote)

x-intercept(s)

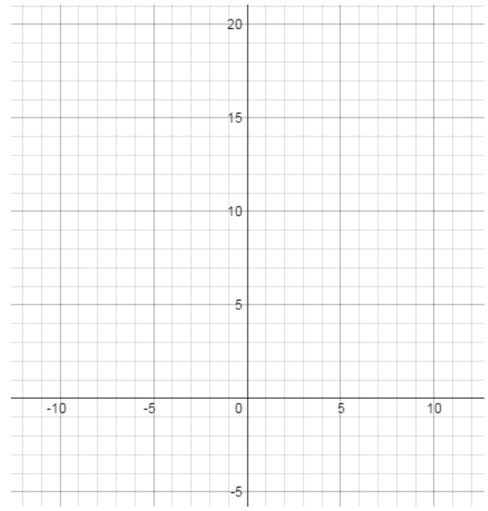
y-intercept

Domain

"Hole"?

Does the graph cross over the Horizontal/End Behavior Asymptote?

3) $y = \dfrac{3x^2 + 7x - 20}{x - 1}$



Vertical Asymptote:

Horizontal Asymptote:

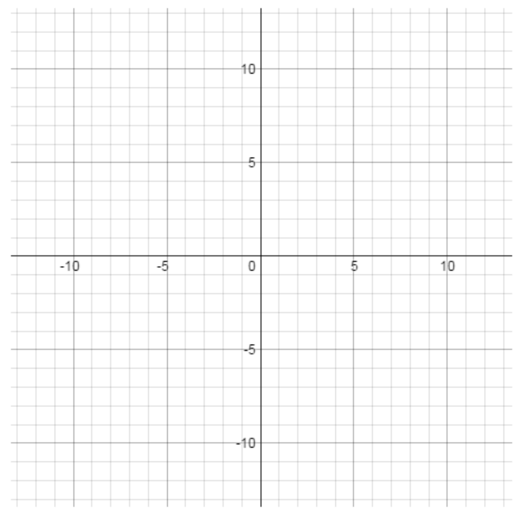
Slant/Oblique Asymptote:

x-intercept(s):

y-intercept:

Does curve cross 'end behavior asymptote'?

4) $h(x) = \dfrac{2x^2 - 5x - 3}{x^2 - 16}$



Vertical Asymptote:

Horizontal Asymptote:

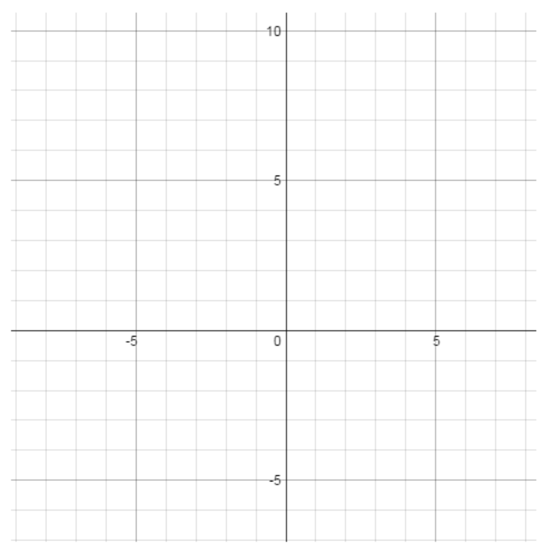
Slant/Oblique Asymptote:

x-intercept(s):

y-intercept:

Does curve cross 'end behavior asymptote'?

5) $f(x) = \dfrac{x^4 - x^3 + 1}{x - 1}$



Vertical Asymptote:

Horizontal Asymptote:

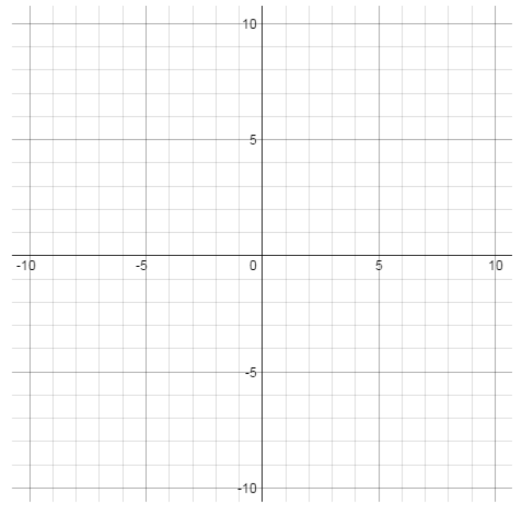
End Behavior Asymptote:

x-intercept(s):

y-intercept:

Does curve cross 'end behavior asymptote'?

