Conics VI: Honors and Analytic Geometry Examples

Topics include tangent lines, coordinate geometry, properties of conics, parametric equations, and more.

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Conics Combinations

1) The foci of the conic $4x^2 + 16y^2 + 8x - 64y - 188 = 0$ are the endpoints of a circle's diameter.

What is the equation of the circle?

2) The segment joining the vertices of $x^2 - 4y^2 + 6x - 7 = 0$ is the latus rectum of a parabola.

Write an equation for the parabola.

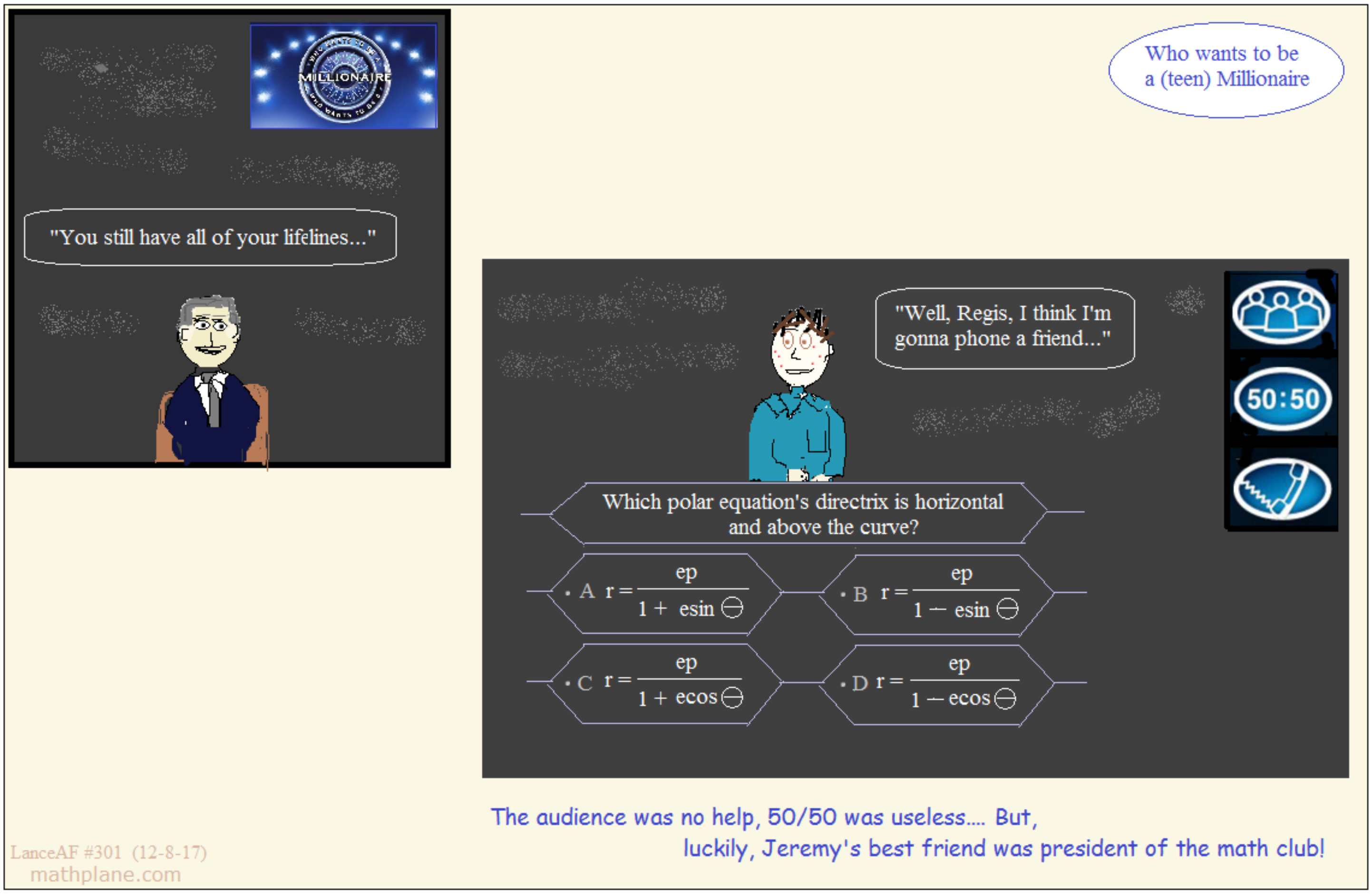
Conics Combinations

3) The vertex of $y^2 - 8x - 6y = 15$ is the center of a circle.
If the circle is tangent to the conic's directrix,

what is the equation of the circle?

4) The foci of $y^2 - x^2 = 100$ are the vertices of an ellipse with eccentricity 3/4.

What is the equation of the ellipse?



Who wants to be a (teen) Millionaire
"You still have all of your lifelines..."
"Well, Regis, I think I'm gonna phone a friend..."
50:50
Which polar equation's directrix is horizontal and above the curve?
A $r = \dfrac{ep}{1 + e\sin\theta}$
B $r = \dfrac{ep}{1 - e\sin\theta}$
C $r = \dfrac{ep}{1 + e\cos\theta}$
D $r = \dfrac{ep}{1 - e\cos\theta}$
The audience was no help, 50/50 was useless.... But,
luckily, Jeremy's best friend was president of the math club!
LanceAF #301 (12-8-17)
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Solutions-→

Conics Combinations

SOLUTIONS

1) The foci of the conic $4x^2 + 16y^2 + 8x - 64y - 188 = 0$

are the endpoints of a circle's diameter.
What is the equation of the circle?

Step 1: Write equation in standard form

$$4x^2 + 8x \qquad 16y^2 - 64y \qquad = 188$$

$$4(x^2 + 2x \quad) + 16(y^2 - 4y \quad) = 188$$

$$4(x^2 + 2x + 1) + 16(y^2 - 4y + 4) = 188 + 4 + 64$$

$$4(x+1)^2 + 16(y-2)^2 = 256$$

$$\boxed{\frac{(x+1)^2}{64} + \frac{(y-2)^2}{16} = 1}$$

Step 2: Indentify the foci

center: (-1, 2)

$a^2 = 64 \quad b^2 = 16 \quad \Rightarrow \quad c^2 = 48$

foci: $(-1 + \sqrt{48}, 2)$ and $(-1 - \sqrt{48}, 2)$

Step 3: Find the equation of the circle...

We need the center and the radius...

center is the midpoint of the foci --- (-1, 2)

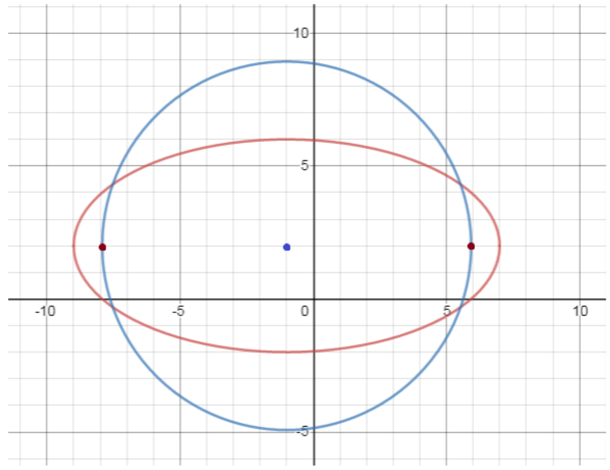
to find the radius, we can use the distance formula...

(-1, 2) $\qquad (-1 + \sqrt{48}, 2)$

center to focus

$$\sqrt{(-1 + \sqrt{48} - (-1))^2 + (2-2)^2} = \sqrt{48} \quad \text{(radius)}$$

$$(x-h)^2 + (y-k)^2 = r^2 \quad \Rightarrow \quad \boxed{(x+1)^2 + (y-2)^2 = 48}$$



2) The segment joining the vertices of $x^2 - 4y^2 + 6x - 7 = 0$
is the latus rectum of a parabola.
Write an equation for the parabola.

Step 1: Write the equation of conic in standard form

$$x^2 + 6x \quad - 4y^2 = 7$$

$$x^2 + 6x + 9 \quad - 4y^2 = 7 + 9$$

$$(x+3)^2 - 4(y-0)^2 = 16$$

$$\frac{(x+3)^2}{16} - \frac{(y-0)^2}{4} = 1$$

Step 2: Idenify needed parts of hyperbola

This is a horizontal hyperbola (opening left and right)

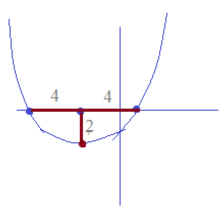
The center is (-3, 0)

and, the vertices are (-7, 0) and (1, 0)

Step 3: Find an equation of a parabola

the endpoints of the latus rectum are (-7, 0) and (1, 0)

the focus of the parabola would be the midpoint: (-3, 0)

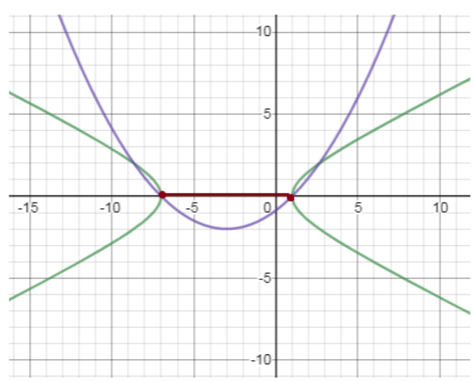


Equation of an upward opening parabola

$$(x-h)^2 = 4p(y-k)$$

$p = 2 \qquad (h, k) = (-3, -2)$

$$\boxed{(x+3)^2 = 8(y+2)} \quad \boxed{y = \frac{1}{8}(x+3)^2 - 2}$$



Note: another answer would be a downward facing parabola with vertex at (-3, 2)

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Conics Combinations

SOLUTIONS

3) The vertex of $y^2 - 8x - 6y = 15$ is the center of a circle. If the circle is tangent to the conic's directrix, what is the equation of the circle?

