Calculus Review 1

Random Topics include limits/asymptotes, max/min, implicit differentiation, related rates, instantaneous rate of change, graphing, integrals, increasing/decreasing intervals, Newton's Cooling, and more.

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Limits and Asymptotes Puzzle

Use the limits and asymptotes "clues" to figure out the function!

A certain rational function $f(x)$ has quadratic functions in both its numerator and denominator.

Also, it has these characteristics:

$f(x)$ has a vertical asymptote at $x = 5$

$f(x)$ has one x-intercept, at $x = 3$

$f(x)$ is (removably) discontinuous at $x = 1$, with $\lim_{x \to 1} f(x) = \frac{-1}{7}$

a) What is the function $f(x)$?

b) $f(0) =$

c) Sketch a graph of the function.

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a) Using the "clues",

vertical asymptote at $x = 5$ $\quad f(x) = \frac{\quad}{(x-5)}$

x-intercept at $x = 3$ $\quad f(x) = \frac{(x-3)}{(x-5)}$

removable discontinuity at $x = 1$ ("hole") $\quad f(x) = \frac{(x-3)(x-1)}{(x-5)(x-1)}$

**The shape of the function is determined by the "a" value $\quad f(x) = \frac{a\,(x-3)(x-1)}{(x-5)(x-1)}$

To find the a value, we need another point, or use the limit!

$$\frac{a(1-3)}{(1-5)} = \frac{-1}{7} \qquad \frac{-2a}{-4} = \frac{-1}{7} \qquad -14a = 4 \qquad a = \frac{-2}{7}$$

$$f(x) = \frac{\frac{-2}{7}(x-3)(x-1)}{(x-5)(x-1)} \quad \text{or,} \quad \frac{-2x^2 + 8x - 6}{7x^2 - 42x + 35}$$

b) $f(0)$ is the y-intercept... $\frac{-6}{35}$

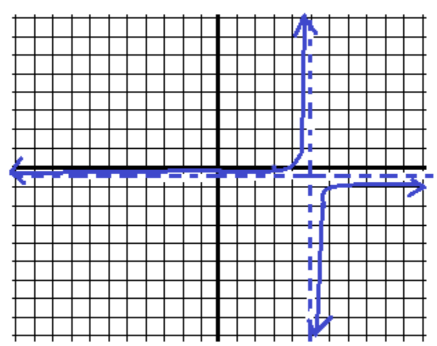
c) To sketch, we'll utilize the intercepts, asymptote, and limits....

horizontal asymptote: $y = -2/7$

vertical asymptote: $x = 5$

x-intercept: (3, 0)

x	f(x)
-3	-3/14
0	-6/35
3	0
4	2/7
6	-6/7
7	-4/7



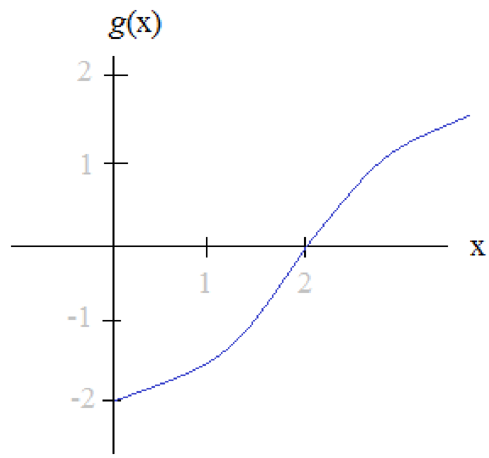
(sketch)

Determine if the value is < > or = to zero:

a) $g(1)$

b) $g'(1)$

c) $g''(1)$



Answers and explanations:

a) For any function,
if the output is positive, it is above the x-axis
if the output is negative, it is below the x-axis
and, if the output is 0, it is on the x-axis....
Since the output of g(1) is below the x-axis, it is negative.
< 0

"position"

b) The first derivative is the function's instantaneous rate of change (slope).
If the slope is positive (upward), then the value is positive.
If the slope is negative (downward), then the derivative is negative.
and, if the slope is 0 (horizontal), then the derivative equals 0.
Since the output of g(1) is on an upward sloping part of the curve,
the first derivative g'(1) is > 0

"slope"

c) The second derivative represents the 'acceleration' or rate the slope is changing..
If the slope is increasing, the curve is concave up and the second derivative is positive.
If the slope is decreasing, the curve is concave down and the second derivative is negative.
If it is at a 'point of inflection', the curve is neither concave up nor down, so, the second derivative is zero.

Since the output of g(1) is on a part that is concave up, the second derivative g''(1) must be > 0

"concavity"

Example: Two particles that move along a horizontal axis have the following models:

$$x(t) = 3\cos(\frac{\pi}{4}t) \qquad s(t) = t^3 - 6t^2 + 9t + 4$$

On the interval $0 \le t \le 6$, when do the particles move in the same direction?

Find the intervals where each particle increases and decreases...

First derivative.... $x'(t) = -3\sin(\frac{\pi}{4}t) \cdot \frac{\pi}{4}$ $\qquad s'(t) = 3t^2 - 12t + 9 + 0$

Then, set equal to zero (to find where particle changes direction)

$$-3\frac{\pi}{4}\sin(\frac{\pi}{4}t) = 0$$

$$\sin(\frac{\pi}{4}t) = 0$$

$t = 4k$ (where k is any integer)

So, in interval [0, 6], 0 and 4

$$3t^2 - 12t + 9 = 0$$

$$3(t^2 - 4t + 3) = 0$$

$$(t-3)(t-1) = 0$$

t = 1 and 3

Then, test each sub-interval to determine whether increasing or decreasing...

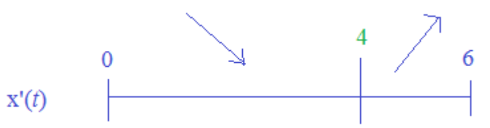
$$x'(1) = -3\frac{\pi}{4}\sin(\frac{\pi}{4}1) < 0$$

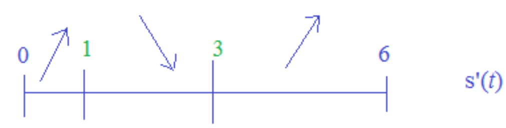
$$x'(5) = -3\frac{\pi}{4}\sin(\frac{\pi}{4}5) > 0$$

$$s'(1/2) = 3(1/2 - 3)(1/2 - 1) > 0$$

$$s'(2) = 2(2-3)(2-1) < 0$$

$$s'(4) = 2(4-3)(4-1) > 0$$

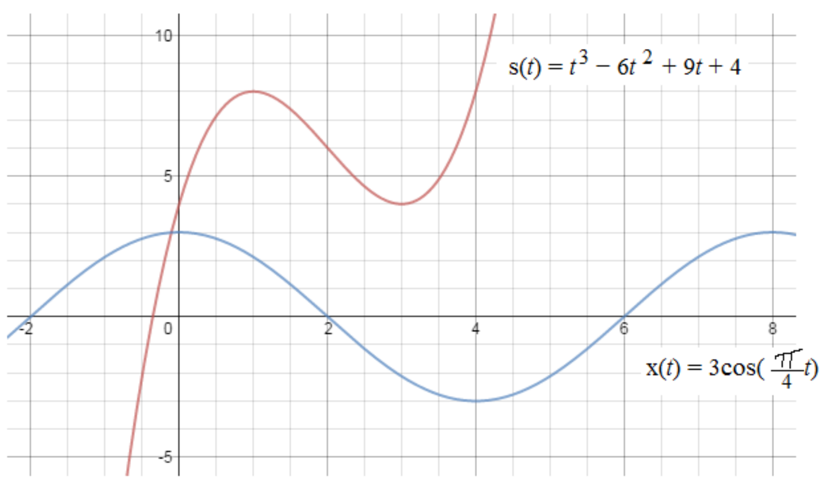




Finally, determine the sub-intervals where x(t) and s(t) move in the same direction....

Interval (1, 3) where both are decreasing (i.e. moving to the left)

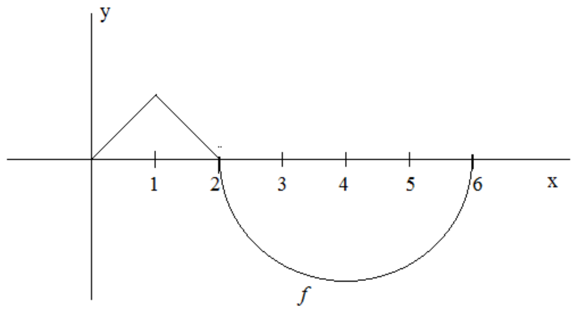
and, Interval (4, 6] where both are increasing (i.e. moving to the right)



Calculus Review Question

The graph of f consists of two line segments and a semicircle.