6) $f(x) = \dfrac{x^3 - 16x}{-2x^2 + 2x + 12}$



Vertical Asymptote:

Horizontal Asymptote:

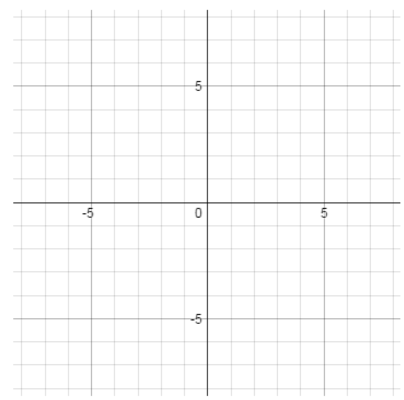
Slant/Oblique Asymptote:

x-intercept(s):

y-intercept:

Does curve cross 'end behavior asymptote'?

7) $g(x) = \dfrac{x - 2}{x^2 - 9}$



Vertical Asymptote:

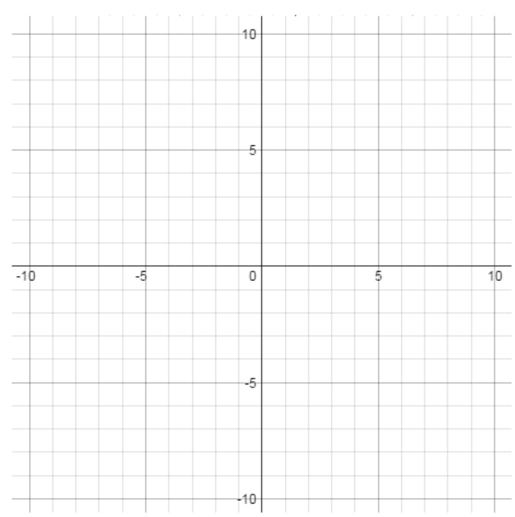
Horizontal Asymptote:

x-intercept(s):

y-intercept:

Does curve cross 'end behavior asymptote'?

8) $h(x) = \dfrac{3x^2 + 5x - 2}{x^2 - 16}$



Vertical Asymptote:

Horizontal Asymptote:

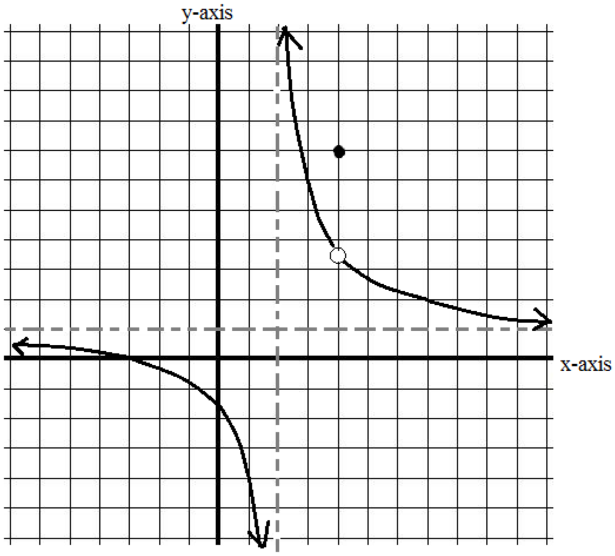
x-intercept(s):

y-intercept:

Does curve cross 'end behavior asymptote'?

More Rational Expression Topics

A)



$\lim_{x \to 2^+} f(x) =$

$\lim_{x \to 2^-} f(x) =$

$\lim_{x \to 2} f(x) =$

$\lim_{x \to \infty} f(x) =$

$\lim_{x \to -\infty} f(x) =$

$\lim_{x \to 4} f(x) =$

$\lim_{x \to 0^+} f(x) =$

$f(1) =$

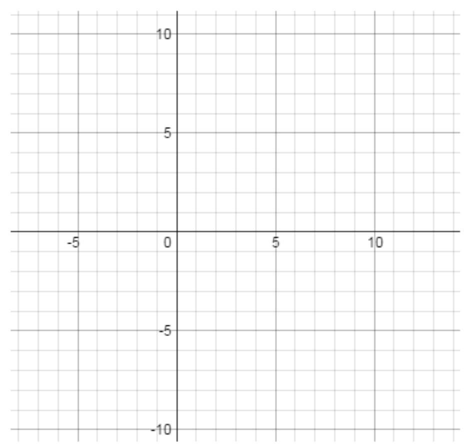
$f(4) =$

$f(2) =$

$f(0) =$

BONUS: Can you determine the rational expression from the graph?