Step 1: Find the characteristics of the conic (parabola)

$$y^2 - 6y = 8x + 15$$

$$y^2 - 6y + 9 = 8x + 15 + 9$$

$$(y-3)^2 = 8(x+3)$$ parabola that opens out to the right...

vertex: (-3, 3)

$p = 2$

focus: (-1, 3)

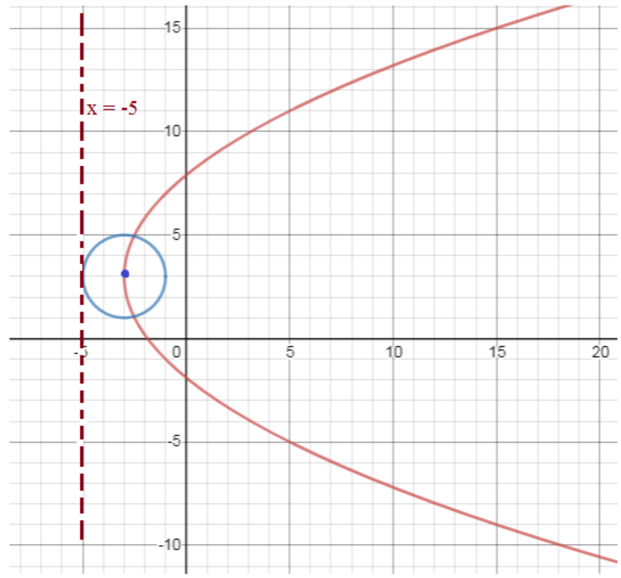
directrix: $x = -5$

Step 2: apply parts to the circle....

center of the circle: (-3, 3)

since the directrix is $x = -5$, the radius is 2...

$$\boxed{(x+3)^2 + (y-3)^2 = 4}$$



4) The foci of $y^2 - x^2 = 100$ are the vertices of an ellipse with eccentricity 3/4.

What is the equation of the ellipse?

Step 1: find foci of hyperbola

$$\frac{y^2}{100} - \frac{x^2}{100} = 1$$

center is (0, 0)

vertices are (0, 10) and (0, -10)

foci are $(0, 10\sqrt{2})$ and $(0, -10\sqrt{2})$

Step 2: find the ellipse, using parts

center is (0, 0) midpoint of the vertices

vertices are $(0, 10\sqrt{2})$ and $(0, -10\sqrt{2})$

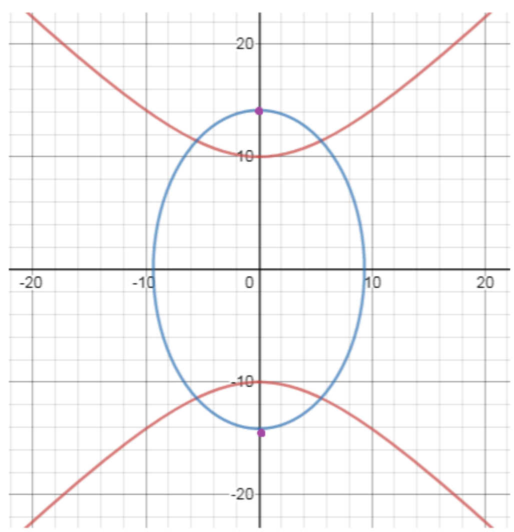
$$\frac{y^2}{200} + \frac{x^2}{b^2} = 1$$

eccentricity = c/a $\qquad \frac{3}{4} = \frac{c}{10\sqrt{2}} \qquad c = \frac{15\sqrt{2}}{2}$

And, we know $c^2 = a^2 - b^2$

$$\frac{450}{4} = 200 - b^2$$

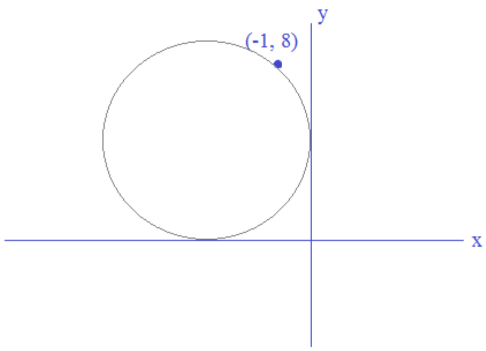
$$b^2 = \frac{350}{4}$$



$$\boxed{\frac{y^2}{200} + \frac{x^2}{\frac{350}{4}} = 1} \qquad \boxed{\frac{y^2}{200} + \frac{2x^2}{175} = 1}$$

Example: Find the equation of a circle that is tangent to both axes and goes through the point $(-1, 8)$

Step 1: Draw a diagram



Step 2: Identify theorems and formulas that may help

Equation of a circle: $(x - h)^2 + (y - k)^2 = r^2$

where (h, k) is the center
and r is the radius

Geometry theorem: "all radii are congruent"

Distance Formula: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
(optional)

Step 3: Apply concepts and theorems to set up equation

$$(x - h)^2 + (y - k)^2 = r^2$$

Because the circle is tangent to BOTH axes, the center will be some coordinate $(-r, r)$

Using substitution:

we'll take the point $(-1, 8)$ on the circle
and the center $(-r, r)$

$$(-1 - (-r))^2 + (8 - r)^2 = r^2$$

$$(r - 1)^2 + (8 - r)^2 = r^2$$

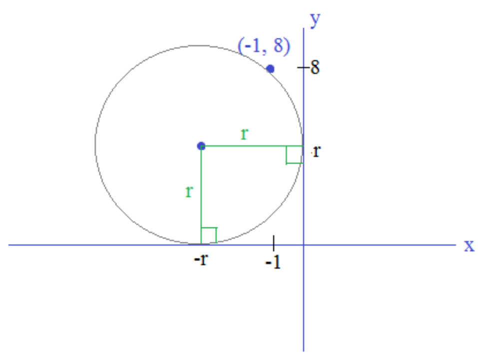
$$r^2 - 2r + 1 + 64 - 16r + r^2 = r^2$$

$$r^2 - 18r + 65 = 0$$

$$(r - 5)(r - 13) = 0$$

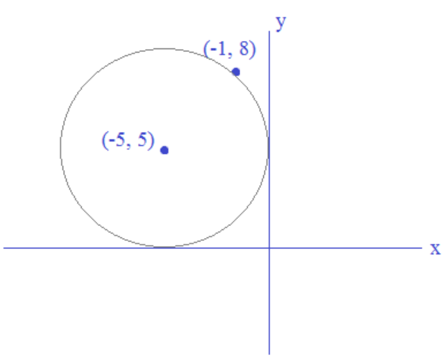
$$r = 5, 13$$

There are 2 circles!



Step 4: check answer

$(x + 5)^2 + (y - 5)^2 = 25$



$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

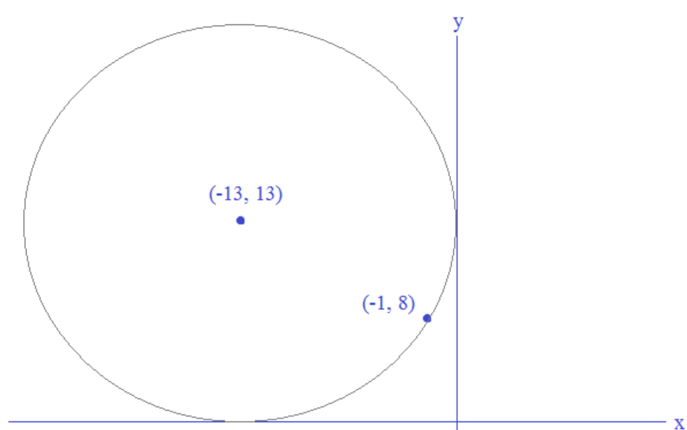
radius is $r = 5$

distance from $(-5, 5)$ to $(-1, 8)$ is

$$\sqrt{(-5 + 1)^2 + (5 - 8)^2}$$

$= 5$ ✓

$(x + 13)^2 + (y - 13)^2 = 169$



radius is $r = 13$

distance from $(-13, 13)$ to $(-1, 8)$ is

$$\sqrt{(-13 + 1)^2 + (13 - 8)^2}$$

$= 13$ ✓

Example: A parabola has a zero at 24 and a horizontal latus rectum of length 12 and an endpoint at (6, 9).

What is the equation of the parabola?

There are two possible answers.

1) latus rectum extends to the left from (6, 9) to (-6, 9)

Since latus rectum is horizontal, the parabola opens up/down..

$x^2 = 4py$

Length of latus rectum is 12, so p = 3

Focus is midpoint of LR: (0, 9)

We know (24, 0) is a point on the curve, so parabola would face down..

Vertex is (0, 12), because it is 3 units above the focus...

$\boxed{x^2 = -12(y-12)}$

When we test the vertex (0, 12), the point works..

$(0)^2 = -12((12)-12)$

$0 = 0$ ✓

When we test the point (24, 0), the point does NOT work...

$(24)^2 = -12((0)-12)$

$576 = 144$ ✗

2) latus rectum extends to the right from (6, 9) to (18, 9)

Since latus rectum is horizontal, the parabola opens up/down..

