Vectors 2

Examples and Practice Questions (with solutions)

Topics include bearings, force, unit vectors, orthogonal vectors, graphing, angle vector theorem, and more.

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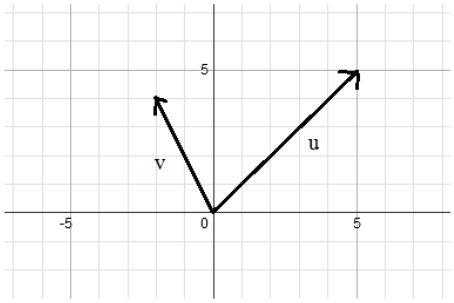
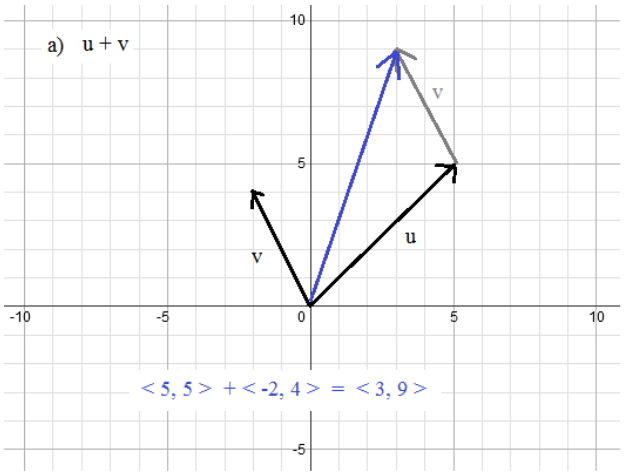
Example: Given the graph of vectors u and v:

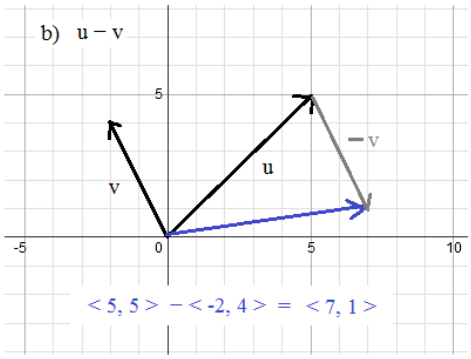


Find a) u + v
b) u − v
c) v − u

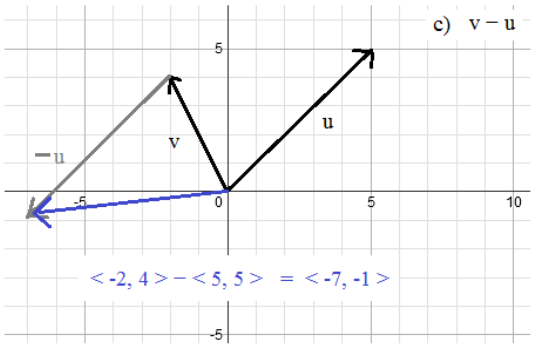
Show answer graphically and using the components...



< 5, 5 > + < -2, 4 > = < 3, 9 >



< 5, 5 > − < -2, 4 > = < 7, 1 >

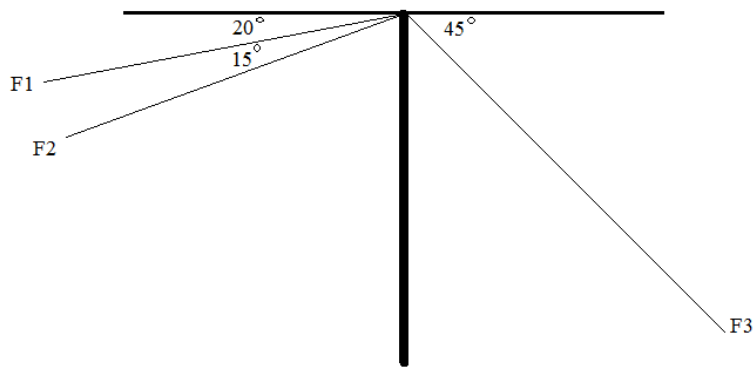


< -2, 4 > − < 5, 5 > = < -7, -1 >

Example: 3 forces act at the top of a pole...
The magnitude of F1 is 250 lbs
and the magnitude of F2 is 300 lbs

If the resultant vector of the 3 forces is directly vertically downward,

what is the magnitude of F3?



x = rcos⊖
y = rsin⊖

since it's a 45 degree angle, x and y will be the same...
Let's search for r, using the x values...

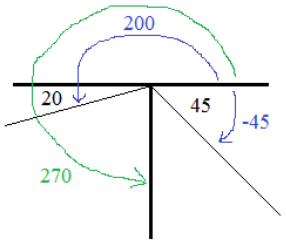
F1 + F2 + F3 must result in x = 0

F1 x = rcos⊖ x = 250cos(200) x = −234.9 (quadrant 2 is negative)
F2 x = rcos⊖ x = 300cos(215) x = −245.7
F3 x = rcos⊖ x = rcos(-45)

250cos(200) + 300cos(215) + rcos(-45) = rcos(270)
−234.9 −245.7 + rcos(-45) = r(0)
− 480.6 = − rcos(-45)

r = 679.6

NOTE: we apply the forces in the diagram to a coordinate plane

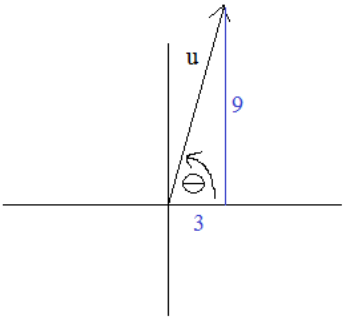


Finding the angle between 2 vectors in a plane

Example: Given vector u = < 3, 9 > and vector v has a magnitude of 8 and direction 60 degrees.
Find the angle between u and v.

