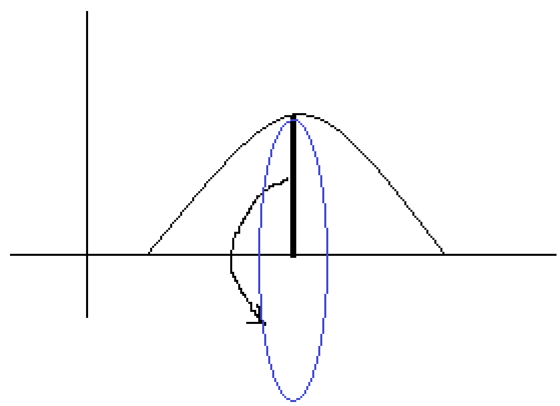
Calculus: Integrals, Area, and Volume

Notes, Examples, Formulas, and Practice Test (with solutions)



Topics include definite integrals, area, “disc method”, volume of a solid from rotation, and more.

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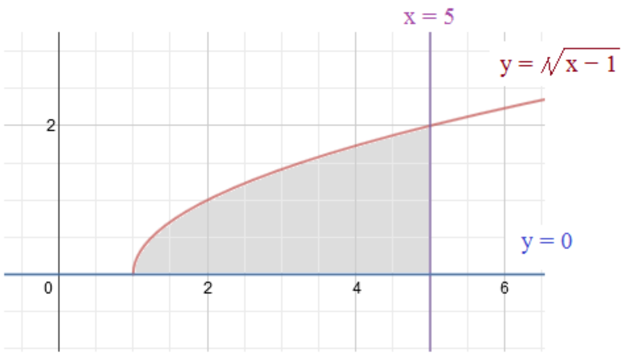
We've learned that the area under a curve can be found by evaluating a definite integral.

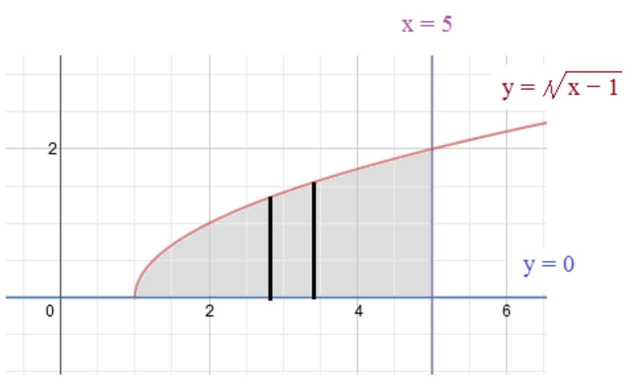
Example: Find the area in the region bounded by $x = 5$
$y = 0$
$y = \sqrt{x-1}$



Area *under* the curve:
(Shaded Area)

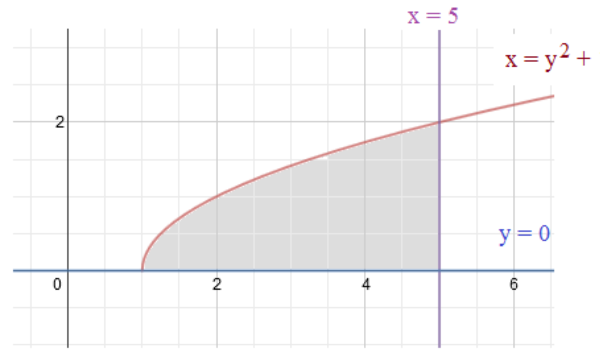
$$\int_0^5 \sqrt{x-1}\;dx - \int_0^5 0\,dx$$

$$\frac{2}{3}(x-1)^{\frac{3}{2}}\Big|_0^5 - 0 = \frac{16}{3}$$



The area was found by taking vertical partitions.

$$\text{Area} = \int_a^b f(x)\;dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

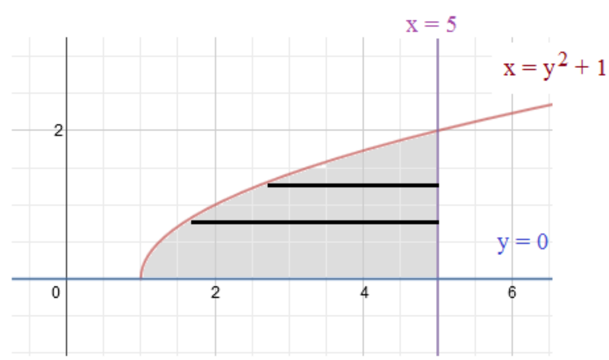


Area *right* of the curve:
(Shaded Area)

$$\int_0^2 5\,dy - \int_0^2 y^2 + 1\;dy$$

$$5y\Big|_0^2 - \left(\frac{y^3}{3} + y\Big|_0^2\right)$$

$$10 - 0 - \left(\frac{8}{3} + 2 - 0 - 0\right) = \frac{16}{3}$$



The area was found by taking horizontal partitions.

$$\text{Area} = \int_c^d f(y)\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(y_i)\,\Delta y$$

Area = (length)(width)

Volume = (length)(width)(depth) = Area(depth)

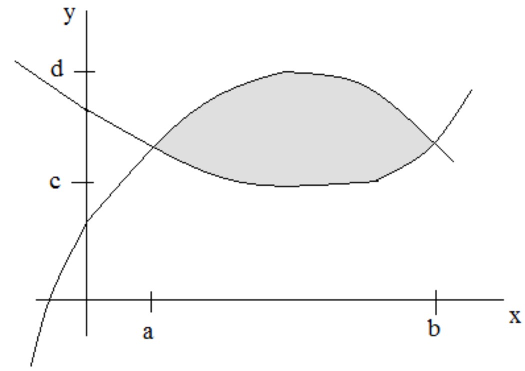
Since

$$\text{Area} = \int_a^b f(x)\,dx \quad \text{or} \quad \int_c^d f(y)\,dy$$

then,

$$\text{Volume} = \int_a^b A(x)\,dx \qquad \text{Volume} = \int_c^d A(y)\,dy$$

where A(x) or A(y) is a cross-section area of a solid



Area: 2-dimensional

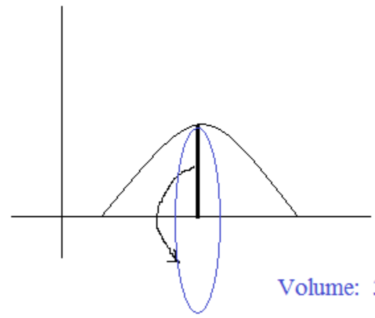
"Disc Method"

If we rotate each segment, we get a sequence of circles...

(Circle) $\text{Area} = \pi\,(\text{radius})^2$

So, the volume of the solid will be the sum of all the circles!

(The *radius* (length) *of each circle* is determined by the output of the function)

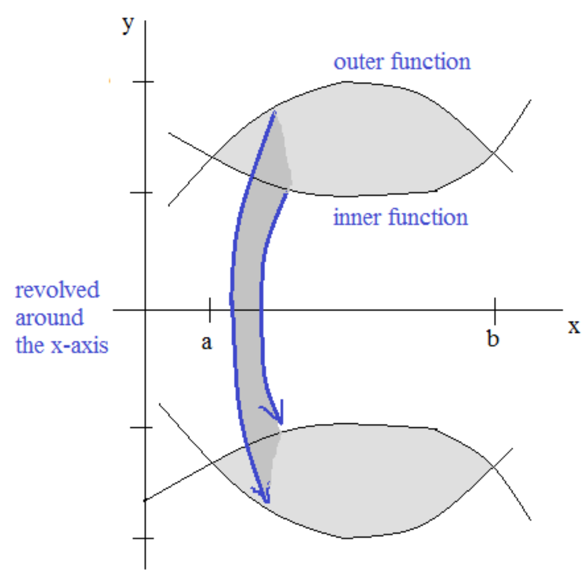


Volume: 3-dimensional

$$\text{Volume} = \int_a^b \pi\,(\text{function})^2\,dx$$

And, if you get a "ring",

$$\text{Volume} = \int_a^b \pi\,(\textit{outer}\text{ function})^2\,dx - \int_a^b \pi\,(\textit{inner}\text{ function})^2\,dx$$



Volume of a cone from rotated line segment

Example: If a portion of the line $y = \frac{1}{2}x$ lying in Quadrant I is rotated around the x-axis, a solid cone is generated.
Find the volume of the cone extending from x = 0 to x = 6.

The length (height) of the cone will extend from 0 to 6

$$\int_0^6 \quad dx$$

The area from the segments will be from the function $\frac{1}{2}x$

(These are the 'radii')

$$\int_0^6 \frac{1}{2}x \, dx$$

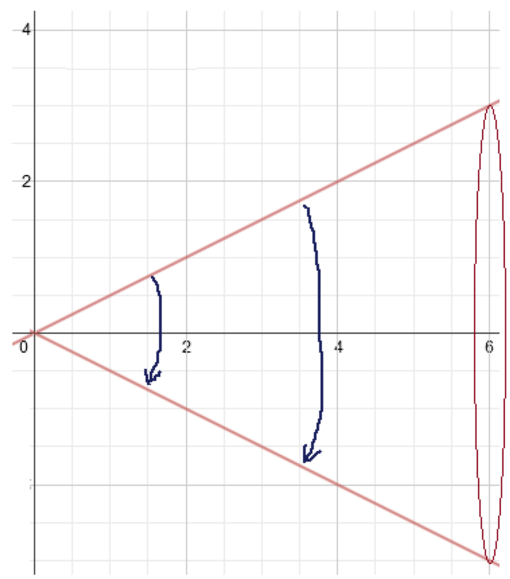
And, the volume of the solid from rotation (revolution) will be from the total area of the segments (radii)

(These are the round 'discs')

$$\int_0^6 \pi\left(\frac{1}{2}x\right)^2 dx$$

$$\pi \int_0^6 \frac{x^2}{4} dx$$

$$\pi \left.\frac{x^3}{12}\right|_0^6 = 18\pi$$



Quick Check: Volume of a cone: $\frac{1}{3}\pi\,(\text{radius})^2\,(\text{height})$

This (sideways) cone: Radius = 3
Height = 6

Volume = $\frac{1}{3}\pi\,(3)^2\,(6) = 18\pi$

Calculus and Area Rotation

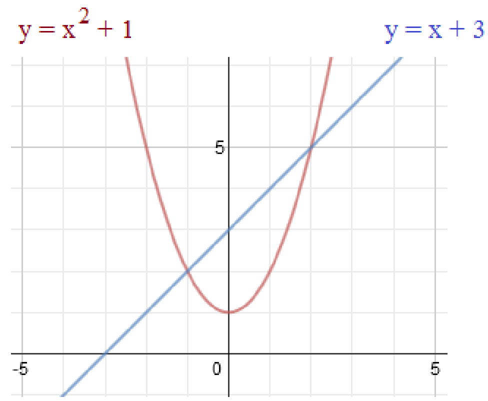
Find the volume of the figure
where the cross-section area is bounded by

$y = x^2 + 1$

$y = x + 3$

and revolved around the x-axis.