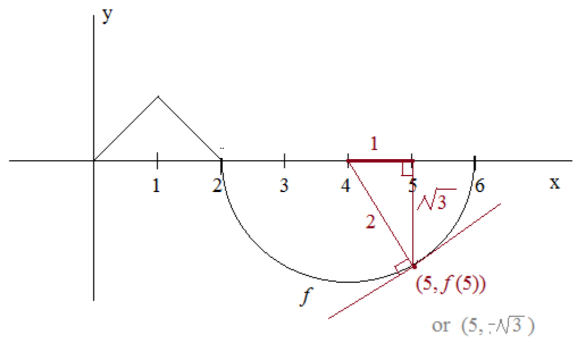
Find $f'(5)$



Solution 1: Recognizing that $f'(x)$ is the slope at a given point x, find the slope of a line tangent to the semicircle at x = 5

Since the span of the semicircle is 4 units, we know the radius would be 2 units.

Since the distance from 4 to 5 is one unit and we have constructed a right triangle, the vertical line segment length is $\sqrt{3}$



Slope of radius = $\frac{\text{"rise"}}{\text{"run"}} = \frac{\Delta y}{\Delta x} = \frac{-\sqrt{3}}{1}$

Slope of perpendicular line = $\frac{1}{\sqrt{3}}$

$$f'(5) = \frac{1}{\sqrt{3}}$$

(geometry note: a tangent line and the radius that shares the common point are perpendicular)

Solution 2: Find the derivative of f, and plug in 5

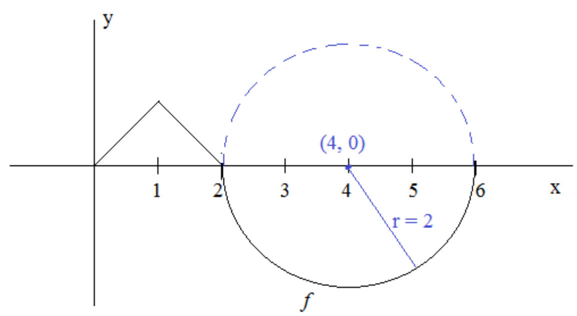
Since x = 5 involves the semicircle, ignore the line segments and describe the equation of the semicircle.

Standard form of a circle: $(x-h)^2 + (y-k)^2 = r^2$

where the radius = r
center = (h, k)

$(x-4)^2 + y^2 = 4$ where $y \le 0$

(because it's a semicircle)



Find derivative of $(x-4)^2 + y^2 = 4$

a) $(x-4)^2 + y^2 = 4$ (solve for y)

$y^2 = 4 - (x-4)^2$

$y = \pm\sqrt{4-(x-4)^2}$

$y = -\sqrt{4-(x-4)^2}$

(since the range is ≤ 0, eliminate the positive solutions)

(find dy/dx)

$$\frac{dy}{dx} = -\frac{1}{2}\left(4-(x-4)^2\right)^{\frac{-1}{2}}\left(-2(x-4)\right)$$

$$= \frac{(x-4)}{\left(4-(x-4)^2\right)^{\frac{1}{2}}}$$

$$f'(5) = \frac{(5-4)}{\left(4-(5-4)^2\right)^{\frac{1}{2}}} = \frac{1}{\sqrt{3}}$$

b) Implicit differentiation

$(x-4)^2 + y^2 = 4$

$2(x-4) + 2yy' = 0$

$2yy' = -2(x-4)$

$y' = \frac{-(x-4)}{y}$

plug in the point $(5, -\sqrt{3})$

$$f'(5) = \frac{-(5-4)}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$$

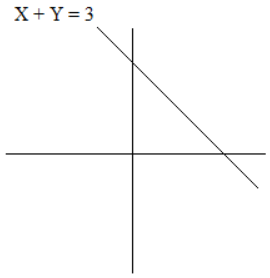
Calculus Review Question

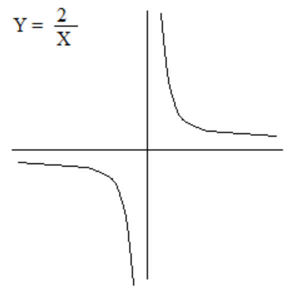
Find the area between the following equations: $X + Y = 3$

$Y = \frac{2}{X}$ And, sketch a graph.

$X + Y = 3 \longrightarrow Y = 3 - X$

$Y = \frac{2}{X}$





(Find where the equations intersect)

$3 - X = \frac{2}{X}$ (multiply by X)

$3X - X^2 = 2$ (factor and solve)

$X^2 - 3X + 2 = 0$

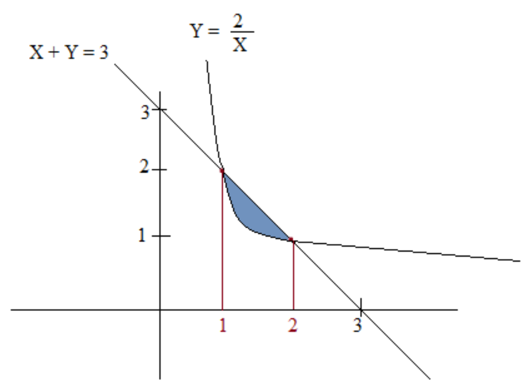
$(X - 1)(X - 2) = 0$

$X = 1, 2$

For $X = 1$, $(1) + Y = 3$, $Y = 2$ (plug X values into original equations to get Y values)

For $X = 2$, $(2) + Y = 3$, $Y = 1$

Equations intersect at (1, 2) and (2, 1)



Set up the definite integral to determine the area:

$$\int_1^2 (3 - X)\,dx - \int_1^2 \frac{2}{X}\,dx$$

(area under the line) (area under the curve)

$$3X - \frac{X^2}{2}\Big|_1^2 - 2\ln X\Big|_1^2$$

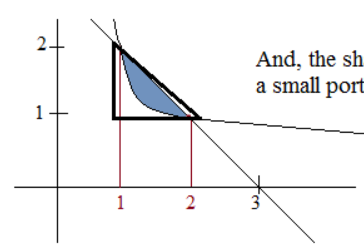
$$3(2) - \frac{(2)^2}{2} - \left(3(1) - \frac{(1)^2}{2}\right) - \left[2\ln(2) - 2\ln(1)\right]$$

$$6 - 2 - (3 - \frac{1}{2}) - 2\ln 2 + 0$$ $\ln(1) = 0$

$$\boxed{\frac{3}{2} - \ln 4}$$ $2\ln 2 = \ln(2)^2$

$$\boxed{1.5 - 1.386 = .114}$$

Check if reasonable:



The area of the triangle is 1/2 or .5

And, the shaded blue area is a small portion of that triangle.

Find the point on the graph $y = 3x^2$ on [1, 2] at which the tangent to the graph has *the same slope* as the line that passes through the endpoints of the closed interval.

Note: this problem is an application/verification of the mean value theorem!

Answer:

Step 1: Find the slope of the line that passes through the endpoints.

at $x = 1$, $y = 3$
and,
at $x = 2$, $y = 12$

slope between (1, 3) and (2, 12) is 9

Step 2: Find the point on the curve where the slope is 9

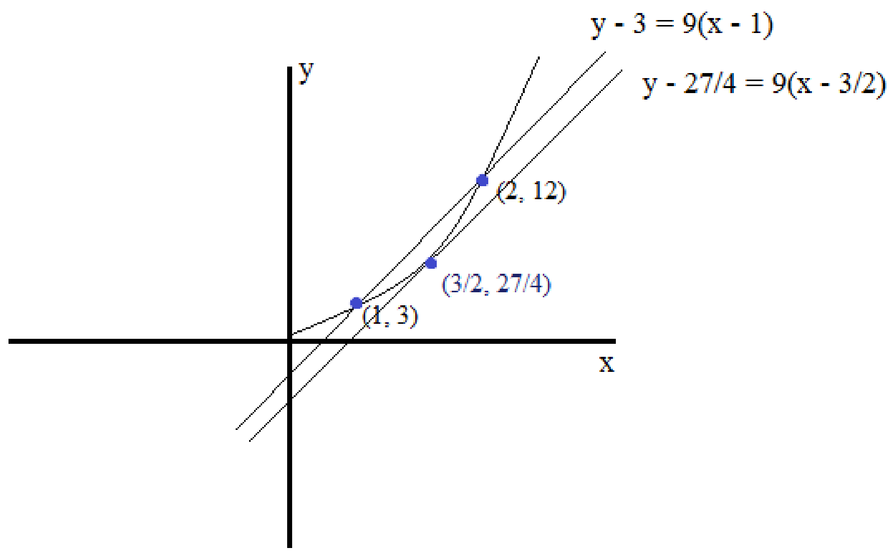
$y' = 6x$

$6x = 9$ when $x = 3/2$

at $x = 3/2$, $y = 3(3/2)^2 = 27/4$

The instantaneous rate of change/slope at (3/2, 27/4) = 9

Step 3: graph equations to verify



Find the $\lim_{x \to 0} \frac{\sin 2x}{x}$

Solutions:

method 1: trig identities

$\sin 2x = 2\sin x\cos x \qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\lim_{x \to 0} \frac{\sin 2x}{x} \longrightarrow \lim_{x \to 0} \frac{2\sin x\cos x}{x} = \lim_{x \to 0} \frac{2\cos x \cdot \sin x}{x}$$

$$\lim_{x \to 0} 2\cos x \cdot \lim_{x \to 0} \frac{\sin x}{x}$$

$$= 2 \times 1 = 2$$

method 2: calculus (l'hospital's rule)

