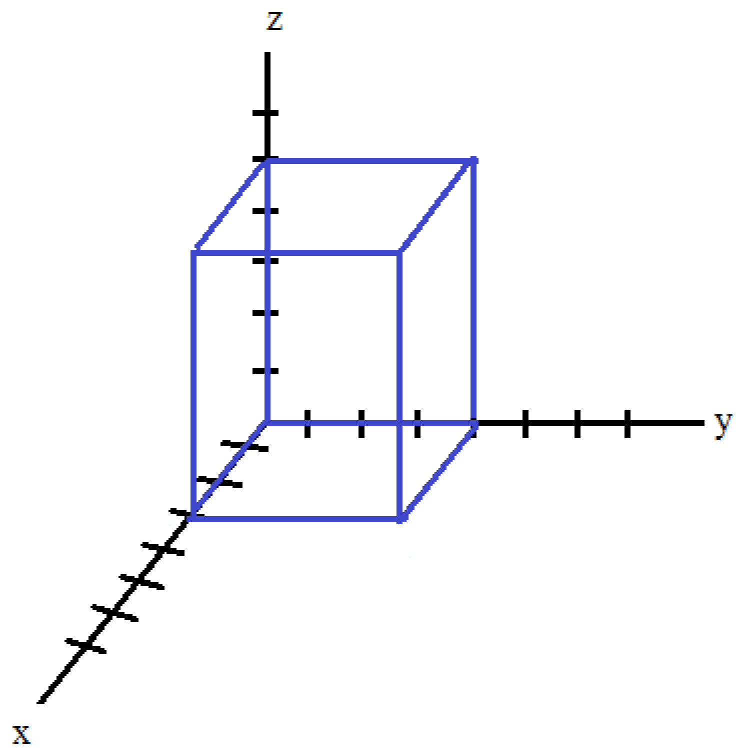
3-Dimensional Space and Vectors

Notes, Examples, and Practice Quiz (with Answers)



Topics include 3-D coordinate plane, area, vector dot product and cross product, equation of a plane, and more.

3-D space: vector and Trigonometry Application

Example: Find the area of the triangle whose vertices are

A (2, 3, -5)

B (-2, -2, 0)

C (3, 0, 6)

Method 1: Using vectors and cross product

Step 1: Select and identify 2 vectors that share the same initial or terminal point

$\overrightarrow{BA}$ (2, 3, -5) − (-2, -2, 0) -----> < 4, 5, -5 > = v

$\overrightarrow{CA}$ (2, 3, -5) + (3, 0, 6) -----> < -1, 3, -11 > = w

$$\text{Area} = \frac{1}{2}\| v \times w \|$$

Step 2: Find cross product

$$v \times w = \begin{vmatrix} i & j & k \\ 4 & 5 & -5 \\ -1 & 3 & -11 \end{vmatrix} = \underbrace{\begin{vmatrix} 5 & -5 \\ 3 & -11 \end{vmatrix}}_{-40} i - \underbrace{\begin{vmatrix} 4 & -5 \\ -1 & -11 \end{vmatrix}}_{-49} j + \underbrace{\begin{vmatrix} 4 & 5 \\ -1 & 3 \end{vmatrix}}_{17} k = -40i - (-49)j + 17k$$

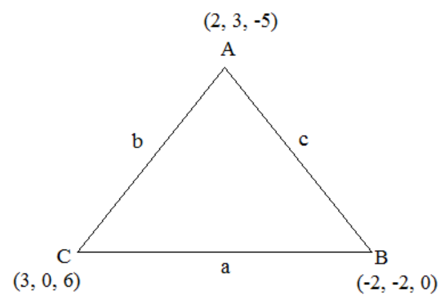
or < -40, 49, 17 >

Step 3: Insert into area formula...

$$\text{Area} = \frac{1}{2}\| v \times w \| = \frac{1}{2}\| < -40, 49, 17 > \| = \frac{1}{2}\sqrt{4290} = \frac{1}{2} \cdot 65.5$$

approx. 32.75

Method 2: Using law of cosines, sines, and trig formulas



Step 1: Find lengths of sides (using distance formula)

side c (distance between A and B) $\sqrt{(2 - (-2))^2 + (3 - (-2))^2 + (-5 - 0)^2}$

$\sqrt{16 + 25 + 25} = \sqrt{66}$

side a (distance between B and C) $\sqrt{(-2 - 3)^2 + (-2 - 0)^2 + (0 - 6)^2}$

$\sqrt{25 + 4 + 36} = \sqrt{65}$

side b (distance between C and A) $\sqrt{(3 - 2)^2 + (0 - 3)^2 + (6 - (-5)^2}$

$\sqrt{1 + 9 + 121} = \sqrt{131}$

Step 2: Find length of 3rd side using law of cosines

$$c^2 = a^2 + b^2 - 2ab(\text{Cos}C)$$

$$66 = 65 + 131 - 2(\sqrt{65})(\sqrt{131})\text{Cos}C$$

$$\frac{-130}{-2\sqrt{8515}} = \text{Cos}C \qquad C = 45.22^{\circ}$$

$$\text{Area of triangle} = \frac{1}{2}ab\text{Sin}C$$

Step 3: Insert into area formula

$$\text{Area} = \frac{1}{2}\sqrt{65}\sqrt{131}\ \text{Sin}(45.22^{\circ}) = 32.75$$

