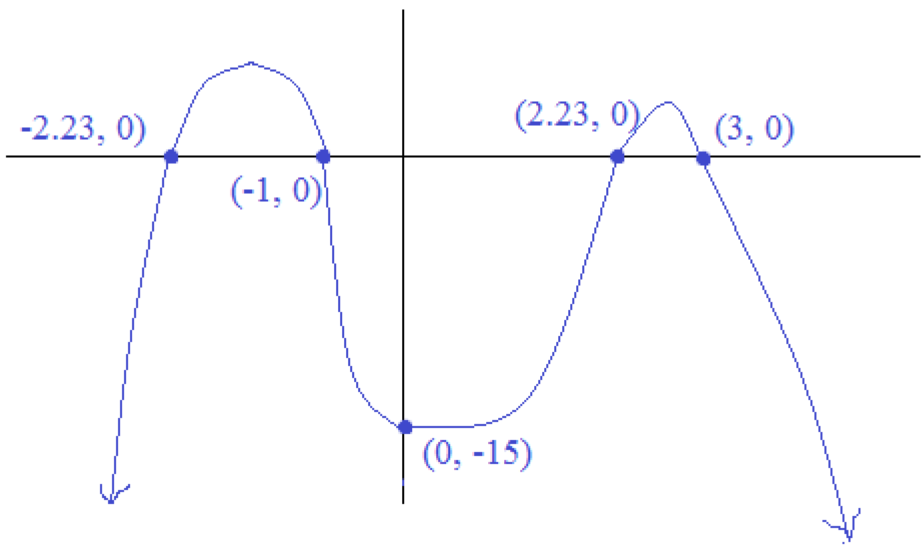
Polynomials: Factors, Roots, and Theorems

Notes, Definitions, Examples, and Practice Test (w/solutions)



Includes intercepts, Factor, Remainder & Rational Root Theorems, conjugates, synthetic division, and more…

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Methods of Factoring

$Y = AX^2 + BX + C$

(Since a quadratic's lead term has an exponent 2, there will be 2 solutions)

Greatest Common Factor

$6X^2 - 3X = 0$

$3X(2X - 1) = 0$ -- Take out greatest common factor 3X

$3X = 0$, $2X - 1 = 0$ → $X = 0$, $X = 1/2$ -- Solve each piece..

$6(0) - 3(0) = 0$ ✓ -- Check solutions

$6(1/2)^2 - 3(1/2) = 6/4 - 3/2 = 0$ ✓

Finding 2 Linear Binomials

$X^2 - 7X + 6 = 0$

$(X - 1)(X - 6) = 0$ -- Find 2 numbers whose product is 6 (the constant) & whose sum is -7 (the middle coefficient)

$(X - 1) = 0$, $(X - 6) = 0$ → $X = 1$, $X = 6$ -- Solve each piece..

$(1)^2 - 7(1) + 6 = 0$ ✓ -- Check solutions

$(6)^2 - 7(6) + 6 = 0$ ✓

Completing the Square

$X^2 + 8X - 84 = 0$

$X^2 + 8X = 84$ -- Isolate the "X terms"

$X^2 + 8X + 16 = 84 + 16$ -- Divide the coefficient of the 2nd term by 2 and square it. (Add this number to both sides)

$(X + 4)(X + 4) = 100$ -- Factor and solve

$\sqrt{(X + 4)^2} = \sqrt{100}$

$X + 4 = 10$, $X + 4 = -10$ → $X = 6$, $X = -14$ -- Check solutions

$(6)^2 + 8(6) - 84 = 0$ ✓

$(-14)^2 + 8(-14) - 84 = 0$ ✓

Quadratic Formula

$X^2 - 7X + 11 = 0$

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1 \quad b = -7 \quad c = 11$ -- Identify coefficients

$$X = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(11)}}{2(1)}$$ -- plug into quadratic formula

$$X = \frac{7 + \sqrt{5}}{2} \cong 4.618$$ -- Simplify

$$X = \frac{7 - \sqrt{5}}{2} \cong 2.382$$ -- check solutions

$(2.4)^2 - 7(2.4) + 11 = -.04$

$(4.6)^2 - 7(4.6) + 11 = -.04$

(this "rough check" supports our solutions)

Difference of Squares

$X^2 - 16 = 0$

X^2 and 16 are perfect squares

Identify the perfect squares

$(X - 4)(X + 4) = 0$

Factor: "(Square root of first minus square root of second) x (square root of first plus square root of second)"

$X = -4, 4$

Solve and check

$(-4)^2 - 16 = 0$ ✓

$(4)^2 - 16 = 0$ ✓

Sum of squares DOES NOT FACTOR!!

$X^2 + 49 = 0$

X^2 and 49 are perfect squares, but it does not factor...

$X^2 + 49 \neq (X + 7)(X + 7)$

$(X + 7)(X + 7) = X^2 + 14X + 49$

Solution:

$X^2 + 49 = 0$

$X^2 = -49$

$X = 7i$ or $-7i$

(i is an imaginary number)

Difference of Cubes

Factor: $X^3 - 8$

X^3 and 8 are perfect cubes

X and 2 are the cube roots

$(X - 2)(X^2 + 2X + 4)$

Identify perfect cubes.

Determine cube roots. Then,

Factor using "SOAP" (signs are Same, Opposite, Always Positive)

$(A - B)(A^2 + AB + B^2)$

S O AP

Sum of Cubes

$X^3 + 27 = 0$

Since it is X^3, we're looking for 3 solutions...

X^3 and 27 are perfect cubes

X and 3 are the cube roots

Identify perfect cubes.

Determine cube roots. then,

Factor using "SOAP" (signs are Same/Opposite/AlwaysPositive)

$(A + B)(A^2 - AB + B^2)$

S O AP

$(X + 3)(X^2 - 3X + 9) = 0$

$(X + 3) = 0$ $X = -3$ — Real Solution

$(X^2 - 3X + 9) = 0$

(Use Quadratic Formula)

$$\frac{3 \pm \sqrt{9 - 36}}{2} = 0$$

Complex/Imaginary Solutions

$$X = \frac{3 + 3i\sqrt{3}}{2}$$

$$X = \frac{3 - 3i\sqrt{3}}{2}$$

Factoring (4 term) Polynomials: Grouping

> *Factor by 'Grouping'*
>
> 1) Separate polynomial into groups
> 2) Factor each group (using Greatest Common Factor)
> 3) Merge and re-group

Example 1: $y^3 + 2y^2 - 81y - 162$

Solution A: $y^3 + 2y^2 \quad - 81y - 162$ — *Separate the polynomial*

$y^2(y + 2) \quad - 81(y + 2)$ — *Factor each group* (using GCF)

$(y^2 - 81)(y + 2)$ — *Merge and re-group*

$(y - 9)(y + 9)(y + 2)$

Solution B: $y^3 - 81y \quad + 2y^2 - 162$

$y(y^2 - 81) \quad + 2(y^2 - 81)$

$(y + 2)(y^2 - 81)$

$(y + 2)(y + 9)(y - 9)$

Note: Although Solutions A and B approach the polynomial differently, the outcome is the same!

Example 2: $b^3 + b^2 = 64b + 64$

$b^3 + b^2 \quad - 64b - 64 = 0$ — *Write equation (setting polynomial equal to zero)*

$b^2(b + 1) \quad - 64(b + 1) = 0$ — *Separate into groups and find GCF's*

$(b^2 - 64)(b + 1) = 0$ — *Merge and regroup*

$(b + 8)(b - 8)(b + 1) = 0$ — *Factor further*

$b = -8, 8, -1$ — *Solve*

Then, check your solutions:

Substitute into the original equation

$b = -8$: $(-8)^3 + (-8)^2 = 64(-8) + 64$

$-512 + 64 = -512 + 64$ ✓

$b = 8$: $(8)^3 + (8)^2 = 64(8) + 64$

$512 + 64 = 512 + 64$ ✓

$b = +1$: $(-1)^3 + (-1)^2 = 64(-1) + 64$

$-1 + 1 = -64 + 64$ ✓

Graphing Polynomials: 2 examples

Quadratic Function:

$f(x) = X^2 - 7X + 10$

Identify y-intercept

(0, ?) is the y-intercept

$f(0) = 0^2 - 7(0) + 10 = 10$

Find x-intercepts (the roots)

(?, 0) are the x-intercepts

$f(x) = 0: \quad X^2 - 7X + 10 = 0$

$(X - 5)(X - 2) = 0$

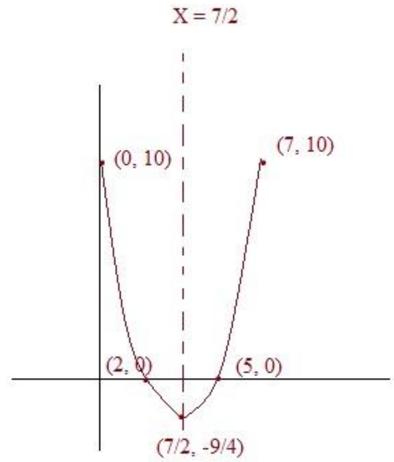
$X = 2, 5$ ("roots")

Plot points and recognize the axis of symmetry and vertex

Find midpoint of 2 and 5 to determine axis of symmetry.. $X = 7/2$

$f(7/2) = 49/4 - 49/2 + 10 = -9/4$

Vertex is (7/2, -9/4)



(Since the coefficient of the X is positive, the parabola faces up.. The vertex is the function's minimum.. There is no maximum)

Cubic Function:

$f(x) = X^3 - 4X^2 - 11X + 30$

Identify y-intercept

$f(0) = 30$ (0, 30) is the y-intercept

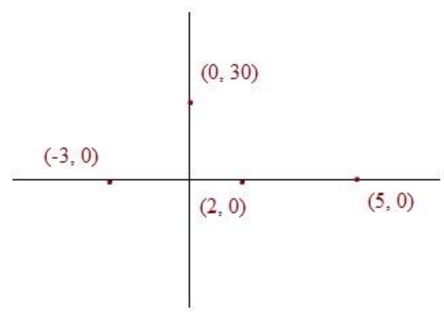
Find the x-intercepts. Since it is a cubic, there should be 3 roots --- 3 intercepts...