Method 1: Find angle of vector u and compare to vector v

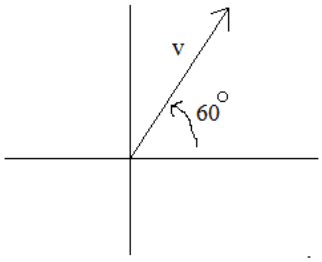
vector u = < 3, 9 >



$\tan(\ominus) = \frac{9}{3} = 3$

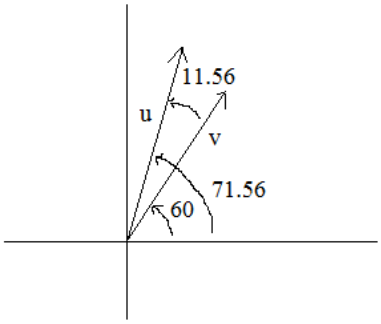
$\tan^{-1}(3) = 71.56^\circ$

angle of vector v is 60°



The angle between the vectors is

$71.56^\circ - 60^\circ = 11.56^\circ$



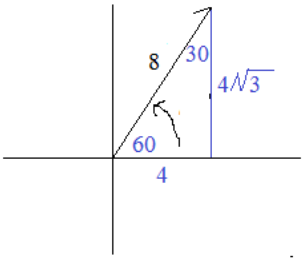
Method 2: Use angle formula $\cos\ominus = \frac{u \cdot v}{\|u\|\,\|v\|}$ where $\ominus$ is the angle between vectors u and v

u = < 3, 9 >

$\|u\| = \sqrt{3^2 + 9^2} = 3\sqrt{10}$

Since $\|v\| = 8$
and, the degree measure is 60,

vector v = $< 4, 4\sqrt{3} >$



$$\cos\ominus = \frac{u \cdot v}{\|u\|\,\|v\|} = \frac{3 \cdot 4 + 9 \cdot 4\sqrt{3}}{3\sqrt{10} \cdot 8} = \frac{74.35}{75.89}$$

$$\ominus = \cos^{-1}\frac{74.35}{75.89} = 11.56$$

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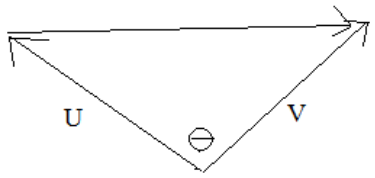
Deriving the Angle Vector equation:

$$\cos\ominus = \frac{U \cdot V}{|U|\,|V|}$$

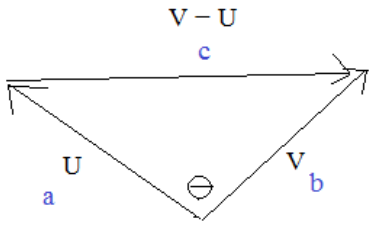
or $$\frac{U \cdot V}{\|U\|\,\|V\|}$$



Law of Cosines: $c^2 = a^2 + b^2 - 2ab(\cos\ominus)$

Vector Dot Product & Length Property

$$\|\vec{w}\|^2 = \vec{w} \cdot \vec{w}$$



$$\|V-U\|^2 = \|U\|^2 + \|V\|^2 - 2\|U\| \cdot \|V\|(\cos\ominus)$$

$$(V-U) \cdot (V-U) = \|U\|^2 + \|V\|^2 - 2\|U\| \cdot \|V\|(\cos\ominus)$$

$$V \cdot V - U \cdot V - V \cdot U + U \cdot U = \|U\|^2 + \|V\|^2 - 2\|U\| \cdot \|V\|(\cos\ominus)$$

$$\|V\|^2 - 2\,U \cdot V + \|U\|^2 = \|U\|^2 + \|V\|^2 - 2\|U\| \cdot \|V\|(\cos\ominus)$$

$$-2\,U \cdot V = -2\|U\| \cdot \|V\|(\cos\ominus)$$

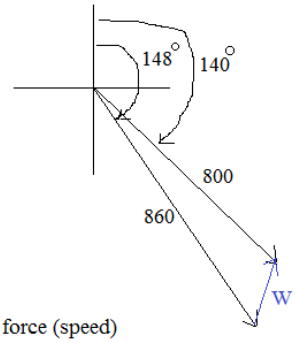
$$\frac{U \cdot V}{\|U\| \cdot \|V\|} = (\cos\ominus)$$

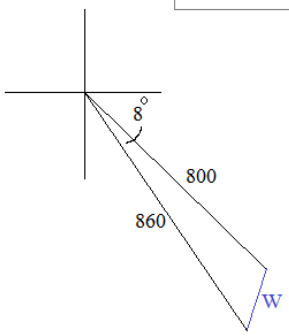
Example: A plane flying at 860 km per hour at a bearing of 148° has a ground speed of 800 km per hour and a bearing of 140°. What is the wind direction and force?

Air speed + Wind Speed = Ground Speed

Method 1: Using Law of Cosines/Sines

Step 1: Draw a diagram





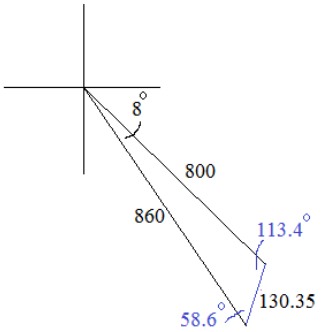
Step 2: Use law of cosines to find wind force (speed)

$W^2 = 800^2 + 860^2 - 2(800)(860)\cos(8^\circ)$

$W^2 = 1379600 - 1376000(.990)$

$W^2 = 16991$

$W = 130.35$



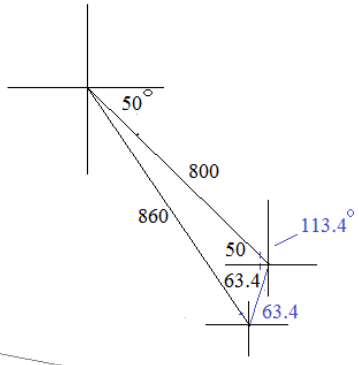
Step 3: Use law of sines to get angles (to find bearing)

$\frac{\sin(8)}{130.35} = \frac{\sin(x)}{800}$ $\sin(x) = \frac{800\sin(8)}{130.35} = .854$ $x = 58.6^\circ$

Then, $58.6 + 8 + y = 180$ so, the other angle is 113.4°

Using geometry properties and theorems, the angle is 63.4 degrees...

Therefore, the bearing is 26.6°

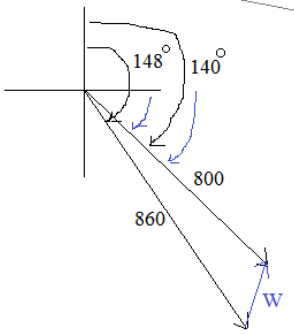


Method 2: Use component vectors

Step 1: Change bearings into standard angle measures...

140 degrees ----> −50 degrees (or 310 degrees)

148 degrees ----> −58 degrees (or 302 degrees)



Step 2: Find the component vectors

$\langle r\cos\theta, r\sin\theta \rangle$

Ground speed: $\langle 800\cos 310, 800\sin 310 \rangle$

$\langle 514.2, -612.8 \rangle$

Air speed: $\langle 860\cos 302, 860\sin 302 \rangle$

$\langle 455.7, -729.3 \rangle$

Step 3: Solve Air speed + Wind Speed = Ground Speed

$\langle 455.7, -729.3 \rangle + \langle i, j \rangle = \langle 514.2, -612.8 \rangle$

Wind speed $= \langle 58.5, 116.5 \rangle$

Force (magnitude) $= \sqrt{58.5^2 + 116.5^2} = 130.36$

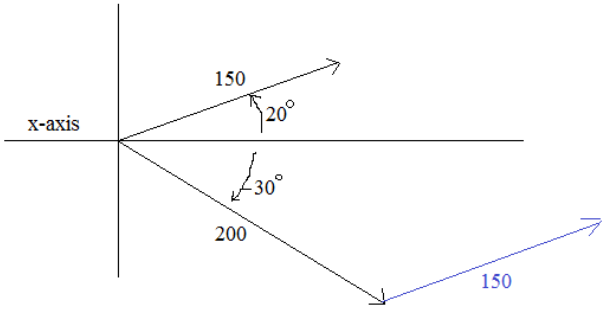
direction: $\tan^{-1}\frac{116.5}{58.5} = 63.33$ ** then, convert back to bearings..