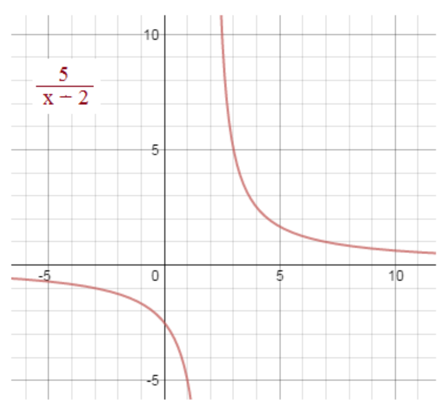
B) Use long division and transformations to graph $y = \frac{x+2}{x-5}$

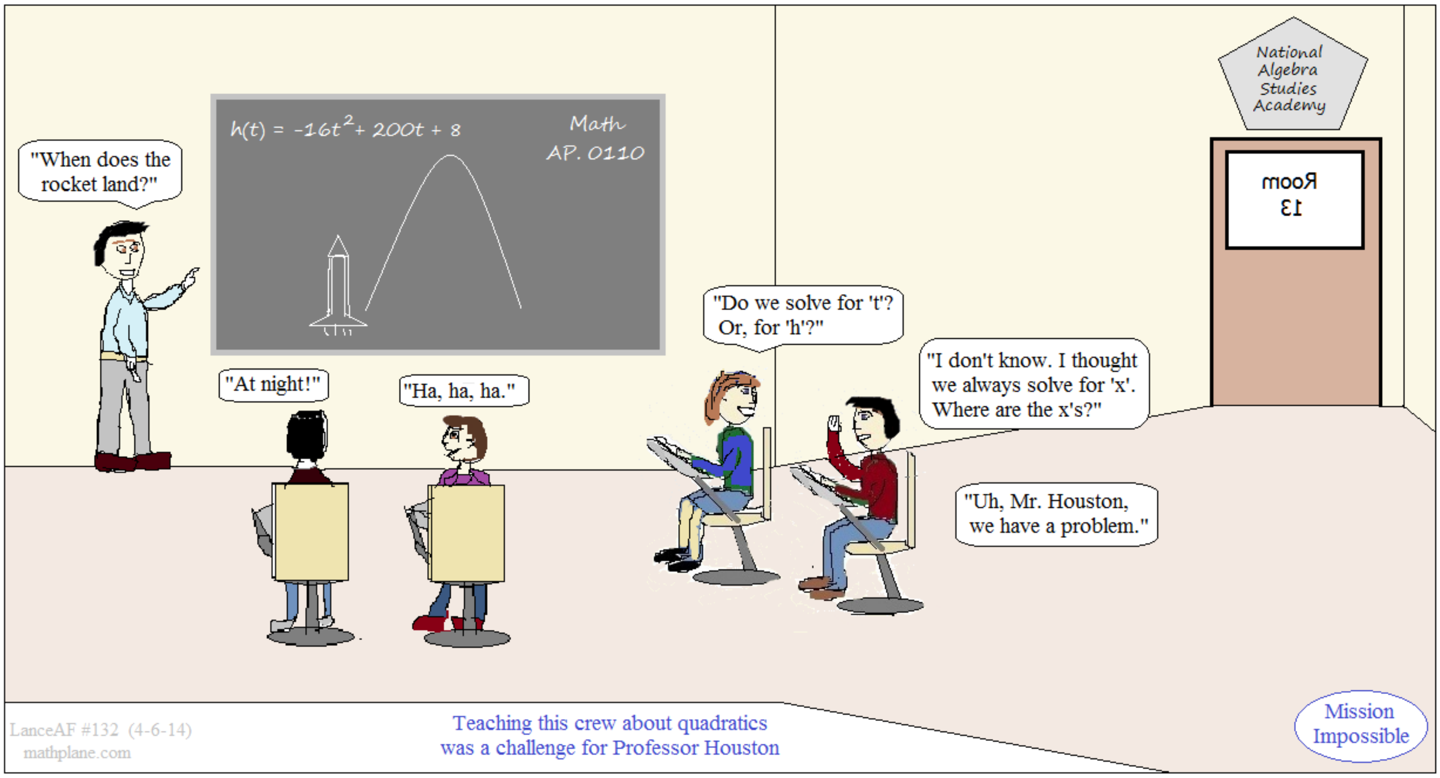


C) $f(x) = \frac{5}{x-2}$

What is the Average Rate Of Change on the interval [3, 5] ?

What is the AROC between 1 and 3?