Example: Find V x U. then, verify that the result is orthogonal to both V and U

U = < 3, 2, -8 >

V = < -1, 6, 7 >

note: V x U is NOT the same as U x V

$$V \times U = \begin{vmatrix} i & j & k \\ -1 & 6 & 7 \\ 3 & 2 & -8 \end{vmatrix} = \begin{vmatrix} 6 & 7 \\ 2 & -8 \end{vmatrix} i - \begin{vmatrix} -1 & 7 \\ 3 & -8 \end{vmatrix} j + \begin{vmatrix} -1 & 6 \\ 3 & 2 \end{vmatrix} k$$

note: the second group, the j component is (-)

$$(-48 - 14)i - (8 - 21)j + (-2 - 18)k$$

$$< -62, 13, -20 >$$

$< -62, 13, -20 > \cdot < 3, 2, -8 > = -186 + 26 + 160 = 0$ ✓

$< -62, 13, -20 > \cdot < -1, 6, 7 > = 62 + 78 - 140 = 0$ ✓

(since dot product equals zero, the vectors are orthogonal)

Example: Find equation of a plane that contains these points: (2, 3, -2) (3, 4, 2) (1, -1, 0)

We need the normal vector and a point... (we have 3 points to choose from)

Step 1: Find 2 vectors

A (2, 3, -2)
B (3, 4, 2) $\quad \overrightarrow{AB} \quad$ < 1, 1, 4 >

A (2, 3, -2)
C (1, -1, 0) $\quad \overrightarrow{AC} \quad$ < -1, -4, 2 >

Step 2: Use the cross product to find the normal vector
(the cross product is orthogonal to BOTH vectors)

$$\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 4 \\ -1 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ -4 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ -1 & -4 \end{vmatrix} k = 18i - 6j - 3k$$

Step 3: Plug your point into the normal vector

A(2, 3, -2) $\quad 18(2) - 6(3) - 3(-2) = 24$ $\qquad 18x - 6y - 3z = 24$

Step 4: Check the answer
(substitute each point into the equation of the plane!)

A(2, 3, -2) Obviously, this one will work: $18(2) - 6(3) - 3(-2) = 24$ ✓

B(3, 4, 2) $\quad 18(3) - 6(4) - 3(2) = 54 - 24 - 6 = 24$ ✓

C(1, -1, 0) $\quad 18(1) - 6(-1) - 3(0) = 18 + 6 - 0 = 24$ ✓

Example: Find the area of parallelogram ABCD whose vertices are

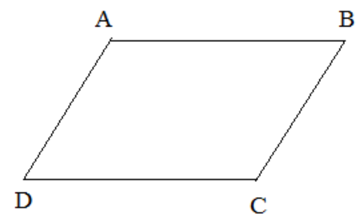
A (5, -6, 3)

B (-2, -9, 8)

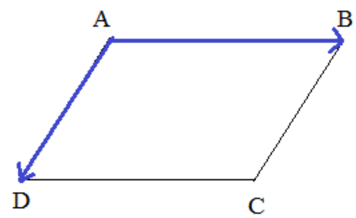
C (2, -5, 5)

D (9, -2, 0)

Step 1: Sketch a diagram



Step 2: Find 2 vectors that share a common vertex...



$\overrightarrow{AB}$ $<(-2-5), (-9-(-6)), (8-3)>$

$<-7, -3, 5>$

$\overrightarrow{AD}$ $<(9-5), (-2-(-6)), (0-3)>$

$<4, 4, -3>$

Step 3: Use Cross product

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} i & j & k \\ -7 & -3 & 5 \\ 4 & 4 & -3 \end{vmatrix} = \begin{vmatrix} -3 & 5 \\ 4 & -3 \end{vmatrix} i - \begin{vmatrix} -7 & 5 \\ 4 & -3 \end{vmatrix} j + \begin{vmatrix} -7 & -3 \\ 4 & 4 \end{vmatrix} k$$