$f(x) = 0: \quad X^3 - 4X^2 - 11X + 30 = 0$

(Using factoring techniques, we find)

$(X + 3)(X - 2)(X - 5) = 0$

$X = -3, 2, 5$ (-3, 0)
(2, 0)
(5, 0) are the x-intercepts

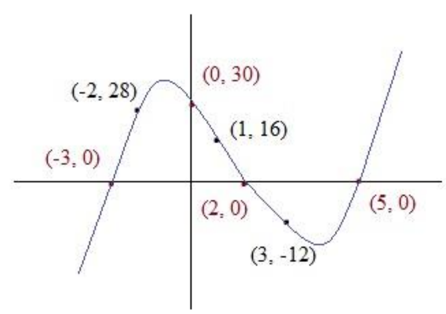


Plot points and determine end behavior

Leading term is X^3

Therefore, the curve's end behavior will be "up to the right" and "down to the left"

To check our intercepts and make a more accurate graph, we add points:
$f(1) = 16$
$f(-2) = 28$
$f(3) = -12$



Fundamental Theorem of Algebra: Any polynomial of degree n will have exactly n roots

What is a root?
For a polynomial $P(x)$, if r is a root, then $P(r) = 0$

$X^2 + 3X + 2$ — degree n = 2 — Two roots: -1, -2

$-3X^2 - 10X + 24 + X^3$ — degree n = 3 (the largest exponent is 3) — Three roots: 2, -3, 4

Factor and find the roots:

$X^4 + 5X^2 - 36$ — Recognize that 9 and -4 add up to 5 and multiply to -36

$(X^2 - 4)(X^2 + 9)$ — Notice that the first term is "difference of squares"

$(X + 2)(X - 2)(X^2 + 9)$ — Set factors equal to zero to find roots

$(X + 2) = 0$ → -2
$(X - 2) = 0$ → 2
$(X^2 + 9) = 0$ → $3i$, $-3i$ where $i^2 = -1$

Since the polynomial is degree 4, there are 4 roots (in this example: 2 are real; 2 are imaginary)

Rational Root Test: A polynomial with leading coefficient 'a' and constant 'b' can have rational roots only of the form

$$\pm \frac{p}{q}$$

where p is a factor of b and q is a factor of a

Note: the Rational Root Test will identify possible roots. You must test the candidates.

$f(X) = -X - 30 + X^3 + 6X^2$ $\quad X^3 + 6X^2 - X - 30$

Write polynomial in standard form (order). Then, identify a (coefficient of first term) and b (the constant)

$a = 1 \quad b = -30$

Determine factors of each term. Then, identify all the possible roots by listing $\pm \frac{p}{q}$

factors of 1: $q = 1$ — factors of -30: $p = 1, 2, 3, 5, 6, 10, 15, 30$

$\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$

(Use synthetic division to) test the candidates to find a root.

$$\begin{array}{r|rrrr} 1 & 1 & 6 & -1 & -30 \\ & & 1 & 7 & 6 \\ \hline & 1 & 7 & 6 & -24 \end{array}$$

(1 is not a root)

$$\begin{array}{r|rrrr} 2 & 1 & 6 & -1 & -30 \\ & & 2 & 16 & 30 \\ \hline & 1 & 8 & 15 & 0 \end{array}$$

2 is a root; and, $(X - 2)$ is a factor of the polynomial...

You may factor the remaining polynomial to find the other 2 roots.

$X^2 + 8X + 15$
$(X + 5)(X + 3)$

-3 and -5 are the other roots...

To check your answer, confirm that $f(2) = f(-3) = f(-5) = 0$

Rational Root Test & Factoring (continued)

$f(X) = 6X^3 + 11X^2 - 3X - 2$

$a = 6$
$b = -2$

factors of a: $q = 1, 2, 3, 6$
factors of b: $p = 1, 2$

possible roots: $\pm\frac{p}{q}$ $\quad \pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{2}{3}, \pm\frac{2}{6}$

Notice: there are 16 candidates. and, three of them are roots.. So, there is a 3/16 chance of randomly selecting a root the first time...

Is 1 a root? $f(1) = 6(1)^3 + 11(1)^2 - 3(1) - 2 = 12 \neq 0$ NO

Is 2 a root? $f(2) = 6(2)^3 + 11(2)^2 - 3(2) - 2 = 84 \neq 0$ NO

Is -2 a root? $f(-2) = 6(-2)^3 + 11(-2)^2 - 3(-2) - 2 = 0$ YES!!

Using the Factor/Remainder Theorems, we search for a root...

-2	6	11	-3	-2
		-12	2	2
	6	-1	-1	0

Using synthetic division, we can break down the polynomial...

Then, factor and set equal to 0 to find the other roots...

$(X - (-2))(6X^2 - X - 1)$

$(X + 2)(3X + 1)(2X - 1) = 0$

$X = \boxed{-2, -1/3, 1/2}$

What is the y-intercept? $f(0) = -2$ $\quad (0, -2)$
What are the x-intercepts? $f(x) = 0$ $\quad (-2, 0), (-1/3, 0), (1/2, 0)$

Fundamental Theorem of Algebra (continued):

It guarantees that any polynomial of degree n will have exactly n roots.. (** You must "double count" the double roots..)

$f(X) = X^3 - 3X^2 + 4$

The polynomial has degree 3, so there will be exactly 3 roots..

$f(X) = (X - 2)(X - 2)(X + 1)$

$(X - 2)^2$ produces a double root.

Roots are 2, 2, -1

$Y = X^3 - 3X^2 + 3X - 1$

According to Fundamental Theorem of Algebra, there will be 3 roots (i.e. 3 zeros)

$Y = (X - 1)(X - 1)(X - 1)$

This is an example of a "triple root"

Roots are 1, 1, 1

Factors and Remainders:

Is 3 a factor of 1284?

Yes, because $1284 \div 3 = 428$

Is 7 a factor of 1284?

No, because there is a remainder..

(It isn't evenly divisible)

$$
\begin{array}{r}
183 \text{ remainder } 3 \\
7 \overline{)1284} \\
-7 \\
\hline
58 \\
-56 \\
\hline
24 \\
-21 \\
\hline
3
\end{array}
$$

Is $(X-8)$ a factor of $X^3 - 7X^2 + 14X - 8$?

$$
\begin{array}{rl}
 & X^2 + X + 22 \quad \text{remainder } 168 \\
(X-8) & \overline{) X^3 - 7X^2 + 14X - 8} \\
 & -\underline{X^3 - 8X^2} \\
 & X^2 + 14X \\
 & -\underline{X^2 - 8X} \\
 & 22X - 8 \\
 & -\underline{22X - 176} \\
 & 168
\end{array}
$$

⟸ Polynomial Long Division

Synthetic Division ⟹

8	1	-7	14	-8
		8	8	176
	1	1	22	(168)

No, $(X - 8)$ is not a factor..

Is $(X - 1)$ a factor?

$$
\begin{array}{rl}
 & X^2 - 6X + 8 \\
X - 1 & \overline{) X^3 - 7X^2 + 14X - 8} \\
 & -\underline{X^3 - X^2} \\
 & -6X^2 + 14X \\
 & -\underline{-6X^2 + 6X} \\
 & 8X - 8 \\
 & -\underline{8X - 8} \\
 & 0
\end{array}
$$

⟸ Polynomial Long Division

Synthetic Division ⟹

1	1	-7	14	-8
		1	-6	-8
	1	-6	8	(0)

Yes, $(X - 1)$ is a factor!

$X^2 - 6X + 8 = (X - 4)(X - 2)$

What is $f(8)$? $(8)^3 - 7(8)^2 + 14(8) - 8 =$

$512 - 448 + 112 - 8 = 168$

It's the same as the remainder!!

What is $f(1)$? $(1)^3 - 7(1)^2 + 14(1) - 8 = 0$

The root has no remainder; It's a factor...

> Remainder Theorem: If a polynomial function $f(x)$ is divided by a linear term $(x - a)$ and the remainder is r, then $f(a) = r$
>
> This implies the
>
> Factor Theorem: If a polynomial function $f(x)$ has a factor $(x - a)$, then $f(a) = 0$
> In other words, there is no remainder..

Since $(X - 4)$
$(X - 2)$
$(X - 1)$ are factors,
$f(4) = f(1) = f(2) = 0$

Given the roots 2 and -4
What is the polynomial function?