Step 1: Draw a sketch



Step 2: Determine the span of the integral

$y = x^2 + 1$

$y = x + 3$

$x^2 + 1 = x + 3$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0 \qquad x = -1, 2$

The boundaries of the area are [-1, 2]

Step 3: Write the integral(s)

The bounded area will revolve around the x-axis

$$\int_{-1}^{2} \pi (x+3)^2 \, dx \; - \; \int_{-1}^{2} \pi (x^2+1)^2 \, dx$$

Area under the line from -1 to 2

Area under the curve from -1 to 2

Step 4: Evaluate the integrals

$$\int_{-1}^{2} \pi (x+3)^2 \, dx \; - \; \int_{-1}^{2} \pi (x^2+1)^2 \, dx$$

$$\pi \int_{-1}^{2} x^2 + 6x + 9 \, dx \; - \; \pi \int_{-1}^{2} x^4 + 2x^2 + 1 \, dx$$

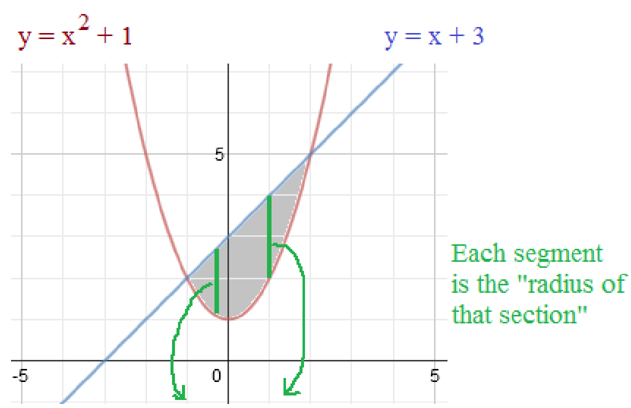
$$\pi \left(\frac{x^3}{3} + 3x^2 + 9x \Bigg|_{-1}^{2} \right) - \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \Bigg|_{-1}^{2} \right)$$

$$\pi \left(32\tfrac{2}{3} - (-6\tfrac{1}{3}) \right) - \pi \left(\frac{206}{15} - \left(\frac{-28}{15}\right) \right)$$

$$39\pi - \frac{234}{15}\pi =$$

$$39\pi - \frac{78}{5}\pi$$

$$\boxed{\frac{117}{5}\pi}$$



NOTE: Volume = $\int_{a}^{b} A(x) \, dx$

$\int \pi r^2$ "where r is the function that is being revolved"

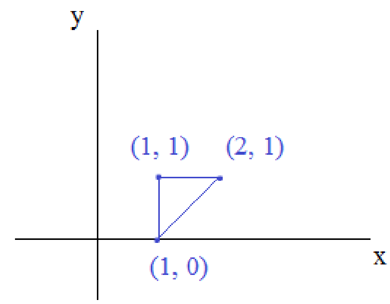
Calculus and Volume (of solids from rotation)

A triangle with vertices (1, 0) (2, 1) and (1, 1)

is rotated around the *y-axis*.

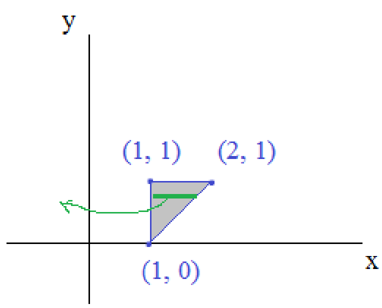
What is the volume of the solid?

Step 1: Draw a sketch



Step 2: Determine the boundaries of the integral

Since the rotation is *around the y-axis*,
the boundaries will be between y = 0 and y = 1



Step 3: Write the integrals

The line connecting (1, 0) and (2, 1)
is $y = x - 1$

or, $x = y + 1$

And, the line connecting (1, 0) and (1, 1) is $x = 1$

$$\int_0^1 \pi (y+1)^2 \, dy \; - \; \int_0^1 \pi (1)^2 \, dy$$

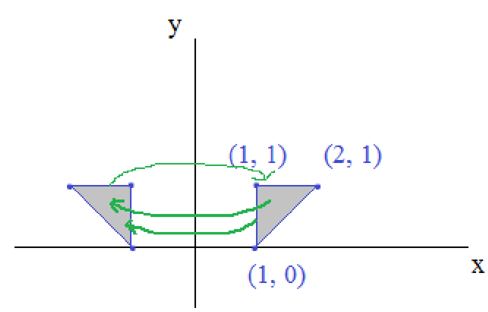
Step 4: Evaluate integrals to find volume

$$\int_0^1 \pi (y+1)^2 \, dy \; - \; \int_0^1 \pi (1)^2 \, dy$$

$$\int_0^1 \pi (y^2 + 2y + 1) \, dy \; - \; \int_0^1 \pi \, dy$$

$$\pi \left(\frac{y^3}{3} + y^2 + y \Bigg|_0^1 \right) - \pi \left(y \Bigg|_0^1 \right)$$

$$2\tfrac{1}{3}\pi - \pi = \boxed{\tfrac{4}{3}\pi}$$



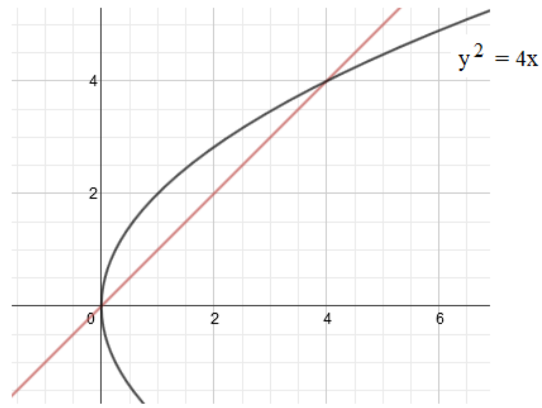
NOTE: Volume = $\int_a^b A(y) \, dy$ ("integral of Area")

$\int \pi r^2$ "where r is the function that is being rotated"

Example: Find the volume of the solid formed by the region bounded by

$$y = x$$
$$y^2 = 4x$$

a) rotated around the x-axis
b) rotated around the y-axis
c) rotated around $x = 4$



a) Since the region is rotated around the x-axis, we'll use 'vertical partitions'.
The left boundary will be $x = 0$
and the right boundary will be $x = 4$

The upper boundary will be $y^2 = 4x$ → $y = 2\sqrt{x}$

The 2-dimensional area of the region would be the integral

$$\int_0^4 2\sqrt{x}\,dx \;-\; \int_0^4 x\,dx \;=$$

area from curve to x-axis — *area from line to x-axis* — *area of enclosed region*

But, the volume adds another dimension... Each segment in the area is rotated to form a disc (circle)
(and, the segments are the radii of all the discs in the solid!)

$$\int_0^4 \pi(2\sqrt{x})^2\,dx \;-\; \int_0^4 \pi(x)^2\,dx$$

Area of circle = $\pi\,(\text{radius})^2$

$$\pi\int_0^4 4x - x^2\,dx$$

$$\pi\left(2x^2 - \frac{x^3}{3}\right)\Bigg|_0^4 = \left(32 - \frac{64}{3}\right)\pi = 10\frac{2}{3}$$

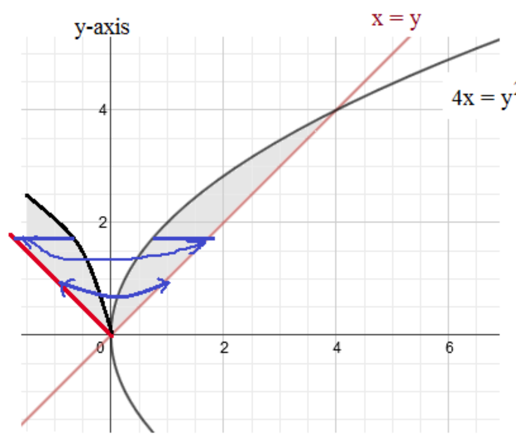
Volume = $\int_a^b \pi\,(\text{function})^2\,dx$ ('sum of vertical discs')

b) Since this is rotated around the y-axis, we'll use 'horizontal partitions'
The lower boundary will be $y = 0$.
And, the upper boundary will be $y = 4$.

The right boundary ('outer radius') will be $x = y$
And, the left boundary ('inner radius') will be $x = \frac{y^2}{4}$ ← $y^2 = 4x$

Volume = $\int_c^d A(y)\,dy$

where $A(y)$ is a cross-section area of a solid



$$\int_0^4 \pi(y)^2\,dy \;-\; \int_0^4 \pi\left(\frac{y^2}{4}\right)^2 dy$$

volume of line revolved around y-axis (forms a cone) — *volume of curve revolved around y-axis (forms a 'funnel')*

$$\pi \int_0^4 y^2\, dy \; - \; \pi \int_0^4 \frac{y^4}{16}\, dy$$

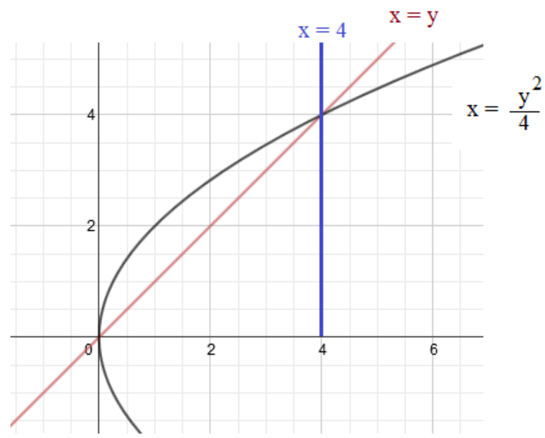
volume of outer part (cone) *volume of inner part (funnel)*

$$\pi \left. \frac{y^3}{3} \right|_0^4 - \pi \left. \frac{y^5}{80} \right|_0^4 = \frac{64}{3}\pi - \frac{64}{5}\pi = \boxed{\frac{128}{15}\pi}$$

c) In this case, the region is rotated around x = 4 (instead of an axis)
We'll use 'horizontal partitions' (*dy*) from y = 0 to y = 4

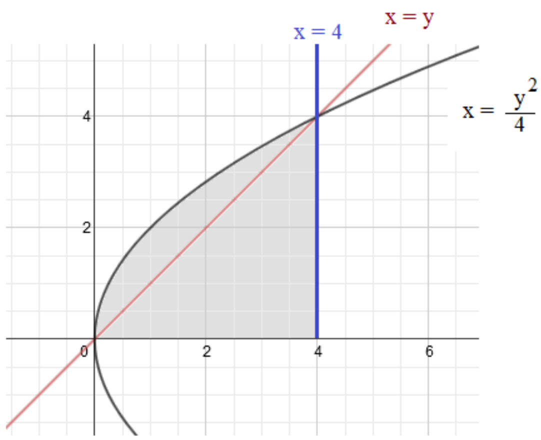
The volume integrals are:

$$V = \int_0^4 \pi \left(4 - \frac{y^2}{4}\right)^2 dy \; - \; \int_0^4 \pi\,(4 - y)^2\, dy$$

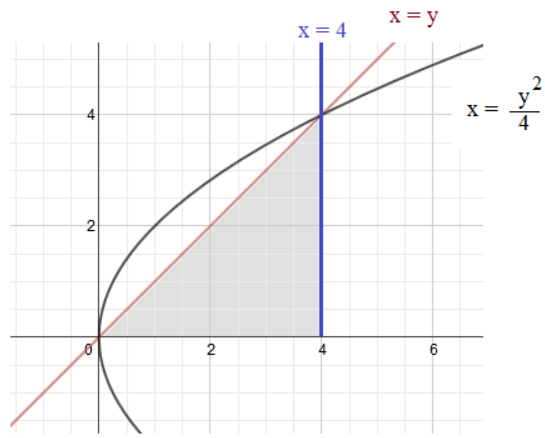


Observe where the area functions came from:
(the difference is the bounded region!)