$90 + 63.33 = 26.67^\circ$

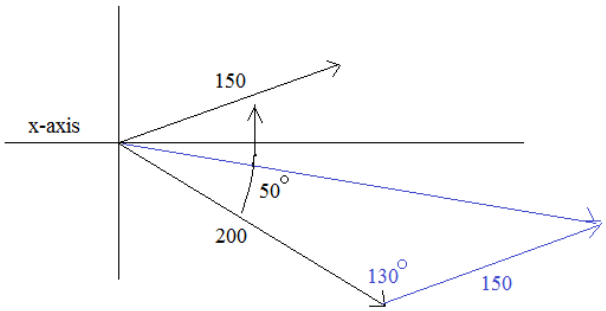
Example: Two forces pull on an object.
The first force pulls at 150 lbs at a 20 degree angle (relative to the x-axis)
The second force pulls at 200 lbs at a −30 degree angle (relative to the x-axis)

What is the total force and which direction is the object pulled?

Method 1: Using Law of cosines and Law of Sines



Use "tip to tail"...



Property of parallelogram:
consecutive angles are supplementary...

$(50^\circ + 130^\circ = 180^\circ)$

To find force, use law of cosines...

$a^2 = 200^2 + 150^2 - 2(200)(150)\cos(130^\circ)$

$a^2 = 62500 - 60000(-.643)$

$a^2 = 101{,}067$

$a = 317.9$ lbs

To find direction, we can use law of sines...

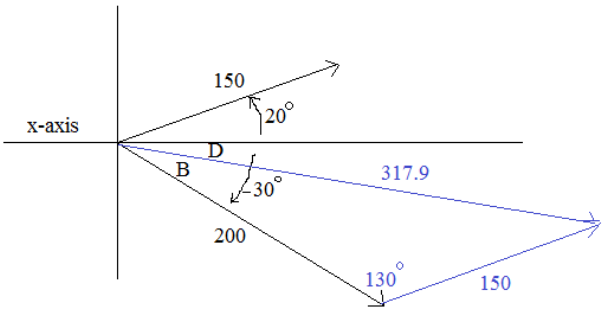
$$\frac{\sin(130)}{317.9} = \frac{\sin B}{150}$$

$$\sin B = \frac{150\sin(130)}{317.9} = .361$$

$B = 21.2$ degrees

Since $D + B = -30$,

direction $D = -8.8^\circ$



Method 2: Use Component Vectors

$$< r\cos\ominus, r\sin\ominus >$$

$< 150\cos(20), 150\sin(20) >$ $< 200\cos(-30), 200\sin(-30) >$

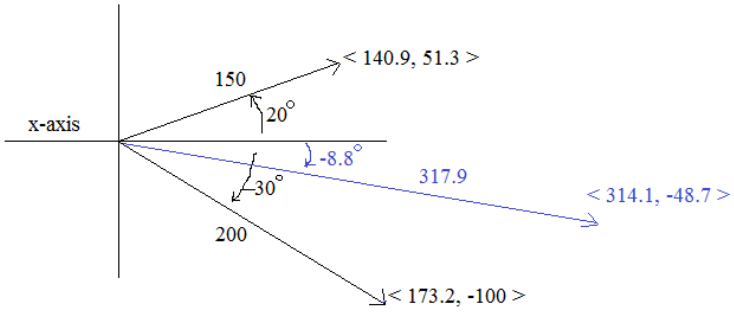
$< 140.9, 51.3 >$ $< 173.2, -100 >$

Add components to get resultant vector:

$< 140.9 + 173.2, 51.3 + (-100) > = < 314.1, -48.7 >$

magnitude of vector: $\sqrt{314.1^2 + (-48.7)^2} = 317.9$

direction of vector: $\tan^{-1}\left(\frac{-48.7}{314.1}\right) = -8.8^\circ$



Example: The sum of 2 vectors is $< 3, 7 >$.

If one vector is parallel to $< 4, 1 >$
and
one vector is perpendicular to $< 4, 1 >$,

what are the 2 vectors?

If a vector is parallel to $< 4, 1 >$, it will have the same ratio (i.e. slope)... $< 4a, a >$

If a vector is perpendicular to $< 4, 1 >$, it will have the opposite reciprocal... $< -1b, 4b >$

$< 4a, a > + < -1b, 4b > = < 3, 7 >$

$$\begin{aligned} 4a - b &= 3 \\ a + 4b &= 7 \end{aligned}$$

$$\begin{aligned} 16a - 4b &= 12 \\ a + 4b &= 7 \end{aligned}$$

$$17a = 19$$

$$a = 19/17$$

$$b = 25/17$$

Since $a = 19/17$ and $b = 25/17$,

the 2 vectors are $< 76/17, 19/17 >$ and $< -25/17, 100/17 >$

Notice the sum of the vectors is $< 3, 7 >$

$< 76/17, 19/17 >$ is parallel to $< 4, 1 >$

(same ratio..)

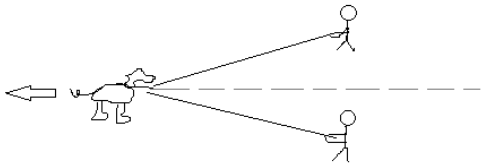
$< -25/17, 100/17 >$ is perpendicular to $< 4, 1 >$

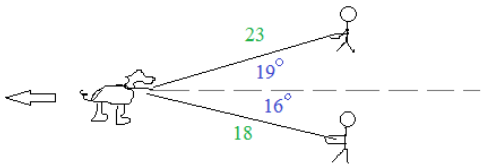
(the dot product is 0)

Example: My dog Norway is tugging on two ropes held by me and my friend.

If I'm pulling on a rope with a force of 23 at an angle of 19 degrees,
and my friend is pulling on the other rope with a force of 18 at an angle of 16 degrees,

what is the force my dog is pulling?





Step 1: Draw a diagram

Step 2: Convert the angles and force into vectors

$x = r\cos\ominus$ $< 23\cos(19) , 23\sin(19) >$

$y = r\sin\ominus$ $< 21.75, 7.49 >$

$< 18\cos(-16), 18\sin(-16) >$

$< 17.30, -4.96 >$

Step 3: Find the resultant vector

$< 21.75, 7.49 > + < 17.30, -4.96 > = < 39.05, 2.53 >$

The resultant vector of me
and my friend combined...