Since the roots are 2 and -4,
the zeros (x-intercepts) are 2 and -4.

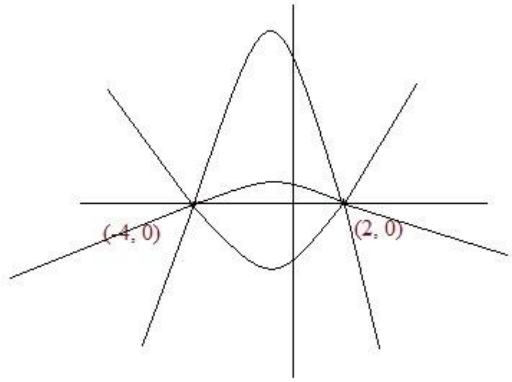
So, the factors are (X - 2) and (X + 4)

$$f(X) = (X - 2)(X + 4)$$

$$= X^2 + 2X - 8$$

However, that is only 1 possibility.
The graph shows other curves with
roots of 2 and -4.

So, we need more information --
another point -- to determine the
specific function.



All 3 are parabolas that have zeros at
2 and -4

Now, suppose you're given roots: 2, -4
and given y-intercept: 3

Use the formula $Y = a(X - r_1)(X - r_2)$

$$Y = a(X + 4)(X - 2)$$

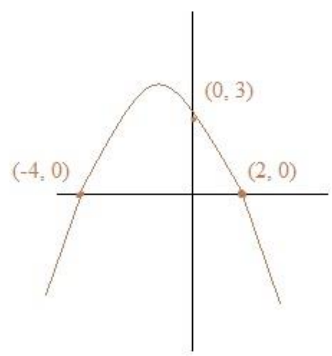
(2, 0) and (-4, 0) both work...

Now, insert (0, 3)

$$3 = a(0 + 4)(0 - 2)$$
$$3 = a(4)(-2)$$

$$a = \frac{-3}{8}$$

$$Y = \frac{-3}{8}(X + 4)(X - 2)$$



"Sum & Product of Roots of a Polynomial"

For the Polynomial $P(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X^1 + C$

The Sum of the roots is: $-\dfrac{a_{n-1}}{a_n}$

The Product of the roots is: If n is odd, $-\dfrac{C}{a_n}$ If n is even, $\dfrac{C}{a_n}$

$f(X) = X^2 - 10X + 21$

Sum: $\dfrac{-(-10)}{1}$ Product: $\dfrac{21}{1}$

$f(X) = (X - 3)(X - 7)$ so, roots are 3, 7

Sum: $3 + 7 = 10$ Product: $3 \times 7 = 21$

$f(X) = 2X^3 - 9X^2 - 11X + 30$

Sum: $\dfrac{-(-9)}{2}$ Product: $\dfrac{-30}{2}$

$f(X) = (2X - 3)(X - 5)(X + 2)$ so, roots are 3/2, 5, -2

Sum: $3/2 + 5 + -2 = 9/2$ Product: $3/2 \times 5 \times (-2) = -15$

"Conjugate Pair Theorem"

If a polynomial has real coefficients, then any complex zeros occur in conjugate pairs. In other words, if $a + bi$ is a zero, then, $a - bi$ is a zero..

$f(X) = X^2 + 4$

$X^2 + 4 = 0$

$X^2 = -4$

$X = \pm 2i$

$f(X) = X^3 - 3X + 52$

Suppose we know $(X - (2 + 3i))$ is a factor. then, $2 + 3i$ is a root...

By the conjugate pair theorem, $2 - 3i$ must be a root, too...

$$(2 - 3i)^3 - 3(2 - 3i) + 52 = (4 - 12i - 9)(2 - 3i) - 6 + 9i + 52$$

$$= (-5 - 12i)(2 - 3i) + 9i + 46$$

(-4 is the other root)

$$= -10 - 24i + 15i - 36 + 9i + 46$$

$$= 0$$

Sketching Polynomials

Generally, when *sketching* a polynomial, we try to provide as much information as possible or demanded:

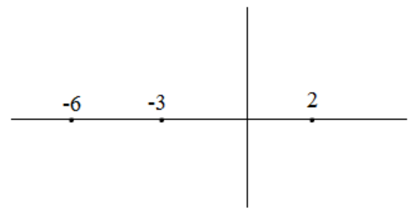
zeros (x-intercepts), end behavior, y-intercept, additional points

Example: Sketch the function $f(x) = -4(x-2)^3(x+3)^2(x+6)$

Step 1: Identify the zeros (x-intercepts)

Since the function is written in "factored form" ("intercept form"), the zeros can be identified easily:

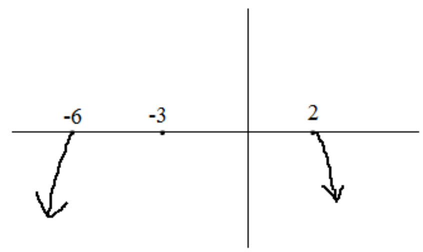
$f(x) = 0 = -4(x-2)^3(x+3)^2(x+6)$ at $x = 2, -3$, and -6



Step 2: Recognize the *end behavior*

What is the "degree" of the polynomial? It is NOT 3; If you were to multiply the terms (i.e. FOIL the parts), the first term would be $-4x^6$

Since the exponent is even, the function's end behavior is the same in either direction. And, since the lead coefficient is *negative* four, the end behavior is "down"

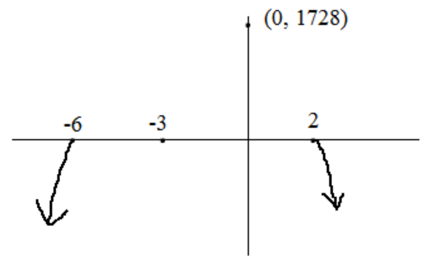


Step 3: Find the y-intercept (or any other "easy" points)

Since the y-intercept is easy to find -- simply plug in 0 -- it's a great way to solidify and check your sketch!

$f(0) = -4(0-2)^3(0+3)^2(0+6) = -4(-8)(9)(6) = 1728$

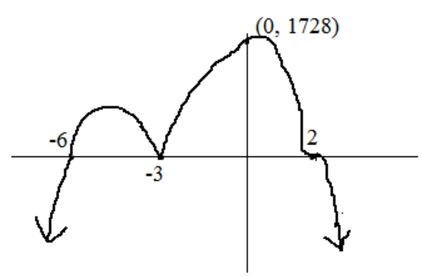
(0, 1728)



Step 4: Fill in the rest of the graph (applying *multiplicity* and "bounces")

since (x - 2) is to the 3rd power, "the zero 2 has multiplicity" of 3

since (x + 3) is to the 2nd power, "the zero -3 has multiplicity" of 2 (it will "bounce")



Step 5: Quick check

You may pick specific points to add to the graph. And, you can do quick checks ---

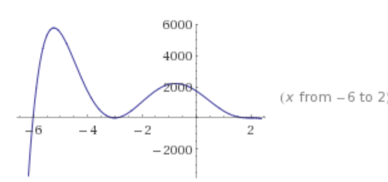
for example, if we test -10:

$f(-10) = -4(-10-2)^3(-10+3)^2(-10+6)$ the answer will be *negative* so our end behavior on the left is correct...

— — + —

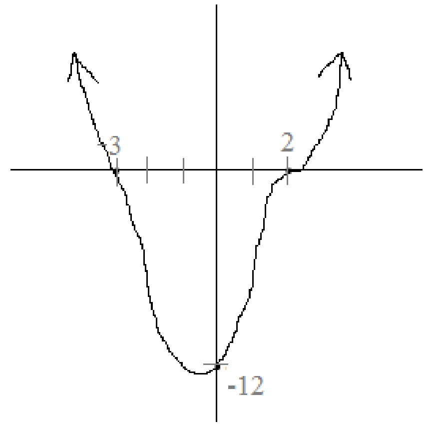
or, if we test -1: $-4(-1-2)^3(-1+3)^2(-1+6)$ the answer will be *positive*, so the point is above the x-axis...

— — + +



Graph from WolframAlpha

Example: Describe the following *4th degree* polynomial:



Step 1: Identify the zeros (x-intercepts)

(-3, 0) and (2, 0) are points on the graph...

therefore, zeros include -3 and 2

$$y = (x+3)\ (x-2)$$

Step 2: Consider degree (and multiplicity)

Since this is a 4th degree polynomial, we need to add more zeros...

Also, note the "pause" in the graph at x = 2

Therefore, we need to add (x - 2) terms...

$$y = (x+3)\ (x-2)^3$$

Step 3: Specify the graph by determining the "a" value

There are an infinite number of polynomials that pass through -3 and 2...