The shaded area is $\int_0^4 4 - \frac{y^2}{4}\, dy$



The shaded area is $\int_0^4 4 - y\; dy$



$$\text{Volume} = \int_0^4 \pi \left(4 - \frac{y^2}{4}\right)^2 dy \; - \; \int_0^4 \pi\,(4 - y)^2\, dy$$

$$\pi \int_0^4 16 - 2y^2 + \frac{y^4}{16}\; dy \; - \; \pi \int_0^4 16 - 8y + y^2\; dy$$

$$\text{Volume} = \int_c^d \pi\,(\text{function})^2\, dy$$

(sum of the horizontal discs)

$$\pi \int_0^4 \frac{y^4}{16} - 3y^2 + 8y\; dy$$

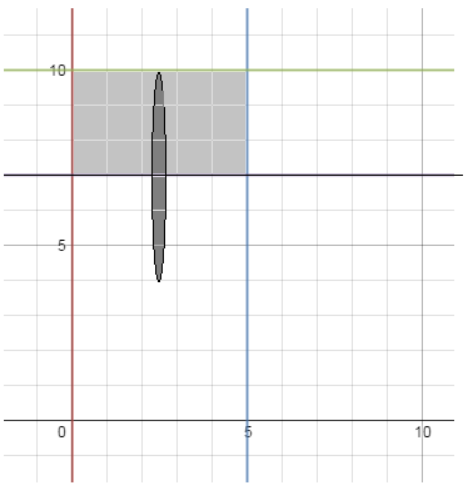
$$\pi \left(\frac{y^5}{80} - y^3 + 4y^2 \Big|_0^4 \right) = (12.8 + 64 + 64)\pi = \boxed{12.8\pi}$$

Example: The area R is between the y-axis, x = 5, y = 10, and y = 7.
Find the volume of the solid obtained by revolving R around the x-axis.

WRONG

$$\int_0^5 \pi(10-7)^2\,dx$$

Each partition has a radius of 3...

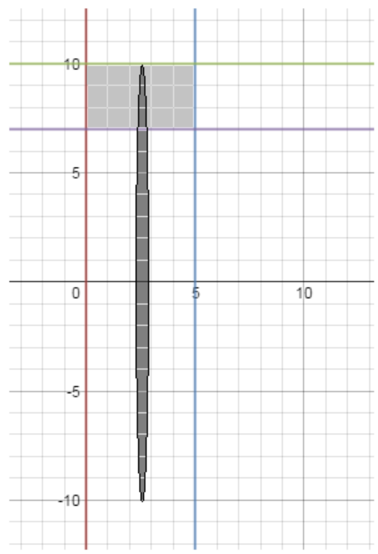


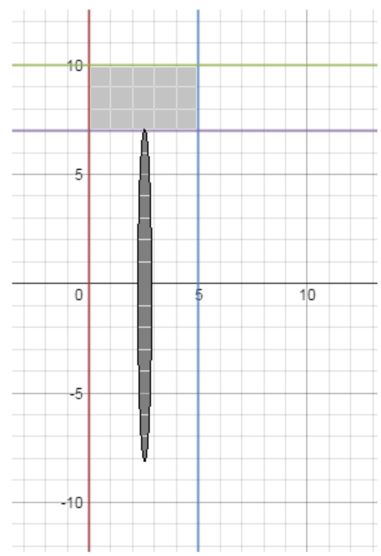
Volume = 45 π

CORRECT

$$\int_0^5 \pi(10)^2\,dx - \int_0^5 \pi(7)^2\,dx$$

Each partition has radius 10.... — Each inner partition has radius 7....

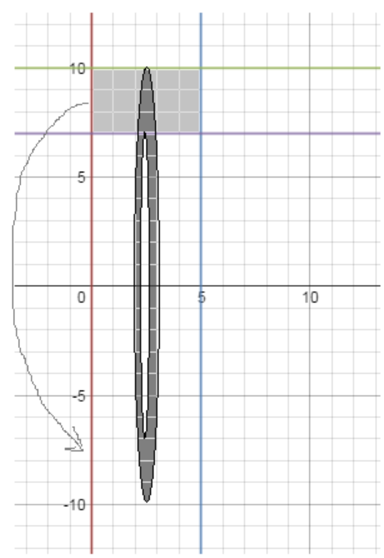


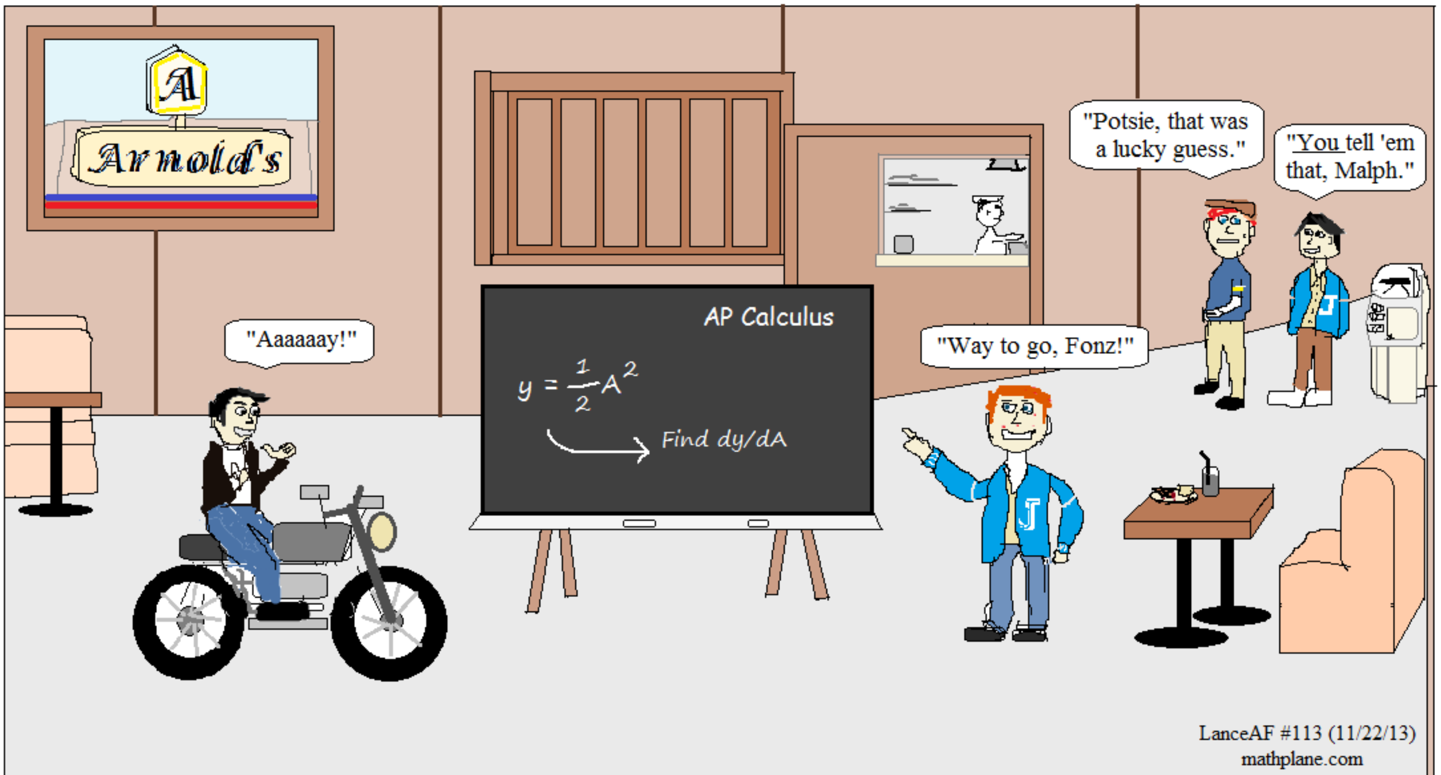


Each partition of the area revolved around the x-axis creates a ring with a hollow center..

Volume = 500 π − 245 π

= 255 π





Happy Days

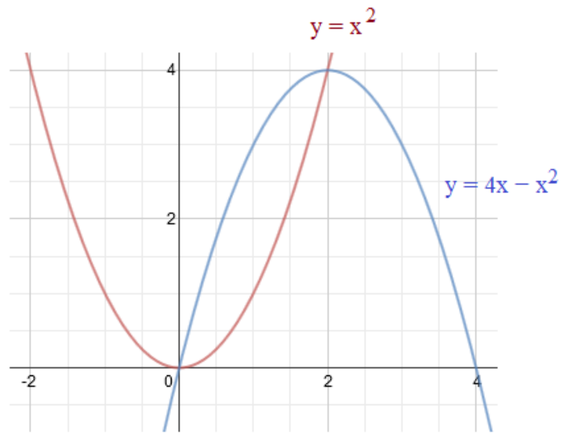
Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

Practice Test -→

Volume of Solids Practice Test

1) Find the volume of the solid formed by revolving the region bounded by

$y = x^2$ and $y = 4x - x^2$



a) about the x-axis

b) about the line $y = 6$

2) Given the area bounded by $y = \sqrt{x}$

$y = 2$

$x = 0$

Find the volume of the solid from rotation

a) about the x-axis

b) about the y-axis

c) around $y = 2$

Volume of Solids Practice Test

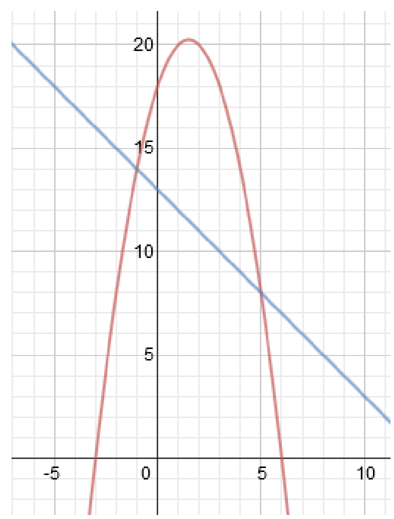
3) The left half of the ellipse $9x^2 + 25y^2 = 225$ is revolved around the y-axis to form a spheroid. Find its volume.