$h(t) = -16t^2 + 200t + 8$
Math
AP. 0110
National
Algebra
Studies
Academy
Room
13
"When does the rocket land?"
"At night!"
"Ha, ha, ha."
"Do we solve for 't'? Or, for 'h'?"
"I don't know. I thought we always solve for 'x'. Where are the x's?"
"Uh, Mr. Houston, we have a problem."
LanceAF #132 (4-6-14)
mathplane.com
Teaching this crew about quadratics
was a challenge for Professor Houston
Mission
Impossible

SOLUTIONS-→

SOLUTIONS

Rational Expressions 2 Graphing Exercises

1) $y = \dfrac{x^2(x-7)}{(x-4)^2(x+1)}$

Vertical Asymptote(s) $x = 4 \quad x = -1$

Horizontal Asymptote
(or, End Behavior Asymptote)
Degree of numerator: 3; Degree of numerator: 3
therefore, look at lead coefficients: $y = 1$

x-intercept(s) (0, 0) with multiplicity of 2.. (7, 0)

y-intercept (0, 0)

Domain all real numbers, except $x \neq 0, 7$

Does the graph cross over the Horizontal/End Behavior Asymptote?

In other words, does the graph intersect y = 1

$$1 = \frac{x^2(x-7)}{(x-4)^2\,(x+1)}$$

cross multiply and expand...

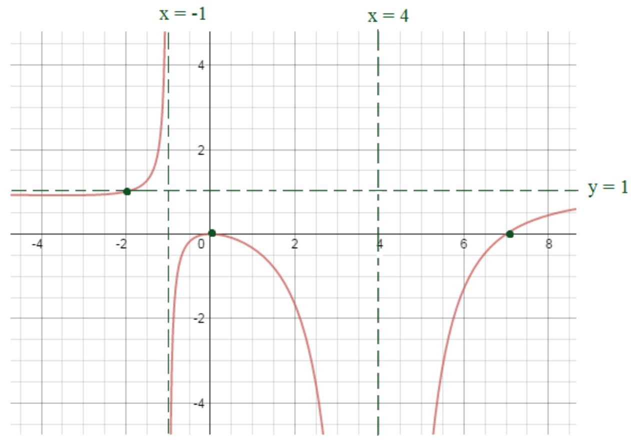
$$x^3 - 7x^2 = (x^2 - 8x + 16)(x + 1)$$

$$x^3 - 7x^2 = x^3 - 7x^2 + 8x + 16$$

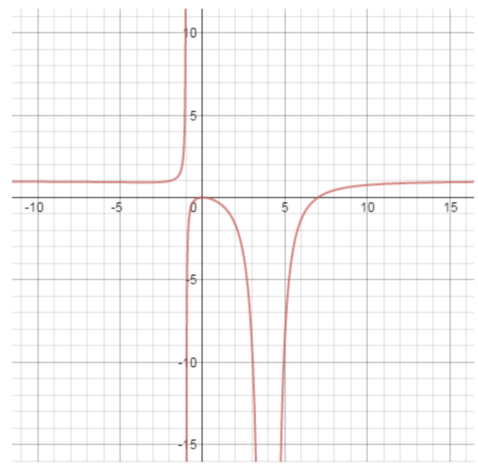
$$0 = 8x + 16$$

$$x = -2$$

YES!
the graph crosses over at (2, 1)



Note: the x-intercept (0, 0) has a multiplicity of 2, so there is a "bounce"
the cross-over point is at (-2, 1)
These and the asymptotes help sketch the curves...



2) $f(x) = \dfrac{2x^2 - x - 1}{x^2 + 2x - 3}$ $\qquad \dfrac{(2x+1)(x-1)}{(x+3)(x-1)}$

Vertical Asymptote(s) $x = -3$

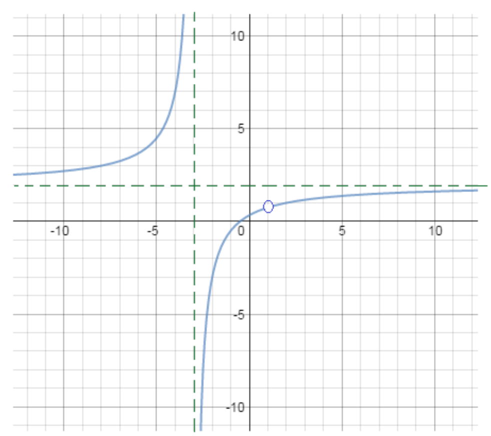
Horizontal Asymptote
(or, End Behavior Asymptote) $y = 2$

x-intercept(s) (-1/2, 0)

y-intercept (0, 1/3)

Domain All real numbers except $x \neq 1, -3$

"Hole"? (1, 3/4)



Does the graph cross over the Horizontal/End Behavior Asymptote?

$$2 = \frac{2x^2 - x - 1}{x^2 + 2x - 3} \quad \text{or} \quad 2 = \frac{(2x+1)(x-1)}{(x+3)(x-1)}$$

$$2x^2 - x - 1 = 2x^2 + 4x - 6$$

$$5 = 5x$$

the solution x = 1 occurs at the "hole"..
So, there is no cross-over point...