But, by adding a coefficient, we express a unique equation (*that includes the point (0, -12)*)..

$$y = a(x+3)(x-2)^3$$

plug in (0, -12)

$$-12 = a(0+3)(0-2)^3$$

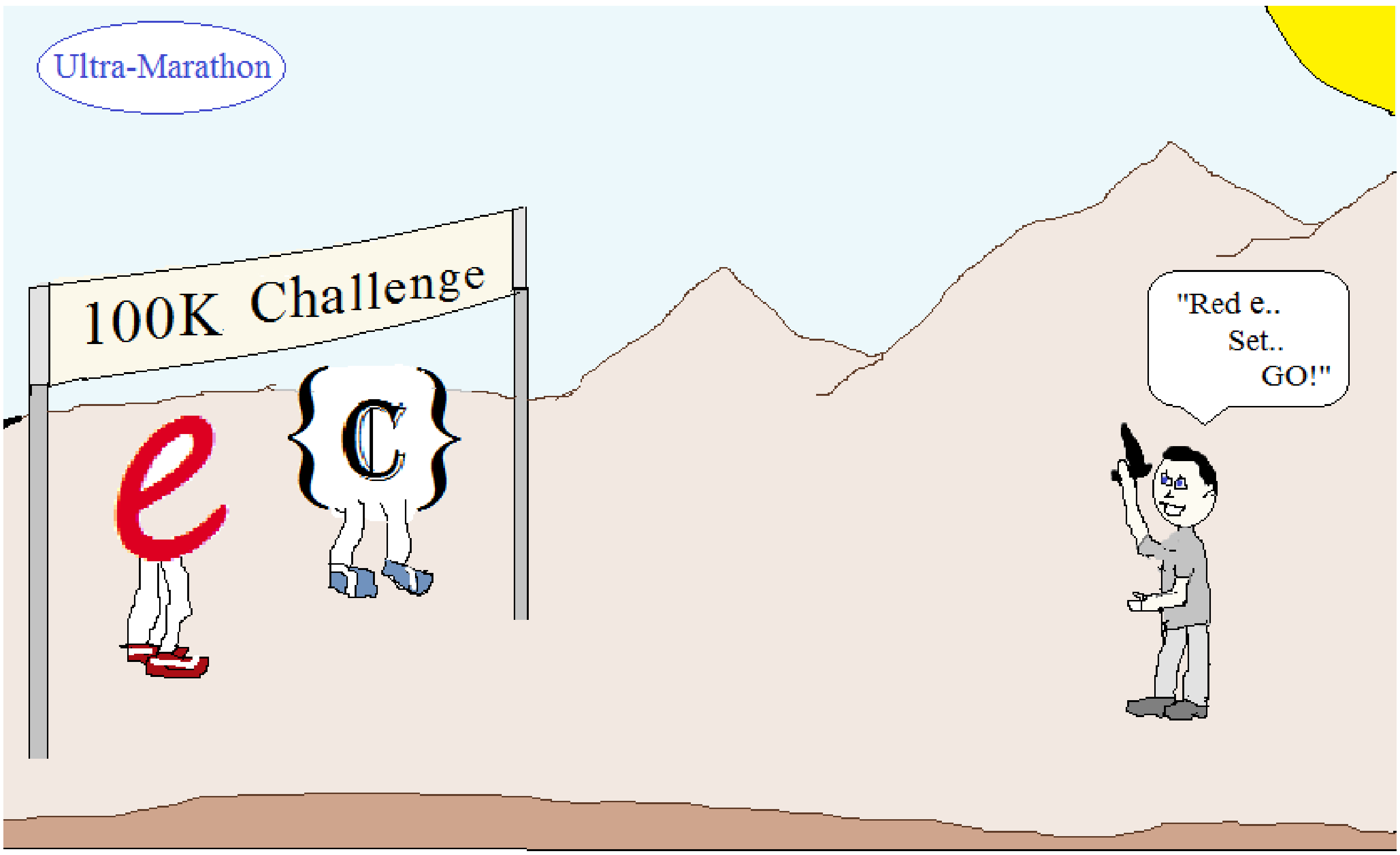
$$-12 = a(3)(-8)$$

$$-12 = -24a$$

$$a = \frac{1}{2}$$

$$y = \frac{1}{2}(x+3)(x-2)^3$$

or, $\frac{x^4}{2} - \frac{3x^3}{2} - 3x^2 + 14x - 12$



Testing the limits of endurance,
these math figures will run on and on...

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PRACTICE Exercises (w/SOLUTIONS)-→

Classifying Polynomials

For each polynomial, determine the degree, the lead coefficient, and classify:

a) $2x^3 + 3x + 6$

Degree: 3

Lead Coefficient: 2

Classification: Cubic Trinomial

b) x

Degree:

Lead Coefficient:

Classification:

c) $3 - 4x^8$

Degree:

Lead Coefficient:

Classification:

d) $p^3 - 2p + 2p^3$

Degree:

Lead Coefficient:

Classification:

e) $t^3 - 3t^2 + t^5 - t^6$

Degree:

Lead Coefficient:

Classification:

f) $3x^4y^3 + 5x^3y + x^2 + y^3 + 9$

Degree:

Lead Coefficient:

Classification:

Why are these NOT polynomials?

a) $5n^2 + \sqrt{n} + 6 - 3mn$

b) $3x + 5x^3 - \frac{7}{x} + xy$

c) $6t^5 + 4t^3 + \sqrt{2}\,t + 3^t$

d) $\frac{x^2 - 4}{x + 3}$

e) $\frac{xy}{z} + 3xz$

f) $3x^5 + 14x^3 + \pi x + x^{-1} + 5x^{-2}$

Determine the end behavior, identify the intercepts, and then sketch the polynomial...

Polynomials Sketching Exercises

1) $f(x) = (x-1)(x+5)^2$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

2) $y = x^3 - 10x^2 - 11x$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

3) $g(x) = -(x+2)^3(x-3)$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

Determine the end behavior, identify the intercepts, and then sketch the polynomial...

Polynomials Sketching Exercises

4) $y = 2x^3 + 6x^2 - x - 3$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

5) $y = (x + 2)^2 (x - 3)(x - 1)$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

6) $y = (x - 4)^2 (x^2 + 4)$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

Polynomials Sketching Exercises

Determine the end behavior, identify the intercepts, and then sketch the polynomial...

7) $h(x) = (x + 3)^3 (2x - 5)$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

8) $y = -\frac{3}{4}x^4 + \frac{3}{4}$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

9) $y = (x + 1)(x - 3)^2(x + 5)^3$

end behavior:
(degree and lead coefficient)

y-intercept:

x-intercept(s):

multiplicity:
('bounces' and 'pauses')

Polynomials and Roots Test

I. General Topics

A) $f(x) = x^3 - 3x^2 - 6x + 8$

Classify the polynomial:

What are the 'p' values?

What are the 'q' values?

List all <u>possible</u> rational zeros:

What are the x-intercepts?

What is the y-intercept?

Sketch the function:

B) $g(x) = 2x^3 + 13x^2 + 5x - 6$

What is the degree of the polynomial?

List all <u>possible</u> rational zeros:

Identify the zeros:

What are the factors?

What is the remainder of $g(x) \div (x + 5)$?

Sketch the function:

C) $h(x) = -x^4 + 2x^3 + 8x^2 - 10x - 15$

Classify the polynomial:

Describe the end behavior:

What are the factors?

What is the y-intercept?

Sketch the function:

Polynomials and Roots Test

II. Factoring, Synthetic Division, and Roots

Factor and identify all roots:

A) $x^5 - x^4 + 9x^3 - 9x^2$

B) $x^4 - 2x^2 + 1$

C) $x^3 + 4x^2 + 9x + 36$

III. Determining the Polynomial

A) Write a cubic function whose graph passes through

(-2, 0) (2, 0) (-4, 0) (-1, 3)

B) Write <u>at least two</u> quadratic functions that have x-intercepts (3, 0) and (8, 0)

Write a polynomial of least degree that has real coefficients, lead coefficient of 1, and the given zeros:

A) -2, -2, 2

B) 2, 5, *i*

C) 4, 2, -3*i*

D) 2 + *i*, 2 − *i*, 3

E) 0, 1, -3

Polynomials and Roots Test

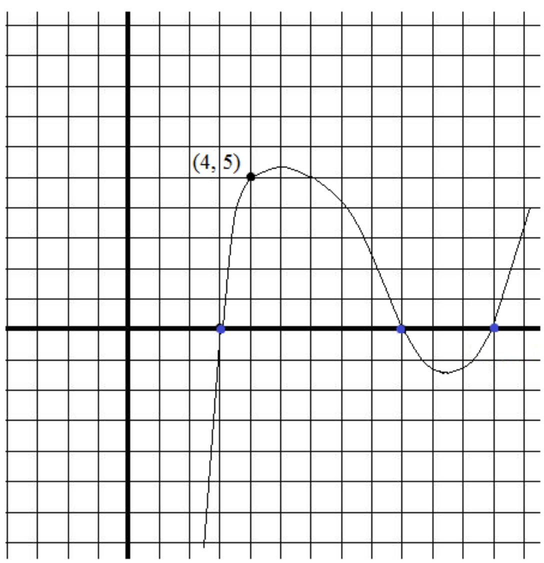
IV. Applying Concepts and Theorems

A) What is the remainder of $10x^5 + 3x^4 - 7x^3 + 2x - 6 \div (x - 1)$?
(Remainder Theorem)

B) Verify that $x^{20} - 1$ has a factor of $(x + 1)$.
(Factor Theorem)

C) Three roots of $x^4 - 5x^3 - 33x^2 + 113x + 140$ are -1, -5, 7... What is the 4th root?
(Sum and Products of Roots)

D) Find the y-intercept of the following sketch:



Classifying Polynomials

SOLUTIONS

For each polynomial, determine the degree, the lead coefficient, and classify:

a) $2x^3 + 3x + 6$

Degree: 3

Lead Coefficient: 2

Classification: Cubic Trinomial

b) x

Degree: 1

Lead Coefficient: 1

$1x^1$

Classification: Linear Monomial

c) $3 - 4x^8$

Degree: 8

$-4x^8 + 3$

Lead Coefficient: -4

Classification: Binomial of degree 8

d) $p^3 - 2p + 2p^3$

Degree: 3

$3p^3 - 2p$

Lead Coefficient: 3

Classification: Cubic Binomial

e) $t^3 - 3t^2 + t^5 - t^6$

Degree: 6

Lead Coefficient: -1

$-t^6 + t^5 + t^3 - 3t^2$

Classification: Polynomial of degree 6

f) $3x^4y^3 + 5x^3y + x^2 + y^3 + 9$

$4 + 3 = 7$

$3x^4y^3 + 5x^3y + x^2 + y^3 + 9$

Degree: 7

Lead Coefficient: 3

Classification: Polynomial of degree 7

Why are these NOT polynomials?

a) $5n^2 + \sqrt{n} + 6 - 3mn$ $\quad \sqrt{n} = n^{\frac{1}{2}}$ (exponent cannot be a fraction)

b) $3x + 5x^3 - \frac{7}{x} + xy$ $\quad \frac{7}{x} = 7x^{-1}$ (exponent cannot be negative)

c) $6t^5 + 4t^3 + \sqrt{2}\,t + 3^t$ $\quad 3^t$ (exponent cannot be a variable)

Exponent must be a whole number...

d) $\frac{x^2 - 4}{x + 3}$ $\quad x - 3 + \frac{5}{x + 3}$ (variable doesn't have whole number exponent)

e) $\frac{xy}{z} + 3xz$ variable in denominator -- i.e. variable with negative exponent

f) $3x^5 + 14x^3 + \pi x + x^{-1} + 5x^{-2}$ negative exponents of variables

Determine the end behavior, identify the intercepts, and then sketch the polynomial...

SOLUTIONS

Polynomials Sketching Exercises

1) $f(x) = (x-1)(x+5)^2$

end behavior:
(degree and lead coefficient)

degree is 3 (odd), lead coefficient is 1 (positive)
so "up to the right and down to the left"

y-intercept:

If $x = 0$, then $f(0) = (-1)(5)^2 = -25$

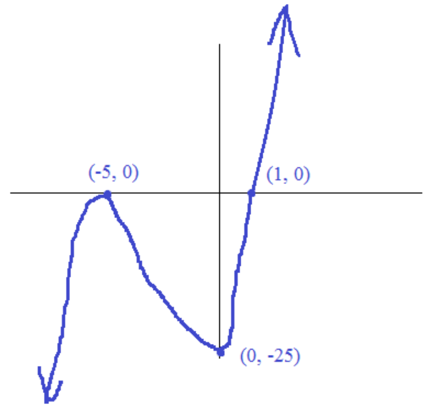
(0, -25)

x-intercept(s):

If $f(x) = 0$, then $x = 1$ or $x = -5$

multiplicity:
('bounces' and 'pauses')

Since $(x + 5)$ is squared, the root -5 has a multiplicity of 2 ----> a "bounce" off the x-axis



2) $y = x^3 - 10x^2 - 11x$

$y = x(x^2 - 10x - 11)$

$y = x(x-11)(x+1)$

end behavior:
(degree and lead coefficient)

degree is 3 and lead coefficient is positive 1...
So, as $x \to \infty$, $y \to \infty$

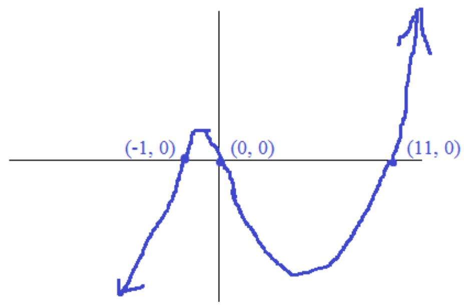
as $x \to -\infty$, $y \to -\infty$

y-intercept: (0, 0)

x-intercept(s): (0, 0) (11, 0) (-1, 0)

multiplicity:
('bounces' and 'pauses')

None



3) $g(x) = -(x+2)^3(x-3)$

end behavior:
(degree and lead coefficient)

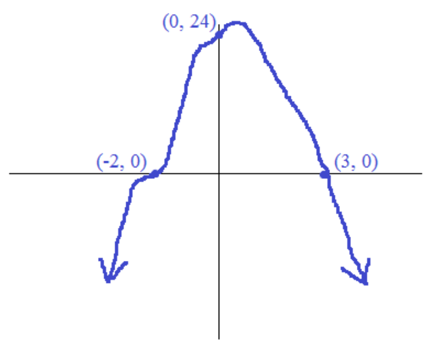
Degree is 4, and lead coefficient is NEGATIVE 1
so, end behavior is "down and down"..

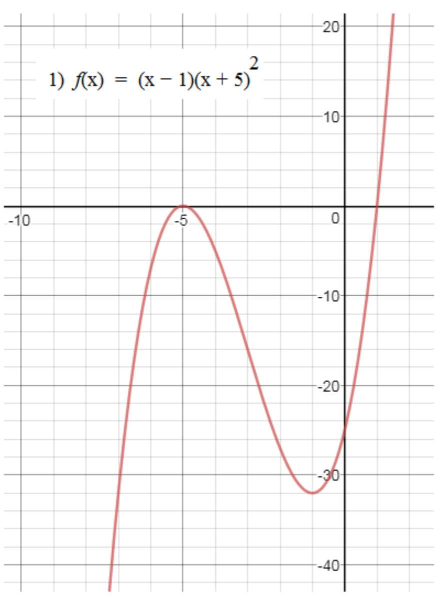
y-intercept: when $x = 0$, $g(0) = -(8)(-3) = 24$

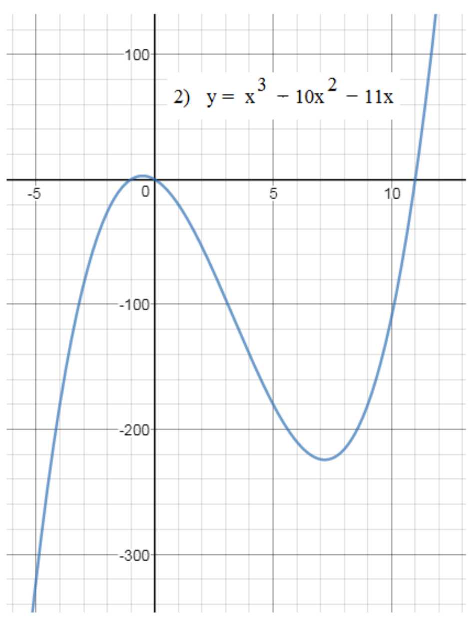
x-intercept(s): when $g(x) = 0$?? when $x = -2$ or $x = 3$

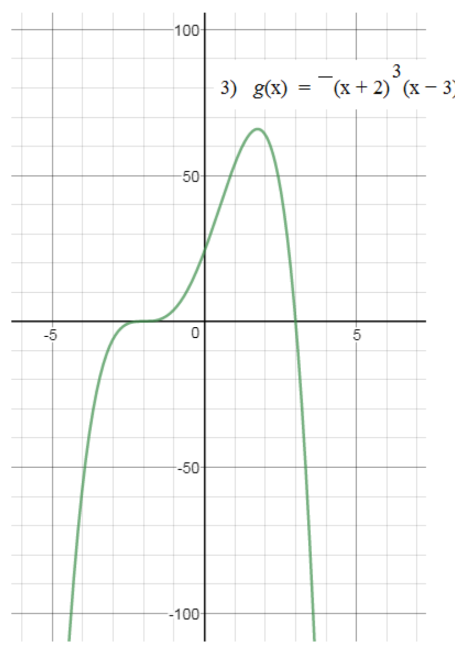
multiplicity:
('bounces' and 'pauses')

the x-intercept (or zero) is -2, there is a "pause" (multiplicity of 3)









Determine the end behavior, identify the intercepts, and then sketch the polynomial...