4) Find the volume of the solid formed by the region bounded by

$$y = -x^2 + 3x + 18$$

$$x + y = 13$$

and revolved around the x-axis



5) Find the volume of the solid when the region R bounded by

$y = x^2 + 1$ and $y = x + 3$

is revolved around the line $y = 5$

6) The region R is bordered by the y-axis, $y = 4$, and the function $y = x^2$.
Determine the volume of the solid generated by rotating the region R around the line $y = -1$.

7) Find the volume of the solid with a region bounded by

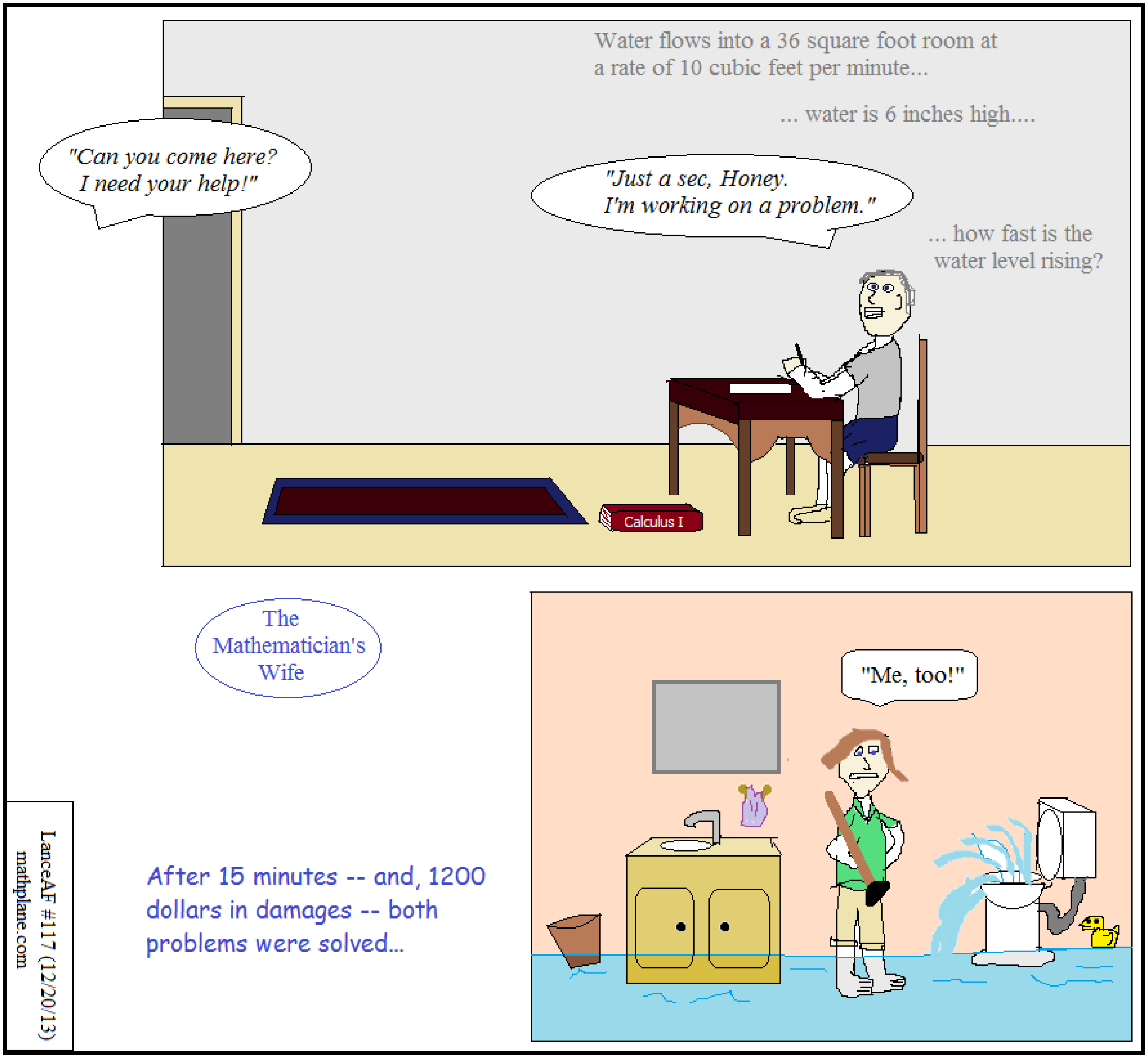
$y = x^2 + 1$
$y = -x^2 + 2x + 5$ and, revolved around the x-axis
$x = 0$
$x = 3$

8) The region R is bounded by the x-axis, y-axis, and $x + 2y = 4$...
Find the volume of the solid whose base is R, and

a) the cross sections are isosceles right triangles perpendicular to the y-axis

b) the cross sections are isosceles right triangles perpendicular to the x-axis

c) the cross sections are semicircles whose diameters lie on the region R perpendicular to the y-axis



Water flows into a 36 square foot room at a rate of 10 cubic feet per minute...
... water is 6 inches high....
"Can you come here? I need your help!"
"Just a sec, Honey. I'm working on a problem."
... how fast is the water level rising?
Calculus I
The Mathematician's Wife
"Me, too!"
After 15 minutes -- and, 1200 dollars in damages -- both problems were solved...
LanceAF #117 (12/20/13)
mathplane.com

ANSWERS -→

Volume of Solids Practice Test

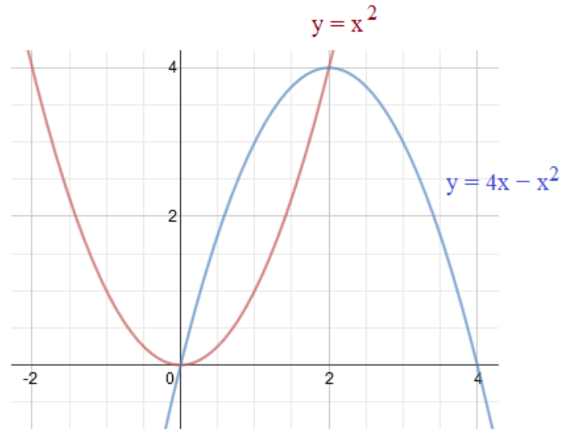
SOLUTIONS

1) Find the volume of the solid formed by revolving the region bounded by

$y = x^2$ and $y = 4x - x^2$

a) about the x-axis

b) about the line $y = 6$



a) The graph shows the 2 parabolas intersecting at $x = 0$ and $x = 2$

(algebraically) $x^2 = 4x - x^2$

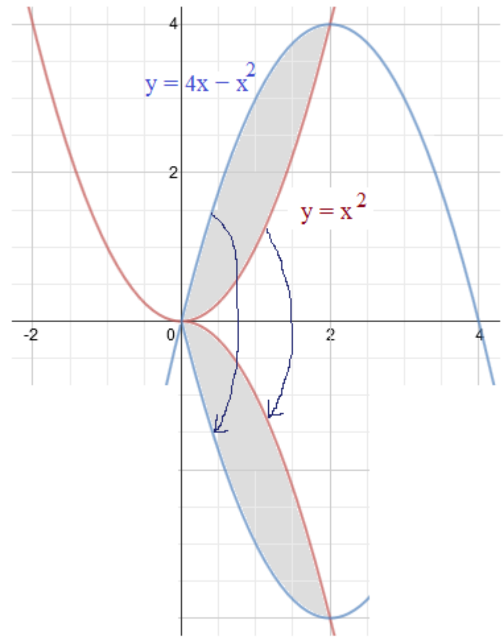
$2x^2 - 4x = 0$ $\qquad x = 0, 2$

$2x(x - 2) = 0$

Since the rotation is about the x-axis,

the *outer* (radius) boundary will be $4x - x^2$

the *inner* (radius) boundary of the solid will be x^2



$$\int \pi (4x - x^2)^2\, dx - \int \pi (x^2)^2\, dx$$

And, the intersections indicate the ends of definite integral

$$\int_0^2 \pi (4x - x^2)^2\, dx - \int_0^2 \pi (x^2)^2\, dx$$

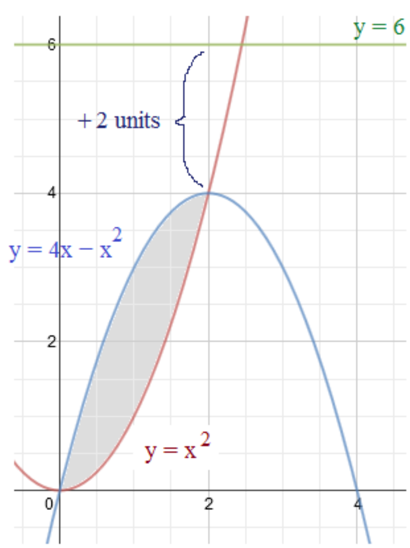
$$\pi \int_0^2 16x^2 - 8x^3 + \cancel{x^4}\, dx - \pi \int_0^2 \cancel{x^4}\, dx$$

$$\pi \left(\frac{16x^3}{3} - 2x^4 \Big|_0^2 \right) = \boxed{10\tfrac{2}{3}\pi}$$

b) Since the rotation is about the line $y = 6$,

the *outer* (radius) boundary will be $x^2 + 2$

the *inner* (radius) boundary of the solid will be $(4x - x^2) + 2$



$$\int_0^2 \pi (x^2 + 2)^2\, dx - \int_0^2 \pi (-x^2 + 4x + 2)^2\, dx$$

$$\pi \int_0^2 \cancel{x^4} + 4x^2 + \cancel{4}\, dx - \pi \int_0^2 \cancel{x^4} - 8x^3 + 12x^2 + 16x + \cancel{4}\, dx$$

$$\pi \int_0^2 8x^3 - 8x^2 - 16x\, dx = \pi\, 2x^4 - \frac{8}{3}x^3 - 8x^2 \Big|_0^2 = \boxed{\frac{64}{3}\pi}$$