$$2 = \frac{(2x+1)}{(x+3)}$$

$$2x + 6 = 2x + 1$$

$$6 = 1$$

No Solution

3) $y = \dfrac{3x^2 + 7x - 20}{x - 1}$ $\qquad \dfrac{(3x - 5)(x + 4)}{(x - 1)}$

$$\begin{array}{r|rrr} 1 & 3 & 7 & -20 \\ & & 3 & 10 \\ \hline & 3 & 10 & -10 \end{array}$$

3x + 10 is the slant asymptote

Vertical Asymptote: x = 1

Horizontal Asymptote: none

Slant/Oblique Asymptote: y = 3x + 10

x-intercept(s): (5/3, 0) (-4, 0)

y-intercept: (0, 20)

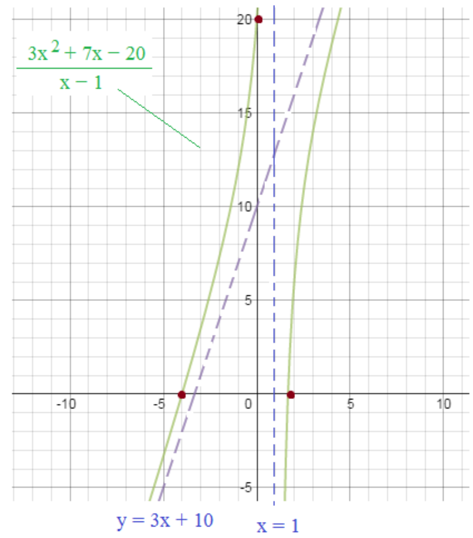
Does curve cross 'end behavior asymptote'? NO

$$3x + 10 = \frac{3x^2 + 7x - 20}{x - 1}$$

$$3x^2 + 7x - 10 = 3x^2 + 7x - 20$$

$$-10 \neq -20$$

no solution, so curve never crosses the slant asymptote



4) $h(x) = \dfrac{2x^2 - 5x - 3}{x^2 - 16}$ $\qquad \dfrac{(2x + 1)(x - 3)}{(x + 4)(x - 4)}$

Vertical Asymptote: x = 4 and x = −4

Horizontal Asymptote: y = 2

Slant/Oblique Asymptote: None

x-intercept(s): (-1/2, 0) and (3, 0)

y-intercept: (0, 3/16)

Does curve cross 'end behavior asymptote'? YES!

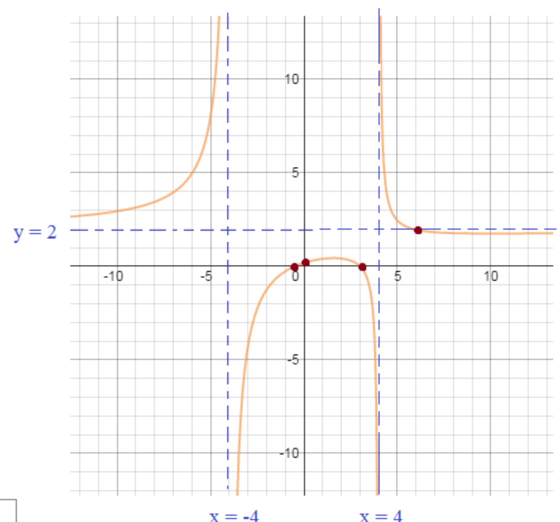
degree of numerator: 2
degree of denominator: 2
same, so use coefficients: 2/1

y = 2 $\qquad 2 = \dfrac{2x^2 - 5x - 3}{x^2 - 16}$

$$2x^2 - 32 = 2x^2 - 5x - 3$$

$$-29 = -5x$$

$$x = 5.8$$



5) $f(x) = \dfrac{x^4 - x^3 + 1}{x - 1}$

$x^4 - x^3 = -1$

If x > 1, the answer is positive
If x < -1, the answer is positive
If 0 < x < 1. the answer is negative (but not -1)
If -1 < x < 0, the answer is positive
∴ there is no x-intercept

Vertical Asymptote: x = 1

Horizontal Asymptote: None (because it's "top heavy")
degree of numerator > degree of denominator

End Behavior Asymptote: x^3

x-intercept(s): None

y-intercept: (0, −1)

Does curve cross 'end behavior asymptote'?