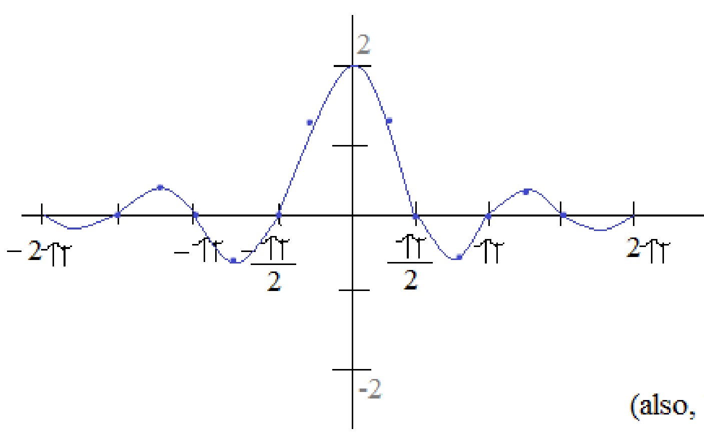
derivative of sin2x = 2cos2x
derivative of x = 1

$\lim_{x \to 0} \frac{\sin 2x}{x}$ using substitution: $\frac{0}{0}$ (since $\frac{0}{0}$, can use l'hospital's rule)

(after taking the 1st derivative) $\lim_{x \to 0} \frac{2\cos 2x}{1} = 2$

method 3: sketch graph and determine behavior

x	y
$-\pi$	0
$-3\pi/4$	$-.424$
$-\pi/2$	0
$-\pi/4$	1.27
0	?
.78	1.27
1.57	0
2.35	$-.424$
3.14	0
3.92	.255
4.71	0



(also, $\lim_{\pm\infty} \frac{\sin(2x)}{x} = 0$)

X	0	1	2	3
f	0	2	0	−2
f'	3	0	DNE	−3
f''	0	−1	DNE	0

X	$0 < X < 1$	$1 < X < 2$	$2 < X < 3$
f	+	+	–
f'	+	–	–
f''	–	–	–

The charts represent the function f (X) on the interval (0, 3)

a) What are the absolute extrema?

b) What are the point(s) of inflection?

c) Sketch the graph of f (X)

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a) The function increases from 0 to 1, then it decreases from 1 to 3. (and, $f' = 0$ at x = 1).
Therefore, the absolute maximum in the interval [0, 3] occurs at x = 1 (the coordinate (1, 2))

And, the minimum will occur at either x = 0 or x = 3...
Since $f(0) = 0$ and $f(3) = -2$, the absolute minimum occurs at x = 3 (the coordinate (3, -2))

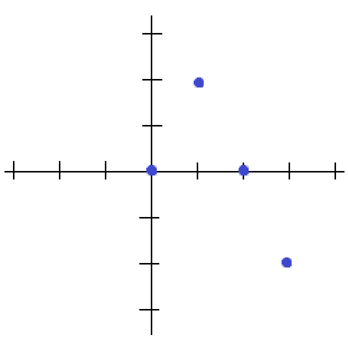
b) A point of inflection occurs when the second derivative equals zero.

On the interval (0, 3), there are no points of inflection.

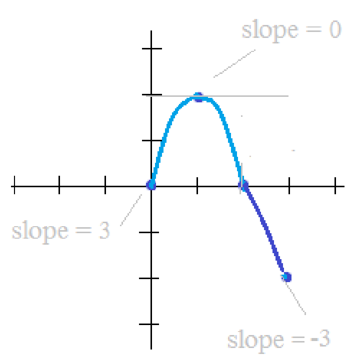
If the domain of the function were extended, there would be points of inflection at x = 0 and x = 3

c) to sketch the graph, start with the function:
Coordinates will include

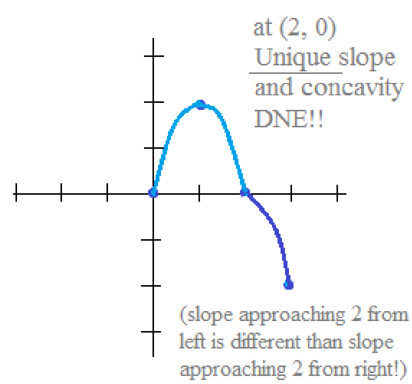
(0, 0) (1, 2) (2, 0) (3, -2)



then, use the first derivative f' to identify the instantaneous slope...



Use the 2 charts and second derivatives to smooth the curves....



Example:

The length of a rectangle is decreasing at a rate of 2 feet/minute.
The width of a rectangle is increasing at a rate of 2 feet/minute.

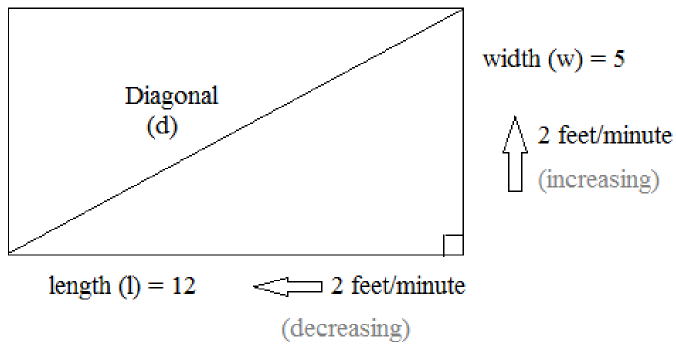
If the length is 12 feet and the width is 5 feet find the rates of the change of the:

a) Area

b) Perimeter

c) Diagonal Length

Step 1: Draw a picture and label given values



rates of change (with respect time (t))

$\frac{dw}{dt} = 2$ ft/min

$\frac{dl}{dt} = -2$ ft/min

Step 2: Write equations (that show how the variables relate to each other)

Area = length x width

Perimeter = 2(length) + 2(width)

Diagonal = $\sqrt{(\text{length})^2 + (\text{width})^2}$ (Pythagorean Theorem)

Step 3: Solve using (implicit) differentiation

a) To find the change of area with respect to time,

$$\frac{dA}{dt} = \frac{dl}{dt}w + \frac{dw}{dt}l \quad \text{(product rule)}$$

$$\frac{dA}{dt} = -2 \text{ ft/min } (5) + 2 \text{ ft/min } (12) \quad \text{(substitution)}$$

$$\boxed{\frac{dA}{dt} = 14 \text{ feet/minute}}$$

b) To find the change in perimeter with respect to time,

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$

$$\boxed{\frac{dP}{dt} = 2(-2 \text{ ft/min}) + 2(2 \text{ ft/min}) = 0 \text{ feet/minute}}$$

c) To find the change in each diagonal with respect to time,

$$\frac{dD}{dt} = \frac{1}{2}(l^2 + w^2)^{-\frac{1}{2}}\left(2l\frac{dl}{dt} + 2w\frac{dw}{dt}\right) \quad \text{(power rule/chain rule)}$$

$$\frac{dD}{dt} = \frac{1}{2}(144 + 25)^{-\frac{1}{2}}(-48 + 20)$$

$$\boxed{\frac{dD}{dt} = \frac{-28}{2(13)} = -\frac{14}{13} \text{ feet/minute}}$$

approx. -1.08

Step 4: Check for reasonableness

one minute ago: length = 14
width = 3
area = 42

$\triangle$ area = 18

now: length = 12
width = 5
area = 60

14 is in between!

$\triangle$ area = 10

one minute later: length = 10
width = 7
area = 70

That makes sense... As the lengths increase 2 feet each, the widths decrease 2 feet each. Although the shape is changing, the perimeter does not change.

one minute ago: length = 14
width = 3
diagonal $\approx$ 14.3

$\triangle$ diagonal = -1.3

now: length = 12
width = 5
diagonal = 13

-1.08 is in between!

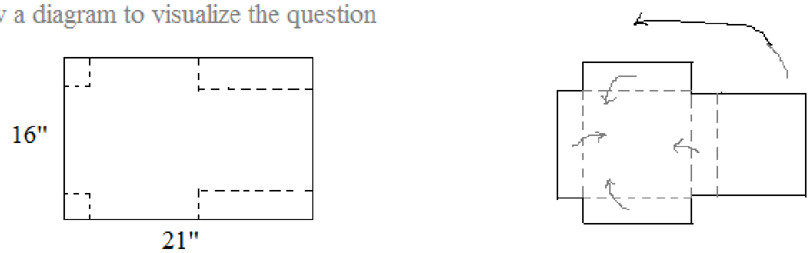
one minute later: length = 10
width = 7
diagonal $\approx$ 12.2

$\triangle$ diagonal = -.8

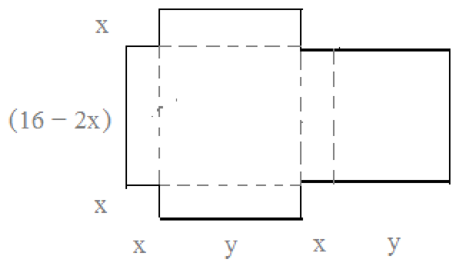
You are given a 16" x 21" cardboard sheet.
After cutting out the corners, you can fold up 3 of the sides.
Then, the fourth side will be folded up and extended over the other 3 to form a lid.