$x^2 = 4py$

Length of latus rectum is 12, so p = 3

Focus is midpoint of LR: (12, 9)

Vertex: (12, 12)

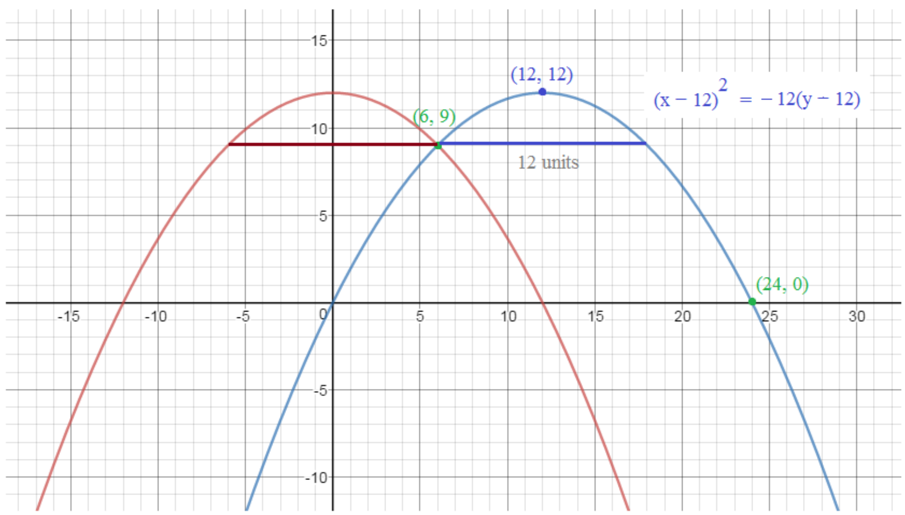
Directrix: y = 15

Parabola faces down, so p value is negative

$\boxed{(x-12)^2 = -12(y-12)}$

Does (24, 0) work?

Yes!!



Example: A circle that passes through (7, 3) and (5, 5) has its center on the line y = 4x + 1.

What is the equation of the circle?

(h, k) is the center

$k = 4h + 1$

$(x-h)^2 + (y-k)^2 = r^2$

Substitute each point:

$(7-h)^2 + (3-k)^2 = r^2$

$(5-h)^2 + (5-k)^2 = r^2$

$(7-h)^2 + (3-k)^2 = (5-h)^2 + (5-k)^2$

$k = 4h + 1$

$(7-h)^2 + (3-(4h+1))^2 = (5-h)^2 + (5-(4h+1))^2$

$49 - 14h + h^2 + 4 - 16h + 16h^2 = 25 - 10h + h^2 + 16 - 32h + 16h^2$

$49 - 14h + \cancel{h^2} + 4 - 16h + \cancel{16h^2} = 25 - 10h + \cancel{h^2} + 16 - 32h + \cancel{16h^2}$

$53 - 30h = 41 - 42h$

$h = -1$ then, $k = -3$

$\boxed{(x+1)^2 + (y+3)^2 = 100}$

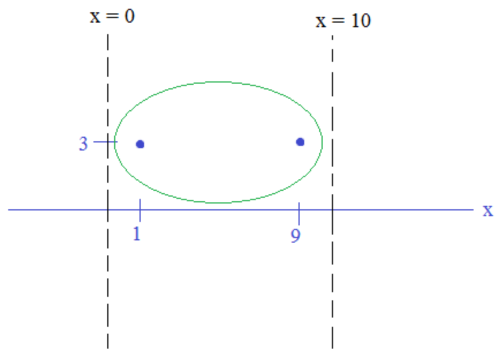
to check and find radius, determine the distance from center to each point...

(-1, -3) to (5, 5) ----> 10

(-1, -3) to (7, 3) ----> 10

Example: Write the equation of an ellipse with foci at (1, 3) and (9, 3) and directrices of $x = 0$ and $x = 10$

Step 1: Draw a sketch with given information



Step 2: Use formulas and equations

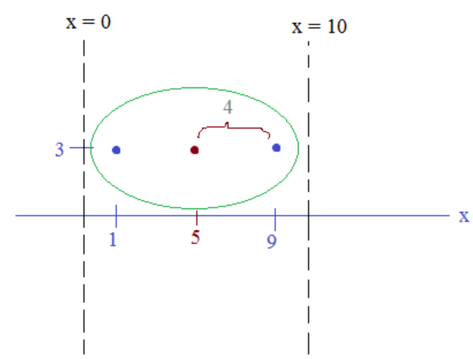
Directrix: $\frac{a^2}{c}$ $\qquad a^2 - b^2 = c^2$

Center is midpoint of foci

Equation of Ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

Step 3: Start filling in parts

center is (5, 3) $\qquad \frac{(x-5)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$



distance from center to each focus is $c = 4$

Directrix: $\frac{a^2}{c}$ ----> distance from center to directrix

$$\frac{a^2}{4} = 5$$

$$a^2 = 20$$

$$\frac{(x-5)^2}{20} + \frac{(y-3)^2}{b^2} = 1$$

$$\boxed{\frac{(x-5)^2}{20} + \frac{(y-3)^2}{4} = 1}$$

$$a^2 - b^2 = c^2$$

$$20 - b^2 = 16$$

$$b^2 = 4$$

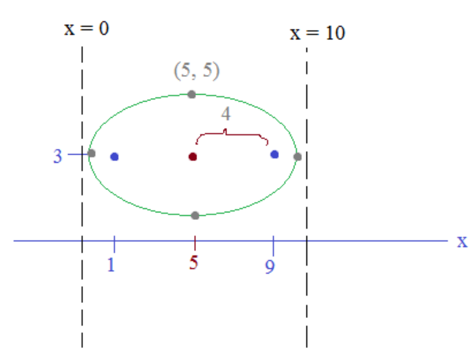
center: (5, 3)

foci: (1, 3) and (9, 3)

vertices: $(5 + \sqrt{20}, 3)$ and $(5 - \sqrt{20}, 3)$

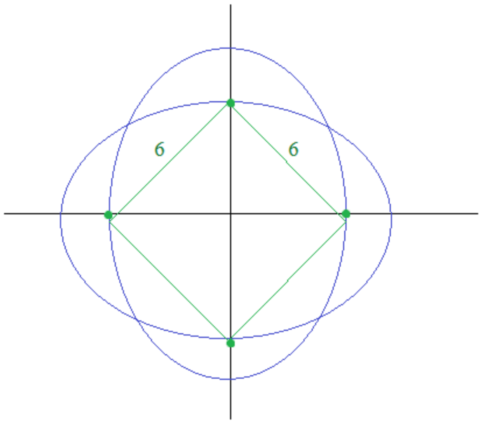
covertices: (5, 5) and (5, 1)

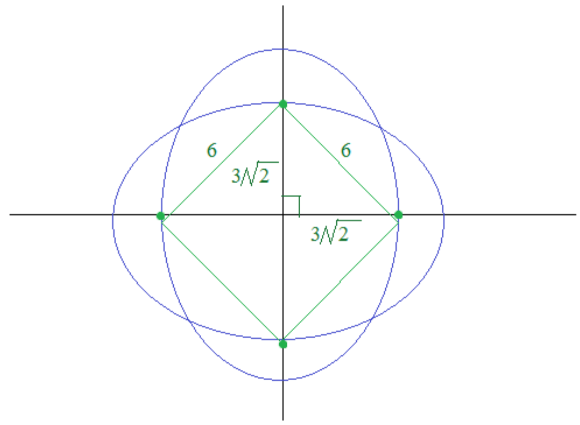
directrices: $x = 0$ and $x = 10$



Area of an Ellipse

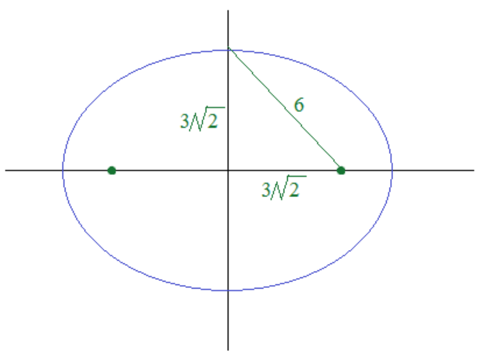
Examples: An ellipse with major axis parallel to the x-axis intersects another ellipse with major axis parallel to the y-axis. Each ellipse passes through the foci of the other ellipse, which form the vertices of a square. If the square has area of 36, what is the area enclosed by one of the ellipses?





Foci: $(3\sqrt{2},\ 0)$ and $(-3\sqrt{2},\ 0)$

Co-Vertices: $(0,\ 3\sqrt{2})$ and $(0,\ -3\sqrt{2})$



$C = 3\sqrt{2}$

$B = 3\sqrt{2}$

$C^2 = A^2 - B^2$

$(3\sqrt{2})^2 = A^2 - (3\sqrt{2})^2$

$18 = A^2 - 18$

$A = 6,\ -6$

$$\frac{x^2}{36} + \frac{y^2}{18} = 1$$

$\boxed{\text{Area of an ellipse} = AB\pi}$

$= (6)(3\sqrt{2})\pi$

$= \boxed{18\sqrt{2}\pi}$

Example: $x_1 = 3\cos t$ $\quad x_2 = -3 + \cos t$

$y_1 = 2\sin t$ $\quad y_2 = 1 + \sin t$

a) Graph the conics, and determine their intersection...

b) If the path of a particle (x_1, y_1) is the first equation, and the path of a particle (x_2, y_2) is the second equation, do the particles collide?

Removing the Parameter to graph

$x_1 = 3\cos t \qquad y_1 = 2\sin t$

$\cos t = \frac{x}{3} \qquad \sin t = \frac{y}{2}$

$\sin^2 + \cos^2 = 1$

$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$x_2 = -3 + \cos t \qquad y_2 = 1 + \sin t$

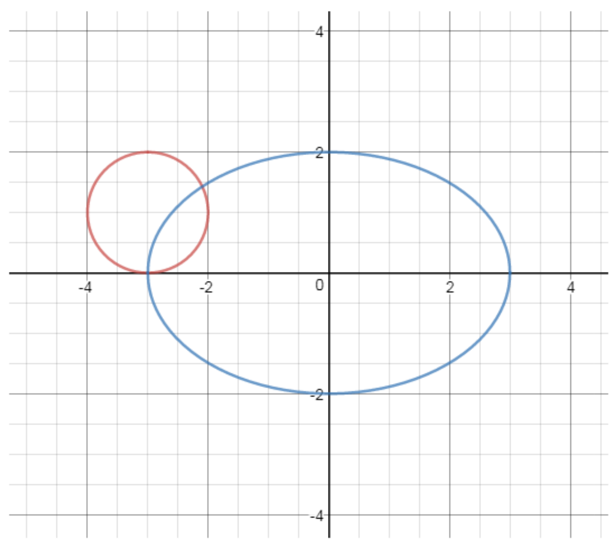
$\cos t = x + 3 \qquad \sin t = y - 1$

$\sin^2 + \cos^2 = 1$

$(y-1)^2 + (x+3)^2 = 1$

intersection:

(-3, 0) and (-2.1, 1.4)



$x_1 = 3\cos t \qquad x_2 = -3 + \cos t$

$y_1 = 2\sin t \qquad y_2 = 1 + \sin t$

Set the x's and y's equal to each other

$3\cos t = -3 + \cos t \qquad 2\sin t = 1 + \sin t$

$2\cos t = -3 \qquad \sin t = 1$

$\cos t = -3/2 \qquad t = 90^{\circ}$

no solution

at $t = 90^0$

$x_1 = 3(0) \qquad x_2 = -3 + (0)$

$y_1 = 2(1) \qquad y_2 = 1 + (1)$

(0, 2) $\qquad$ (-3, 2)

particles will never collide...