$$x^3 = \frac{x^4 - x^3 + 1}{x - 1} \qquad \text{NO}$$

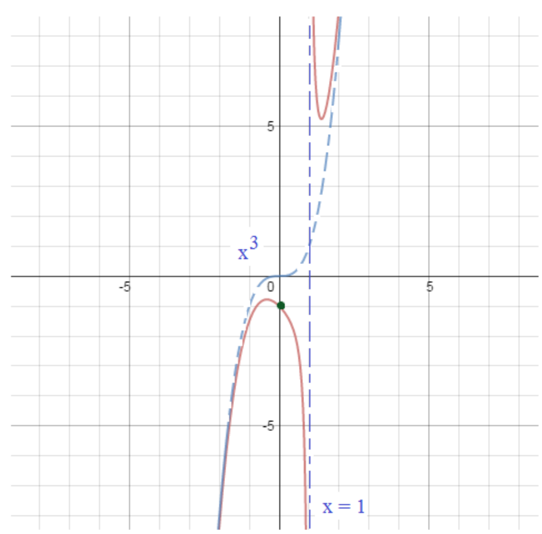
$$x^4 - x^3 = x^4 - x^3 + 1$$

$$0 = 1$$

Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & 0 \\ \hline & 1 & 0 & 0 & 0 & 1 \end{array}$$

$$x^3 + \frac{1}{x - 1}$$



6) $f(x) = \dfrac{x^3 - 16x}{-2x^2 + 2x + 12}$ $\quad \dfrac{x(x^2 - 16)}{-2(x^2 - x - 6)}$ $\quad \dfrac{x(x + 4)(x - 4)}{-2(x - 3)(x + 2)}$

x-intercept gives clue that middle curve goes through the oblique asymptote..

$$\begin{array}{r} -\frac{1}{2}x - \frac{1}{2} \\ -2x^2 + 2x + 12 \overline{\big)\, x^3 + 0x^2 - 16x} \\ -\ (x^3 - x^2 - 6x) \\ \hline x^2 - 10x \\ -\ (x^2 - x - 6) \\ \hline -9x + 6 \end{array}$$

Vertical Asymptote: $x = -2$ and $x = 3$

Horizontal Asymptote: None

Slant/Oblique Asymptote: $y = -\frac{1}{2}x - \frac{1}{2}$

x-intercept(s): (0, 0) (−4, 0) (4, 0)

y-intercept: (0, 0)

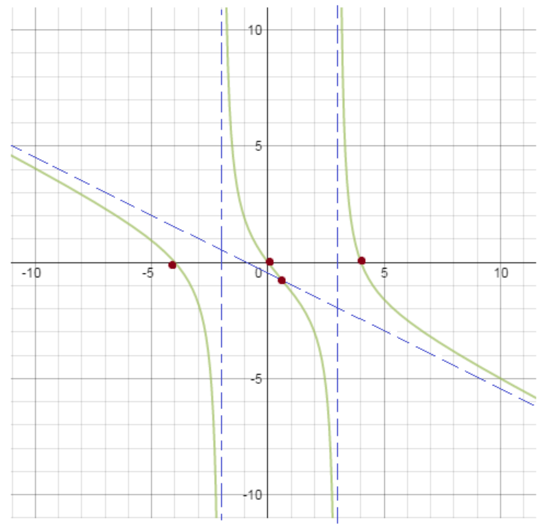
Does curve cross 'end behavior asymptote'?

$$-\frac{1}{2}x - \frac{1}{2} = \frac{x^3 - 16x}{-2x^2 + 2x + 12}$$

$$x^3 - x^2 - 6x + x^2 - x - 6 = x^3 - 16x$$

$$-7x - 6 = -16x$$

YES at $x = 2/3$



Note: graphs may cross through end behavior asymptotes.
(However, they may not cross through a vertical asymptote.)

7) $g(x) = \dfrac{x - 2}{x^2 - 9}$

$(x + 3)(x - 3)$

Vertical Asymptote: $x = 3$ and $x = -3$

Horizontal Asymptote: $y = 0$ (because expression is "bottom heavy"
deg. of numerator < deg. of denominator)

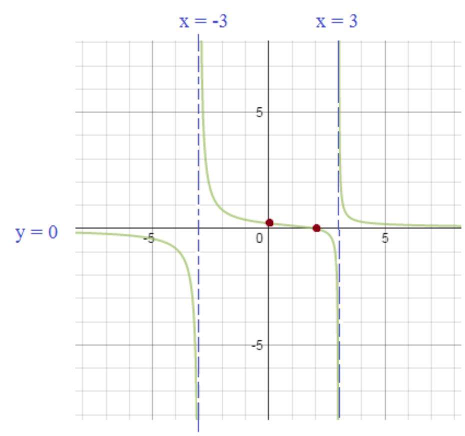
x-intercept(s): (2, 0)

y-intercept: (0, 2/9)

Does curve cross 'end behavior asymptote'?