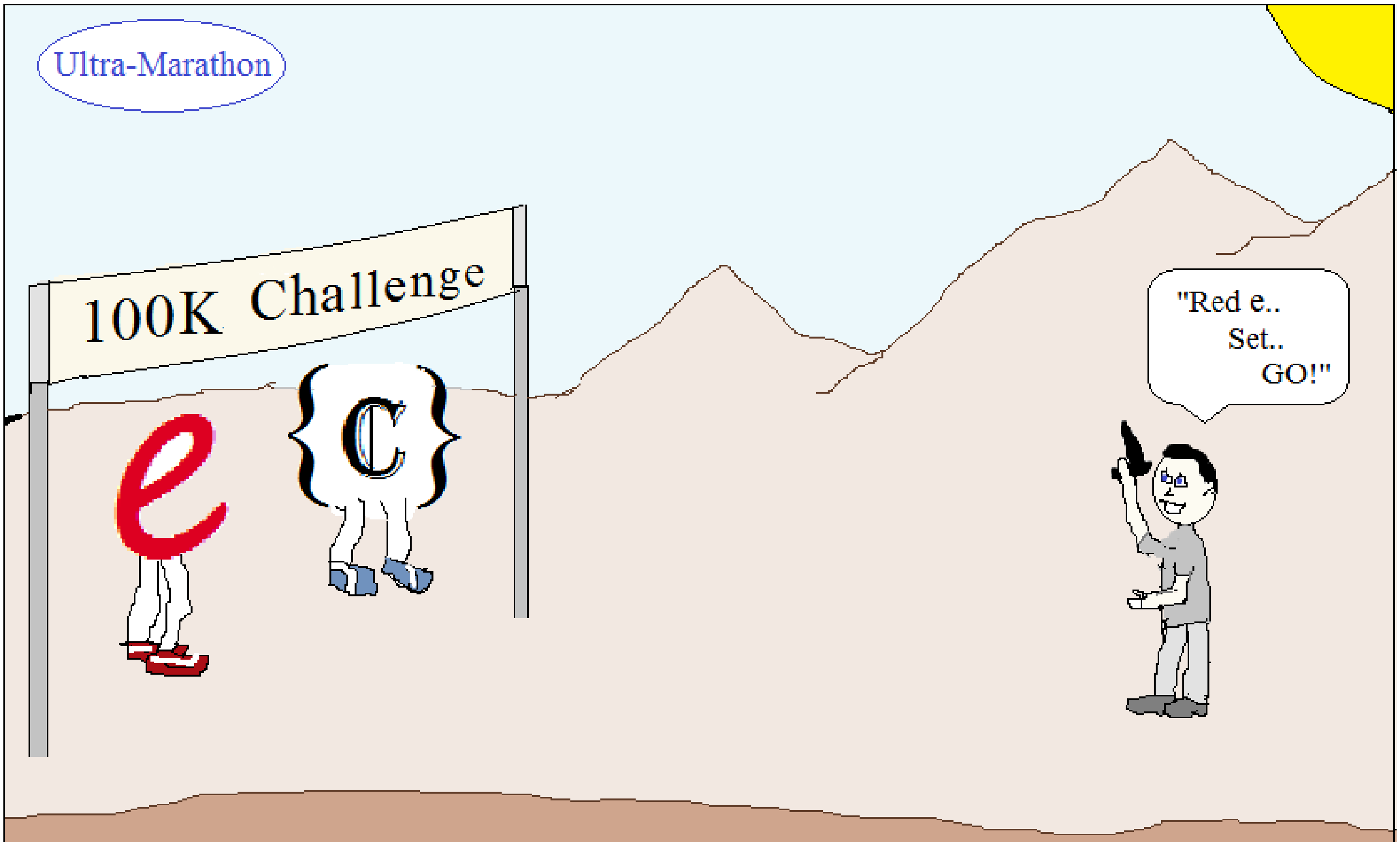
$$-11i - 1j + -16k \qquad <-11, -1, -16>$$

Step 4: Use Area formula for Parallelogram...

$$\left\| \overrightarrow{AB} \times \overrightarrow{AD} \right\| = \left\| <-11, -1, -16> \right\|$$

Area of parallelogram = $\| u \times v \|$

$$\sqrt{121 + 1 + 256} = \sqrt{378} = 3\sqrt{42} = 19.44$$



Testing the limits of endurance,
these math figures will run on and on...

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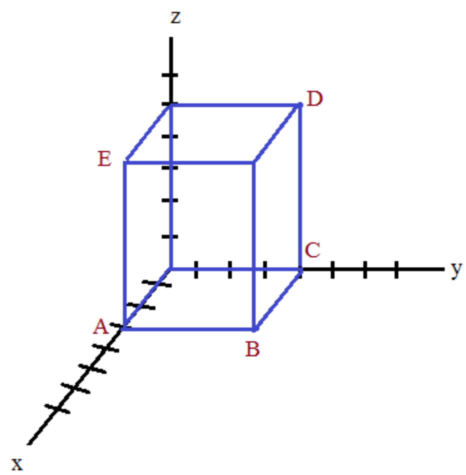
Practice Quiz-→

1) Three-Dimensional Coordinates

Three-Dimensional Space Quiz



Identify the coordinates of the following points:

A:

B:

C:

D:

E:

Determine the midpoints of:

A and B

D and C

E and C

E and B

Find the distance between:

A and B

A and E

A and D

2) Spheres

A) Find the center and radius of the following sphere:

$$x^2 + y^2 + z^2 - 8y + 2z = 8$$

Identify any 3 points that lie on the sphere:

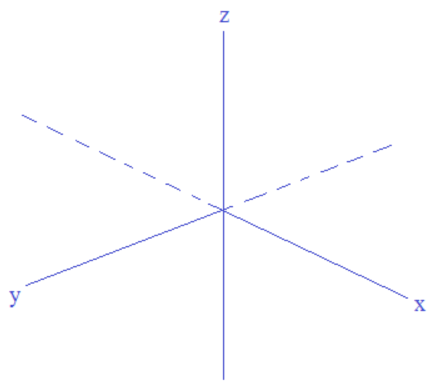
Is the origin (0, 0, 0) inside the sphere?