Step 4: Find the force of the dog

Since the vector of me and my friend is $< 39.05, 2.53 >$,

the force ('magnitude') is $\sqrt{39.05^2 + 2.53^2} = 39.13$

therefore, the force of Norway the dog is 39.13...

$< 0, -39.13 >$ is the vector

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Vectors and Angles

$\cos\theta = \dfrac{a \cdot b}{\|a\|\,\|b\|}$ where θ is the angle between vectors a and b.

Dot Product		
	> 0	ACUTE
	< 0	OBTUSE
	= 0	RIGHT (Orthogonal Vectors)

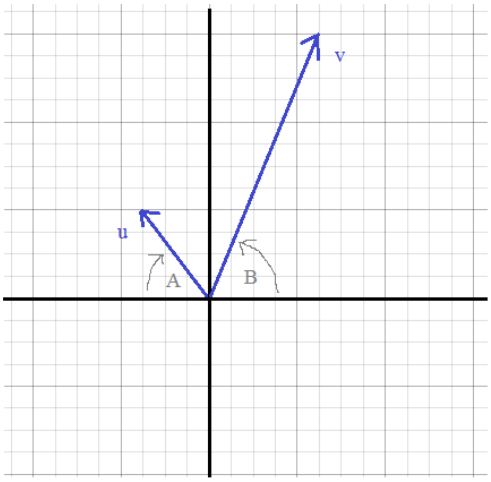
Example: $v = \langle 5, 12 \rangle$

$u = \langle -3, 4 \rangle$

Find the angle between vectors u and v,

a) using the vector formula

b) using geometry and trigonometry



Using the formula, $\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$

$\cos\theta = \dfrac{33}{(5)(13)}$

59.5 degrees

Using geometry and trigonometry,

$\tan A = \dfrac{4}{3}$ $A = 53.1$

$\tan B = \dfrac{12}{5}$ $B = 67.4$

$A + B + \text{Angle} = 180$

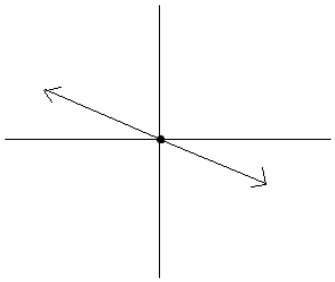
$53.1 + 67.4 + \text{Angle} = 180$

Angle = 59.5

Example: Find the angle between $\langle 3, -1 \rangle$ and $\langle -3, 1 \rangle$

$\cos\theta = \dfrac{-10}{10} = -1$

θ = 180 degrees



Example. $|u| = 8$ "Bearing direction" 115 degrees

$|v| = 10$ "Bearing direction" 50 degrees

Draw the vectors. Then find $|u + v|$, using a) component vectors b) law of cosines and geometry

a) using component vectors

u: $x = 8\cos(-25^\circ)$ 7.25

$y = 8\sin(-25^\circ)$ -3.4

$\langle 7.25, -3.4 \rangle$

v: $x = 10\cos(40^\circ)$ 7.66

$y = 10\sin(40^\circ)$ 6.4

$\langle 7.66, 6.4 \rangle$

$u + v = \langle 14.91, 3 \rangle$

$|u + v| = 15.2$

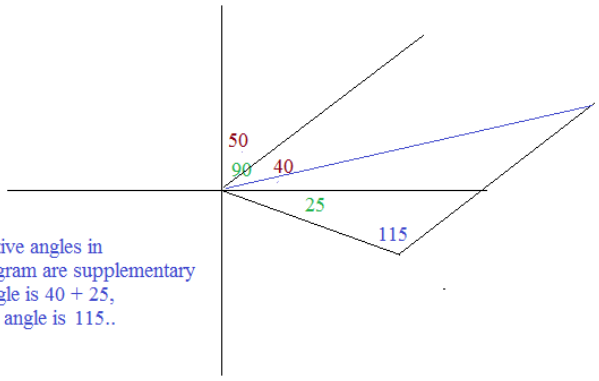
b) using law of cosines and geometry

$c^2 = 8^2 + 10^2 - 2(8)(10)\cos(115)$

$c^2 = 164 - 160\cos(115)$

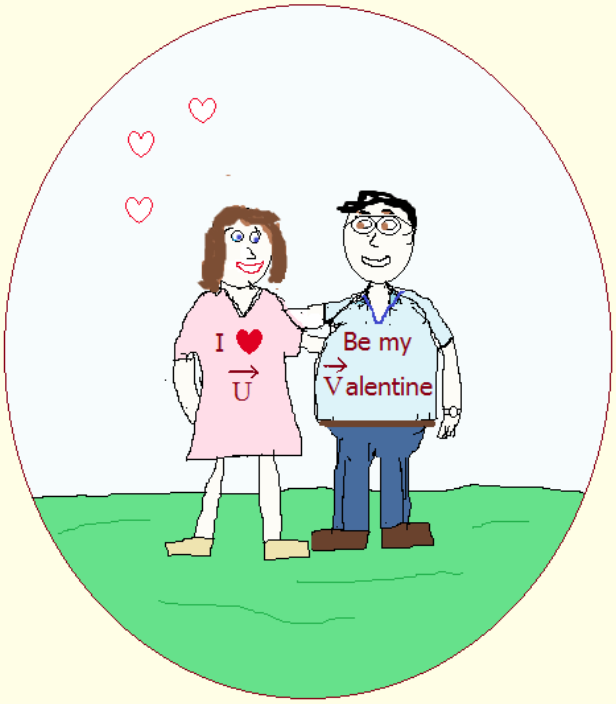
$c^2 = 164 - -67.6$

c = 15. 2



consecutive angles in parallelogram are supplementary
Since angle is 40 + 25,
the other angle is 115..

A relationship of significant magnitude: Dot and Norm



(Their embarrassed kids, *ike, jay,* and *kay*, were nowhere to be found...)

LanceAF #177 2-14-15
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Math Romance II

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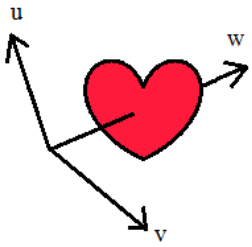

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on-line dating that gets resultants....