$x_1 = 3\cos t$

$y_1 = 2\sin t$

at (-3, 0) ----> $3\cos t = -3$

$2\sin t = 0$

$t = 180^{\circ}$ (or π)

$x_2 = -3 + \cos t$

$y_2 = 1 + \sin t$

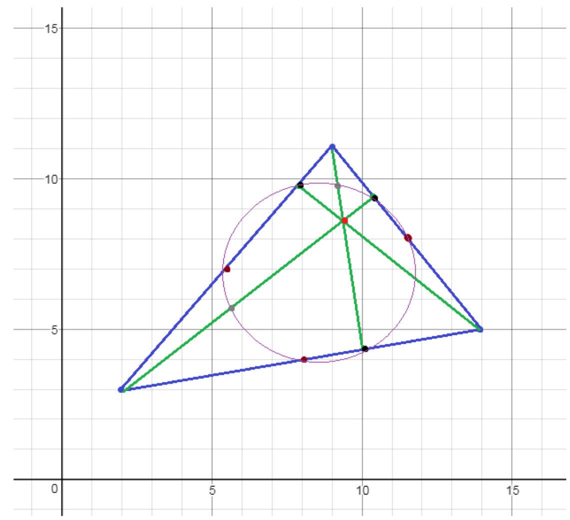
if $t = 180^{\circ}$ (or π) then $-3 + \cos(180^{\circ}) = -4$

$1 + \sin(180^{\circ}) = 1$

(-4, 1)

"The 9-Points Circle"

Example: Find the 9-points circle of a triangle with vertices (2, 3) (9, 11) and (14, 5)



Determine the midpoint of each side...

(5.5, 7) (11.5, 8) (8, 4)

The midpoints are 3 points of the circle...

Draw the 3 altitudes...

The base of the altitudes are 3 points of the circle...

The orthocenter is the intersection of the 3 altitudes..

Identify the midpoint between the 3 vertices and the orthocenter

The 3 vertex/orthocenter midpoints are 3 points of the circle...

circle: $(x-h)^2 + (y-k)^2 = r^2$

(2, 3) (9, 11) and (14, 5)

$(2-h)^2 + (3-k)^2 = r^2$

$(9-h)^2 + (11-k)^2 = r^2$

$(14-h)^2 + (5-k)^2 = r^2$

Finding circle that circumscribes the triangle

after substituting each point on the circle into the standard form of a circle, we have 3 equations with 3 unknowns...

h = 7.84 k = 4.95 r = +6.16 or ~~-6.16~~

center: (7.84, 4.95) radius: 6.16

***Note: To find the orthocenter (exactly), you must find the intersection of 2 altitudes...
(In other words, find equation of 2 altitudes, then use system of linear equations to find intersection)

(2, 3) and (14, 5) slope: 2/12 = 1/6

$\Rightarrow$ slope of (perpendicular) altitude: -6

altitude runs through point (9, 11)

$\boxed{y - 11 = -6\,(x - 9)}$

(9, 11) and (14, 5) slope: -6/5

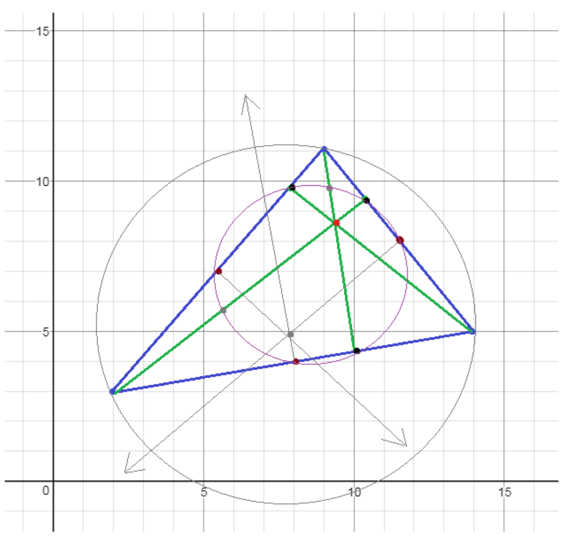
$\Rightarrow$ slope of altitude: 5/6

altitude runs through point (2, 3)

$\boxed{y - 3 = \frac{5}{6}(x - 2)}$

system of equations ----> orthocenter: of triangle (9.32, 9.10)

Quick note: The orthocenter in the graph appears to be around (9.4, 8.7)... This difference is due to the graph being a "hand sketch", so the altitudes may not be "exact"...



NOTE: radius of 9 points circle is HALF the radius of the circle that circumscribes the triangle

To find the circle that circumscribes a triangle:
Draw 3 perpendicular bisectors from the sides
Find the intersection of the 3 perpendicular bisectors (circumcenter)
**The circumcenter is equidistant to the 3 vertices of the triangle!
Therefore, it is the center of the circumscribed circle..

3 gray perpendicular bisectors...

the gray intersection is the circumcenter...

the gray large circle is circumscribed around the triangle..

approx. radius is 6.2

therefore, the radius of 9-points circle would be approx. 3.1

Finding equation of 9-points circle

circle: $(x-h)^2 + (y-k)^2 = r^2$

(5.5, 7) (11.5, 8) (8, 4)

(using the 3 midpoints of the triangle sides)

after substituting each point on the circle into the standard form of a circle, we have 3 equations with 3 unknowns...

Equation of 9 points circle: we know the radius is approx. 3.1 (half of circumscribed circle) so, we need the center...

$(5.5-h)^2 + (7-k)^2 = r^2$

$(11.5-h)^2 + (8-k)^2 = r^2$

$(8-h)^2 + (4-k)^2 = r^2$

h = 8.57

k = 7.02

r = 3.08 or ~~-3.08~~

center: (8.57, 7.02)

radius: 3.08

Example: Find an equation of the line tangent to $y = x^2$ at (1, 1)

Analytic Geometry

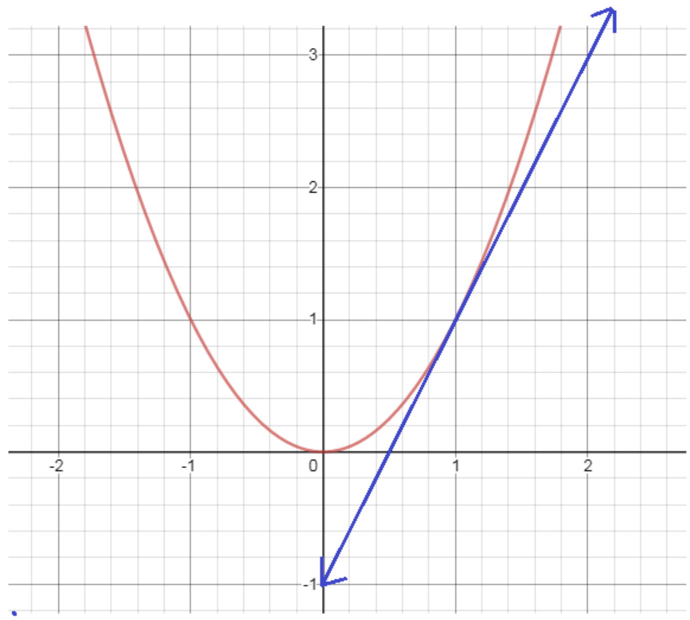
Using Calculus:

$$y = x^2$$

the derivative $y' = 2x$

therefore, the slope of the tangent line at x = 1 is 2

equation of tangent line: $y - 1 = 2(x - 1)$



Using Analytic Geometry:

Equation of a line: $y - 1 = m(x - 1)$

$\Longrightarrow y = mx - m + 1$

Equation of parabola:

$\Longrightarrow y = x^2$

Now, let's solve the system!

$$mx - m + 1 = x^2$$

Rearrange....

$$x^2 - mx + (m - 1) = 0$$

Since we are looking for ONE solution, this quadratic must yield a discriminant

$$B^2 - 4AC = 0$$

A = 1

B = $-m$

C = $m - 1$

$$(-m)^2 - 4(1)(m-1) = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m - 2)^2 = 0$$

slope $m = 2$

$$y - 1 = 2(x - 1)$$

NOTE: A tangent line has only one point of intersection. (i.e. one solution)

If you drew a line that intersected (1, 1) that was NOT tangent, it would have 2 solutions!
(2 intersections)

Example: Find the equation of the parabola with focus (0, 0) and directrix $x + y = 4$

First, let's find the vertex....

We know the vertex is equidistant from the focus and directrix....

focus: (0, 0)

directrix: $y = -x + 4$

Using geometry, we can determine the vertex...