3) Planes

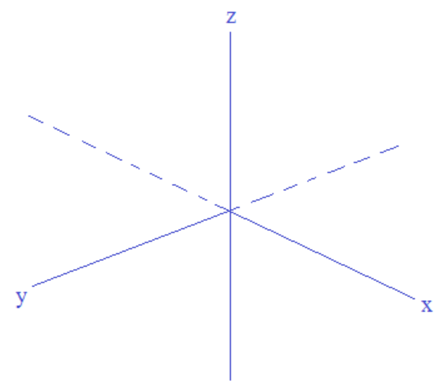
Three-Dimensional Space Quiz

A) Find the intercepts and sketch the following planes:

1) $x - 3y + 2z = 6$



2) $x + 2y + 3z = 6$



B) Find the equation of a plane that contains these points: (1, 2, 3) (0, -4, -1) (6, 1, 5)

4) Intersections and the 'trace'

A) The center of a sphere is (2, 4, 3).
If the radius is 3, describe the intersection with the

1) xy-plane

2) xz-plane

3) yz-plane

B) $(x - 9)^2 + (y + 6)^2 + (z - 11)^2 = r^2$

a) Find possible values of r that have NO trace in the xy, yz, and xz planes..

b) Find possible values of r where the sphere intersects ONLY the xz plane..

c) Describe the traces if $r = 7$

1) xz trace

2) yz trace

3) xy trace

Three-Dimensional Space Quiz

5) 3-Dimensional Vectors

A) What is $u \cdot v$?

$u = 3i - 7j$

$v = 8j + 9k$

B) What is $v \times u$?

C) Vector w lies on the yz-plane.
It has a magnitude of 6 and makes a 45-degree angle with the negative y-axis.

What is vector w?

D) Vector N lies on the xz-plane. The magnitude is 10.
If it's a 60-degree angle from the z-axis, what is the vector?

6) Vectors and Parametric Concepts

Three-Dimensional Space Quiz

A) Write the parametric equation of a line that crosses the z-axis at $z = 6$ and crosses the xy-plane when $x = 2$ and $y = 5$.

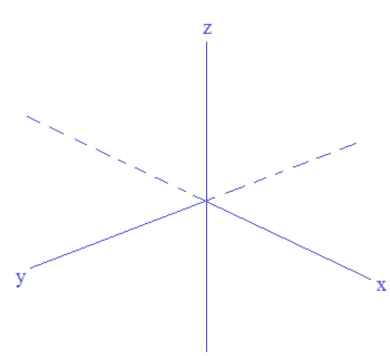
B) Are the vectors $< 2, 3 >$ and $< -4, -9 >$ parallel, perpendicular, or neither?

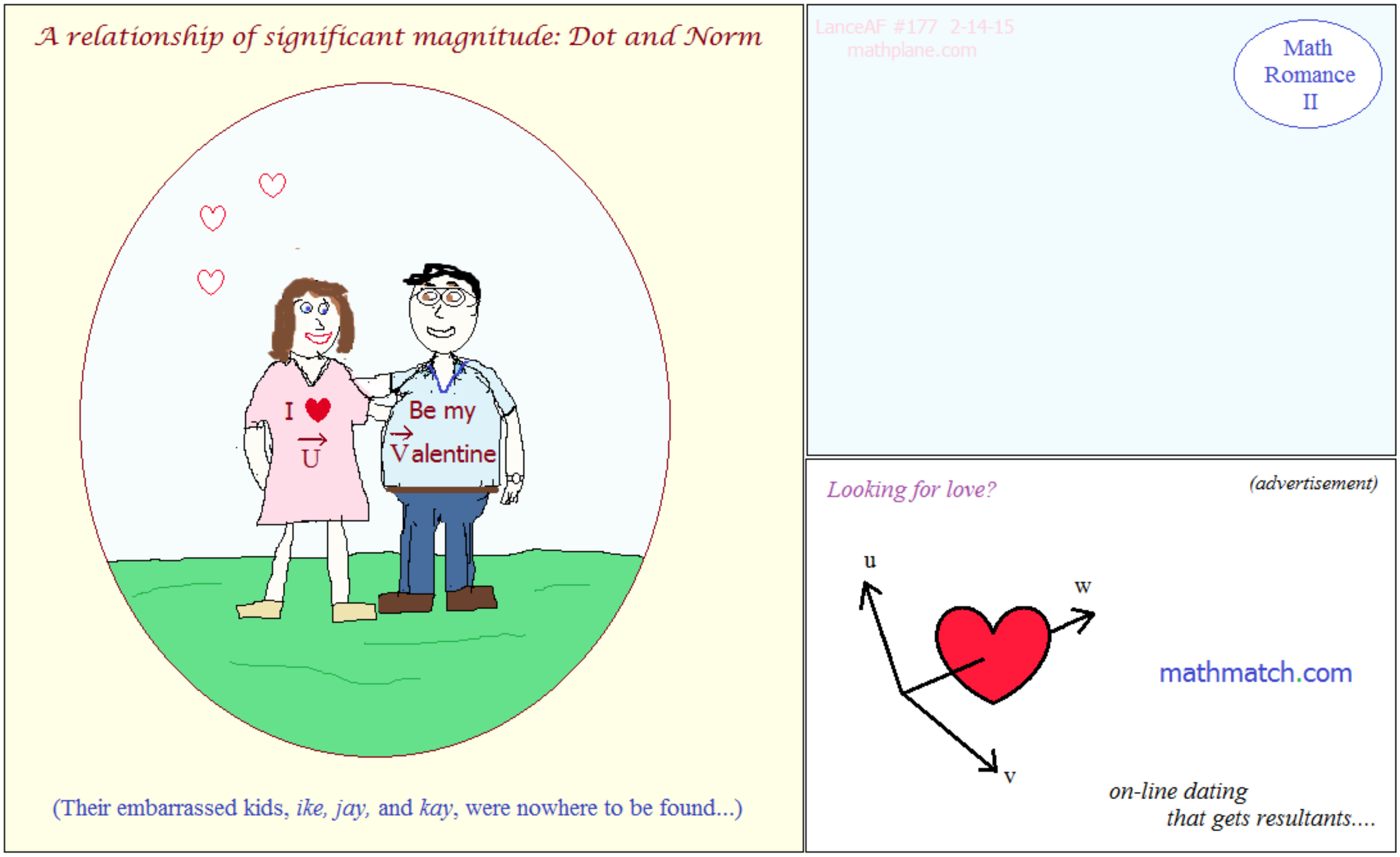
C) Find *2 unit vectors* that are perpendicular to $j + 3k$ amd $i - 2j + 4k$.

