Counting Methods

Formulas, examples, and Practice Questions (and, Solutions)

Topics include combinations, permutations, nested sums, balls and urns, objects and dividers, and more…

Counting: "Theater Row of Seats"

Example: An usher brings 10 people to the front row of a theater.
How many ways can he seat the 10 people?

This is an 'arrangement' of 10 people, so order does matter. (permutation)

10 choices	9	8	7	6	5	4	3	2	1

The usher can pick from 10 people for the aisle seat...
Then, he can pick from 9 people for the next seat..
Then, 8 people for the 3rd seat... Until he finishes...

10 x 9 x 8 x 7 .... x 1 = 10!

$\boxed{\text{3,628,800 ways}}$

Jack and Jill are friends and want to sit next to each other..
If the usher grants their request, how many ways can he seat the 10 people
(with Jack and Jill next to each other)?

To solve, we'll 'batch' Jack and Jill together... Now, we have an arrangement of 9 'people' (where Jack and Jill are 1 'person')

Here is one arrangement....

9	8	7	6 Jack Jill	5	4	3	2	1

The number of arrangements is 9 x 8 x 7 x ... x 2 x 1 = 9!

***Then, we have to remember that Jack and Jill can be seated in 2 ways: Jack on the left; or Jill on the left..

$\Rightarrow$ 9! x 2 $\boxed{\text{725,760 ways}}$

Among the 10 people, Lucy and Ethel are angry at each other and do NOT want to be seated next to each other.
How many ways can the usher seat the 10 people (and keeping Lucy and Ethel separated)?

The best approach is to use the above information...

The number of possible arrangements is 10!...
and, the number of ways to seat 2 people next to each other is 9! x 2...

Therefore, the number of ways NOT to seat 2 people together is 10! − (9! x 2) $\boxed{\text{2,903,040 ways}}$

Larry, Moe, and Curly ask the usher to be seated together.
How many ways can the 10 people be seated, where Larry, Moe, and Curly sit together
(and, everyone else is randomly placed)?

Again, we'll batch Larry, Moe, and Curly into one group...
We then have 8! ways to seat everyone (where L, M, and C are a 'person').

Here is one possible arrangement

8	7	6	5	4	3 LMC	2	1

And, again, we have to remember that Larry, Moe, and Curly can be seated in different orders:

LMC LCM CLM CML MCL MLC 3! or 6 ways....

So, the total ways is 8! x 3! $\boxed{\text{241920 ways}}$

"Salad Bar, Pizza, and Sandwiches"

Example: At a restaurant, you order the salad bar for $10.99. With an empty plate in hand, you look over the 12 items you can select. (Lettuce, tomatoes, cucumbers, pasta, and more.) How many possible salads can you create?

In this case, we'll reorient or thinking toward the individual items...
There are 2 choices for each: yes or no...
Lettuce: yes or no.
Tomatoes: yes or no.
etc..

$2 \times 2 \times 2 \times \dots \times 2$ (12 items with 2 choices each)

Number of possible salads is 2^{12}

BUT, we assume we won't include an empty plate.

So, $\boxed{2^{12} - 1}$

Example: Your friends order a large pizza. The pizzeria offers 4 toppings: pepperoni, sausage, mushroom, and spinach. How many possible types of pizza can your friends order?

In this case, we will separate into number of toppings:

0 toppings (plain cheese)	1 possible way (selecting none)	${}_4C_0$
1 topping only	4 possible ways (just P, just S, just M, or just Sp)	${}_4C_1$
2 toppings	6 ways (PS, PM, PSp, SM, SSp, MSp)	${}_4C_2$
3 toppings (or leaving off 1 topping)	4 ways (PSM, PSSp, PMSp, SMSp)	${}_4C_3$
4 toppings	1 way	${}_4C_4$

Note the relation to Pascal's Triangle:

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ \boxed{1 \quad 4 \quad 6 \quad 4 \quad 1} \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

Total: 16 possible types of pizza

Note: We could use the other approach. Each topping is either Yes or No...

4 toppings ----> $2^4 = 16$

Now, suppose your friends can't decide on one type of pizza. Instead, they order a large pizza, divided into 2 halves with different combinations. How many possible 2-half pizzas could the friends have ordered?