Practice Test-→

Vectors Questions

1) Find the unit vector of $\vec{v} = 2i - 5j$

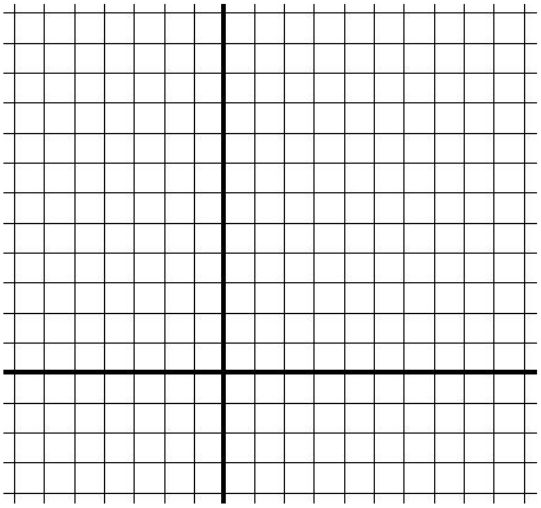
2) For the given vector $\vec{u} = \langle -4, 9 \rangle$

a) What is the length of the vector?

b) Find the unit vector of $\vec{u}$

c) Determine a vector in the same direction with length 5...

d) Graph the vectors...



3) For the given vectors $v = \langle 3, -4 \rangle$ $w = \langle 1, K \rangle$

a) If v and w are parallel, what is K?

b) If v and w are orthogonal (or perpendicular), what is K?

4) Find the vector with the given magnitude and direction:

$\| v \| = 8$ in the direction $(5, 6)$

5) Given the following magnitudes:

$\| F_1 \| = 72$ $\| F_2 \| = 38$

$\| F_1 + F_2 \| = 93$

what is the angle between the vectors?

Vectors Questions

6) Given the vectors in component form

$u = \langle -4, -4 \rangle \quad v = \langle 4, 2 \rangle$

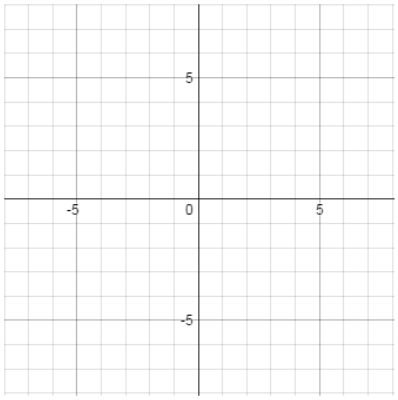
Find a) $u + v$

b) $u + 3v$

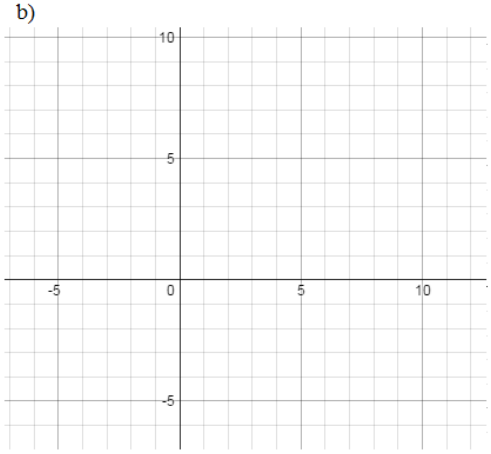
c) $u - 2v$

Answer both graphically and using the components...

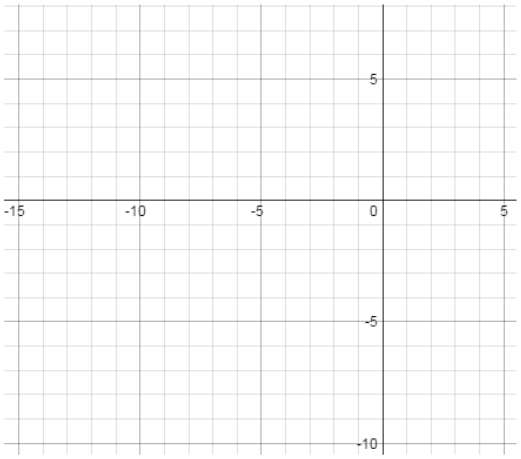
a)



b)



c)



7) A plane flying N52W at an *air speed* of 340 miles per hour has a *ground speed* of 325 miles per hour going N47W. What is the bearing and speed of the wind?

AIR SPEED + WIND SPEED = GROUND SPEED

Vector Questions

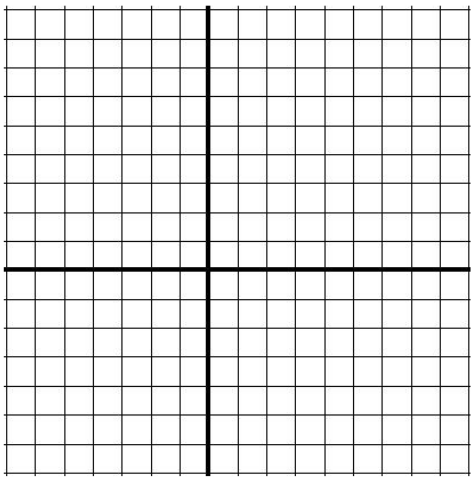
8) $u = \langle -3, 2 \rangle$ $v = \langle 4, 8 \rangle$

find $(u \cdot v)v$

9) Graph the vectors and find the angle between them...