What are the dimensions of the enclosed box with the largest volume?

Step 1: Draw a diagram to visualize the question



Step 2: Label diagram, establish variables, and write equations



since $2x + 2y = 21$, $y = \dfrac{21 - 2x}{2}$

Volume = (length)(width)(height)

length = $(16 - 2x)$

height = x

width = $\dfrac{(21 - 2x)}{2}$

$$V = (16 - 2x)\left(\frac{(21 - 2x)}{2}\right)(x)$$

Step 3: Solve.

To find the maximum (or minimum) volume, find dV/dx and set it equal to 0...

$$V = (16x - 2x^2)\left(\frac{(21 - 2x)}{2}\right)$$

$$V = (8x - x^2)(21 - 2x)$$

$$V = 168x - 16x^2 - 21x^2 + 2x^3$$

$$V = 2x^3 - 37x^2 + 168x$$

$$\frac{dV}{dx} = 6x^2 - 74x + 168$$

then, set derivative equal to zero...

$$6x^2 - 74x + 168 = 0$$

$$3x^2 - 37x + 84 = 0$$

$$x = 3 \text{ or } 28/3$$

Step 4: Answer question and check solutions

If $x = 28/3$, height = 9.33
length = $(16 - 2(9.33)) = -2.66$

Impossible... $x \neq 28/3$

If $x = 3$, height = 3
length = $(16 + 2(3)) = 10$
width = $\dfrac{(21 + 2(3))}{2} = 7.5$

The dimensions of the box (with lid) are

10" x 7.5" x 3"

Check:

If x = 2, then dimensions are 12" x 8.5" x 2" 204 cubic inches
If x = 2.5, then dimensions are 11" x 8" x 2.5" 220 cubic inches
If x = 3, then dimensions are 10" x 7.5" x 3" 225 cubic inches ⇐
If x = 3.5, then dimensions are 9" x 7" x 3.5" 220.5 cubic inches
If x = 4, then dimensions are 8" x 6.5" x 4" 208 cubic inches

The change in temperature is directly proportional to the difference of the item and the surrounding temperature...

Newton's Law of Cooling

Example: A thermometer reading 85 degrees is brought into a room of 72 degrees.
After 2 minutes, the thermometer reads 83 degrees...

When will the thermometer's temperature read 75 degrees? 65 degrees?

$$\frac{dT}{dt} = k\,(T_0 - T_M)$$

change in temperature as it relates to time	constant of proportion	difference of temperature and medium...

We'll use separation of variables to establish the function....

$$\frac{dT}{dt} = k(T - 72)$$

$$\frac{1}{(T-72)}\,dT = k\,dt$$

$$\int \frac{1}{(T-72)}\,dT = \int k\,dt$$

$$\ln|T - 72| = kt + C$$

$$T - 72 = e^{kt + C}$$

$$T = Ce^{kt} + 72$$

$$85 = Ce^{k(0)} + 72$$ ⇦ At time (t) = 0, the temperature (T) = 85

$C = 13$ (the initial difference)

$$T = 13e^{kt} + 72$$

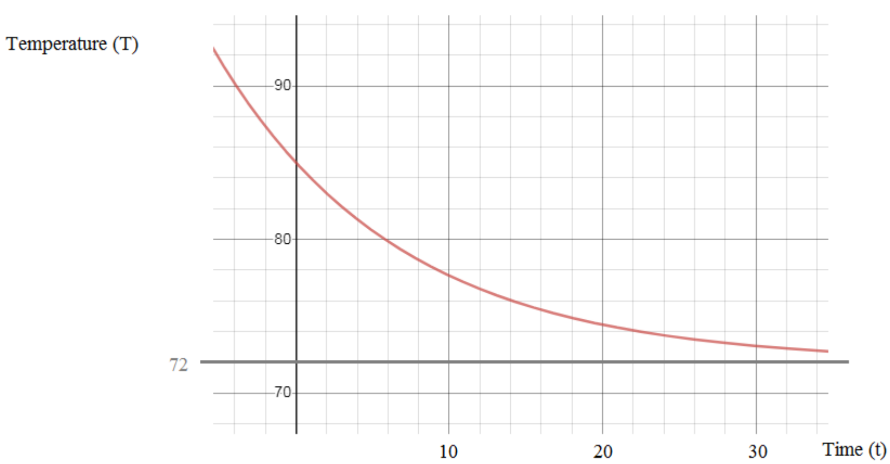
$$83 = 13e^{2k} + 72$$ ⇦ To find k, we plugged in a known amount... at t = 2, the temperature is T = 83

$$\frac{11}{13} = e^{2k}$$

$$\ln\left(\frac{11}{13}\right) = 2k$$

$$k \cong -.08353$$

$$T = 13e^{-.08353t} + 72$$



Let T = 75:

t is approximately 17.5 minutes

Let T = 65: there is no solution...

It'll never reach 65 degrees

Examples: $\displaystyle\int \frac{x^2 - 2x - 2}{x^3 - 1}\,dx$

Factor the denominator, then decompose into partial fractions

Using Partial Fractions...

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2+x+1)}$$

Solve the rational equation with common denominators...

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A\,(x^2+x+1)}{(x-1)\,(x^2+x+1)} + \frac{(Bx + C)\,(x-1)}{(x^2+x+1)\,(x-1)}$$

$$x^2 - 2x - 2 = A\,(x^2+x+1) + (Bx + C)(x-1)$$

$$x^2 - 2x - 2 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$x^2 = (A + B)x^2$$

$$-2x = (A - B + C)x$$

$$-2 = (A - C)$$

Solve the system:

$A + B = 1$

$A - B + C = -2$

$A - C = -2$

$2A - B = -4$

$3A = -3$

$A = -1$

$B = 2$

$C = 1$

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx + C}{(x^2+x+1)}$$

$$= \frac{-1}{(x-1)} + \frac{2x+1}{(x^2+x+1)}$$

$$\int \frac{x^2 - 2x - 2}{x^3 - 1}\,dx = \int \frac{-1}{(x-1)}\,dx + \int \frac{2x+1}{(x^2+x+1)}\,dx$$

$$-\ln|x-1| + \ln|x^2+x+1| + C$$

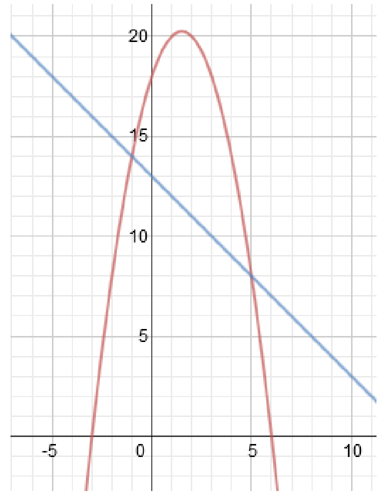
Find the volume of the solid formed by the region bounded by

$$y = -x^2 + 3x + 18$$

$$x + y = 13$$

and revolved around the x-axis

Integration: Volume of a Solid



When we graph the equations, we observe an upside down parabola that is intersected by a line.

The boundary (from left to right) are the points of intersection.

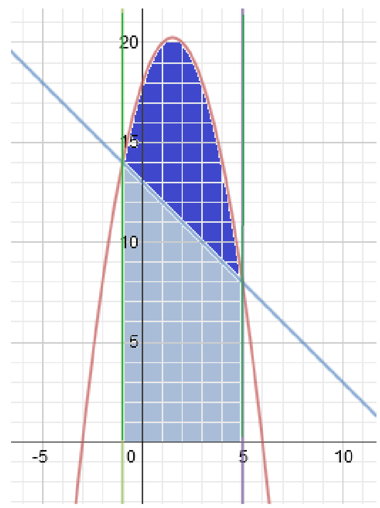
$$\begin{cases} y = -x^2 + 3x + 18 \\ y = -x + 13 \end{cases} \quad \text{substitution}$$

$$-x + 13 = -x^2 + 3x + 18 \quad \text{collect terms}$$

$$x^2 - 4x - 5 = 0 \quad \text{factor}$$

$$(x - 5)(x + 1) = 0 \quad \text{solve}$$

$x = -1$ and $x = 5$



The outer radius will be from the parabola
The inner radius will be from the line