D) What is the angle between vectors $\vec{v}$ and $\vec{w}$?

$\vec{v} = < 1, 2, -5 >$

$\vec{w} = < 0, 3, 3 >$





A relationship of significant magnitude: Dot and Norm
I ♥ $\vec{U}$
Be my $\vec{V}$alentine
(Their embarrassed kids, *ike, jay,* and *kay*, were nowhere to be found...)
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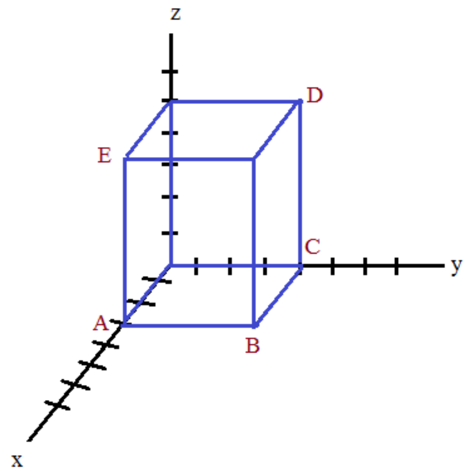
that gets resultants....

Solutions -→

1) Three-Dimensional Coordinates

SOLUTIONS

Three-Dimensional Space Quiz



Identify the coordinates of the following points:

A: (3, 0, 0)

B: (3, 4, 0)

C: (0, 4, 0)

D: (0, 4, 5)

E: (3, 0, 5)

Determine the midpoints of:

A and B $x = \frac{3+3}{2}$ $y = \frac{0+4}{2}$ $z = \frac{0+0}{2}$ (3, 2, 0)

D and C $x = \frac{0+0}{2}$ $y = \frac{4+4}{2}$ $z = \frac{5+0}{2}$ (0, 4, 2.5)

E and C (1.5, 2, 2.5) or (3/2, 2, 5/2)

E and B (3, 2, 2.5)

Find the distance between:

A and B $\sqrt{(3-3)^2 + (4-0)^2 + (0-0)^2} = 4$

A and E $\sqrt{(3-3)^2 + (0-0)^2 + (5-0)^2} = 5$

A and D $\sqrt{(3-0)^2 + (0-4)^2 + (0-5)^2} = \sqrt{50} = 5\sqrt{2}$

NOTE: the similarity of the formulas for circles in a plane and spheres in space!

2) Spheres

A) Find the center and radius of the following sphere:

$$x^2 + y^2 + z^2 - 8y + 2z = 8$$

Complete the square to put general equation into standard form...

$$x^2 + y^2 - 8y + 16 + z^2 + 2z + 1 = 8 + 16 + 1$$

$$x^2 + (y-4)^2 + (z+1)^2 = 25$$

The center is $(0, 4, -1)$ and the radius is 5

Identify any 3 points that lie on the sphere:

Three easy points to find are any intercepts:

If y and z are 0, $x^2 + (0)^2 + (0)^2 - 8(0) + 2(0) = 8$

$(\sqrt{8},\ 0,\ 0)$ $(-\sqrt{8},\ 0,\ 0)$

If x and y are 0, $(0)^2 + (0)^2 + z^2 - 8(0) + 2z = 8$

$z^2 + 2z = 8$

$(z+4)(z-2) = 0$

(0, 0, 2) (0, 0, -4)

Is the origin (0, 0, 0) inside the sphere?

Find radius --- that is the length from the center to the sphere...
Then, find distance from origin to point on the sphere..
If distance < radius, then origin is inside the sphere!

distance between origin and center of the sphere:

$$\sqrt{(0-0)^2 + (4-0)^2 + (-1-0)^2} = \sqrt{17}$$

radius = 5

The origin is inside the sphere!

3) Planes

SOLUTIONS

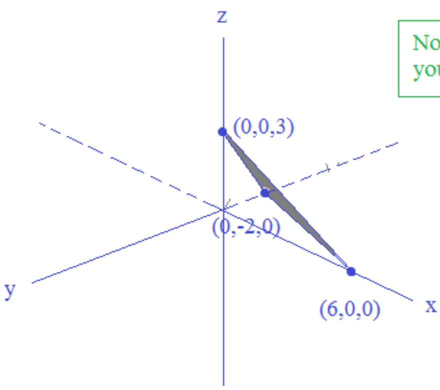
Three-Dimensional Space Quiz

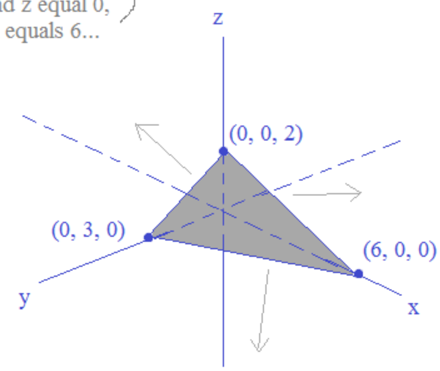
A) Find the intercepts and sketch the following planes:

	x-intercept	y-intercept	z-intercept
1) $x - 3y + 2z = 6$	(6, 0, 0)	(0, -2, 0)	(0, 0, 3)
2) $x + 2y + 3z = 6$	(6, 0, 0)	(0, 3, 0)	(0, 0, 2)

If y and z equal 0, then x equals 6...