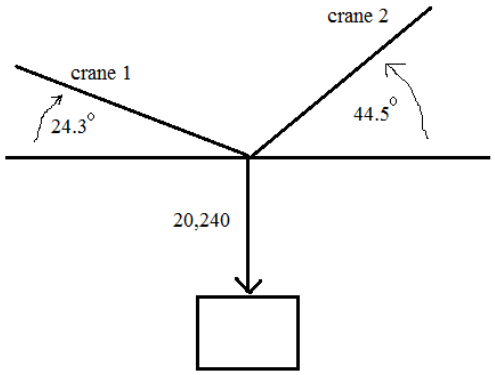
$u = 2i + 4j$

$v = -3i + 5j$



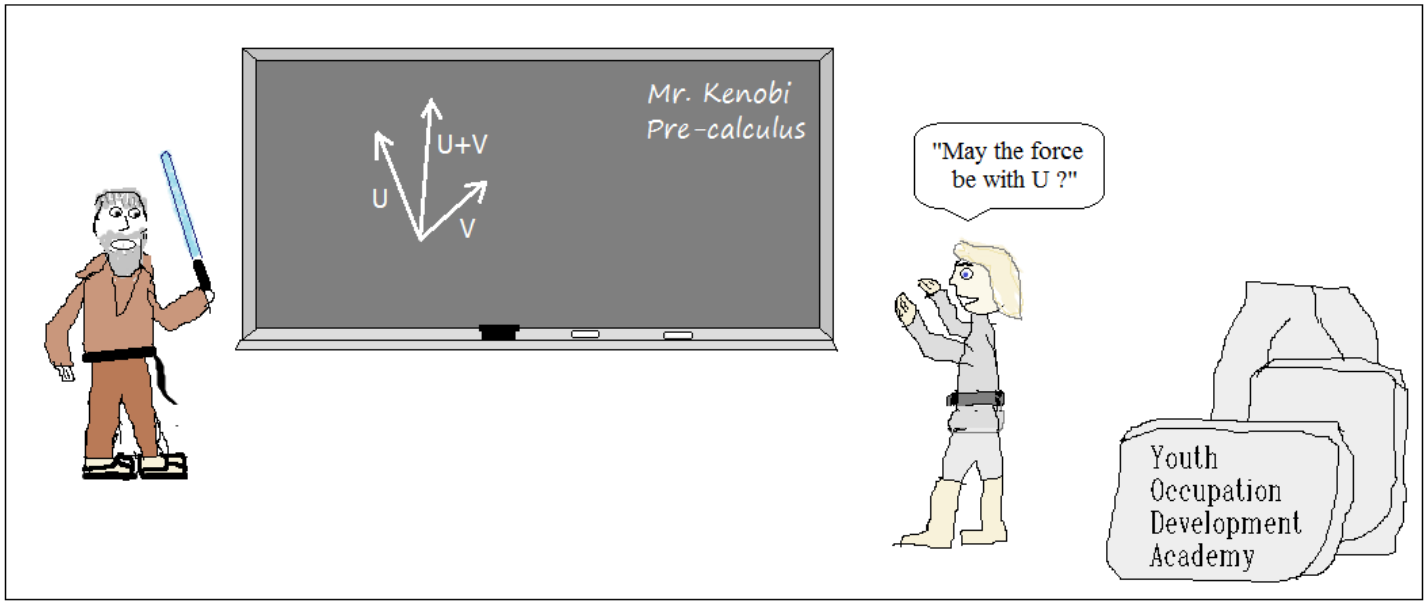
10) Two cranes lift an object. The diagram shows the known angles and weight.

Find the force of each crane:



A long time ago,
in a classroom
far, far away...

Math Lessons from the Jedi



LanceAF #72 2-17-13
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Obi-Wan teaches Luke about resultant vectors and (the) force

Solutions -→

Vectors Questions

1) Find the unit vector of $\vec{v} = 2i - 5j$

SOLUTION

unit vector $\hat{v} = \dfrac{v}{\| v \|}$ $\quad \| v \| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$

$\hat{v} = \dfrac{2}{\sqrt{29}} i - \dfrac{5}{\sqrt{29}} j$

2) For the given vector $\vec{u} = \langle -4, 9 \rangle$

a) What is the length of the vector?

$\| v \| = \sqrt{(-4)^2 + (9)^2} = \sqrt{97}$

b) Find the unit vector of $\vec{u}$

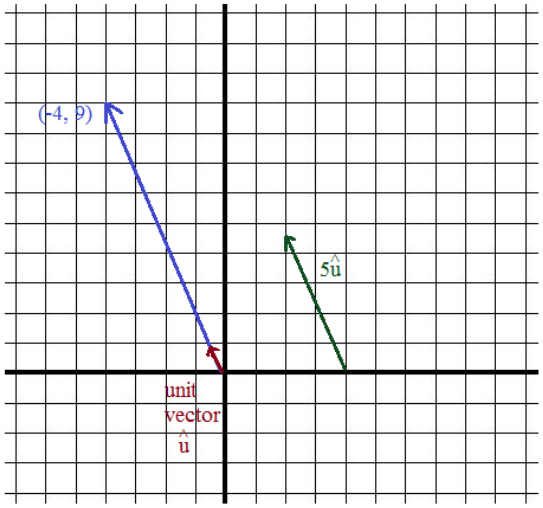
unit vector $= \dfrac{u}{\| u \|}$ $\quad$ unit vector $\hat{u} = \left\langle \dfrac{-4}{\sqrt{97}}, \dfrac{9}{\sqrt{97}} \right\rangle$

c) Determine a vector in the same direction with length 5...

Since unit vector has length 1,
a vector in the same direction with length 5:

$5 \cdot \hat{u} = \left\langle \dfrac{-20}{\sqrt{97}}, \dfrac{45}{\sqrt{97}} \right\rangle$

d) Graph the vectors...



3) For the given vectors $v = \langle 3, -4 \rangle$ $\quad w = \langle 1, K \rangle$

a) If v and w are parallel, what is K?

If the vectors are parallel, they are going in the same direction..
therefore, there 'slopes' are the same...

direction (slope) of v $= \dfrac{-4}{3}$ $\quad$ direction of w $= \dfrac{K}{1}$ $\quad K = -4/3$

b) If v and w are orthogonal (or perpendicular), what is K?

If the vectors are orthogonal, then the dot product equals 0.

$v \cdot w = (3)(1) + (-4)(K) = 0$

$3 - 4K = 0$ $\quad K = 3/4$

4) Find the vector with the given magnitude and direction:

$\| v \| = 8$ in the direction $(5, 6)$

first, find the unit vector in the direction (5, 6): $\left\langle \dfrac{5}{\sqrt{61}}, \dfrac{6}{\sqrt{61}} \right\rangle$ (magnitude = 1)

then, adjust it to the correct magnitude:

$8 \left\langle \dfrac{5}{\sqrt{61}}, \dfrac{6}{\sqrt{61}} \right\rangle = \left\langle \dfrac{40}{\sqrt{61}}, \dfrac{48}{\sqrt{61}} \right\rangle$ or $\dfrac{40}{\sqrt{61}} i + \dfrac{48}{\sqrt{61}} j$

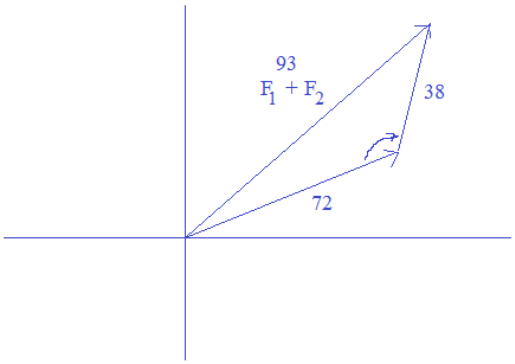
5) Given the following magnitudes:

$\| F_1 \| = 72$ $\quad \| F_2 \| = 38$

$\| F_1 + F_2 \| = 93$

what is the angle between the vectors?

(possible sketch)



Use law of cosines...