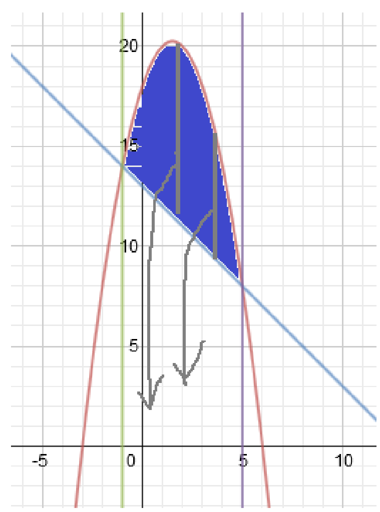
$$\int_{-1}^{5} -x^2 + 3x + 18 \; dx \; - \; \int_{-1}^{5} -x + 13 \; dx$$

area below the parabola — area below the line

$$\int_{-1}^{5} \pi\,(-x^2 + 3x + 18)^2 \, dx \; - \; \int_{-1}^{5} \pi\,(-x + 13)^2 \, dx$$

discs from the parabola — discs from the line

$$\pi \int_{-1}^{5} x^4 - 6x^3 - 27x^2 + 108x + 324 \; dx \; - \; \pi \int_{-1}^{5} x^2 - 26x + 169 \; dx$$

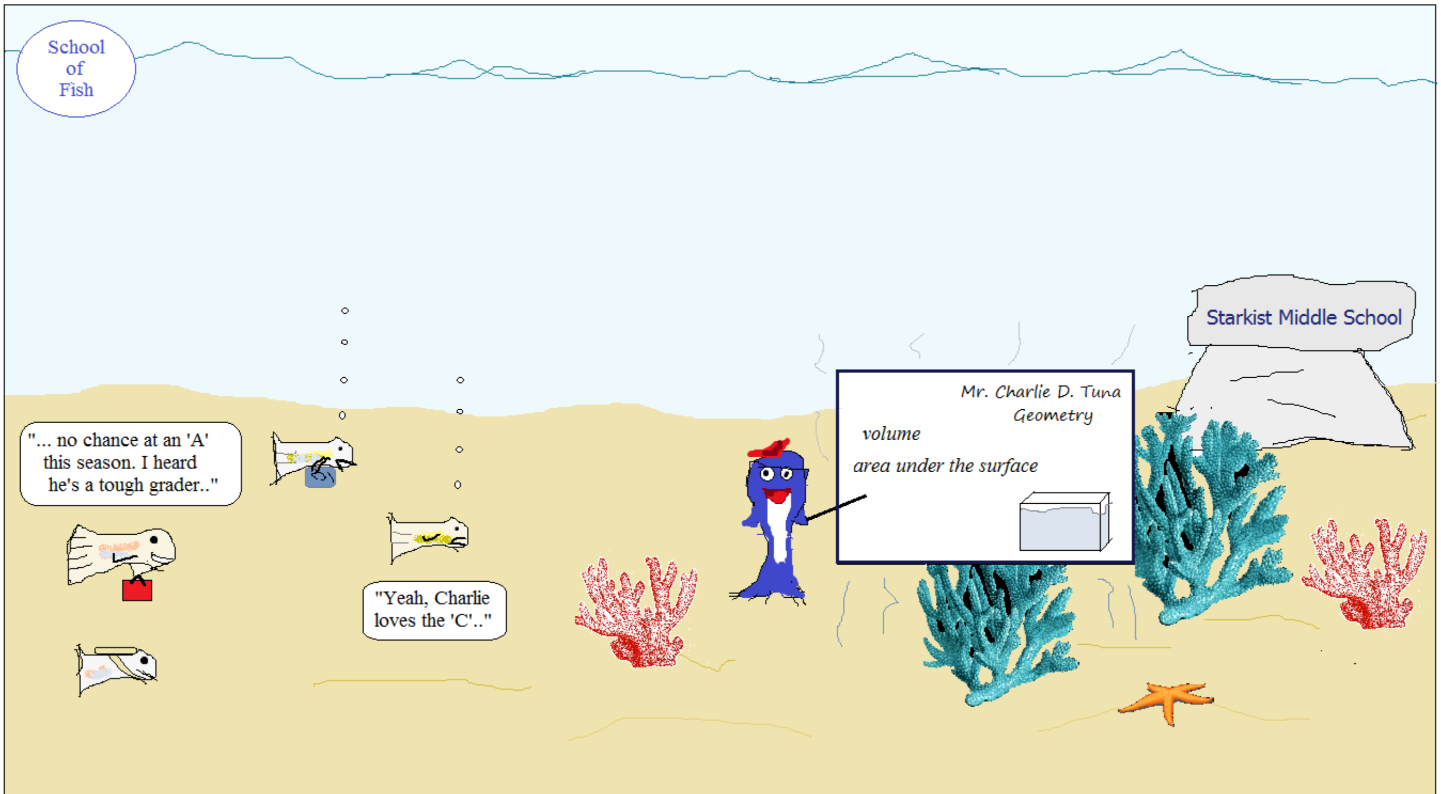


$$\pi \int_{-1}^{5} x^4 - 6x^3 - 28x^2 + 134x + 155 \; dx$$

$$\pi \left(\frac{x^5}{5} - \frac{3x^4}{2} - \frac{28x^3}{3} + 67x^2 + 155x \Big|_{-1}^{5} \right)$$

$$\pi \left(625 - 937.5 - 1166\tfrac{2}{3} + 1675 + 775 - \left(\frac{-1}{5} - \frac{3}{2} - \frac{-28}{3} + 67 - 155 \right) \right)$$

$$970.83 - (-80.36) = \boxed{1051.2\,\pi}$$



After a fun summer, Gil, Fin, Cod, and Sole swim back to school...

LanceAF #374 (8-1-20)
mathplane.com

Practice Questions-→

For the function $f(x) = 3x^2 + 3$

find $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ where $h \neq 0$ for $g(x) = 2x^4$

For what value(s) of B is $f(x)$ continuous?

$$f(x) = \begin{cases} x^2 + 4 & \text{if } x \leq 2 \\ Bx & \text{if } x > 2 \end{cases}$$

Is $f(x)$ differentiable? Why or why not?

Sketch a graph with the following conditions:

1) Local maximum at x = 3, and differentiable at x = 3

2) Local maximum at x = 3, and not continuous at x = 3

3) Local maximum at x = 3, but not differentiable at x = 3

Derivatives formulas: Concepts Quiz

Assume the following: $u(3) = -1$ $u'(3) = 4$

$v(3) = 2$ $v'(3) = 6$

Evaluate the following derivatives at $x = 3$

a) $\frac{d}{dx}(u + v)$

b) $\frac{d}{dx}(uv)$

c) $\frac{d}{dx}(3v)$

d) $\frac{d}{dx}(2u + 4v)$

e) $\frac{d}{dx}\left(\frac{v}{5}\right)$

f) $\frac{d}{dx}\left(\frac{u}{v}\right)$

g) $\frac{d}{dx}\left(\frac{v}{u}\right)$

h) $\frac{d}{dx}\ \frac{a}{v^2}$

i) $\frac{d}{dx}\ (uv)^4$

Answer each question. Then, sketch a graph to verify. (optional)

a) Write the equation of the line tangent to $x^3 + 5$ @ $x = 2$

b) At what point(s) are the tangent lines horizontal for the function

$$f(x) = x^3 - \frac{3x^2}{2} - 6x$$

c) What is the equation of the normal line to the function $-x^2 + 8$ @ $x = 4$

d) given the quadratic $y = x^2 - 6x + 5$

Determine the AROC (average rate of change) on the interval [2, 6]
the IROC (instantaneous rate of change) at the midpoint $x = 4$

Are they the same? If not, where is IROC equal to AROC in the interval?
(according to the Mean Value Theorem, there is at least one point in the interval.)

e) If the line $y = 3x + 2$ is tangent in the 1st quadrant to the curve $y = x^3 - k$, then what is k?

Find the derivative of $(1 + \cos^2 7x)^3$

What value of b gives the function a minimum at x = 2?

$$y = bx + \frac{3}{x}$$

Find the points on the curve $y = x^3 + 7$ where the tangents are parallel to the x-axis

$f(x) = Ax^3 + 18x^2 + Bx$ has a local max at x = 7 and a point of inflection at x = 3

Find A and B.

If $y = (\cos x)(\sin x)$, what is the slope of the graph at $\frac{\pi}{3}$

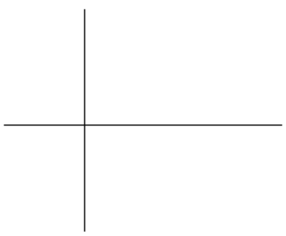
Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $x = 0$

Use it to approximate the value of $\sqrt{.9}$

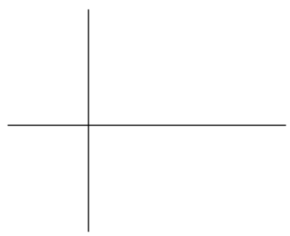
Then, compare the approximate value to the actual value.