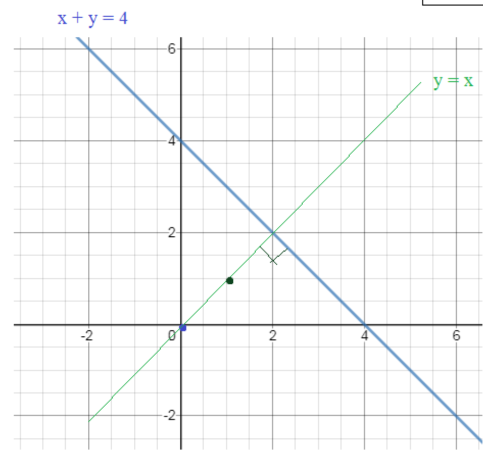
Since directrix slope is -1, we know the slope of a perpendicular line is 1...

and, since the perpendicular line goes through (0, 0), we have $y = x$

then, solving system of equations, we know the intersection of

$y = x$ and $y = -x + 4$ is (2, 2)

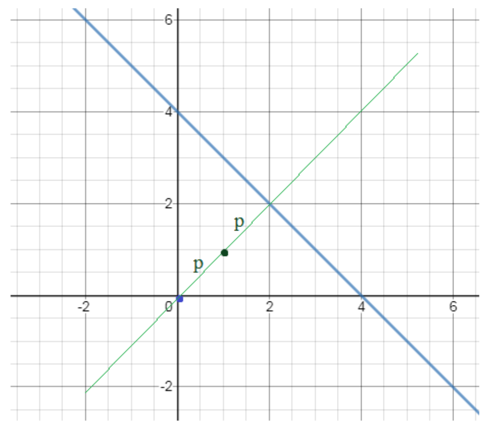
The intersection is (2, 2).... therefore, the midpoint between the focus (0, 0) and the intersection (2, 2) is (1, 1)



We now know the vertex is (1, 1)....

and, we see that the directrix has been rotated 45 degrees....

***definition of parabola: any point on the parabola is equidistant to the focus and directrix!



Let's remember the definition of a parabola....

distance p from vertex to directrix (or focus) is

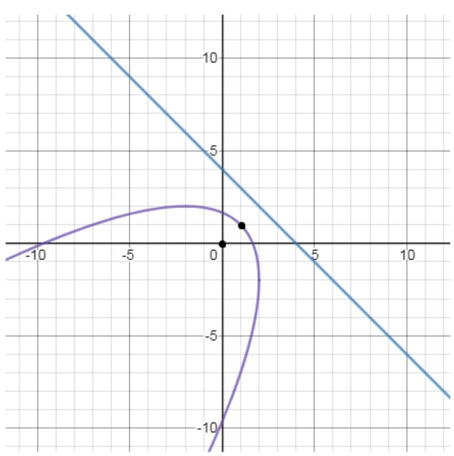
$$p = \sqrt{(1-2)^2 + (1-2)^2} = \sqrt{2}$$

$x + y = 4$ -----> $x + y - 4 = 0$ $\quad$ A = 1, B = 1, C = -4

Distance from a point to a line: (directrix) $\dfrac{Ax + By + C}{\sqrt{A^2 + B^2}}$ -----> $\dfrac{x + y + (-4)}{\sqrt{1^2 + 1^2}}$

Distance from a point to the focus: $\sqrt{(x-0)^2 + (y-0)^2}$

focus: (0, 0) $\quad = \sqrt{x^2 + y^2}$



Distance from any point to the directrix

Distance from any point to the focus

$$\frac{x + y + (-4)}{\sqrt{1^2 + 1^2}} = \sqrt{x^2 + y^2}$$

cross multiply....

$$\sqrt{2x^2 + 2y^2} = x + y - 4$$

square both sides...

$$2x^2 + 2y^2 = (x + y - 4)^2$$

$$2x^2 + 2y^2 = x^2 + xy + (-4x) + xy + y^2 + (-4y) - 4x - 4y + 16$$

collect like terms...

$$x^2 + y^2 = 2xy - 8x - 8y + 16$$

$$\boxed{x^2 - 2xy + y^2 + 8x + 8y - 16 = 0}$$

Example: *Example:* Find the equation of the line(s) tangent to $y^2 = 2(x-5)$ that pass through the point (2, 1)

The equation of a line passing through (2, 1) will be $y - 1 = m(x-2)$

$$y = mx - 2m + 1$$ where m is the slope of the line...

And, the equation of the parabola is $y^2 = 2x - 10$

To find the intersection of the line and parabola, we will solve the system.....

$$\boxed{\begin{aligned} y &= mx - 2m + 1 \\ y^2 &= 2x - 10 \end{aligned}}$$

using substitution: $(mx - 2m + 1)^2 = 2x - 10$

expand

$$m^2x^2 - 2m^2x + mx - 2m^2x + 4m^2 - 2m + mx - 2m + 1 = 2x - 10$$

combine "like" terms

$$m^2x^2 - 4m^2x + 2mx + 4m^2 - 4m + 1 = 2x - 10$$

$$m^2x^2 - 4m^2x + 2mx - 2x + 4m^2 - 4m + 11 = 0$$

separate/factor the coefficients

$$\underset{A}{(m^2)}x^2 + \underset{B}{(-4m^2 + 2m - 2)}x + \underset{C}{(4m^2 - 4m + 11)} = 0$$

Since the intersection of a tangent line is a unique point, the quadratic discriminant

$B^2 - 4AC = 0$

$$(-4m^2 + 2m - 2)^2 - 4(m^2)(4m^2 - 4m + 11) = 0$$

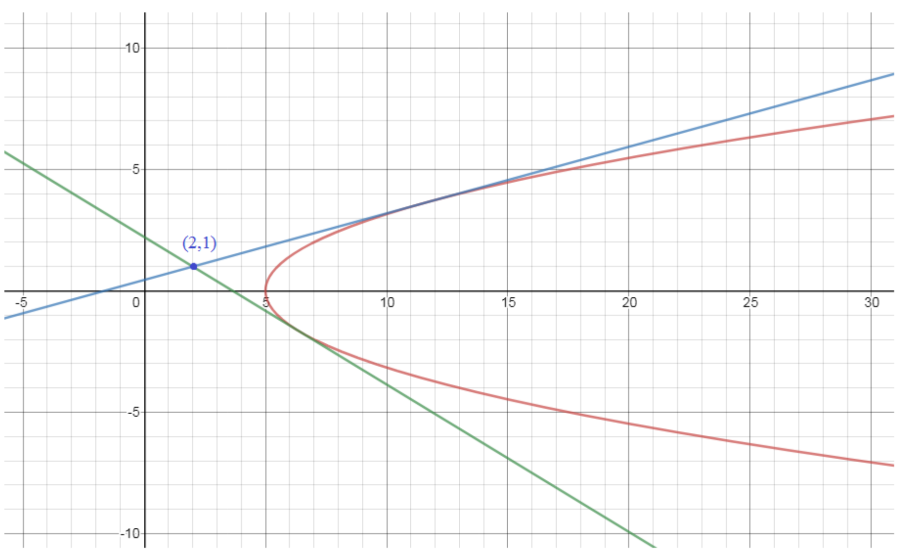
solve (with calculator)

$$m = \frac{\sqrt{7} - 1}{6} \qquad m = \frac{-\sqrt{7} - 1}{6}$$

approx. = .2743 approx. = -0.6076

Therefore, the lines passing through the point (2, 1) would be

$\boxed{y - 1 = .2743(x - 2)}$ and $\boxed{y - 1 = -0.6076(x - 2)}$



Example: Find the equation of the line tangent to $5x^2 + 3y^2 = 32$ @ (1, 3)

Using Analytic Geometry:

Equation of a line: $y - 3 = m(x - 1)$

$\Rightarrow$ $y = mx - m + 3$

Equation of ellipse:

$\Rightarrow$ $5x^2 + 3y^2 = 32$

Now, let's solve the system! (using substitution)

$$5x^2 + 3(mx - m + 3)^2 = 32$$

$$(mx - m + 3)^2$$

$$(mx - m + 3)(mx - m + 3)$$

$$\begin{array}{rrrrrrr} m^2x^2 & - m^2x & + 3mx & & & & \\ & - m^2x & & + m^2 & - 3m & & \\ & & 3mx & & - 3m & + 9 & \\ \hline m^2x^2 & - 2m^2x & + 6mx & + m^2 & - 6m & + 9 & \end{array}$$

$$5x^2 + 3(m^2x^2 - 2m^2x + 6mx + m^2 - 6m + 9) = 32$$

$$5x^2 + 3m^2x^2 - 6m^2x + 18mx + 3m^2 - 18m + 27 = 32$$

Rearrange....

$$(5 + 3m^2)x^2 + (18m - 6m^2)x + (3m^2 - 18m - 5) = 0$$

$Ax^2 + Bx + C$

$A = 5 + 3m^2$

$B = 18m - 6m^2$

$C = 3m^2 - 18m - 5$

Since we are looking for ONE solution, this quadratic must yield a discriminant

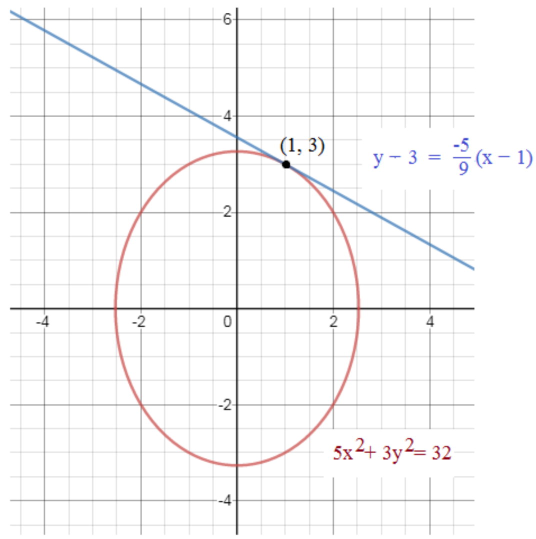
$$B^2 - 4AC = 0$$

$$(18m - 6m^2)^2 - 4(5 + 3m^2)(3m^2 - 18m - 5) = 0$$

$$m = -5/9$$

Equation of line passing through (1, 3)

$$\boxed{y - 3 = \frac{-5}{9}(x - 1)}$$



Using Calculus: $5x^2 + 3y^2 = 32$

the derivative $10x + 6y\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-10x}{6y}$$

therefore, the slope of the tangent line at (1, 3) $\Rightarrow$ $\frac{-10}{18}$