Volume of Solids Practice Test

SOLUTIONS

2) Given the area bounded by $y = \sqrt{x}$

$y = 2$

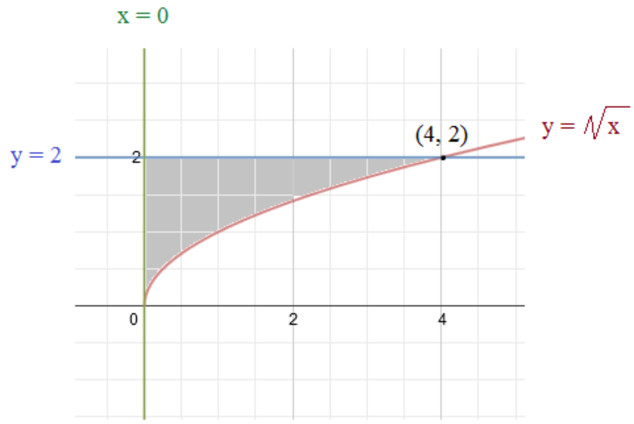
$x = 0$

Find the volume of the solid from rotation

a) about the x-axis

b) about the y-axis

c) around $y = 2$



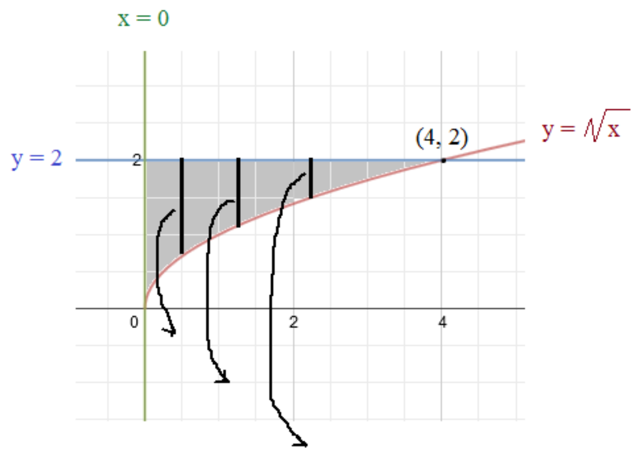
a) Since the rotation (revolution) is about the x-axis, the outer radius will be $y = 2$, and the inner radius will be $y = \sqrt{x}$

Then, the endpoints (or limits of integration) will be 0 and 4

$$\int_0^4 \pi (2)^2 \, dx - \int_0^4 \pi (\sqrt{x})^2 \, dx$$

$$\pi [4x] \Big|_0^4 - \pi \left[\frac{x^2}{2} \right] \Big|_0^4$$

$$16\pi - 8\pi = \boxed{8\pi}$$



b) Since the revolution (rotation) is about the y-axis, we need to rewrite the expression:

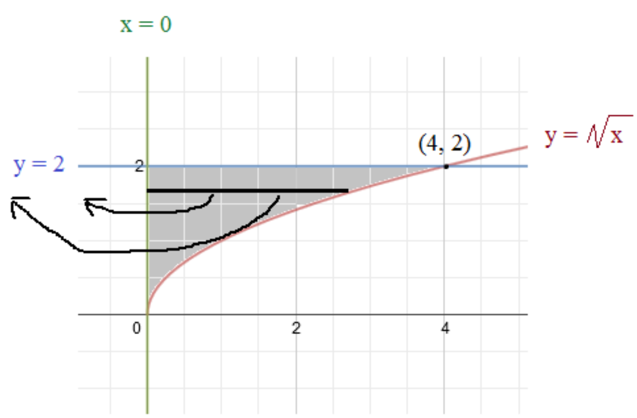
$y = \sqrt{x} \longrightarrow x = y^2$

Then, the endpoints (limits of integration) will be 0 and 2

$$\int_0^2 \pi (y^2)^2 \, dy$$

$$\pi \int_0^2 y^4 \, dy$$

$$\pi \frac{y^5}{5} \Big|_0^2 = \boxed{\frac{32}{5}\pi}$$



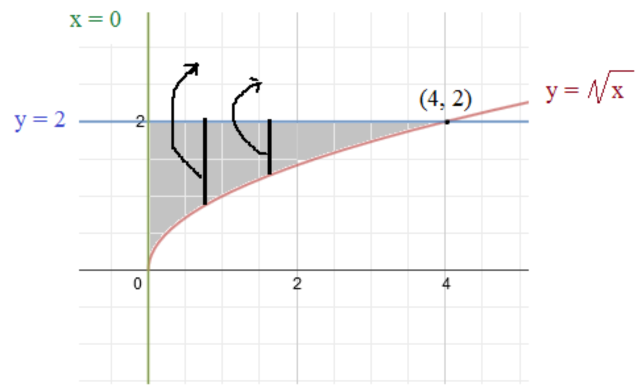
c) Since the rotation is around y = 2, the radius will come from the area between y = 2 and $y = \sqrt{x}$

$$\int_0^4 \pi (2 - \sqrt{x})^2 \, dx$$

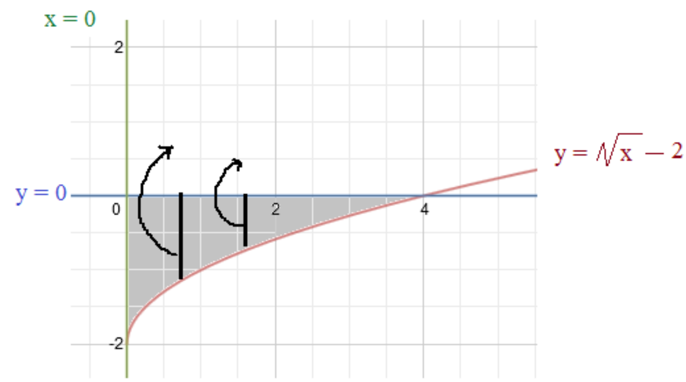
$$\pi \int_0^4 4 - 4\sqrt{x} + x \, dx$$

$$\pi \left[4x - \frac{4x^{3/2}}{3/2} + \frac{x^2}{2} \right] \Big|_0^4$$

$$16\pi - \frac{64}{3}\pi + 8\pi = \frac{8}{3}\pi$$



**The volume of solid would be the same if we *shift the area down 2* units and *rotate it around the x-axis*!



$$\int_0^4 \pi (\sqrt{x} - 2)^2 \, dx$$

$$\pi \int_0^4 x - 4\sqrt{x} + 4 \, dx$$

$$\pi \left(\frac{x^2}{2} - \frac{4x^{3/2}}{3/2} + 4x \Big|_0^4 \right)$$

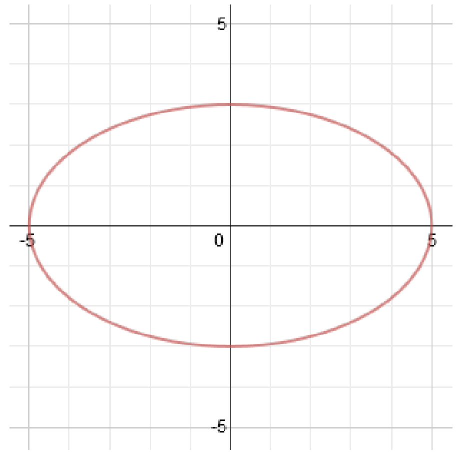
$$\pi \left(8 - \frac{64}{3} + 16 - 0 - 0 - 0 \right) = \frac{8}{3}\pi$$

3) The left half of the ellipse $9x^2 + 25y^2 = 225$ is revolved around the y-axis to form a spheroid. Find its volume.

SOLUTIONS

First, let's graph the ellipse:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$



We can see the left half extends up/down from 3 to -3 and the left half from -5 to 0...

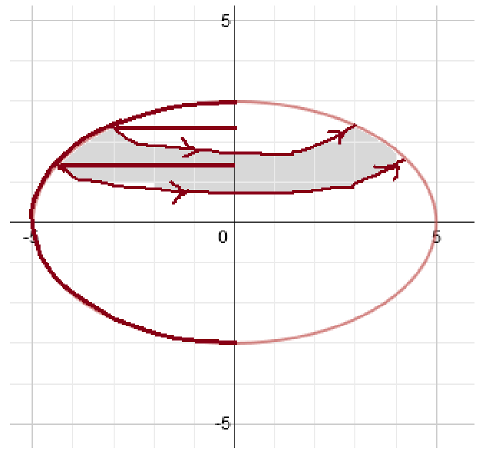
Since the half ellipse is revolved around the y-axis,

$$\int \qquad dy$$

Since the left half is symmetric over the x-axis, we'll find the volume for the top half (and then double it)

$$\int_0^3 \qquad dy$$

"Disc method"



$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

$$x^2 = 25 - \frac{25y^2}{9} \qquad x = \sqrt{25 - \frac{25y^2}{9}}$$

$$\int_0^3 \pi\left(25 - \frac{25y^2}{9}\right) dy$$

$$\text{Volume} = \int \pi\, f(y)^2\, dy$$

$$\pi\left(25y - \frac{25y^3}{27}\Bigg|_0^3\right) = 50\pi$$

Therefore, the full spheroid (top half and bottom half) is $\boxed{100\pi}$

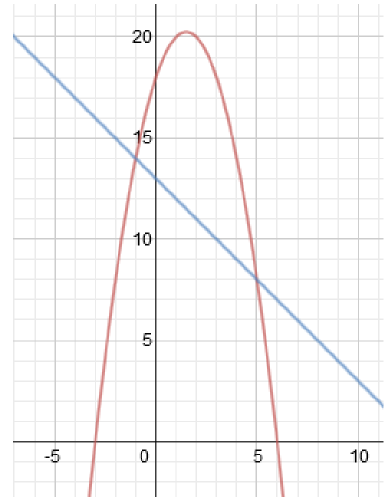
4) Find the volume of the solid formed by the region bounded by

$$y = -x^2 + 3x + 18$$

$$x + y = 13$$

and revolved around the x-axis

SOLUTIONS



When we graph the equations, we observe an upside down parabola that is intersected by a line.