Since there is an x-intercept @ (2, 0)
and the horizontal asymptote is $y = 0$,

then obviously the curve crosses the asymptote at $x = 2$!



8) $h(x) = \dfrac{3x^2 + 5x - 2}{x^2 - 16}$ $\quad \dfrac{(3x - 1)(x + 2)}{(x + 4)(x - 4)}$

Vertical Asymptote: $x = -4$ $x = 4$

Horizontal Asymptote: $y = 3$ (deg. of numerator = deg. of denominator
then, lead cofficients 3/1)

x-intercept(s): (1/3, 0) (-2, 0)

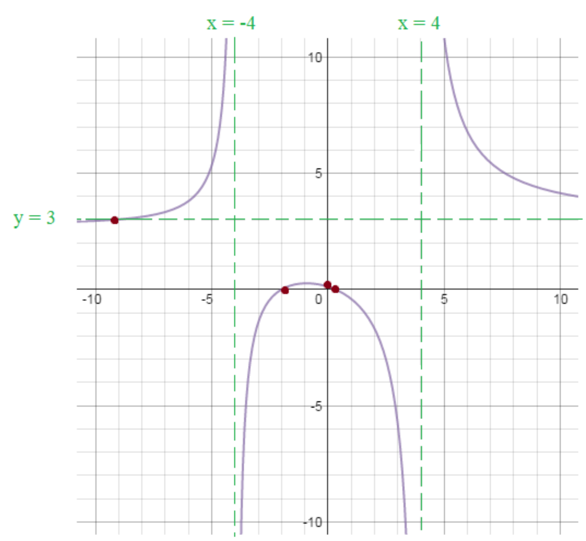
y-intercept: (0, -2/-16) ----> (0, 1/8)

Does curve cross 'end behavior asymptote'? YES.. At, (-9.2, 3)

$$3 = \frac{3x^2 + 5x - 2}{x^2 - 16}$$

$$3x^2 - 48 = 3x^2 + 5x - 2$$

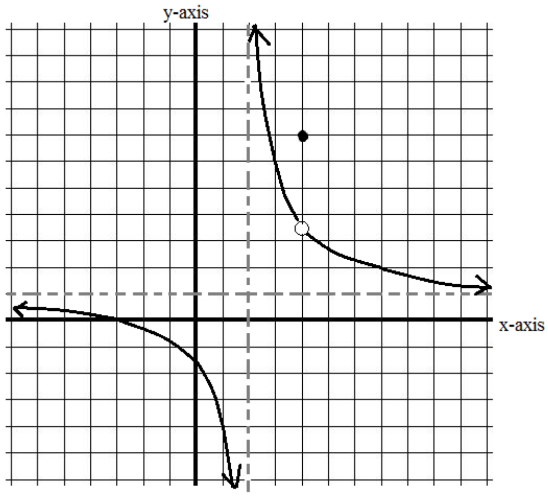
$$-46 = 5x \qquad x = -9.2$$



More Rational Expression Topics

SOLUTIONS

A)



$\lim_{x\to 2^+} f(x) = +\infty$

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 2} f(x) =$ Does not exist (DNE)

$\lim_{x\to \infty} f(x) = 1$

$\lim_{x\to -\infty} f(x) = 1$

$\lim_{x\to 4} f(x) = 7/2$

$\lim_{x\to 0^+} f(x) = -3/2$

$f(1) = -4$

$f(4) = 7$

$f(2) =$ undefined

$f(0) = -3/2$

BONUS: Can you determine the rational expression from the graph?

$$f(x) = \begin{cases} \dfrac{(x+3)(x-4)}{(x-2)(x-4)} & \text{for all } x \neq 4 \\ 7 & \text{if } x = 4 \end{cases}$$

B) Use long division and transformations to graph $y = \dfrac{x+2}{x-5}$

$$\begin{array}{r|l} & 1 \quad + \dfrac{7}{x-5} \\ x-5 & x+2 \\ & -\;x-5 \\ \hline & \quad 7 \end{array}$$

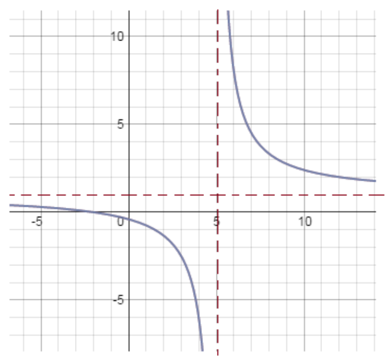
This is a 'reciprocal function'...