SOLUTIONS

4) $y = 2x^3 + 6x^2 - x - 3$

factor by grouping $2x^2(x+3) \;+\; -1(x+3)$

$(2x^2 - 1)(x+3)$

end behavior:
(degree and lead coefficient)

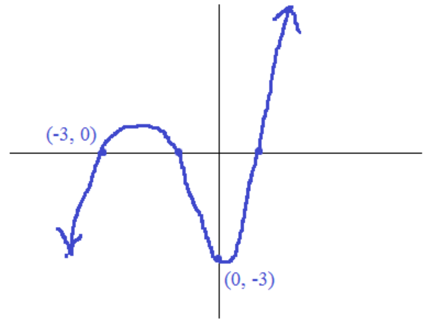
odd degree (3); positive lead coefficient (2)
therefore, ↙ ↗

as x ---> infinity,
y ---> positive infinity
as x ---> negative infinity,
y---> negative infinity

y-intercept: (0, −3)

x-intercept(s): $\left(\sqrt{\frac{1}{2}}, 0\right)$ $\left(-\sqrt{\frac{1}{2}}, 0\right)$ and (-3, 0)

multiplicity:
('bounces' and 'pauses') None



5) $y = (x+2)^2(x-3)(x-1)$

end behavior:
(degree and lead coefficient)

(if you multiply the terms, and put into standard descending order, degree is 4 and lead coefficient is 1) ↖ ↗

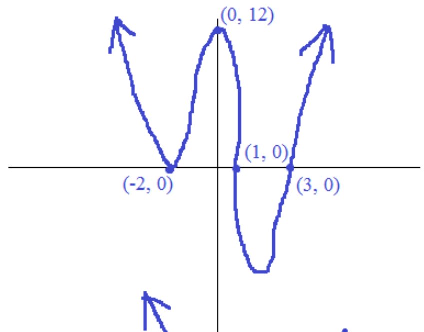
y-intercept:

plug in 0 for x.... (0, 12)

x-intercept(s):

plug in 0 for y.... (-2, 0) (3, 0) (1, 0)

multiplicity:
('bounces' and 'pauses') since $(x+2)$ has degree 2, the zero -2 has multiplicity of 2... (a "bounce" in the graph)



6) $y = (x-4)^2(x^2+4)$

end behavior:
(degree and lead coefficient)

Degree: 4 Lead Coefficient: 1 ↑ ↗

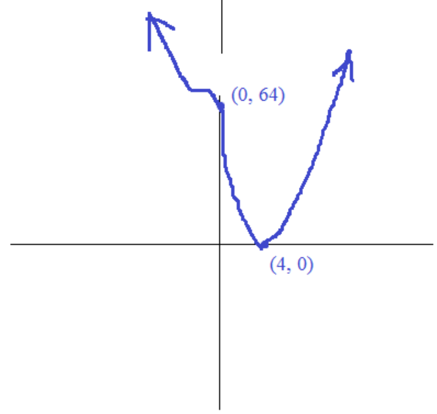
as x ---> inf., y ---> inf.
as x ---> neg. inf., y ---> inf.

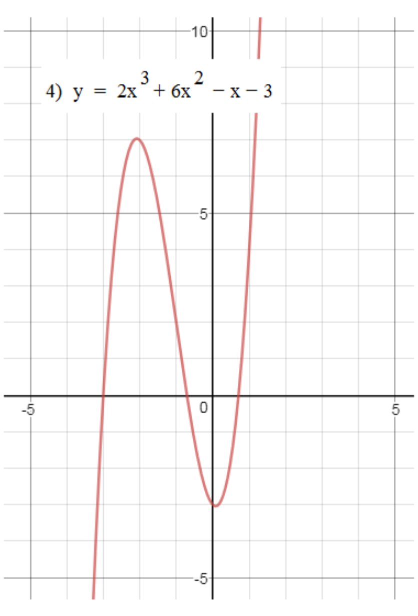
y-intercept: (0, 64)

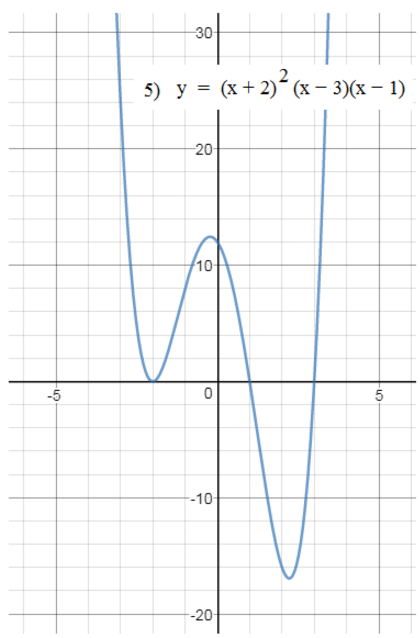
x-intercept(s): (4, 0)

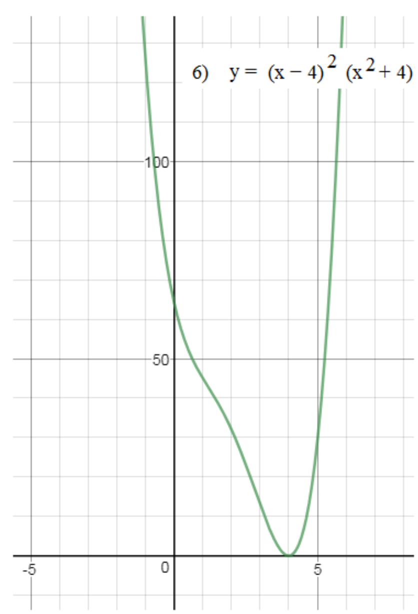
$x^2 + 4 = 0$	*2 imaginary roots!!*
$x^2 = -4$	$x = -2i$ and $2i$

multiplicity:
('bounces' and 'pauses') (x - 4) has a degree of 2, so the root 4 has a multiplicity of 2









Determine the end behavior, identify the intercepts, and then sketch the polynomial... SOLUTIONS

Polynomials Sketching Exercises

7) $h(x) = (x + 3)^3(2x - 5)$

end behavior:
(degree and lead coefficient)

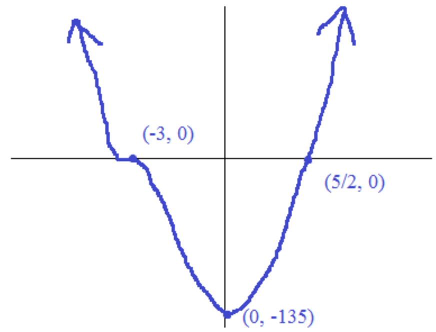
If multiplied/converted to standard form, lead term would be $2x^4$

since degree (4) is even and coefficient (2) is positive,

y-intercept: (0, -135)

x-intercept(s): (-3, 0) and (5/2, 0)

multiplicity:
('bounces' and 'pauses') the root -3 has a multiplicity of 3 ('pause' in the graph)



8) $y = -\frac{3}{4}x^4 + \frac{3}{4}$ $y = \frac{-3}{4}(x^4 - 1)$

factored form

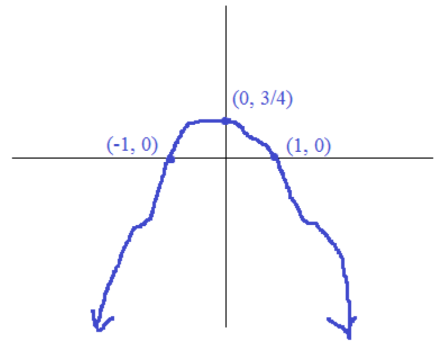
$y = \frac{-3}{4}(x + 1)(x - 1)(x^2 + 1)$

end behavior:
(degree and lead coefficient) lead degree is 4 (even) lead coefficient is negative

y-intercept: (0, 3/4)

x-intercept(s): (1, 0) and (-1, 0) (The other roots *i* and -*i* are imaginary)

multiplicity:
('bounces' and 'pauses') None



9) $y = (x + 1)(x - 3)^2(x + 5)^3$

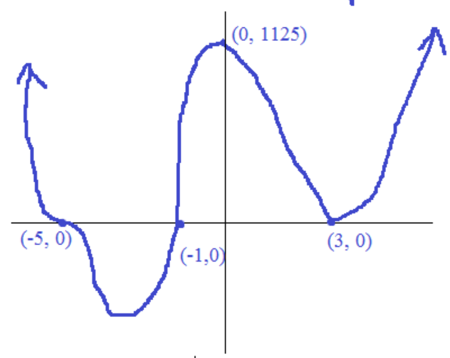
end behavior:
(degree and lead coefficient) degree is 6 and lead coefficient is 1...

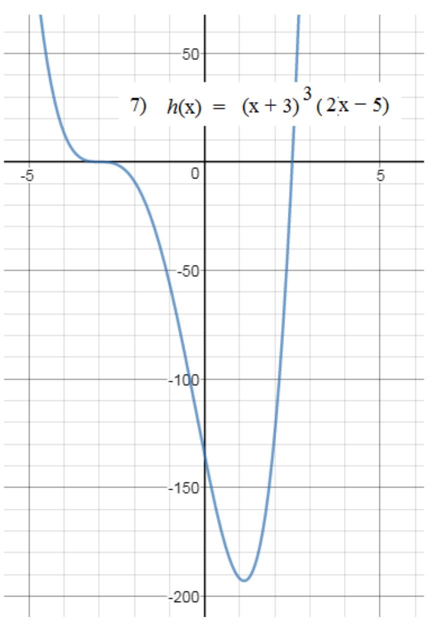
y-intercept: y = (1)(9)(125) = 1125 (0, 1125)

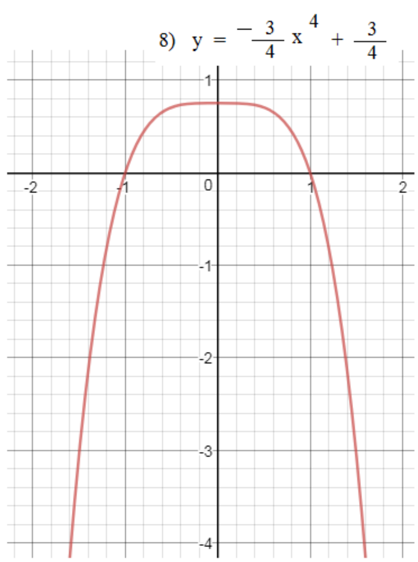
x-intercept(s): (-1, 0) (3, 0) (-5, 0)