Note: Once you have 3 points, you can sketch the plane!





B) Find the equation of a plane that contains these points: A (1, 2, 3) B (0, -4, -1) C (6, 1, 5)

We need the normal vector and a point... (we have 3 points to choose from)

Step 1: Find 2 vectors

$\overrightarrow{AB} \quad \langle (0-1), (-4-2), (-1-3) \rangle = \langle -1, -6, -4 \rangle$

$\overrightarrow{AC} \quad \langle (6-1), (1-2), (5-3) \rangle = \langle 5, -1, 2 \rangle$

Step 2: Use the cross product to find the normal vector
(the cross product is orthogonal to BOTH vectors)

$$\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & -6 & -4 \\ 5 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -6 & -4 \\ -1 & 2 \end{vmatrix} i - \begin{vmatrix} -1 & -4 \\ 5 & 2 \end{vmatrix} j + \begin{vmatrix} -1 & -6 \\ 5 & -1 \end{vmatrix} k = -16i - 18j + 31k$$

Step 3: Plug your point into the normal vector

$A(1, 2, 3) \qquad -16(1) - 18(2) + 31(3) = 41$

$$\boxed{-16x - 18y + 31z = 41}$$

To check: Does (0, -4, -1) lie in the plane?

$-16(0) - 18(-4) + 31(-1) = 41$

$0 - (-72) - 31 = 41$

$41 = 41$ ✓

Does (6, 1, 5) lie in the plane?

$-16(6) - 18(1) + 31(5) = 41$

$-96 - 18 + 155 = 41$

$41 = 41$ ✓

4) Intersections and the 'trace'

SOLUTIONS

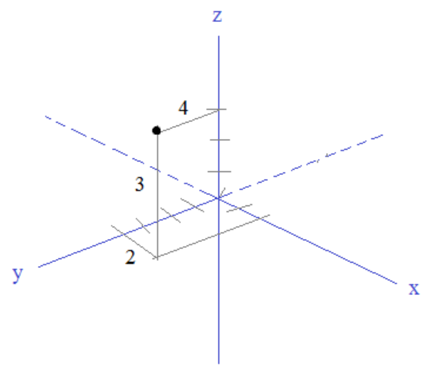
Three-Dimensional Space Quiz

A) The center of a sphere is (2, 4, 3).
If the radius is 3, describe the intersection with the

1) xy-plane point (the sphere will touch (2, 4, 0) in the xy-plane)

2) xz-plane none since the center is 4 units away from the xz-plane, a radius of 3 units will not reach....

3) yz-plane circle



B) $(x - 9)^2 + (y + 6)^2 + (z - 11)^2 = r^2$

a) Find possible values of r that have NO trace in the xy, yz, and xz planes..

Sphere must not intersect either plane...
Therefore, its radius must be less than all x, y, and z points...

$$\boxed{0 < r < 6}$$

(also, r must be positive length)

b) Find possible values of r where the sphere intersects ONLY the xz plane..

$$\boxed{6 \leq r < 9}$$

must be long enough to intersect xz plane... BUT must be short enough not to intersect the other planes...

c) Describe the traces if $r = 7$

1) xz trace

for xz trace, $y = 0$

$$(x - 9)^2 + (0 + 6)^2 + (z - 11)^2 = 7^2$$

$$(x - 9)^2 + 36 + (z - 11)^2 = 49$$

$$(x - 9)^2 + (z - 11)^2 = 13$$

$\boxed{\text{a circle with center } (9, 0, 11) \text{ and radius } \sqrt{13}}$

2) yz trace

$\boxed{\text{NONE}}$ (if the radius is 7, then the sphere won't intersect the yz plane)

$$(0 - 9)^2 + (y + 6)^2 + (z - 11)^2 = 7^2$$

$(y + 6)^2 + (z - 11)^2 = -32$ radius cannot be negative

3) xy trace

$\boxed{\text{NONE}}$ (if the radius is 7, the sphere will not reach the xy plane)

$(x - 9)^2 + (y + 6)^2 + (0 - 11)^2 = 7^2$ (notice, the 'z distance' is greater than the radius)

5) 3-Dimensional Vectors

SOLUTIONS

Three-Dimensional Space Quiz

$u = 3i - 7j$

$v = 8j + 9k$

A) What is $u \cdot v$?

$< 3, -7, 0 >$

$< 0, 8, 9 >$

$3 \cdot 0,\ -7 \cdot 8,\ 0 \cdot 9 \quad = 0 - 56 + 0 = \boxed{-56}$

B) What is $v \times u$?

note: v x u is not
equal to u x v!