The boundary (from left to right) are the points of intersection.

$$\begin{cases} y = -x^2 + 3x + 18 \\ y = -x + 13 \end{cases}$$ substitution

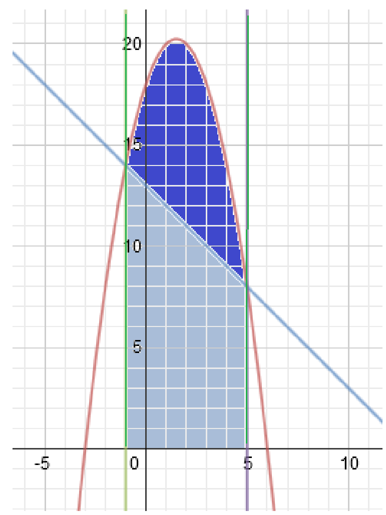
$-x + 13 = -x^2 + 3x + 18$ collect terms

$x^2 - 4x - 5 = 0$ factor

$(x - 5)(x + 1) = 0$ solve

$x = -1$ and $x = 5$

The outer radius will be from the parabola
The inner radius will be from the line



$$\int_{-1}^{5} -x^2 + 3x + 18 \, dx \; - \; \int_{-1}^{5} -x + 13 \, dx$$

area below the parabola — area below the line

$$\int_{-1}^{5} \pi (-x^2 + 3x + 18)^2 \, dx \; - \; \int_{-1}^{5} \pi (-x + 13)^2 \, dx$$

discs from the parabola — discs from the line

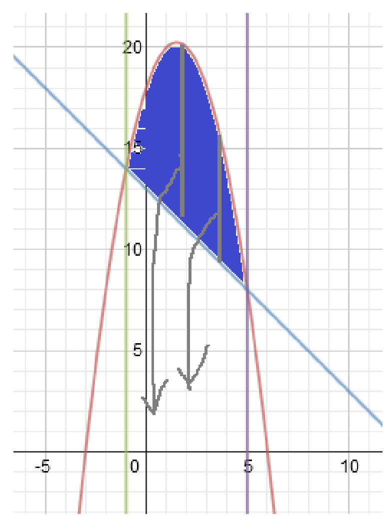
$$\pi \int_{-1}^{5} x^4 - 6x^3 - 27x^2 + 108x + 324 \, dx \; - \; \pi \int_{-1}^{5} x^2 - 26x + 169 \, dx$$

$$\pi \int_{-1}^{5} x^4 - 6x^3 - 28x^2 + 134x + 155 \, dx$$

$$\pi \left(\frac{x^5}{5} - \frac{3x^4}{2} - \frac{28x^3}{3} + 67x^2 + 155x \Big|_{-1}^{5} \right)$$

$$\pi \left(625 - 937.5 - 1166\tfrac{2}{3} + 1675 + 775 - \left(\frac{-1}{5} - \frac{3}{2} - \frac{-28}{3} + 67 - 155 \right) \right)$$

$$970.83 - (-80.36) = \boxed{1051.2\,\pi}$$



SOLUTIONS

5) Find the volume of the solid when the region R bounded by

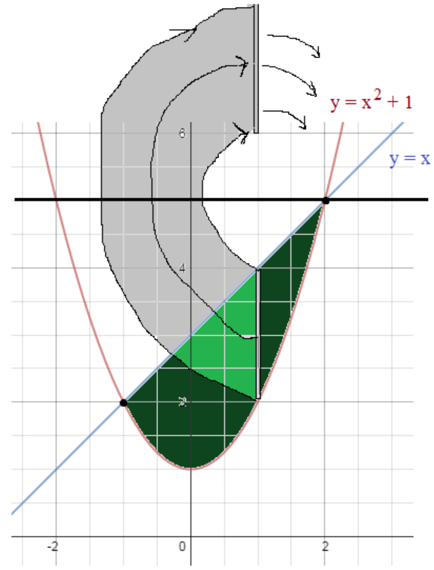
$y = x^2 + 1$ and $y = x + 3$

is revolved around the line y= 5

$x^2 + 1 = x + 3$

$x^2 - x - 2 = 0 \qquad x = -1, 2$

$(x + 1)(x - 2) = 0$



The equations intersect at (-1, 2) and (2, 5) --- the boundaries of the integral

$$\int_{-1}^{2} \pi (5 - (x^2 + 1))^2 \, dx \; - \; \int_{-1}^{2} \pi (5 - (x + 3))^2 \, dx$$

outer radius (entire disk) — inner radius (middle cut out to create washer)

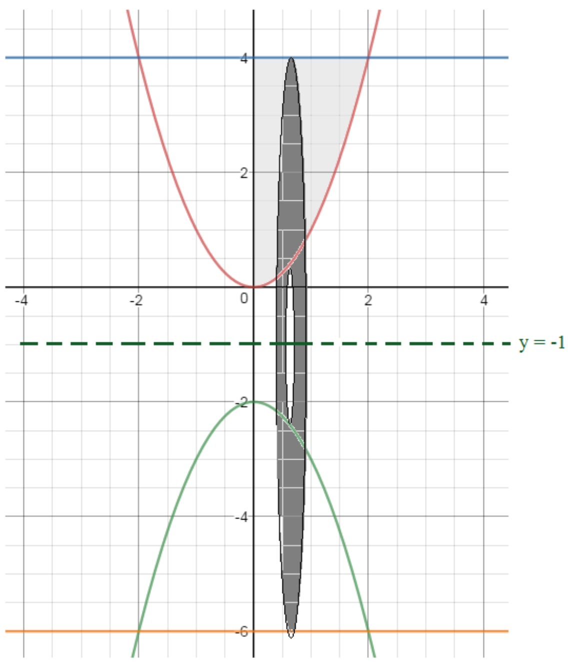
$$\pi \int_{-1}^{2} (5 - (x^2 + 1))^2 - (5 - (x + 3))^2 \, dx$$

$$\pi \int_{-1}^{2} (4 - x^2)^2 - (2 - x)^2 \, dx$$

$$\pi \int_{-1}^{2} 16 - 8x^2 + x^4 - 4 + 4x - x^2 \, dx = \pi \int_{-1}^{2} x^4 - 9x^2 + 4x + 12 \, dx = \pi \left(\frac{x^5}{5} - 3x^3 + 2x^2 + 12x \right) \Big|_{-1}^{2} = \pi \left(\frac{72}{5} - \left(-\frac{36}{5}\right) \right) = \boxed{\frac{108}{5} \pi}$$

6) The region R is bordered by the y-axis, y = 4, and the function $y = x^2$.
Determine the volume of the solid generated by rotating the region R around the line $y = -1$.

Step 1: Sketch the graph...



Step 2: Identify the radius of the entire disc/washer (the 'outer').. R

$y = 4 \; - \; y = -1 \implies 5$

Identify the radius of the entire middle of the washer (the 'inner')... r

$y = x^2 \; - \; y = -1 \implies x^2 \;\; (-1)$

Step 3: Determine the boundary of the integral...

The region R extends from x = 0 to x = 2 (the intersection of y = 4 and $y = x^2$)

$$\int_{0}^{2} R + r \, dx$$

Step 4: Apply the formula for a circle, and write the integral...

$$\int_{0}^{2} \pi (5)^2 \, dx \; - \; \int_{0}^{2} \pi (x^2 + 1)^2 \, dx$$

Step 5: Solve...

$$25x \, \pi \Big|_{0}^{2} \; - \; \pi \int_{0}^{2} x^4 + 2x^2 + 1 \, dx$$

$$25x \, \pi \Big|_{0}^{2} \; - \; \pi \left(\frac{x^5}{5} + \frac{2x^3}{3} + x \Big|_{0}^{2} \right)$$

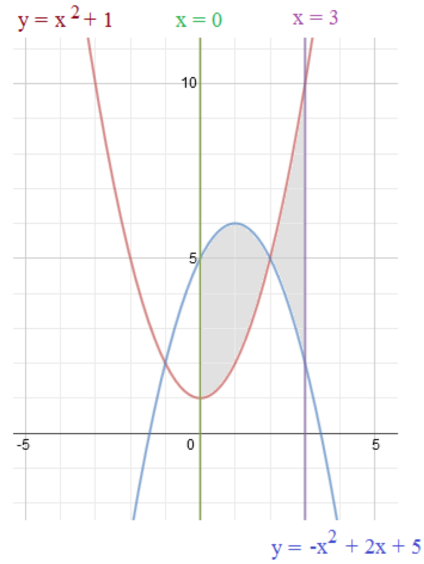
$$50 \pi - \pi \left(\frac{32}{5} + \frac{16}{3} + 2 \right) \implies \frac{750 \pi}{15} - \pi \left(\frac{96}{15} + \frac{80}{15} + \frac{30}{15} \right) \implies \boxed{\frac{544 \pi}{15}}$$

7) Find the volume of the solid with a region bounded by

SOLUTIONS

$$y = x^2 + 1$$
$$y = -x^2 + 2x + 5$$
$$x = 0$$
$$x = 3$$

and, revolved around the x-axis



Step 1: Sketch the graph and identify the boundaries

> Since the functions 'cross over', it'll be helpful to divide the region into 2 parts...

Where do the functions cross? Find the intersection...

$$x^2 + 1 = -x^2 + 2x + 5$$

$$2x^2 - 2x - 4 = 0$$

$$2(x - 2)(x + 1) = 0$$

Intersections occur at (-1, 2) and (2, 5)

We'll evaluate the 'left' region from 0 to 2...
And, the 'right' region from 2 to 3....

Step 2: Determine the integrals

$$\int_0^2 \pi\left(-x^2 + 2x + 5\right)^2 - \pi\left(x^2 + 1\right)^2 \, dx \quad + \quad \int_2^3 \pi\left(x^2 + 1\right)^2 - \pi\left(-x^2 + 2x + 5\right)^2 \, dx$$

left region: since downward parabola is above the upward facing parabola, it goes first...

right region: since upward parabola is above the downward facing parabola, it goes first in the integral..

(otherwise, value would be negative)

Step 3: Evaluate definite integrals (using fund. theorem of calculus)

$$\pi \int_0^2 x^4 - 4x^3 - 6x^2 + 20x + 25 - \left(x^4 + 2x^2 + 1\right) dx + \pi \int_2^3 x^4 + 2x^2 + 1 - \left(x^4 - 4x^3 - 6x^2 + 20x + 25\right) dx$$

$$\pi \int_0^2 -4x^3 - 8x^2 + 20x + 24 \, dx \quad + \quad \pi \int_2^3 4x^3 + 8x^2 - 20x - 24 \, dx$$

$$\pi\left(-x^4 - \frac{8x^3}{3} + 10x^2 + 24x \Big|_0^2\right) \quad + \quad \pi\left(x^4 + \frac{8x^3}{3} - 10x^2 - 24x \Big|_2^3\right)$$

$$\pi\left(-16 - \frac{64}{3} + 40 + 48\right) \quad + \quad \pi\Big((81 + 72 - 90 - 72) - (16 + 64/3 - 40 - 48)\Big)$$

$$\frac{152}{3}\pi \quad + \quad \pi(-9 - -152/3) \quad = \quad \boxed{\frac{277}{3}\pi}$$