$93^2 = 72^2 + 38^2 - 2(72)(38)\cos\ominus$

$8649 = 5184 + 1444 - 5472\cos\ominus$

$2021 = -5472\cos\ominus$ $\quad \ominus = 111.7^\circ$

$-.369 = \cos\ominus$

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Vectors Questions

SOLUTIONS

6) Given the vectors in component form

$u = \langle -4, -4 \rangle \quad v = \langle 4, 2 \rangle$

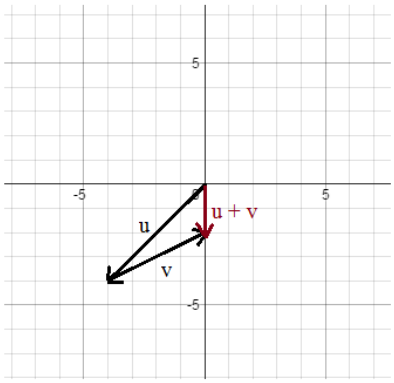
Find a) $u + v$ $\langle -4, -4 \rangle + \langle 4, 2 \rangle = \langle 0, -2 \rangle$

b) $u + 3v$ $\langle -4, -4 \rangle + \langle 12, 6 \rangle = \langle 8, 2 \rangle$

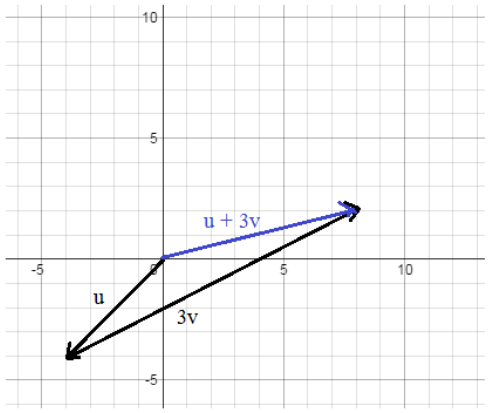
c) $u - 2v$ $\langle -4, -4 \rangle - \langle 8, 4 \rangle = \langle -12, -8 \rangle$

Answer both graphically and using the components...

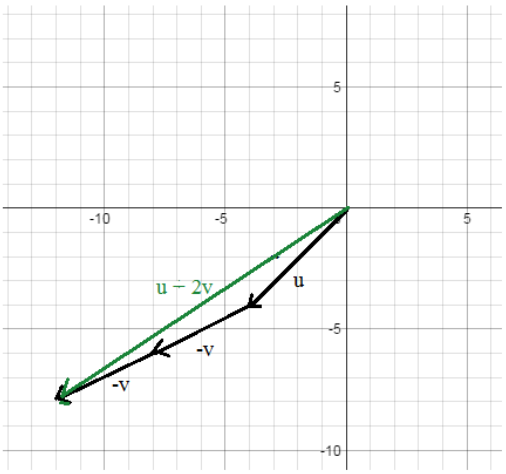
a)



b)



c)



7) A plane flying N52W at an *air speed* of 340 miles per hour has a *ground speed* of 325 miles per hour going N47W. What is the bearing and speed of the wind?

AIR SPEED + WIND SPEED = GROUND SPEED

Step 1: Draw a diagram

Step 2: Determine the component vectors

$\langle r\cos\ominus , r\sin\ominus \rangle$

$\langle 340\cos(142^\circ), 340\sin(142^\circ) \rangle$

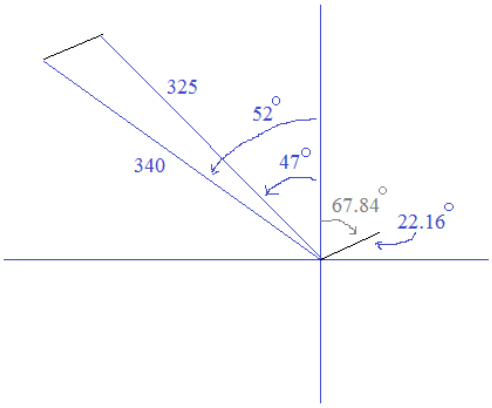
$\langle -267.9, 209.3 \rangle$

$\langle 325\cos(137^\circ), 325\sin(137^\circ) \rangle$

$\langle -237.7, 221.6 \rangle$

NOTE: we changed the orientation from navigation bearings to geometry plane

EX: N52W becomes 142°



Step 3: Find resultant vector and magnitude...

Air speed + Wind speed = Ground Speed

$\langle -267.9, 209.3 \rangle + \langle W_i , W_j \rangle = \langle -237.7, 221.6 \rangle$

Wind vector = $\langle 30.2, 12.3 \rangle$

$\| W \| = \sqrt{30.2^2 + 12.3^2} = 32.6$ mph

Step 4: Find the wind direction

$\tan^{-1} \frac{12.3}{30.2} = \ominus$

$= 22.16^\circ$

since the angle is 22.16, the bearing is

90 - 22.16 ----> N67.84E

8) $u = \langle -3, 2 \rangle$ $v = \langle 4, 8 \rangle$ (order of operations -- parenthesis first)

SOLUTIONS

Vector Questions

find $(u \cdot v)v$

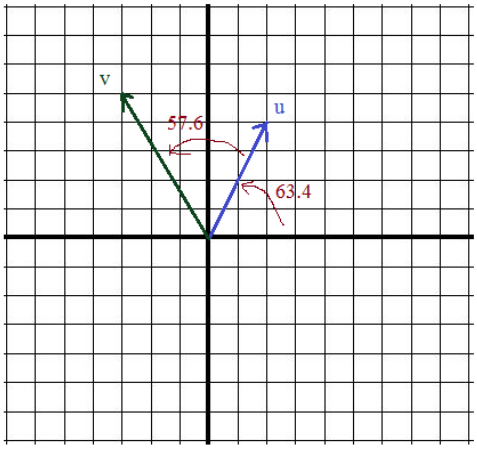
$u \cdot v = (-3 \times 4) + (2 \times 8) = 4$

$4v = \boxed{\langle 16, 32 \rangle}$

9) Graph the vectors and find the angle between them...

$u = 2i + 4j$

$v = -3i + 5j$



$$\cos\theta = \frac{u \cdot v}{|u|\,|v|} = \frac{(2 \times (-3)) + (4 \times 5)}{\sqrt{20} \times \sqrt{34}} = \frac{14}{26.08}$$

$$\theta = \cos^{-1}\left(\frac{14}{26.08}\right) = \boxed{57.5 \text{ degrees}}$$

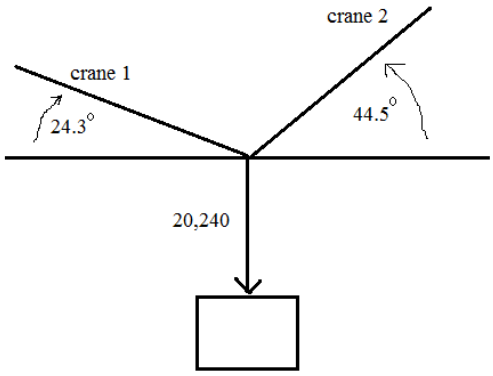
Alternate method:

$u = \langle 2, 4 \rangle$ $\tan^{-1}\left(\frac{4}{2}\right) = 63.4$ degrees → $121 - 63.4 = 57.6$ ✓

$v = \langle -3, 5 \rangle$ $\tan^{-1}\left(\frac{5}{-3}\right) = -59.0 + 180 = 121$ degrees

10) Two cranes lift an object. The diagram shows the known angles and weight.