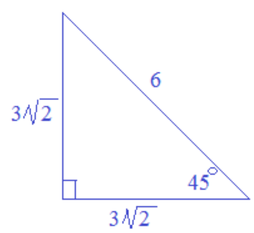
$$\begin{vmatrix} i & j & k \\ 0 & 8 & 9 \\ 3 & -7 & 0 \end{vmatrix} = \begin{vmatrix} 8 & 9 \\ -7 & 0 \end{vmatrix} i - \begin{vmatrix} 0 & 9 \\ 3 & 0 \end{vmatrix} j + \begin{vmatrix} 0 & 8 \\ 3 & -7 \end{vmatrix} k = \boxed{63i + 27j - 24k}$$

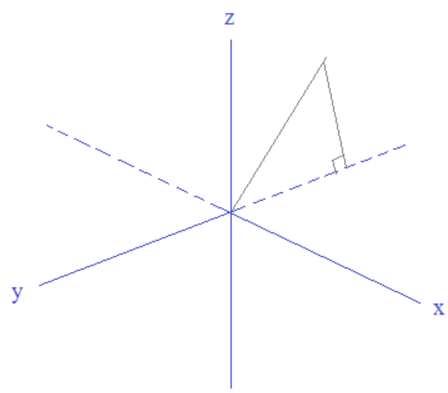
negative

C) Vector w lies on the yz-plane.
It has a magnitude of 6 and makes a 45-degree angle with the negative y-axis.

What is vector w?

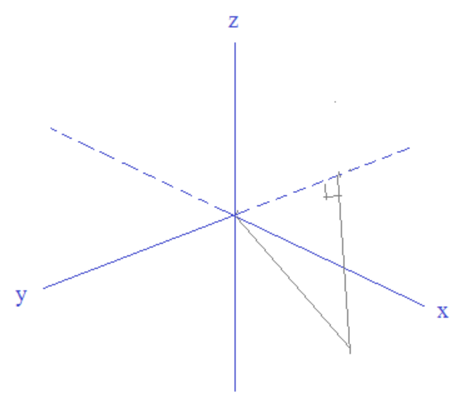
$\boxed{< 0, -3\sqrt{2}, 3\sqrt{2} >}$



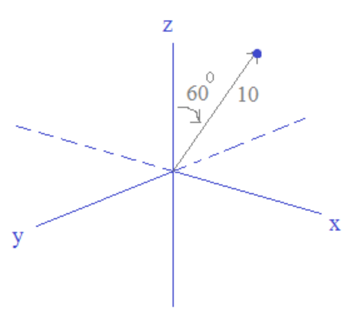


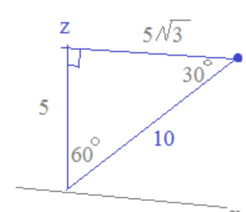
or,

$\boxed{< 0, -3\sqrt{2}, -3\sqrt{2} >}$



D) Vector N lies on the xz-plane. The magnitude is 10.
If it's a 60-degree angle from the z-axis, what is the vector?





$\boxed{< 5\sqrt{3},\ 0,\ 5 >}$ or $\boxed{< -5\sqrt{3},\ 0,\ 5 >}$

6) Vectors and Parametric Concepts

SOLUTIONS

Three-Dimensional Space Quiz

A) Write the parametric equation of a line that crosses the z-axis at z = 6 and crosses the xy-plane when x = 2 and y = 5.

Since the line crosses the z-axis at z = 6, we know (0, 0, 6) is a point on the line...

And, since it crosses the xy-plane when x = 2 and y = 5, we know (2, 5, 0) is a point on the line..

The direction of the line: $(2, 5, 0) - (0, 0, 6) = \langle 2, 5, -6 \rangle$
(slope)

Then, using a point on the line, (2, 5, 0), the parametric equation is

$$\boxed{\begin{aligned} x &= 2 + 2t \\ y &= 5 + 5t \\ z &= -6t \end{aligned}}$$

Quick check: if t = 0, then (x, y, z) = (2, 5, 0)

and, if t = -1, then (x, y, z) = (0, 0, 6)

Both points fit in the parametric equation, and therefore lie on the line..

B) Are the vectors $\langle 2, 3 \rangle$ and $\langle -4, -9 \rangle$ parallel, perpendicular, or neither?

If the vectors are perpendicular, then the dot product equals zero..

Dot product: $(2 \times -4)(3 \times -9) = -35$ Not perpendicular

If the vectors have the same "slope", direction, then they are parallel..
(i.e. do they differ by a scalar value?)

$\frac{3}{2} \neq \frac{-9}{-4}$ Not parallel

There is no value of n where

$n \langle 2, 3 \rangle = \langle -4, -9 \rangle$

$\boxed{\text{The vectors are neither....}}$

C) Find *2 unit vectors* that are perpendicular to $j + 3k$ amd $i - 2j + 4k$.