Volume of Solids Practice Test: Cross Sections

8) The region R is bounded by the x-axis, y-axis, and $x + 2y = 4$...
Find the volume of the solid whose base is R, and

a) the cross sections are isosceles right triangles perpendicular to the y-axis

Since we're using horizontal partitions, $x = 4 - 2y$

$$\int_0^2 \frac{1}{2}(4-2y)^2\,dy$$

$$\frac{1}{2}\int_0^2 16 - 16y + 4y^2\,dy$$

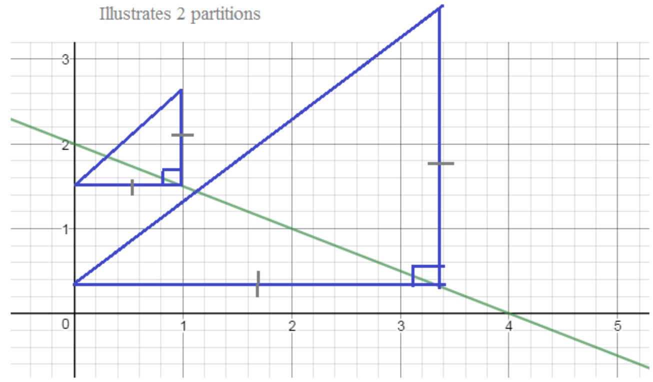
$$\frac{1}{2}\left(16y - 8y^2 + \frac{4y^3}{3}\Bigg|_0^2\right) = \frac{1}{2}(32 - 32 + 32/3) = \boxed{\frac{16}{3}}$$

Area $\triangle = \frac{1}{2}bh$

Isosceles, so base and height are the same

Area $= \frac{1}{2}(\text{base})^2$

Illustrates 2 partitions



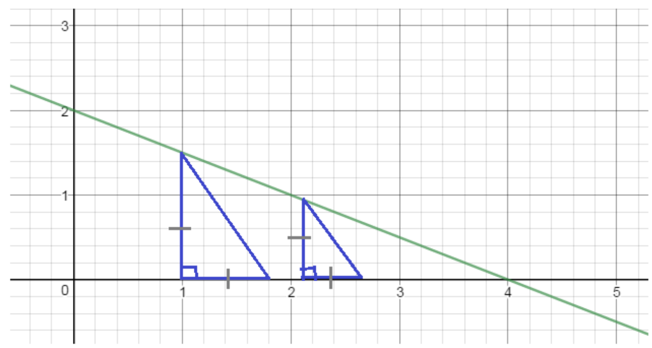
b) the cross sections are isosceles right triangles perpendicular to the x-axis

Since we're using vertical partitions, $y = \frac{4-x}{2}$

$$\int_0^4 \frac{1}{2}\left(\frac{4-x}{2}\right)^2 dx$$

$$\frac{1}{8}\int_0^4 16 - 8x + x^2\,dx$$

$$\frac{1}{8}\left(16x - 4x^2 + \frac{x^3}{3}\Bigg|_0^4\right) = \frac{1}{8}(64 - 64 + 64/3) = \boxed{\frac{8}{3}}$$



c) the cross sections are semicircles whose diameters lie on the region R perpendicular to the y-axis

Since the diameters are perpendicular to the y-axis, we'll use horizontal partitions $x = 4 - 2y$

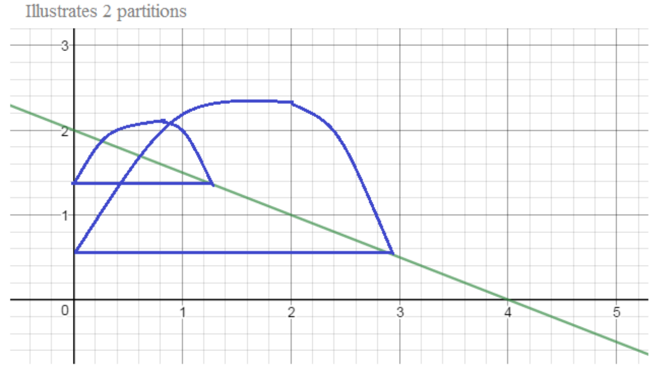
$$\int_0^2 \frac{1}{8}\pi\,(\text{diameter})^2$$

$$\int_0^2 \frac{1}{8}\pi\,(4-2y)^2\,dy$$

$$\frac{1}{8}\pi\int_0^2 16 - 16y + 4y^2\,dy$$

$$\frac{1}{8}\pi\left(16y - 8y^2 + \frac{4y^3}{3}\Bigg|_0^2\right) = \frac{1}{8}\pi\,(32 - 32 + 32/3) = \boxed{\frac{4}{3}\pi}$$

Illustrates 2 partitions



$\pi\,(\text{radius})^2$ = Area of circle

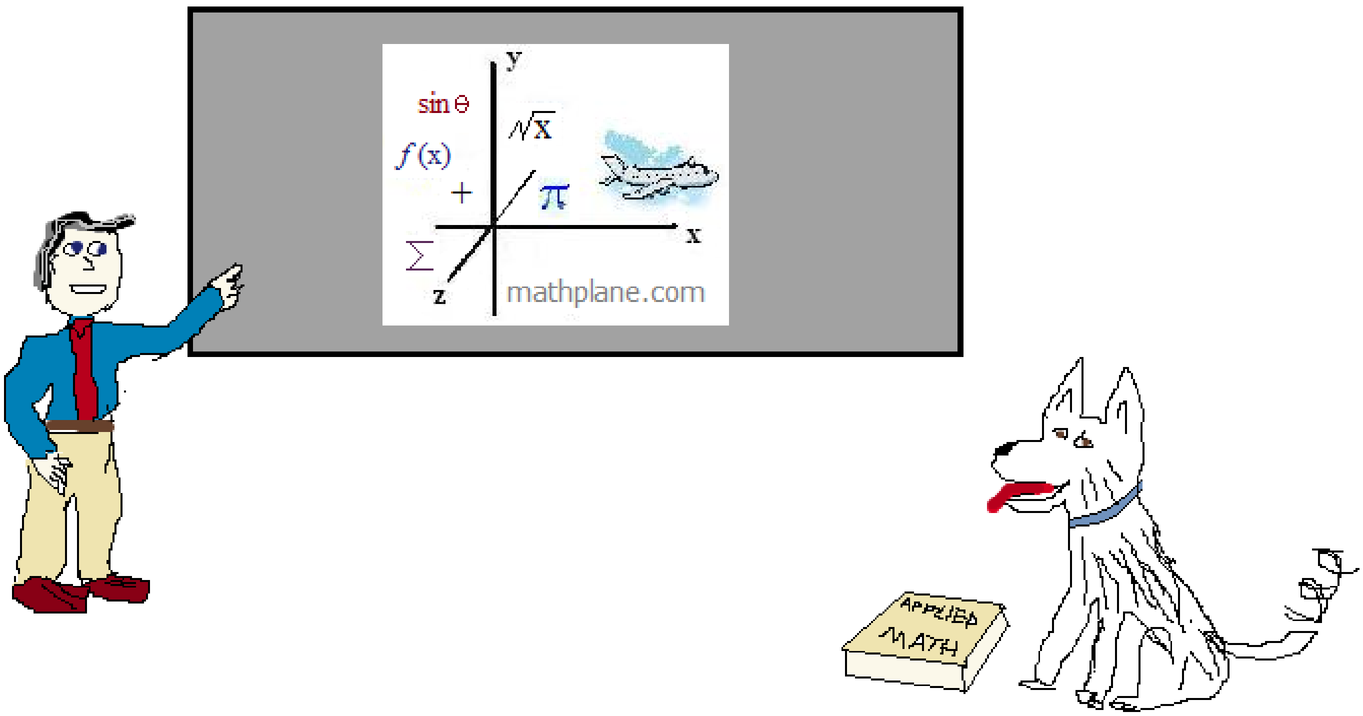
$\frac{1}{2}\pi\,(\text{radius})^2$ = Area of semicircle

$\frac{1}{2}\pi\,(\frac{1}{2}\text{diameter})^2$ = Area of semicircle

$\frac{1}{8}\pi\,(\text{diameter})^2$ = Area of semicircle

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.



Also, mathplane *express* for mobile and tablets at mathplane.org

And, TES, TeachersPayTeachers, and Pinterest

Other cross sections to explore -→

Volume of a Solid with various cross sections

Example: The area of a region is bounded by $y = e^x$, x-axis, y-axis, and $x = 1$

a) Find the area of the region

b) Find the volume of the solid when the region is rotated around the x-axis

c) Find the volume of the solid where the cross sections perpendicular to the x-axis are squares

d) Find the volume of the solid where the cross sections perpendicular to the x-axis are semicircles

---

a) Find the area of the region

Step 1: Determine the span of the integral..

The boundaries go from $x = 0$ to $x = 1$

$$\int_0^1$$

Step 2: Determine the function to evaluate

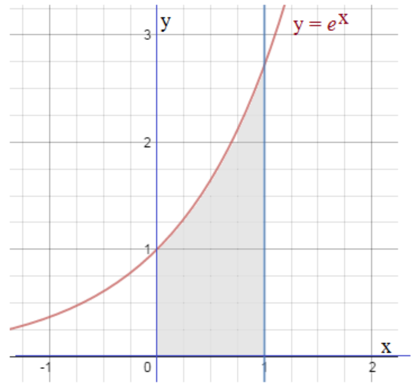
The upper boundary is $y = e^x$ and

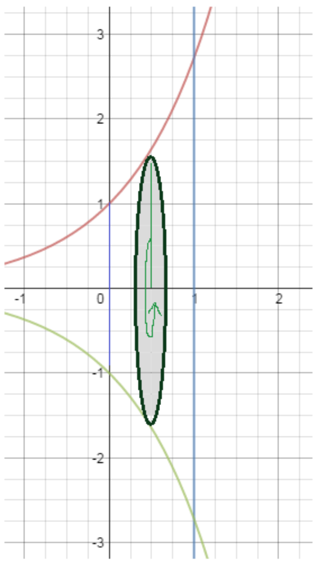
the lower boundary is $y = 0$

$$\int_0^1 e^x - 0 \, dx$$

Step 3: Evaluate

$$e^x \Big|_0^1 = e^1 - e^0 = e - 1 \quad \text{or} \quad 1.72$$





b) Find the volume of the solid when the region is rotated around the x-axis

area of a circle: π (radius)^2

(the radius is the length of the function)

$$\int_0^1 \pi (e^x)^2 \, dx$$

radius of each partition

$$\int_0^1 \pi e^{2x} \, dx = \pi \int_0^1 e^{2x} \, dx$$

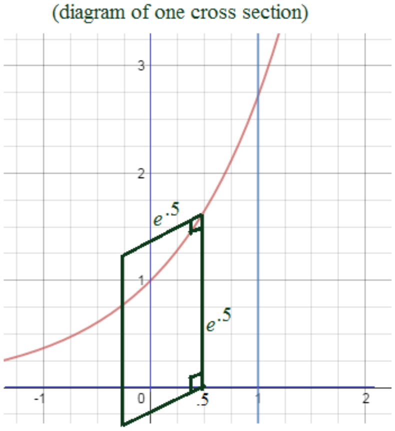
$$= \frac{1}{2}\pi \int_0^1 2e^{2x} \, dx = \frac{1}{2}\pi \cdot e^{2x} \Big|_0^1$$