Find the force of each crane:



c_1 = force of crane 1

c_2 = force of crane 2

Using component vectors:

$\langle c_2 \cos(44.5), c_2 \sin(44.5) \rangle$

+

$\langle c_1 \cos(155.7), c_1 \sin(155.7) \rangle$

=

$\langle 20{,}240\cos(270), 20{,}240\sin(270) \rangle$

$\langle .7133c_2, .7009c_2 \rangle$

+

$\langle -.9114c_1, .4115c_1 \rangle$

=

$\langle 0, -20{,}240 \rangle$

$.7133c_2 + -.9114c_1 = 0$

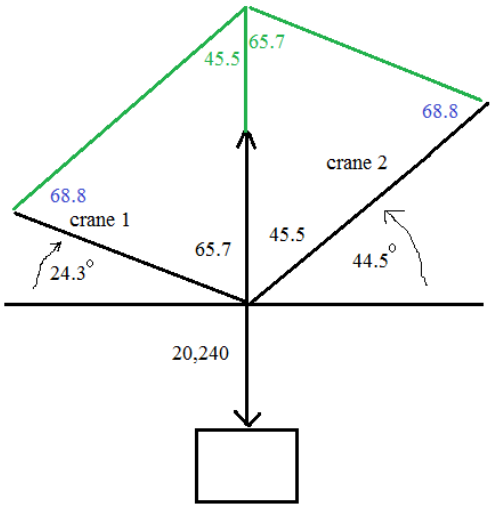
$.7009c_2 + .4115c_1 = -20{,}240$

solve system of equations

$c_2 = -19{,}786$

$c_1 = -15{,}485$

Using trigonometry, geometry, and law of sines:



Note: consecutive angles of parallelogram must be supplementary...

$$\frac{\sin(65.7)}{c_2} = \frac{\sin(68.8)}{20{,}240}$$

$$\frac{\sin(45.5)}{c_1} = \frac{\sin(68.8)}{20{,}240}$$

crane 2 = 19,785

crane 1 = 15, 484

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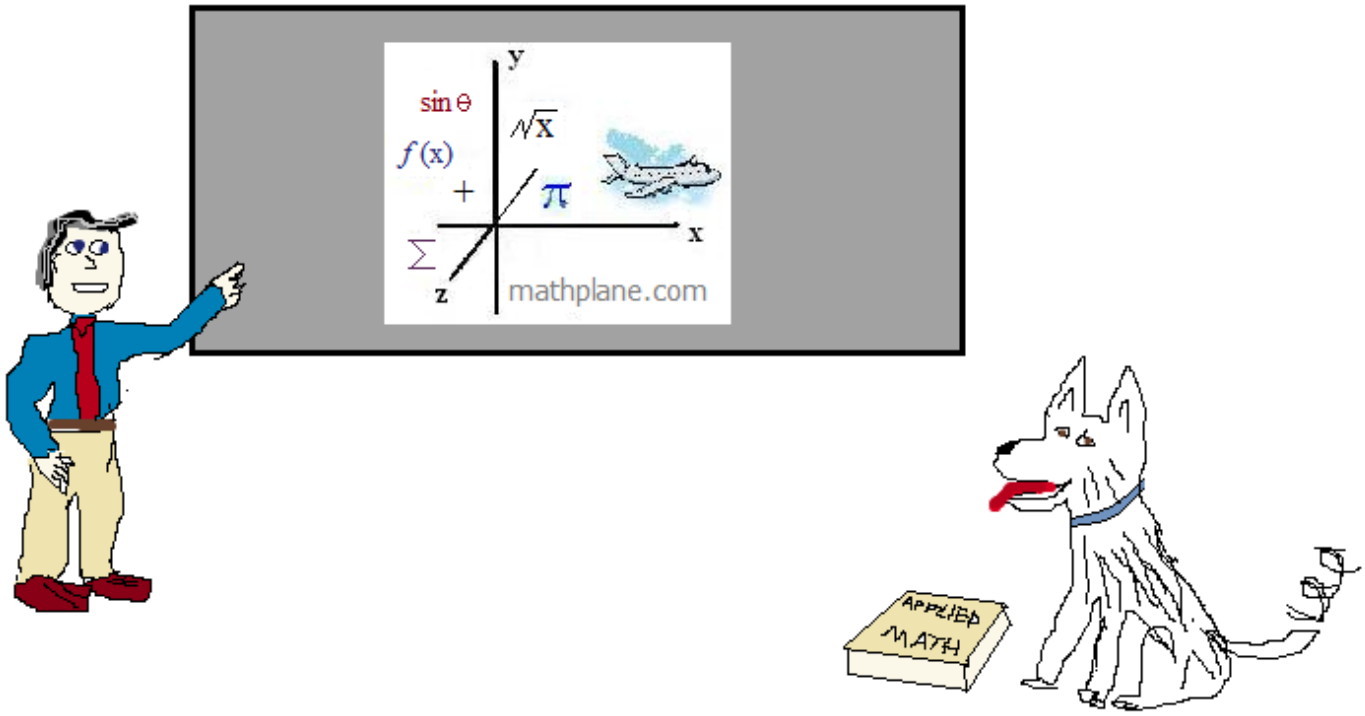
Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

Cheers



Also, at Facebook, Google +, Pinterest, and TeachersPayTeachers…

One more question…

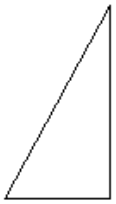
Find the component form of the vector V if ||V|| = 8, and the angle it makes with the *x-axis* is 60 degrees…

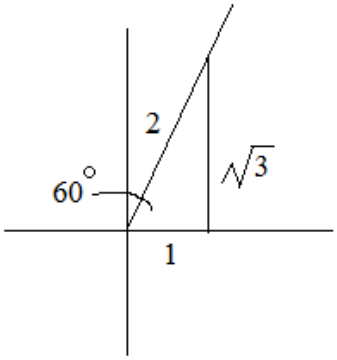
Answer on the next page-→

Find the component form of the vector V if $\|V\| = 8$

and the angle it makes with the *x-axis* is 60 degrees.

30-60-90 right triangle

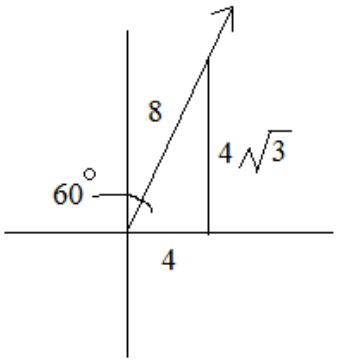




$i + \sqrt{3}\ j = V$

$\|V\| = 2$

Unit vector: $\frac{i + \sqrt{3}\ j}{2}$



Unit vector x 8 = $4i + 4\sqrt{3}j$

60 degree angle and length 8!!