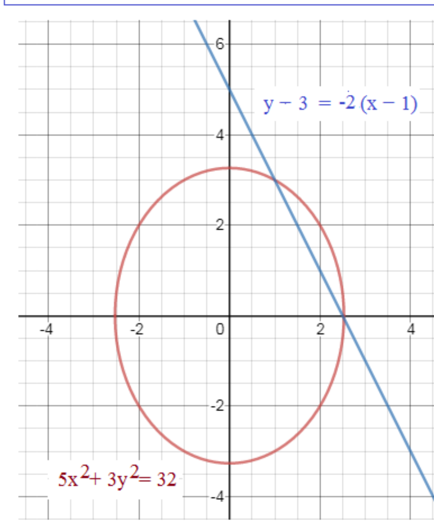
equation of tangent line: $y - 3 = \frac{-5}{9}(x - 1)$

NOTE: A tangent line has only one point of intersection. (i.e. one solution)

If you drew a line that intersected (1, 3) that was NOT tangent, it would have 2 solutions!
(2 intersections)

$m = -2$

$B^2 - 4AC \neq 0$



Example: Find the equation of a circle that passes through the following points:

(1, 1) (3, 9) and (13, 3)

Step 1: draw a quick diagram

Since no pair of points appear to be endpoints of a diameter,
we cannot use midpoint and distance formulas to find center and radius

Method 1: Apply the standard form of a circle $(x-h)^2 + (y-k)^2 = r^2$

Substitute each point: (1, 1) $(1-h)^2 + (1-k)^2 = r^2$

(3, 9) $(3-h)^2 + (9-k)^2 = r^2$

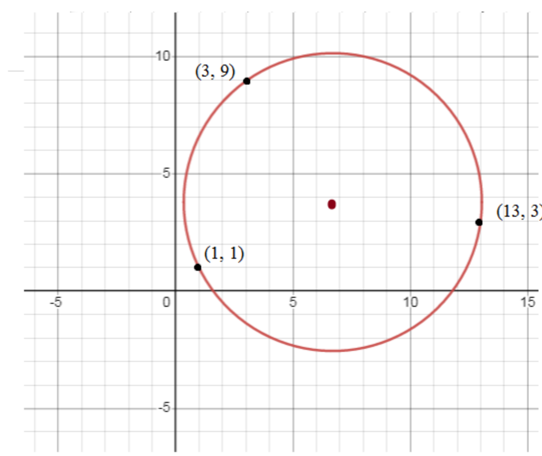
(13, 3) $(13-h)^2 + (3-k)^2 = r^2$

this is a system of 3 equations with 3 unknowns!

(Using a calculator) the solution is $h = 6.7$ $k = 3.8$ $r = 6.35$

$$\boxed{(x-6.7)^2 + (y-3.8)^2 = 6.35^2}$$

$(x-6.7)^2 + (y-3.8)^2 = 6.35^2$



Method 2: Geometry properties of triangles.
The 3 points form the vertices of a triangle
The intersection of the perpendicular bisectors is the circumcenter.
This it the center of the circle!

Perpendicular bisector of (1, 1) and (3, 9)

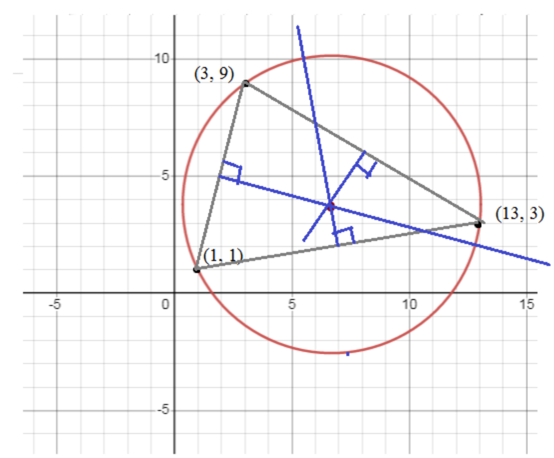
midpoint: (2, 5)
slope of (1, 1) and (3, 9) ----> 4
slope of perpendicular bisector ----> -1/4

$$(y-5) = -\frac{1}{4}(x-2)$$

Perpendicular bisector of (1, 1) and (13, 3)

midpoint: (7, 2)
slope of (1, 1) and (13, 3) ----> 1/6
slope of perpendicular bisector -----> -6

$$(y-2) = -6(x-7)$$



Find the intersection of the two perpendicular bisectors

$$(y-5) = -\frac{1}{4}(x-2)$$

$$(y-2) = -6(x-7)$$

(x, y) = (6.7, 3.8) ----> circumcenter
(h, k) center of circle

Find the distance from (6.7, 3.8) to any of the 3 given points...

(1, 1) to (6.7, 3.8)

$$d = \sqrt{(6.7-1)^2 + (3.8-1)^2} \approx 6.39$$ (radius of the circle)

(1, 1) to (13, 3)

$$d = \sqrt{(6.7-13)^2 + (3.8-3)^2} \approx 6.35$$ (radius of the circle)

Analytic Geometry

1) Find the equation of a line that is tangent to $y = (x - 3)^2 - 2$ @ the point $(0, 7)$

2) Find the line tangent to $2x^2 + 3y^2 = 11$ at the point $(2, 1)$

3) A circle contains the points (5, 2) and (-1, 6)

a) If the center lies on the line $y = x$, then what is the equation of the circle?

b) If the center lies on the line $y = 2x + 3$, then what is the equation of the circle?

4) Find the equation of the line(s) tangent to the hyperbola $x^2 - y^2 - 9 = 0$ passing through the point (12, 12)

5) Find the equation(s) of a circle(s) with radius 5 and tangent to the line $y = 2x - 10$ at $(8, 6)$

Analytic Geometry

6) Find the tangent line(s) of circle $(x - 5)^2 + (y - 7)^2 = 9$ passing through $(10, 12)$.

SOLUTIONS

1) Find the equation of a line that is tangent to $y = (x-3)^2 - 2$ @ the point (0, 7)

We know some line $y = mx + b$ will intersect the parabola $y = x^2 - 6x + 7$ at (0, 7)

If we solve the system of equations....

$$mx + 7 = x^2 - 6x + 7$$

$$x^2 - 6x - mx = 0$$

$$\underset{A}{x^2} + \underset{B}{(-6-m)x} + \underset{C}{0} = 0$$

Since a tangent line meets at one point, and this is a quadratic, we need to find m such that the discriminant is 0

$$B^2 - 4AC = 0$$

$$(-6 - m)^2 + 4(1)(0) = 0$$

$$m = -6$$

The equation of the line is $y = -6x + 7$

Note: we're looking for the 'equation of a line', so we need a point and the slope... We know the point is (0, 7)... We need to find the slope....

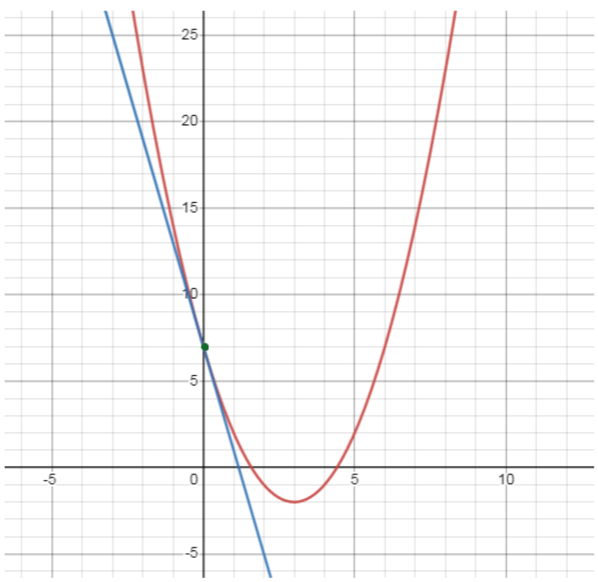
Using calculus: find the derivative of the parabola..

$$y = x^2 - 6x + 7 \quad \text{at } (0, 7)$$

$$y' = 2x - 6$$

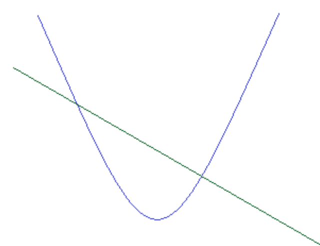
@ $x = 0$, the slope is -6

$$y = -6x + 7$$

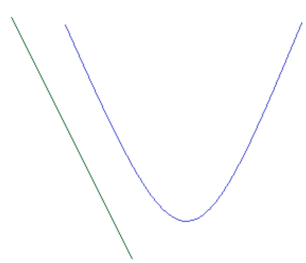


Note: if the discriminant were positive, you might get a slope such as this:

secant line



If the discriminant were negative, you may get a graph that looks like this...



no intersection

2) Find the line tangent to $2x^2 + 3y^2 = 11$ at the point (2, 1)

SOLUTIONS

We're seeking a line through point (2, 1): $y - 1 = m(x - 2)$

$$y = mx - 2m + 1$$

and, it must be on the ellipse: $2x^2 + 3y^2 = 11$

Solve the system of equations....

$$2x^2 + 3(mx - 2m + 1)^2 = 11$$

$$2x^2 + 3m^2x^2 - 12m^2x + 6mx + 12m^2 - 12m + 3 = 11$$

$$(2 + 3m^2)x^2 + (-12m^2 + 6m)x + 12m^2 - 12m - 8 = 0$$

$$(mx - 2m + 1)^2 \Rightarrow \begin{array}{r} mx - 2m + 1 \\ mx - 2m + 1 \\ \hline m^2x^2 - 2m^2x + mx \\ - 2m^2x + 4m^2 - 2m \\ + mx - 2m + 1 \\ \hline m^2x^2 - 4m^2x + 2mx + 4m^2 - 4m + 1 \end{array}$$