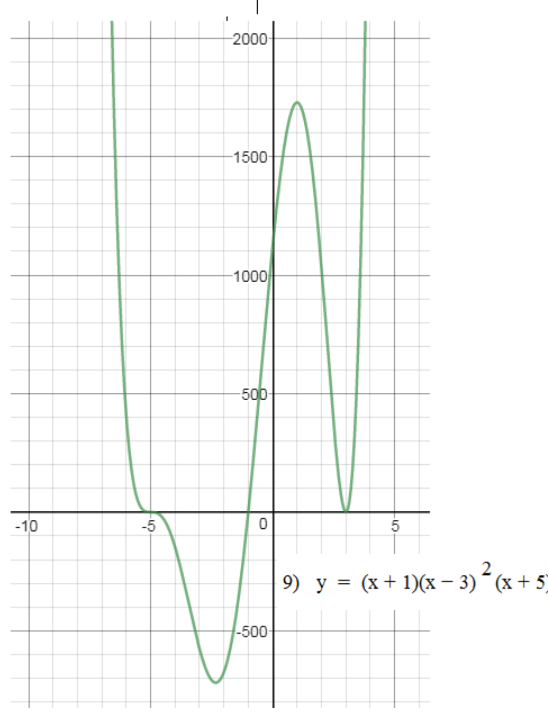
multiplicity:
('bounces' and 'pauses') "bounce" at zero 3

"pause" at zero -5









Polynomials and Roots Test

SOLUTIONS

I. General Topics

A) $f(x) = x^3 - 3x^2 - 6x + 8 \quad = \quad (x - 1)(x - 4)(x + 2)$

Classify the polynomial: (lead degree is 3) -- Cubic polynomial of 4 terms

What are the 'p' values? (factors of the constant) 1, 2, 4, 8

What are the 'q' values? (factors of lead coefficient) 1

List all possible rational zeros: $\frac{\pm p}{q}$ 1, -1, 2, -2, 4, -4, 8, -8

What are the x-intercepts? factor the polynomial: $f(1) = 1 - 3 - 6 + 8 = 0$

(1, 0) (4, 0) (-2, 0)

1 is a root!

synthetic division

1	1	-3	-6	8
		1	-2	-8
	1	-2	-8	0

$x^2 - 2x - 8$ factor

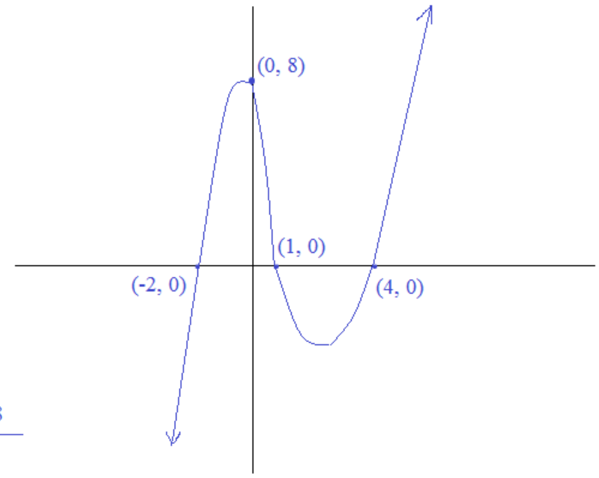
$(x - 4)(x + 2)$

What is the y-intercept?

find $f(0) = 0 - 0 - 0 + 8 = 8$ (0, 8)

Sketch the function:

Note: you can check your points by plugging them into the original polynomial...



B) $g(x) = 2x^3 + 13x^2 + 5x - 6$

What is the degree of the polynomial? 3 (exponent of the lead term)

List all possible rational zeros: $\frac{\mp p}{q}$ p: 1, 2, 3, 6 q: 1, 2

$\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

Identify the zeros: factor the polynomial using factor theorem and synthetic division

-1, 1/2, -6

What are the factors?

$(x + 1)(2x - 1)(x + 6)$

$g(1) = 14$ (not a factor)

$g(-1) = 0$ (factor!)

-1	2	13	5	-6
		-2	-11	6
	2	11	-6	0

$2x^2 + 11x - 6$

$(2x - 1)(x + 6)$

What is the remainder of $g(x) \div (x + 5)$?

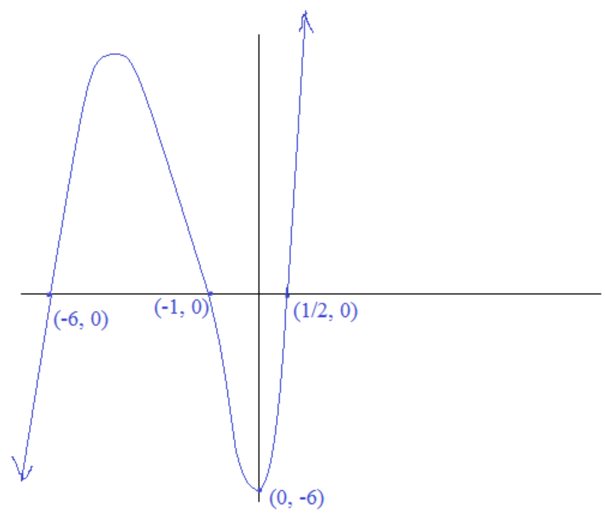
$g(-5) = -250 + 325 - 25 - 6 = 44$

using remainder theorem

-5	2	13	5	-6
		-10	-15	50
	2	3	-10	44

using synthetic division

Sketch the function:



C) $h(x) = -x^4 + 2x^3 + 8x^2 - 10x - 15$

Classify the polynomial: (lead degree is 4) quartic polynomial (of 5 terms)

Describe the end behavior: degree is 4, lead coefficient is negative:

as x goes to $-\infty$, $h(x)$ goes to $-\infty$

as x goes to $+\infty$, $h(x)$ goes to $-\infty$

"polynomial faces down"

What are the factors?

$(x + 1)\ (x - 3)\ (x^2 - 5)$

What is the y-intercept?

$h(0) = 0 + 0 + 0 - 0 - 15 = -15$

(0, -15)

possible real zeros:

1, -1, 3, -3, 5, -5, 15, -15

since $h(-1) = 0$, -1 is a root

-1	-1	2	8	-10	-15
		1	-3	-5	15
	-1	3	5	-15	0

$-x^3 + 3x^2 + 5x - 15$

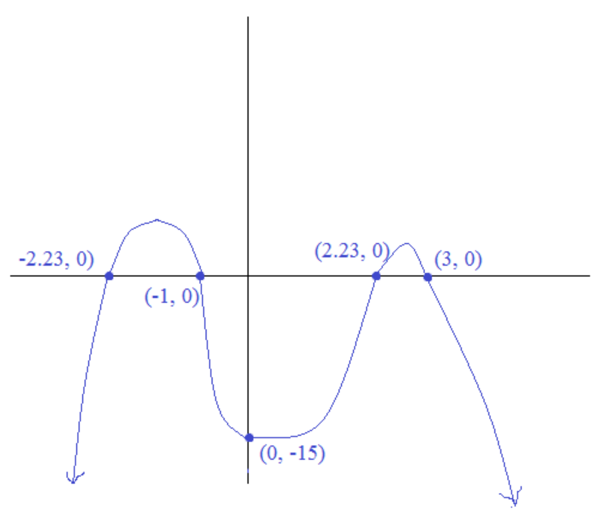
Sketch the function:

$-1(x + \sqrt{5})(x - \sqrt{5})$ 3 is a root:

3	-1	3	5	-15
		-3	0	15
	-1	0	5	0

$-x^2 + 5$

roots are -1, 3, $\sqrt{5}$, $-\sqrt{5}$



Polynomials and Roots Test

SOLUTIONS

II. Factoring, Synthetic Division, and Roots

Factor and identify all roots:

A) $x^5 - x^4 + 9x^3 - 9x^2$

Greatest common factor: x^2

$x^2(x^3 - x^2 + 9x - 9)$

"factor by grouping"

$x^2(x^2(x - 1) + 9(x - 1))$

$x^2(x^2 + 9)(x - 1)$

roots: 0, 0, $3i$, $-3i$, 1

degree is 5; 5 roots...

B) $x^4 - 2x^2 + 1$

$(x^2 - 1)(x^2 - 1)$

"difference of squares"

$(x + 1)(x - 1)(x + 1)(x - 1)$

roots: -1, -1, 1, 1

C) $x^3 + 4x^2 + 9x + 36$

factor by grouping:

$x^3 + 4x^2 \quad + 9x + 36$

$x^2(x + 4) + 9(x + 4)$

regroup:

$(x^2 + 9)(x + 4)$

roots: -4, $-3i$, $+3i$

$x^2 + 9 = 0$

$x^2 = -9$

$x = 3i, -3i$

III. Determining the Polynomial

A) Write a cubic function whose graph passes through

(-2, 0) (2, 0) (-4, 0) (-1, 3)

$y = a(x - x_1)(x - x_2)(x - x_3)$

$y = a(x + 2)(x - 2)(x + 4)$

plug in (-1, 3) to find a

$3 = a(-1 + 2)(-1 - 2)(-1 + 4)$

$3 = -9a$

$a = -1/3$

$f(x) = -\frac{1}{3}(x + 2)(x - 2)(x + 4)$ factored form

general form

test points to check solution:

$= \frac{-x^3}{3} - \frac{4x^2}{3} + \frac{4x}{3} + \frac{16}{3}$

B) Write at least two quadratic functions that have x-intercepts (3, 0) and (8, 0)

a general equation will be

$y = a(x - 3)(x - 8)$

if $a = 1$, then $y = (x - 3)(x - 8)$

$= x^2 - 11x + 24$

if $a = 2$, then $y = 2(x - 3)(x - 8)$

$= 2x^2 - 22x + 48$

the equations have the same x-intercepts, but are shaped differently.. (for example, note: different y-intercepts)

Write a polynomial of least degree that has real coefficients, lead coefficient of 1, and the given zeros:

A) -2, -2, 2

$(x - (-2))(x - (-2))(x - 2) = (x + 2)^2(x - 2)$

B) 2, 5, i

(according to the conjugate pairs rule) since i is a zero, then $-i$ is also a zero.