Sketch a graph of a function with the following:

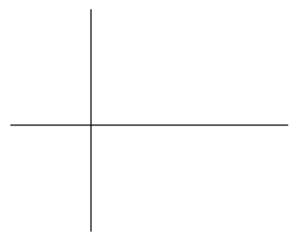
a) $f(x) > 0$
$f'(x) > 0$
$f''(x) > 0$



b) $f(x) < 0$
$f'(x) > 0$
$f''(x) < 0$



c) $f(x) > 0$
$f'(x) < 0$
$f''(x) < 0$



Find the function $f(x)$ where

$f''(x) = 8x + 1$

Graph of $f(x)$ passes through (2, 1)
and is tangent to the line $3x - y - 5 = 0$

$$\int_3^5 2f(x) = 8 \qquad \int_6^5 f(x) = 2 \qquad \text{Find} \int_3^6 f(x) =$$

If the average value of f on the interval $3 \le x \le 6$ is 7,

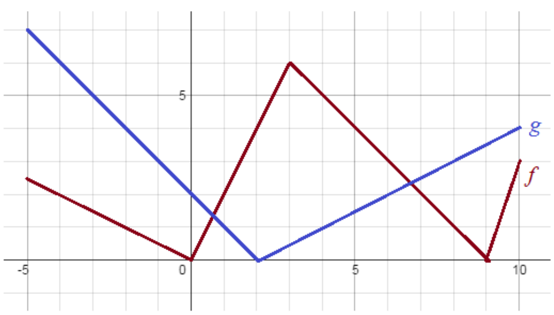
what is

$$\int_3^6 4f(x) + 3 \ dx \quad ?$$

A bug travels along the x-axis.
The model of its velocity is $v(t) = t^2 - 8t + 15$

where $0 \le t \le 18$ minutes, and the initial position $s(0) = 43$ feet

What is the position of the bug when its acceleration is 6 inches per minute?



$U(x) = f(x) \cdot g(x)$ $\qquad V(x) = \dfrac{f(x)}{g(x)}$ $\qquad W(x) = \dfrac{2f(x)}{x^2}$

a) $U'(4) =$

b) $V'(6) =$

c) $W'(1) =$

Find $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

where $h \neq 0$

For the function $f(x) = 3x^2 + 3$

$\lim_{h \to 0} \frac{3(a+h)^2 + 3 - (3a^2 + 3)}{h}$

$\lim_{h \to 0} \frac{\cancel{3a^2} + 6ah + 3h^2 \cancel{+3} - \cancel{3a^2} \cancel{-3}}{h}$

$\lim_{h \to 0} \frac{6ah + 3h^2}{h} = \lim_{h \to 0} 6a + 3h = \boxed{6a}$

For the function $g(x) = 2x^4$

$\lim_{h \to 0} \frac{2(a+h)^4 - 2a^4}{h}$

$\lim_{h \to 0} \frac{\cancel{2a^4} + 8a^3h + 12a^2h^2 + 8ah^3 + 2h^4 - \cancel{2a^4}}{h}$

$\lim_{h \to 0} 8a^3 + 12a^2h + 8ah^2 + 2h^3 = \boxed{8a^3}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

For what value(s) of B is $f(x)$ continuous?

$$f(x) = \begin{cases} x^2 + 4 & \text{if } x \leq 2 \\ Bx & \text{if } x > 2 \end{cases}$$

Is $f(x)$ differentiable? Why or why not?

Limit from left must equal limit from right

$\lim_{x \to 2^-} f(x) = 8$

$\lim_{x \to 2^+} f(x) = 8$ iff $B = 4$

continuous when B = 4

SOLUTIONS

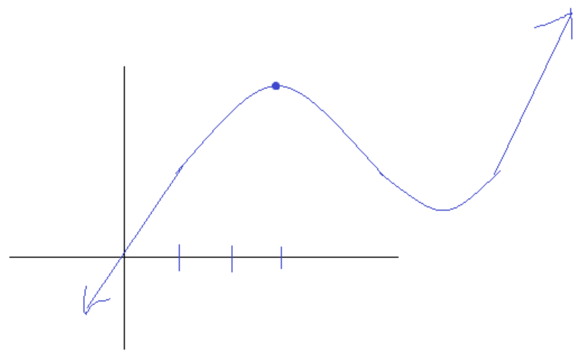
To be differentiable, derivative from the left must equal derivative from the right....

Derivative from the left is 2x ---> @x = 2, it's 4
Derivative from the right is B ---> @x = 2, it's 4

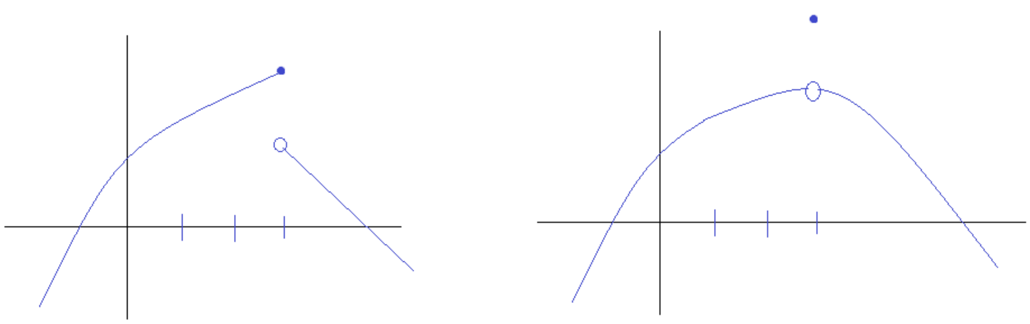
It's differentiable!!

Sketch a graph with the following conditions:

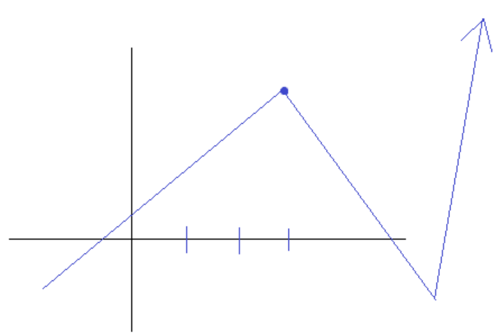
1) Local maximum at x = 3, and differentiable at x = 3



2) Local maximum at x = 3, and not continuous at x = 3



3) Local maximum at x = 3, but not differentiable at x = 3



Derivatives formulas: Concepts Quiz

SOLUTIONS

Assume the following: $u(3) = -1$ $\quad u'(3) = 4$

$v(3) = 2$ $\quad v'(3) = 6$

Evaluate the following derivatives at $x = 3$

a) $\frac{d}{dx}(u + v)$ $\quad u'(3) + v'(3) = 4 + 6 = \boxed{10}$ sum

b) $\frac{d}{dx}(uv)$ $\quad u'v + v'u = 4(2) + 6(-1) = \boxed{2}$ product rule

c) $\frac{d}{dx}(3v)$ $\quad 3v' = 3(6) = \boxed{18}$ scalar

d) $\frac{d}{dx}(2u + 4v)$ $\quad 2u' + 4v' = 8 + 24 = \boxed{32}$

e) $\frac{d}{dx}\left(\frac{v}{5}\right)$ $\quad \frac{1}{5}v' = (1/5)(6) = \boxed{6/5}$

f) $\frac{d}{dx}\left(\frac{u}{v}\right)$ $\quad u'v - v'u = 4(2) - 6(-1) = \boxed{14}$ quotient rule

g) $\frac{d}{dx}\left(\frac{v}{u}\right)$ $\quad v'u - u'v = 6(-1) - 4(2) = \boxed{-14}$

h) $\frac{d}{dx}\frac{a}{v^2}$ $\quad av^{-2} \Rightarrow -2av^{-3}v' = -2a(2)^{-3}(6)$ power rule chain rule

$= \frac{-12a}{8} = \boxed{\frac{-4a}{3}}$

i) $\frac{d}{dx}(uv)^4$ $\quad 4(uv)^3 \cdot \frac{d}{dx}(uv)$

$= 4(-2)^3 \cdot 2 = \boxed{-64}$

power/chain rule

Answer each question. Then, sketch a graph to verify. (optional)

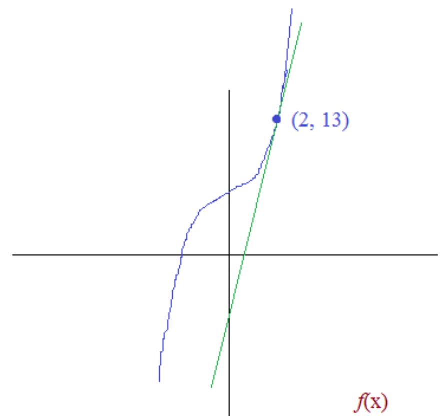
a) Write the equation of the line tangent to $x^3 + 5$ @ $x = 2$

"equation of a line" requires
2 parts: slope and point...

point: $x = 2$ then, $y = 13$ $\quad (2, 13)$

slope: derivative $= 3x^2 + 0$ @ $x = 2$, the slope is 12

$\boxed{y - 13 = 12(x - 2)}$



b) At what point(s) are the tangent lines horizontal for the function

$$f(x) = x^3 - \frac{3x^2}{2} - 6x$$

horizontal lines have slope = 0

$f'(x) = 3x^2 - 3x - 6$

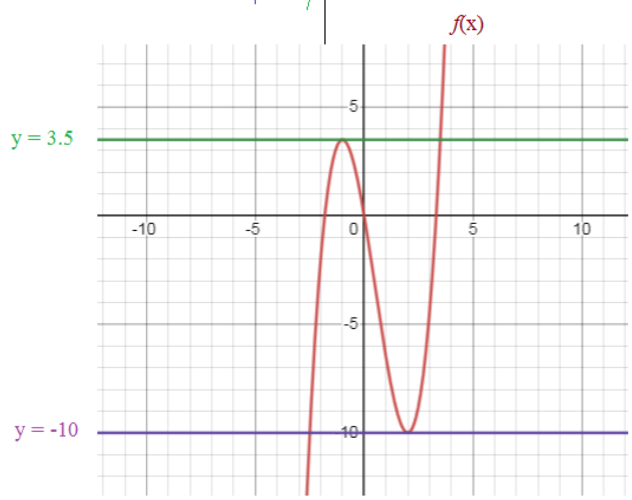
$0 = 3x^2 - 3x - 6$

$0 = x^2 - x - 2$

$0 = (x - 2)(x + 1)$

$x = 2, -1$

$\boxed{(-1, 3.5) \quad (2, -10)}$



c) What is the equation of the normal line to the function $-x^2 + 8$ @ $x = 4$

A normal line is perpendicular to the tangent line...