This is a quadratic where $A = (2 + 3m^2)$

$B = (-12m^2 + 6m)$

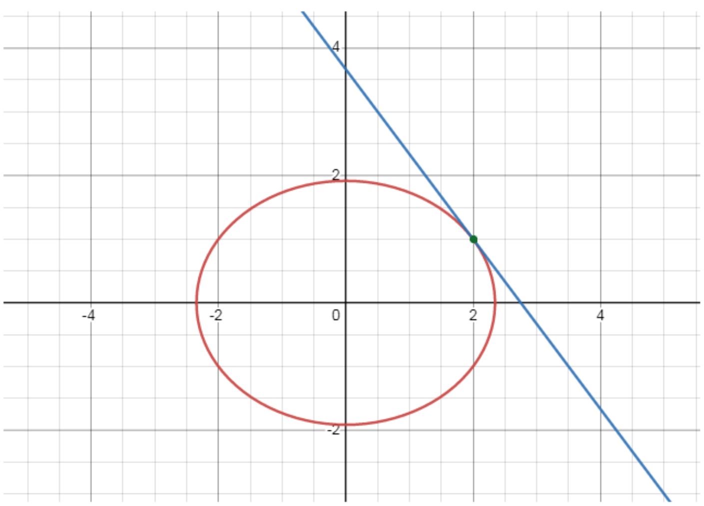
$C = 12m^2 - 12m - 8$

to find ONE solution (i.e. a tangent line only has one intersection)

set discriminant $B^2 - 4AC = 0$

$$(6m - 12m^2)^2 - 4(2 + 3m^2)(12m^2 - 12m - 8) = 0$$

$$m = \frac{-4}{3}$$



Using Calculus (derivatives), we can check our answer....

$$4x + 6y\frac{dy}{dx} = 0$$

$$6y\frac{dy}{dx} = -4x$$

$$\frac{dy}{dx} = \frac{-2x}{3y}$$

Then, at the point (2, 1)....

$$\frac{-2(2)}{3(1)} = \frac{-4}{3}$$

3) A circle contains the points (5, 2) and (-1, 6)

Analytic Geometry

SOLUTIONS

a) If the center lies on the line $y = x$, then what is the equation of the circle?

b) If the center lies on the line $y = 2x + 3$, then what is the equation of the circle?

a) We're seeking the center (h, k), so we need equations that include (h, k)...

equation of the circle will include (5, 2) and (-1, 6)

$$(5 - h)^2 + (2 - k)^2 = r^2$$

$$(-1 - h)^2 + (6 - k)^2 = r^2$$

and, center is on the line $y = x$

$$k = h$$

system of 3 equations with 3 variables

$$\begin{cases} (5 - h)^2 + (2 - k)^2 = r^2 \\ (-1 - h)^2 + (6 - k)^2 = r^2 \\ k = h \end{cases}$$

$$h = -2$$

$$k = -2$$

$$r = \sqrt{65}$$

$$(x + 2)^2 + (y + 2)^2 = 65$$

b) equation of the circle will include (5, 2) and (-1, 6)

$$(5 - h)^2 + (2 - k)^2 = r^2$$

$$(-1 - h)^2 + (6 - k)^2 = r^2$$

and, center in on the line $y = 2x + 3$

$$k = 2(h) + 3$$

system of 3 equations with 3 variables

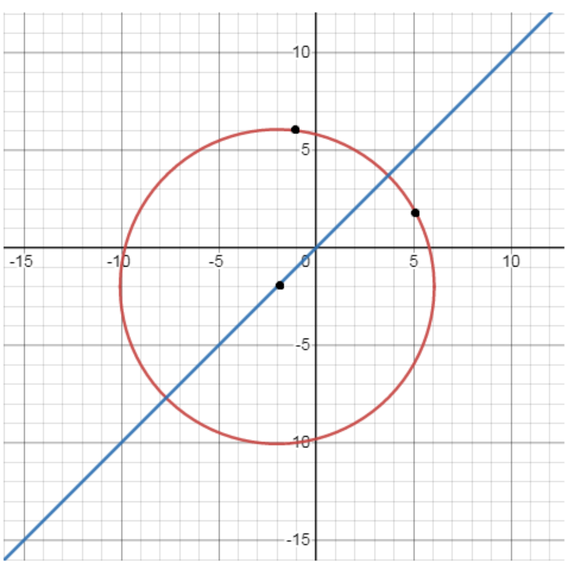
$$\begin{cases} (5 - h)^2 + (2 - k)^2 = r^2 \\ (-1 - h)^2 + (6 - k)^2 = r^2 \\ k = 2h + 3 \end{cases}$$

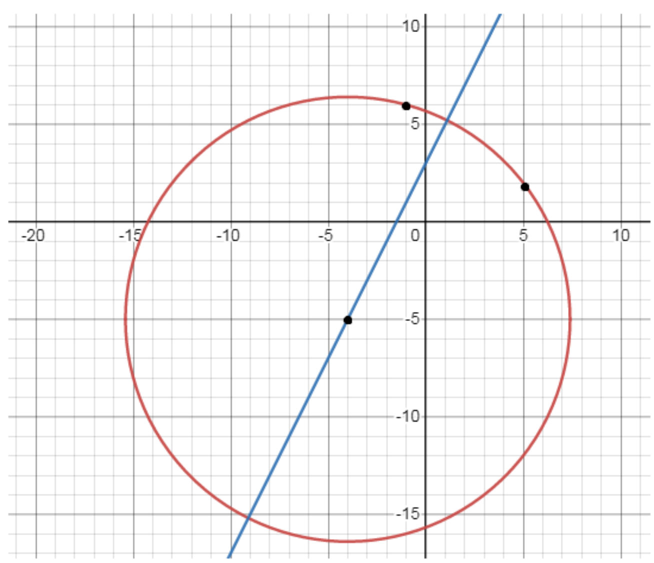
$$h = -4$$

$$k = -5$$

$$r = \sqrt{130}$$

$$(x + 4)^2 + (y + 5)^2 = 130$$





4) Find the equation of the line(s) tangent to the hyperbola $x^2 - y^2 - 9 = 0$ passing through the point (12, 12)

SOLUTIONS

Analytic Geometry

We're seeking a line through (12, 12):

$y - 12 = m(x - 12) \implies y = mx - 12m + 12$

that must be on the hyperbola:

$x^2 - y^2 - 9 = 0 \implies x^2 - y^2 = 9$

Solve the system of equations using substitution...

$x^2 - (mx - 12m + 12)^2 = 9$

$$\boxed{(mx - 12m + 12)^2 =}$$

$$\begin{array}{l} (mx - 12m + 12) \\ (mx - 12m + 12) \\ \hline m^2x^2 - 12m^2x + 12mx \\ \quad - 12m^2x \qquad + 144m^2 - 144m \\ \qquad\qquad 12mx \qquad - 144m + 144 \\ \hline \boxed{m^2x^2 - 24m^2x + 24mx + 144m^2 - 288m + 144} \end{array}$$

$x^2 - (m^2x^2 - 24m^2x + 24mx + 144m^2 - 288m + 144) = 9$

$x^2 - m^2x^2 + 24m^2x - 24mx - 144m^2 + 288m - 153 = 0$

This is a quadratic where the slope m could have 0, 1, or 2 solutions..
Since we want a tangent line (with 1 intersection), we will search for a value with 1 solution ----> discriminant $B^2 - 4AC = 0$

coefficients of x^2 A: $1 - m^2$

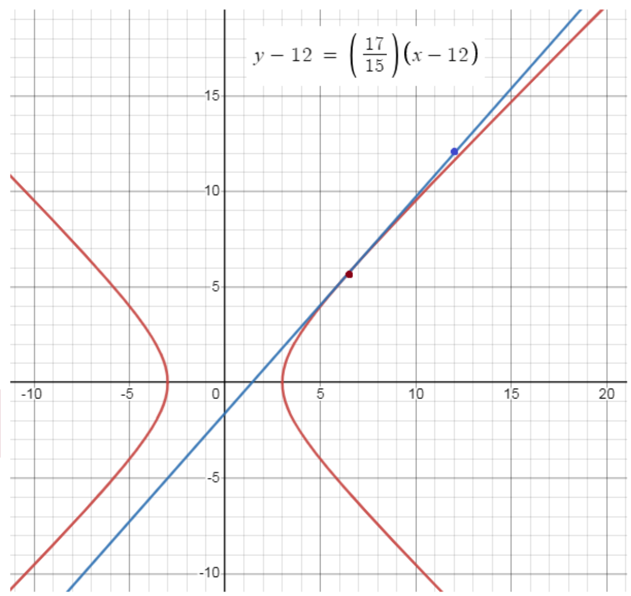
coefficients of x B: $24m^2 - 24m$

constants C: $-144m^2 + 288m - 153$

$(24m^2 - 24m)^2 - 4(1 - m^2)(-144m^2 + 288m - 153) = 0$

slope $m = 1$ or $\frac{17}{15}$ $\implies$ $\boxed{y - 12 = \frac{17}{15}(x - 12)}$

Tangent at the point (51/8, 45/8)



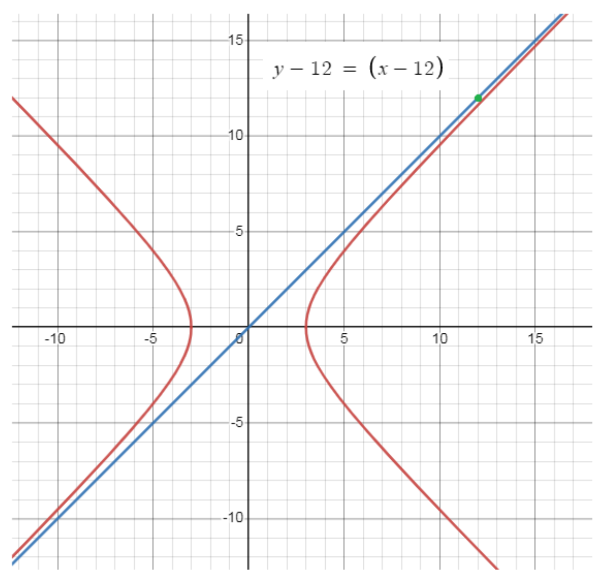
Note: one of the hyperbola's asymptotes is y = x

So, the equation $y - 12 = 1(x - 12)$

$y = x$

is a tangent line that *approaches* the asymptote

(i.e. the tangent intersection doesn't exist...)



hyperbola $x^2 - y^2 - 9 = 0$

calculus check:

$2x - 2y\frac{dy}{dx} = 0$ derivative of hyperbola..