$(x - 2)(x - 5)(x - i)(x + i)$

$(x - 2)(x - 5)(x^2 + 1)$

C) 4, 2, $-3i$

$+3i$ is also a zero...

$(x - 4)(x - 2)(x + 3i)(x - 3i)$

$(x - 4)(x - 2)(x^2 + 9)$

D) $2 + i$, $2 - i$, 3

$(x - 3)(x - (2 + i))(x - (2 - i))$

$(x - 3)[x^2 - 2x + ix - 2x + 4 - 2i - ix + 2i - i^2]$

$(x - 3)[x^2 - 4x + 5]$

test $2 + i$

$(2 + i - 3)[(2 + i)(2 + i) - 4(2 + i) + 5]$

$(i - 1)[4 + 4i + i^2 - 8 - 4i + 5]$

$(i - 1)[4 + (-1) - 3]$

$= 0$ ✓

E) 0, 1, -3

$(x - 0)(x - 1)(x - -3)$

$x(x - 1)(x + 3)$

Polynomials and Roots Test

SOLUTIONS

IV. Applying Concepts and Theorems

A) What is the remainder of $10x^5 + 3x^4 - 7x^3 + 2x - 6 \div (x - 1)$?
(Remainder Theorem)

Using remainder theorem: find $f(1)$: $10 + 3 - 7 + 2 - 6 = 2$

remainder is 2

check with synthetic division:

1	10	3	-7	2	-6
		10	13	6	8
	10	13	6	8	(2)

B) Verify that $x^{20} - 1$ has a factor of $(x + 1)$.
(Factor Theorem)

Using factor/remainder theorem: find $f(-1) = (-1)^{20} - 1 = 0$

remainder is 0, therefore -1 is a root
and (x + 1) is a factor!

C) Three roots of $x^4 - 5x^3 - 33x^2 + 113x + 140$ are -1, -5, 7... What is the 4th root? **The 4th root is 4
(Sum and Products of Roots)

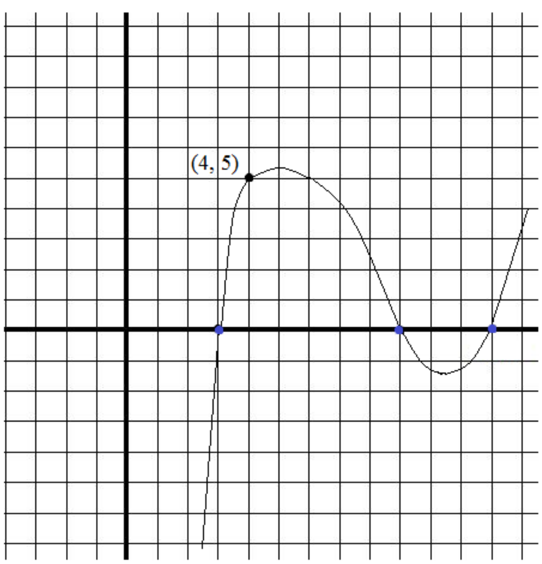
sum of the roots: $-\frac{-5}{1}$ (2nd coefficient) / (1st coefficient)

product of the roots: $\frac{140}{1}$ (constant) / (1st coefficient)

$-1 + -5 + 7 + r = 5$ root = 4

$-1 \times -5 \times 7 \times r = 140$
$r = 4$

D) Find the y-intercept of the following sketch:



the cubic goes through

(3, 0)

(9, 0)

(12, 0)

(4, 5)

$y = a(x - 3)(x - 9)(x - 12)$

substitute (4, 5) into (x, y) to find a

$5 = a(4 - 3)(4 - 9)(4 - 12)$

$5 = 40a$

$a = 1/8$

$y = 1/8(x - 3)(x - 9)(x - 12)$

to find y-intercept, find $x = 0$

$y = 1/8(0 - 3)(0 - 9)(0 - 12)$

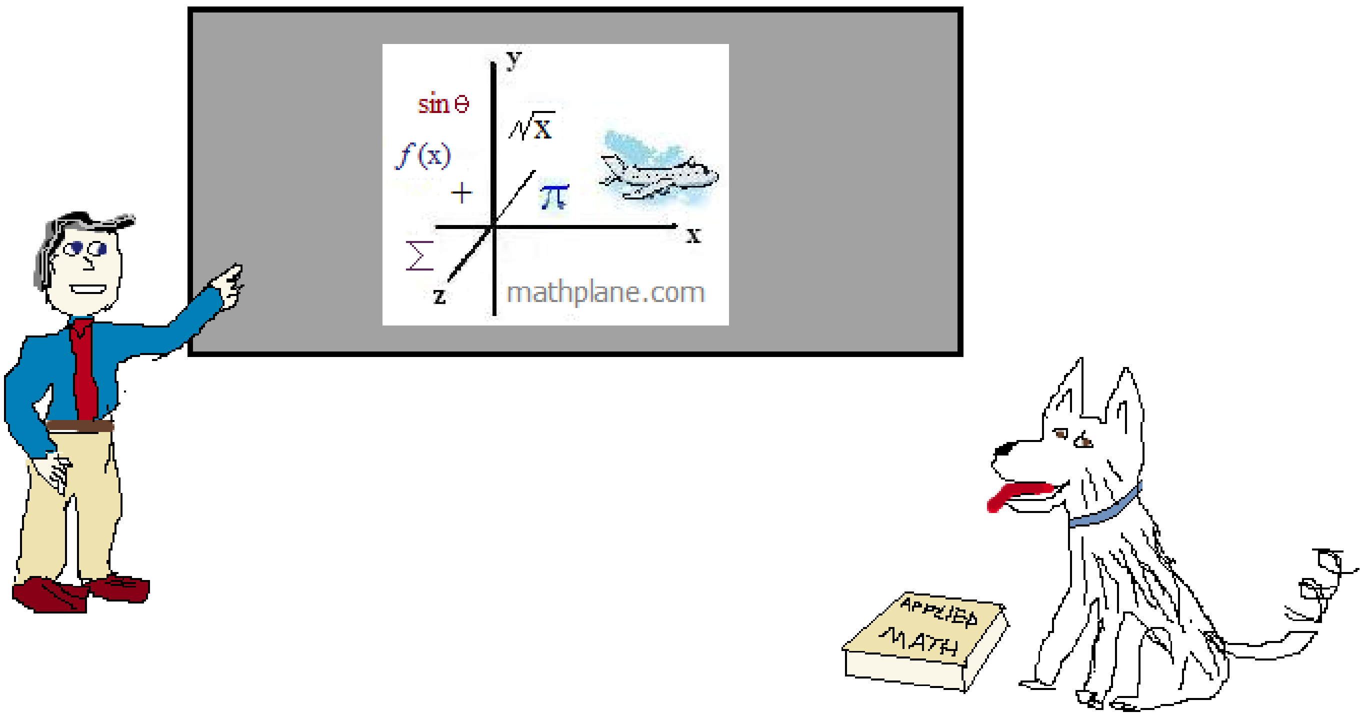
$= 1/8(-324)$

$-\frac{324}{8} = \boxed{\frac{-81}{2}}$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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One more question

If (x + 1) is a factor of $x^4 + 6x^3 + 13x^2 + Kx + 4$

then $K =$

ANSWER-→

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ANSWER

Utilizing the factor theorem:
(if (x + 1) is a factor, then)

$f(-1) = (-1)^4 + 6(-1)^3 + 13(-1)^2 + K(-1) + 4$ (must equal zero)

$1 - 6 + 13 - K + 4 = 0$

$12 - K = 0$

$K = 12$

Using Synthetic Division:

−1	1	6	13	K	4
		-1	-5	-8	(-K + 8)
	1	5	8	(K - 8)	(12 - K)

Since the remainder must be 0,

$12 - K = 0$

therefore, K = 12...

To continue factoring the polynomial....

$(x + 1)(x^3 + 5x^2 + 8x + 4)$

Using the rational root theorem,
possible roots are 1, 2, 4, -1, -2, -4..
And, since all the terms are positive
we can eliminate 1, 2, 4... There is
no way f(x) = 0 if x is positive...

Since f(-1) = 0....
(x + 1) is a factor again....

−1	1	5	8	4
		-1	-4	-4
	1	4	4	0

$(x + 1)(x + 1)(x^2 + 4x + 4)$

then, the factored form is

$(x + 1)^2 (x + 2)^2$