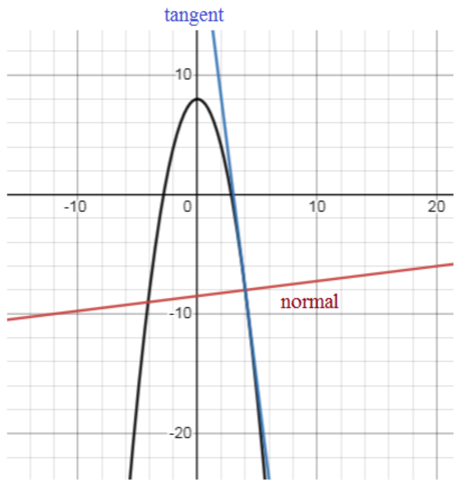
SOLUTIONS

point: at $x = 4$, $(4, -8)$

slope: derivative at $x = 4$ $-2x + 0$ ----> slope of tangent = -8

therefore slope of normal = 1/8

$$y + 8 = \frac{1}{8}(x - 4)$$



d) given the quadratic $y = x^2 - 6x + 5$

Determine the AROC (average rate of change) on the interval [2, 6]
the IROC (instantaneous rate of change) at the midpoint $x = 4$

Are they the same? If not, where is IROC equal to AROC in the interval?
(according to the Mean Value Theorem, there is at least one point in the interval.)

AROC: $x = 2$, $y = -3$ $(2, -3)$
$x = 6$, $y = 5$ $(6, 5)$

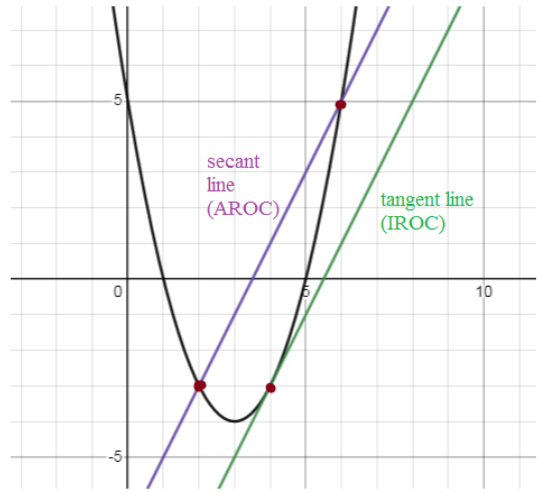
slope = 2

The same!!

IROC: derivative $y' = 2x - 6$ @ $x = 4$, slope is 2

IROC = AROC when $y' = 2$

$2x - 6 = 2$ $x = 4$



e) If the line $y = 3x + 2$ is tangent in the 1st quadrant to the curve $y = x^3 - k$, then what is k?

Since the slope of the tangent line is 3, the slope at the point on the curve is 3...

$$y' = 3x^2 - 0$$

$y' = 3$ so, $x = 1$ or $x = -1$

Since the tangent is in Quadrant I, we can eliminate $x = -1$

In Quad I, if $x = 1$, then, the point on the line is $(1, 5)$....

Then, we can take $(1, 5)$ and substitute it into $y = x^3 - k$

$$5 = (1)^3 - k$$

$$k = -4$$

Quick check: $y = x^3 + 4$

line tangent is $y = 3x + 2$ at the point $(1, 5)$

slope at that point is 3
and, $(1, 5)$ is a common point...

SOLUTIONS

Find the derivative of $(1 + \cos^2 7x)^3$

$$y' = 3(1 + \cos^2 7x)^2 \cdot [0 + 2(\cos 7x)^1(-\sin 7x)(7)]$$

$$= 3(1 + \cos^2 7x)^2 [-14(\cos 7x)(\sin 7x)]$$

$$= -42(1 + \cos^2 7x)^2 (\cos 7x)(\sin 7x)$$

What value of b gives the function a minimum at x = 2?

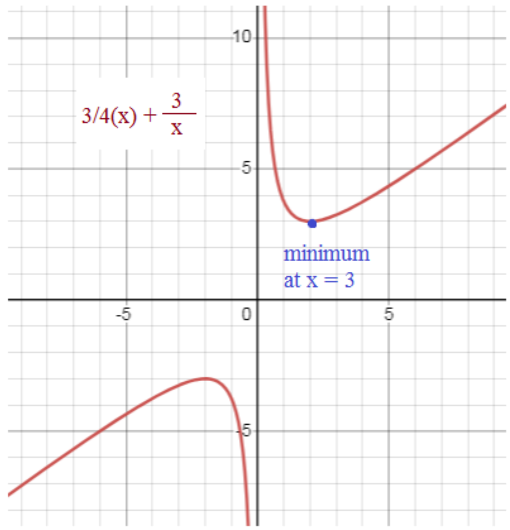
$$y = bx + \frac{3}{x}$$

critical point (min or max) occurs when derivative is zero at the point.

$$y' = b + \frac{-3}{x^2}$$

$$0 = b + \frac{-3}{4}$$

$$b = \frac{3}{4}$$



Find the points on the curve $y = x^3 + 7$ where the tangents are parallel to the x-axis

If tangents are parallel to the x-axis, then they are horizontal lines with slope = 0

$$y' = 3x^2 + 0$$

$y' = 0$ when $x = 0$

(0, 7)

$f(x) = Ax^3 + 18x^2 + Bx$ has a local max at $x = 7$ and a point of inflection at $x = 3$

Find A and B.

$f'(x) = 3Ax^2 + 36x + B$ Since a local max is at 7, $f'(7) = 0$

$$= 3A(49) + 36(7) + B = 0$$

$$147A + B = -252$$

$f''(x) = 6Ax + 36$ Since a point of inflection is at 3, $f''(3) = 0$

$$= 6A(3) + 36 = 0$$

$$18A = -36$$

$$A = -2$$

$$147(-2) + B = -252$$

$$-294 + B = -252$$

$$B = 42$$

$$f(x) = -2x^3 + 18x^2 + 42x$$

If $y = (\cos x)(\sin x)$, what is the slope of the graph at $\frac{\pi}{3}$

$y' = (-\sin x)(\sin x) + (\cos x)(\cos x)$ at $\frac{\pi}{3}$

(product rule)

slope is $(-\sin\frac{\pi}{3})(\sin\frac{\pi}{3}) + (\cos\frac{\pi}{3})(\cos\frac{\pi}{3})$

$$-\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = -1/2$$

Find the linear approximation of the function $f(x) = \sqrt{1 - x}$ at $x = 0$

Use it to approximate the value of $\sqrt{.9}$

Then, compare the approximate value to the actual value.