$\frac{dy}{dx} = \frac{x}{y}$

@ (51/8, 45/8) , the slope is $\frac{51}{45} = \frac{17}{15}$ ✓

$\left(\frac{51}{8}\right)^2 - \left(\frac{45}{8}\right)^2 - 9 = 0$ ✓

5) Find the equation(s) of a circle(s) with radius 5 and tangent to the line $y = 2x - 10$ at $(8, 6)$

SOLUTIONS

Since we're looking for a circle, we need the radius and the center (h, k).
The radius given is $r = 5$

Lines tangent to a circle are perpendicular to the radii.

the slope of the radii will be $-1/2$

We can set up the slope formula: $\dfrac{6 - k}{8 - h} = \dfrac{-1}{2}$

$$12 - 2k = h - 8 \implies h + 2k = 20$$

and, we know the radius is 5: $\text{distance} = \sqrt{(8 - h)^2 + (6 - k)^2} = 5$

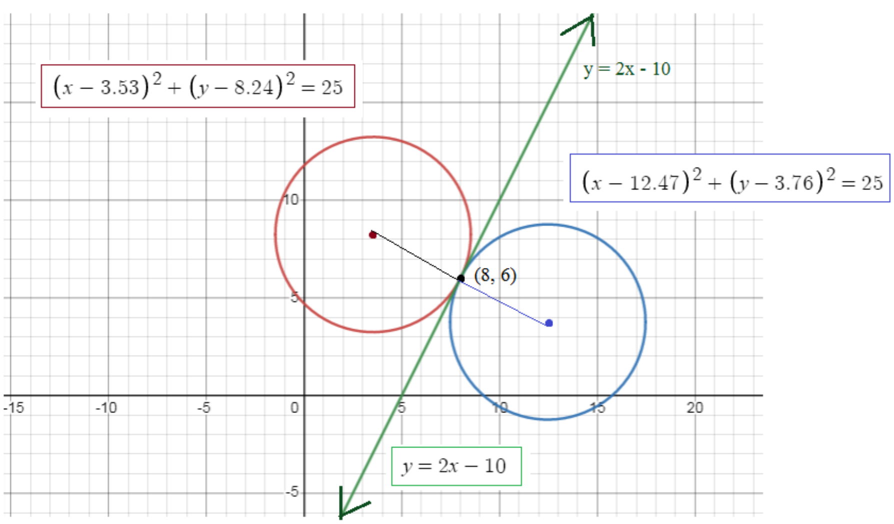
$$\implies (8 - h)^2 + (6 - k)^2 = 25$$

We now have a system of 2 equations with 2 unknowns....

$$h + 2k = 20$$
$$(8 - h)^2 + (6 - k)^2 = 25$$

$(h, k) = (3.53, 8.24)$ or $(12.47, 3.76)$

$$\implies \boxed{\begin{array}{l}(x - 3.53)^2 + (y - 8.24)^2 = 25 \\ (x - 12.47)^2 + (y - 3.76)^2 = 25\end{array}}$$



6) Find the tangent line(s) of circle $(x-5)^2 + (y-7)^2 = 9$ passing through (10, 12).

SOLUTIONS

Analytic Geometry

We know the center of the circle (h, k) is (5, 7)
and, the point(s) we need are (x, y)...

The slope of line (10, 12) to (x, y) is perpendicular to slope of line (5, 7) to (x, y)

(radius is perpendicular to tangent line)

$$\frac{y-12}{x-10} = -\left(\frac{x-5}{y-7}\right)$$ perpendicular is opposite reciprocal

$$\frac{y-12}{x-10} = \frac{5-x}{y-7}$$

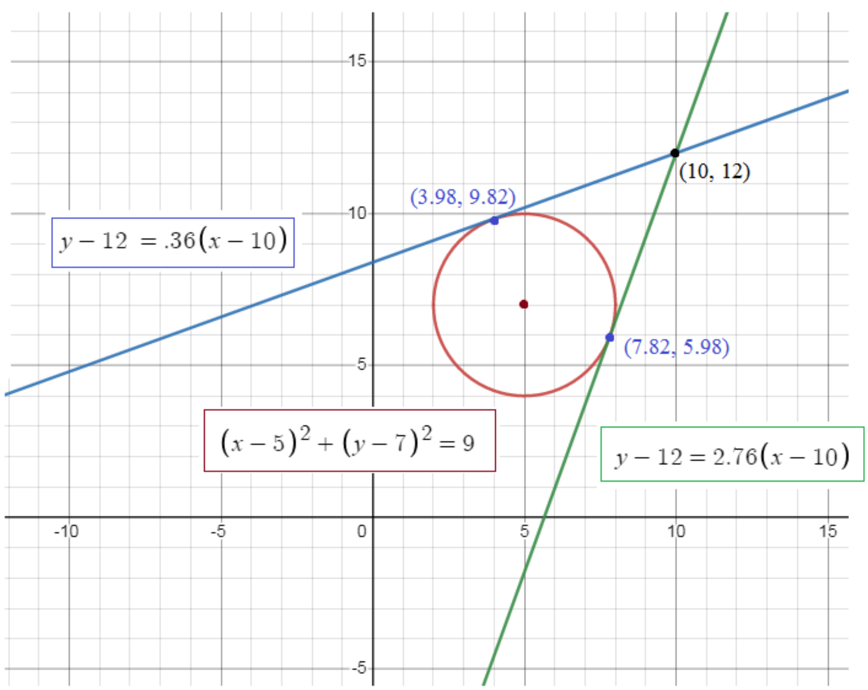
$$y^2 - 19y + 84 = -x^2 + 15x - 50$$

$$x^2 + y^2 - 15x - 19y + 134 = 0 \Longrightarrow x^2 + y^2 - 15x - 19y + 134 = 0$$ points where (x, y) make perpendicular lines

$$(x-5)^2 + (y-7)^2 = 9$$ points on the circle

solve the system with calculator

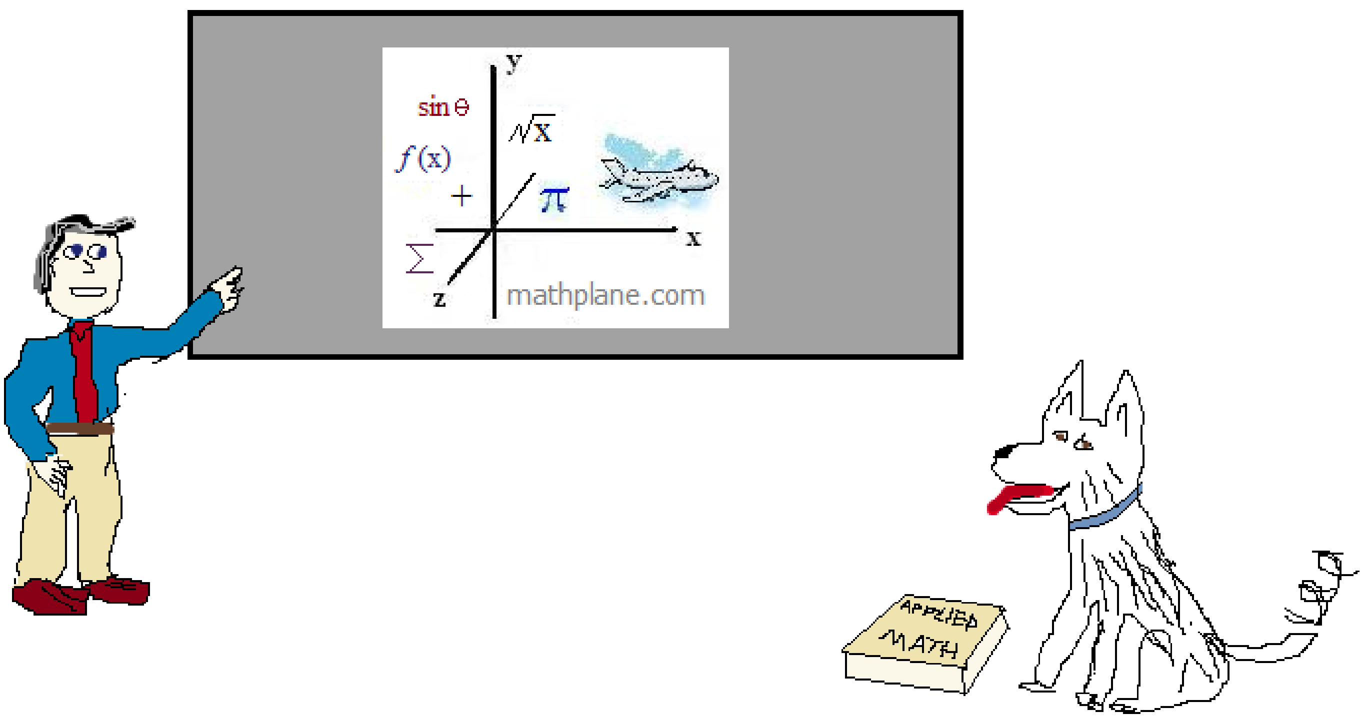
$\boxed{(3.98, 9.82) \text{ and } (7.82, 5.98)}$



Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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