Vertical shift: up 1

Horizontal shift: left 5

Vertical stretch: factor of 7

$\dfrac{7}{x-5} + 1$



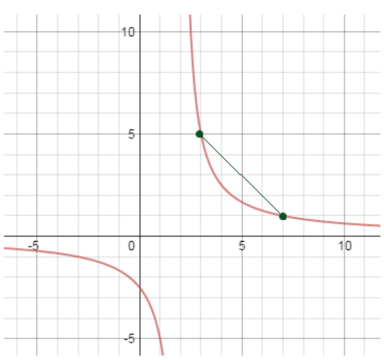
C) $f(x) = \dfrac{5}{x-2}$

What is the Average Rate Of Change on the interval [3, 7] ?

(3, 5) and (7, 1)

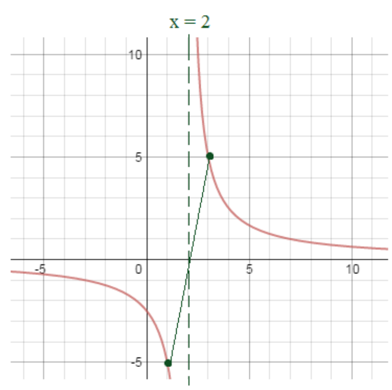
AROC is "slope" between points...

$\dfrac{5-1}{3-7} = -1$



What is the AROC between 1 and 3?

The average rate of change cannot be determined!
(because the interval [1, 3] includes an undefined value..)

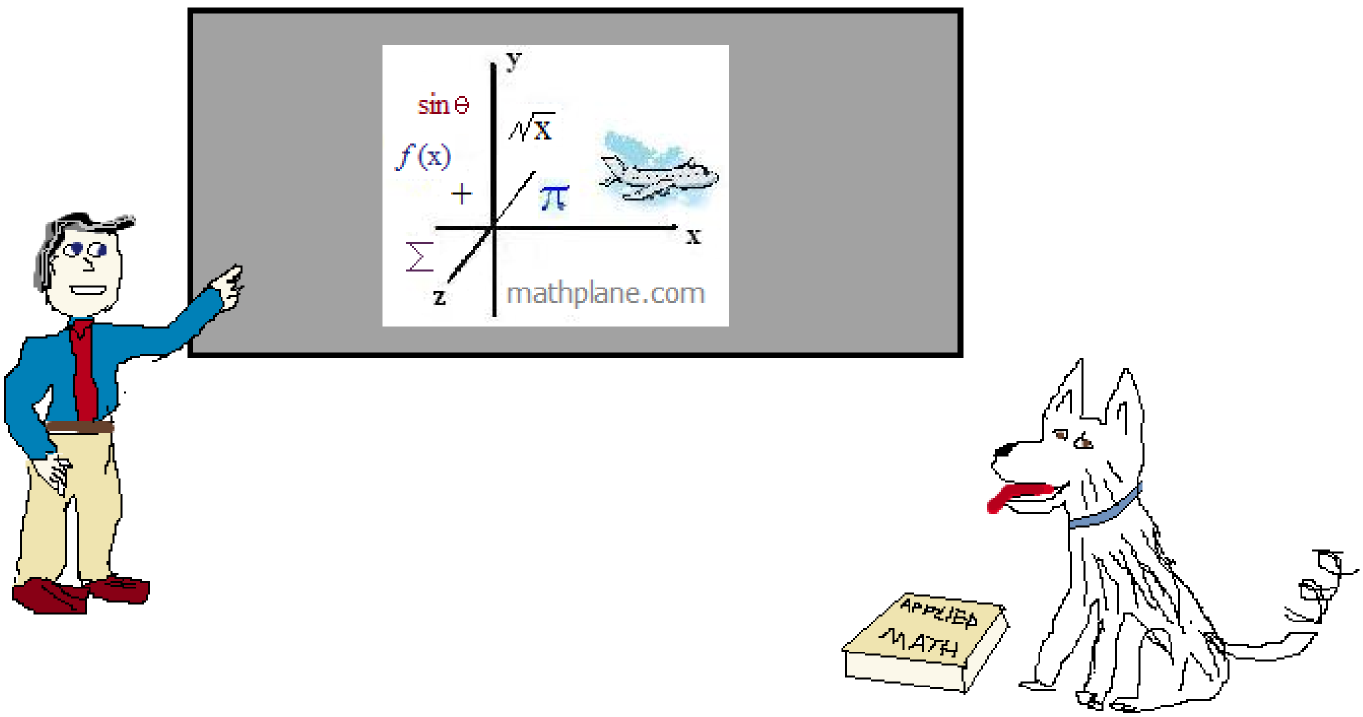


Why can't the AROC be determined?
Because the AROC is also the average of all the instantaneous rates of change.
Since the IROC cannot be determined at x = 2, it's impossible to determine the entire average...

Thanks for visiting. Hope it helped!

If you have questions, suggestions, or requests, let us know.

Cheers



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