For *linear approximation*, we need a point and the slope (instantaneous rate of change)

The point: @ $x = 0$, $f(0) = 1$ (0, 1)

The slope: $f'(x) = \frac{1}{2}(1 - x)^{-\frac{1}{2}}(-1) = \frac{-1}{2\sqrt{1 - x}}$

and, the slope at $x = 0$ is $\frac{-1}{2}$

The linear approximation is $y - 1 = \frac{-1}{2}(x - 0)$

$$y = \frac{-1}{2}x + 1$$

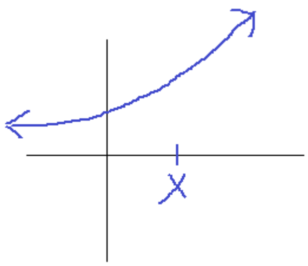
To approximate $\sqrt{.9}$, we let $x = .1$ because $\sqrt{1 - .1} = \sqrt{.9}$

and, if $x = .1$, then (using the linear approximation $y = \frac{-1}{2}x + 1$), the output is $(-.5)(.1) + 1 =$ .95

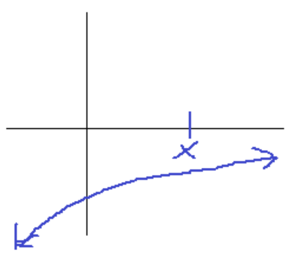
The *actual* value of $\sqrt{.9}$ (using a calculator) is .94868

Sketch a graph of a function with the following:

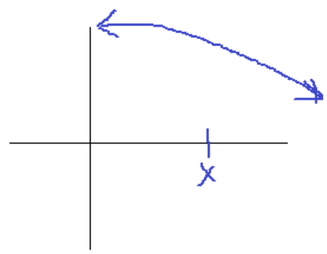
a) $f(x) > 0$ above x-axis
$f'(x) > 0$ slope up
$f''(x) > 0$ concave up



b) $f(x) < 0$ below x-axis
$f'(x) > 0$ slope up
$f''(x) < 0$ concave down



c) $f(x) > 0$ above x-axis
$f'(x) < 0$ slope down
$f''(x) < 0$ concave down



Find the function $f(x)$ where

$f''(x) = 8x + 1$

Graph of $f(x)$ passes through (2, 1)
and is tangent to the line $3x - y - 5 = 0$

$y = 3x - 5$

$f''(x) = 8x + 1$

$f'(x) = \int 8x + 1 \, dx = 4x^2 + x + C$

since the slope of the tangent line is 3, $f'(2) = 4(2)^2 + 2 + C = 3$

$18 + C = 3$

$C = -15$

$f'(x) = 4x^2 + x - 15$

Now, we'll find $f(x) = \int 4x^2 + x - 15 \, dx \Rightarrow \frac{4x^3}{3} + \frac{x^2}{2} - 15x + C$

Then, since it passes through (2, 1)

$1 = \frac{4(2)^3}{3} + \frac{(2)^2}{2} - 15(2) + C$

$1 = 32/3 + 2 - 30 + C$

$29 = 32/3 + C$ $C = 55/3$

$$\frac{4x^3}{3} + \frac{x^2}{2} - 15x + 55/3$$

$\int_3^5 2f(x) = 8$ $\quad \int_6^5 f(x) = 2$ $\quad$ Find $\int_3^6 f(x) =$

SOLUTIONS

$\int_3^5 f(x) = 4 \qquad \int_5^6 f(x) = -2 \qquad$ so, $\boxed{\int_3^6 f(x) = 2}$

If the average value of f on the interval $3 \le x \le 6$ is 7,

what is $\int_3^6 4f(x) + 3 \; dx$?

$$4\int_3^6 f(x)\, dx + \int_3^6 3\, dx = 4(21) + 9 = \boxed{93}$$

since the average value on the interval is 7, the total area is $(6-3)7 = 21$

A bug travels along the x-axis.
The model of its velocity is $v(t) = t^2 - 8t + 15$

where $0 \le t \le 18$ minutes, and the initial position $s(0) = 43$ feet

What is the position of the bug when its acceleration is 6 inches per minute?

To find the acceleration of the bug, we must find v'(t).

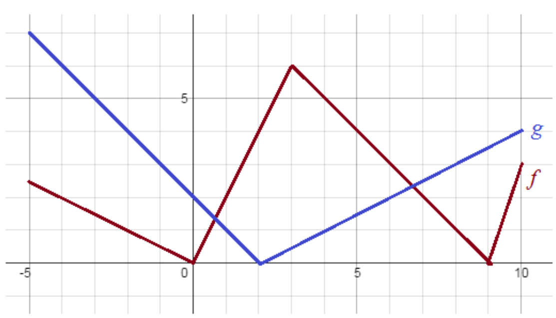
$v'(t) = 2t - 8 \qquad 6 = 2t - 8$, $t = 14$ $\qquad$ The accleration occurs at 14 minutes

Then, to find the position function, we must find the antiderivative of $v(t)$

$$s(t) = \int t^2 - 8t + 15 = \frac{t^3}{3} - 4t^2 + 15t + C$$

since $s(0) = 43$, $C = 43$ $\qquad s(t) = \frac{t^3}{3} - 4t^2 + 15t + 43$

then, $s(14) = 914.67 - 784 + 210 + 43 = \boxed{383.67}$



$U(x) = f(x) \cdot g(x) \qquad V(x) = \frac{f(x)}{g(x)} \qquad W(x) = \frac{2f(x)}{x^2}$

a) $U'(4) =$ $\quad U'(x) = f'(x)g(x) + g'(x)f(x) \Rightarrow \boxed{U'(4) = 3/2}$

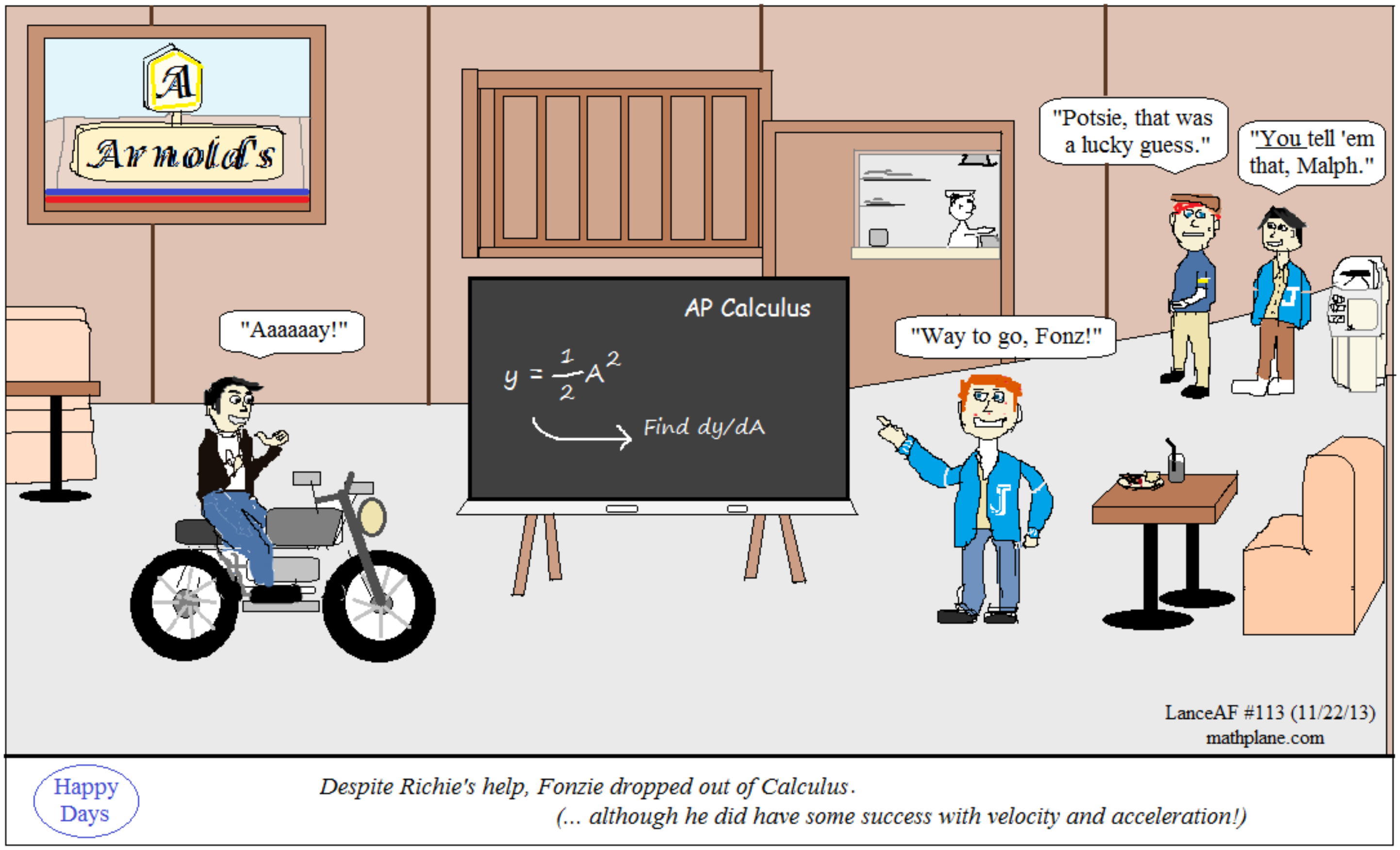
$f'(4) = -1 \quad g(4) = 1 \quad g'(4) = 1/2 \quad f(4) = 5$

b) $V'(6) =$ $\quad V'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} \Rightarrow \boxed{V'(6) = \frac{-2 - 3/2}{4} = \frac{-7}{8}}$

$f'(6) = -1 \quad g(6) = 2 \quad g'(6) = 1/2 \quad f(6) = 3$

c) $W'(1) =$ $\quad W'(x) = \frac{2f'(x)(x^2) - 2x(2f(x))}{(x^2)^2} \Rightarrow \boxed{W'(1) = \frac{2(2)(1) - 2(1)(2)}{1} = 0}$

$f'(1) = 2 \quad x = 1 \quad f(1) = 2$



Thanks for visiting. (Hope it helped!)

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