To find perpendicular vector(s) or normal(s), use the cross product...

$$\begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 1 & -2 & 4 \end{vmatrix} = i \begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} - j \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} + k \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 10i + 3j - 1k$$

To check: (the dot product of perpendicular/orthogonal vectors is 0)

$\langle 0, 1, 3 \rangle \cdot \langle 10, 3, -1 \rangle = 0 + 3 - 3 = 0$

$\langle 1, -2, 4 \rangle \cdot \langle 10, 3, -1 \rangle = 10 - 6 - 4 = 0$

Then, to find the unit vector...

the length of $\langle 10, 3, -1 \rangle = \sqrt{10^2 + 3^2 + (-1)^2} = \sqrt{110}$

so, unit vector is

$$\frac{1}{\sqrt{110}}\left(10i + 3j - 1k\right) = \boxed{\frac{\sqrt{110}}{11} i + \frac{3\sqrt{110}}{110} j - \frac{\sqrt{110}}{110} k}$$

Note: the unit vector has a length of 1

then, another unit vector that is perpendicular
has the same length but is going the opposite direction....

$$\boxed{-\frac{\sqrt{110}}{11} i - \frac{3\sqrt{110}}{110} j + \frac{\sqrt{110}}{110} k}$$

D) What is the angle between vectors $\vec{v}$ and $\vec{w}$?

$\vec{v} = \langle 1, 2, -5 \rangle$

$\vec{w} = \langle 0, 3, 3 \rangle$

$v \cdot w = (1 \times 0) + (2 \times 3) + (-5 \times 3) = -9$

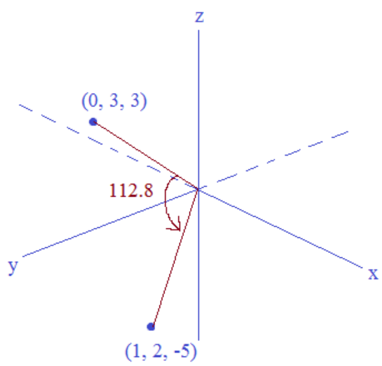
$|v| = \sqrt{1^2 + 2^2 + (-5)^2} = \sqrt{30}$

$|w| = \sqrt{0^2 + 3^2 + 3^2} = 3\sqrt{2}$

$$\cos\theta = \frac{v \cdot w}{|v||w|}$$

$$\cos\theta = \frac{-9}{\sqrt{30} \cdot 3\sqrt{2}} = \frac{-9}{6\sqrt{15}}$$

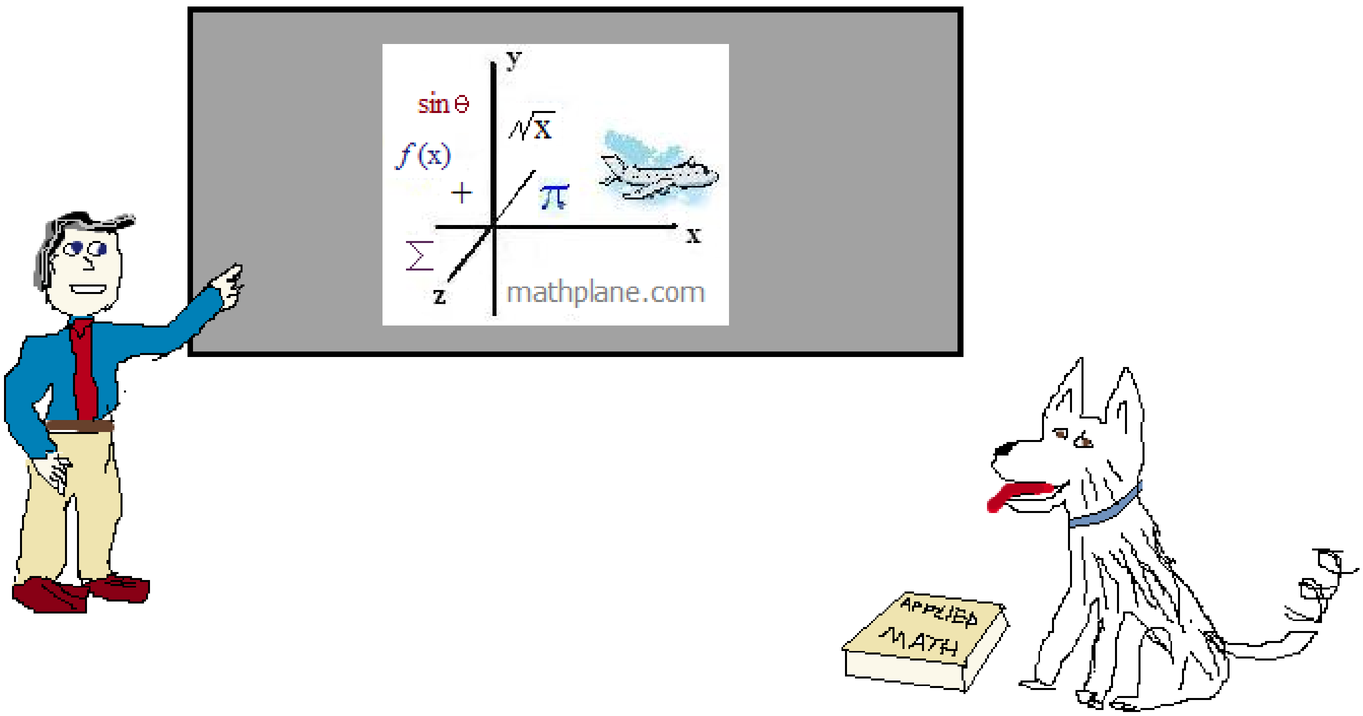
$$\boxed{\theta = 112.8^\circ}$$



Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

Good luck.



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