$$\frac{1}{2}\pi \left(e^2 - e^0 \right) = \frac{1}{2}\pi (e^2 - 1)$$

approx. 10.03

c) cross sections perpendicular to the x-axis are squares

(diagram of one cross section)



area of a square: (side)^2

(the side is the length of the function)

$$\int_0^1 (e^x)^2 \, dx$$

side of each partition

$$\int_0^1 e^{2x} \, dx$$

$$\frac{1}{2} \int_0^1 2e^{2x} \, dx$$

$$\frac{1}{2} \cdot e^{2x} \Big|_0^1 = \frac{1}{2}(e^2 - 1)$$

approx. 3.19

d) cross sections perpendicular to the x-axis are semicircles

$$\int_0^1 \frac{1}{2}\pi \left(\frac{1}{2} e^x\right)^2 dx$$

radius of each partition

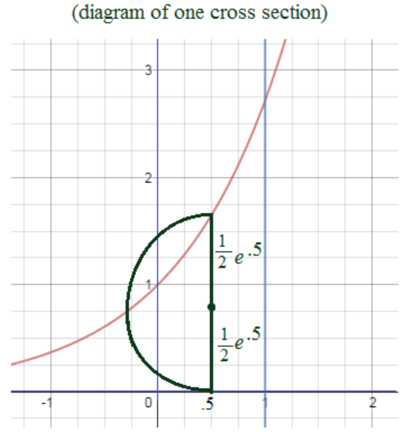
$$\frac{1}{2}\pi \int_0^1 \frac{1}{4} e^{2x} \, dx$$

$$\frac{1}{16}\pi \int_0^1 2e^{2x} \, dx$$

$$\frac{1}{16}\pi \cdot e^{2x} \Big|_0^1$$

$$\frac{1}{16}\pi \left(e^2 - e^0 \right) = \text{approx. } 1.25$$

(diagram of one cross section)



area of a semicircle: $\frac{1}{2}\pi$ (radius)^2

(the radius is 1/2 the length of the function)

Example: Find the volume of the solid from the region bounded by

$y = 8 - x^2$ and $y = x^2$

a) whose partitions are squares that are perpendicular to the x-axis

b) whose partitions are squares that are perpendicular to the y-axis

Step 1: Sketch diagram — The equations are intersecting parabolas (one opening upward and one opening downward)

Step 2: Determine integral boundaries

The graph illustrates the intersections at x = -2 and x = 2

To verify algebraically, set equations equal to each other:

$$8 - x^2 = x^2$$
$$8 = 2x^2 \qquad x = -2 \text{ and } 2$$
$$4 = x^2$$

$$\int_{-2}^{2} \qquad dx$$

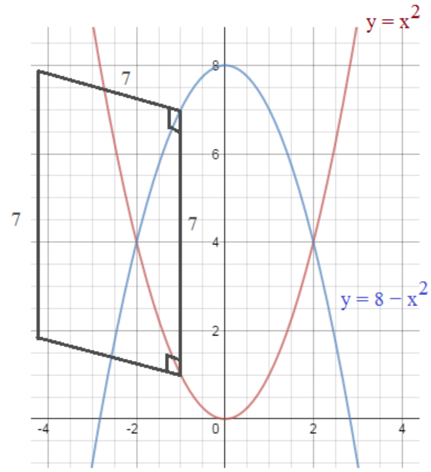
Step 3: Identify the partitions -- write the equation to be integrated

Each partition will be a square whose sides have length $(8 - x^2) - (x^2)$ (upper bound) (lower bound)

Area of a square = $(\text{length})^2$

$$\int_{-2}^{2} \left((8 - x^2) - (x^2) \right)^2 dx$$

Diagram with one square partition at x = -1



Step 4: Solve

$$\int_{-2}^{2} \left((8 - 2x^2) \right)^2 dx$$

$$\int_{-2}^{2} \left((8 - 2x^2) \right)^2 dx = \int_{-2}^{2} 64 - 32x^2 + 4x^4 \, dx$$

$$= 64x - \frac{32x^3}{3} + \frac{4x^5}{5} \Big|_{-2}^{2} = \frac{1024}{15} - \left(-\frac{1024}{15}\right)$$

$$\boxed{\frac{2048}{15} \text{ or } 136.53}$$

Step 1: Sketch diagram

$y = x^2 \longrightarrow x = \pm\sqrt{y} \qquad y = 8 - x^2 \longrightarrow x = \pm\sqrt{8 - y}$

Step 2: Determine integral boundaries

Since we are using horizontal partitions, the integral will include the upper and lower boundaries!

The graph shows boundaries at y = 8 and y = 0 and intersections at y = 4

To verify algebraically, find the vertex of each equation... Then, find the intersection:

$$\sqrt{y} = \sqrt{8 - y} \qquad y = 4$$
$$y = 8 - y$$

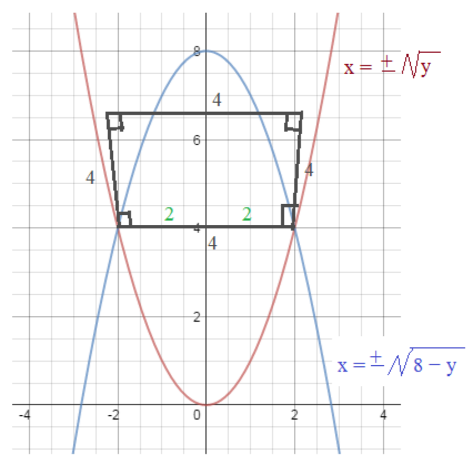
Step 3: Identify the partitions and write the definite integral

Area of a square = $(\text{length})^2$

$$\underbrace{\int_{0}^{4} (2\sqrt{y})^2 \, dy}_{\text{lower half}} + \underbrace{\int_{4}^{8} (2\sqrt{8 - y})^2 \, dy}_{\text{upper half}}$$

Since the parabolas are even functions that are symmetrical over y-axis, we'll find the length of the positive side and double it

Diagram of one square partition at y = 4



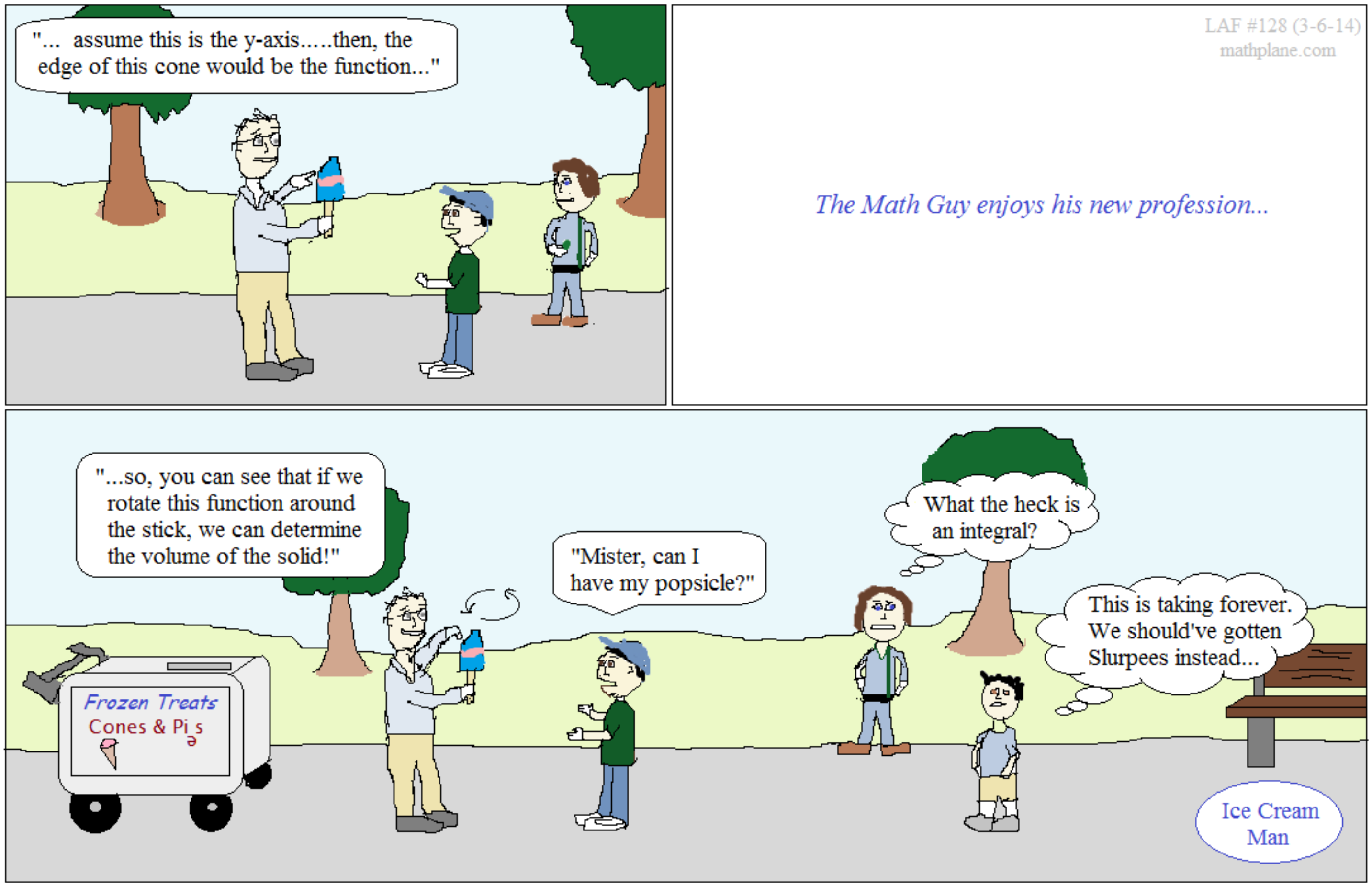
Step 4: Solve

$$\int_{0}^{4} 4y \, dy + \int_{4}^{8} 32 - 4y \, dy$$

$$2y^2 \Big|_{0}^{4} + 32y - 2y^2 \Big|_{4}^{8} = 32 - 0 + 128 - 96$$

$$\boxed{64}$$

Note the volume is less when the squares use lengths from horizontal partitions (rather than lengths from vertical partitions!)


"... assume this is the y-axis.....then, the edge of this cone would be the function..."
LAF #128 (3-6-14)
mathplane.com
The Math Guy enjoys his new profession...
"...so, you can see that if we rotate this function around the stick, we can determine the volume of the solid!"
"Mister, can I have my popsicle?"
What the heck is an integral?
This is taking forever. We should've gotten Slurpees instead...
Frozen Treats
Cones & Pies
Ice Cream Man