We know the number of types of pizzas is 16... Assuming the 2 halves are different, the possibilities are

$$\frac{\underbrace{16}_{\text{1st half choices}} \times \underbrace{15}_{\text{2nd half of the pizza}}}{2}$$

since it's a combination
(i.e. just cheese on 1st half and sausage/pepperoni on the other half
is the SAME as sausage/pepperoni on 1st half and just cheese on 2nd half

120 possible pizzas

Example: A sub shop serves sandwiches with the following options:

4 types of bread
3 cheeses or without cheese
4 meats or no meat
any combo of veggies from the following choices: lettuce, tomato, peppers, onions, and/or avocado

How many different sandwiches are possible?

$$\underbrace{4}_{\text{bread}} \times \underbrace{4}_{\text{3 cheese or none}} \times \underbrace{5}_{\text{4 meats or none}} \times \underbrace{(2 \times 2 \times 2 \times 2 \times 2)}_{\text{yes/no choice for each veggie}}$$

2560 possible ways to construct a sandwich!

"Ice Cream and Gift Exchanges"

Example: How many "words" can you make by using all the letters in 'ICE CREAM'?

First, we have to count how many ways we can arrange 8 letters: 8!

then, we have to consider how many 'double counts' there are:

For example:

I	^{1}C	^{1}E	^{2}C	R	^{2}E	A	M

But, suppose we swap the two C's?

I	^{2}C	^{1}E	^{1}C	R	^{2}E	A	M

It's a different arrangement; however, it's the same word!

so, to eliminate the double counts, we divide by the extra possiblities....

total arrangements

$$\frac{8!}{2!\quad 2!} = \boxed{\text{10,080 different 'words'}}$$

duplicate C's

duplicate E's

Example: A camp counselor buys ice cream for his 15 kids.
He returns with 3 vanilla, 5 chocolate, 4 strawberry, and 3 mint chip cups.
How many ways can the counselor distribute the ice cream to the 15 campers?

This is an application of the above 'word spelling problem'...

There are 15 campers ----> 15-letter word...

$$\frac{15!}{3!\ 5!\ 4!\ 3!} = \boxed{\text{12,612,600 different ways to distribute the flavors to the campers}}$$

duplicate vanilla cups

duplicate chocolate cups

Example: 20 people attend a meeting. If everyone shakes hands with each person at the meeting, how many handshakes?

people at the meeting

each person makes 19 handshakes

$$\frac{20 \times 19}{2} = \boxed{\text{190 unique handshakes}}$$

'double counts'
ex: if A shakes hands with B, then
B shaking hands with A is a duplication

Example: 20 people attend the holiday party. Each person brings a gift for everyone in attendance. How many gifts were exchanged?

Each gift exchange is one direction and unique...

$$20 \times 19 = \boxed{\text{380 gifts are given}}$$

Note: 'A giving a gift to B' is different than 'B giving a gift to A'...
(there are no 'double counts')

Example: You are placing 10 different math books on a shelf.
How many ways can you arrange these books
where the algebra and calculus books are not together?

Ways to arrange the 10 books: 10!

Then, the ways you can arrange the books,
where algebra and calculus are next to each other..

☐ ☐ ☐ ☐ [algebra/ calculus] ☐ ☐ ☐ ☐

9! x 2

because there are 9! ways to arrange these blocks..
Then, "x 2" because algebra/calculus and calculus/algebra are distinct

arrangements where algebra
and calculus are together

$$10! - (9! \times 2) = \boxed{2903040}$$

total
arrangements

arrangements where
algebra and calculus
are NOT together

How many ways can you arrange those books where algebra and
calculus are on each end of the shelf?

$$\boxed{2}\ \underbrace{\boxed{8}\ \boxed{7}\ \boxed{6}\ \boxed{5}\ \boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{1}}\ \boxed{1}$$

2 choices
(algebra or calculus)

ways to arrange the other 8
math books....

1 choice
(the other book;
calculus or algebra)

$$8! \times 2 \times 1 = \boxed{80640 \text{ possible arrangements}}$$

Example: A student moves into his dorm room and unpacks 15 books.
There are 3 book shelves. How many ways can he place the 15
books, assuming he will place at least one book on each shelf?

There are 15 books to arrange: 15!

Imagine lining the 15 books.. Now, he must determine how
to split them up...

We'll use 'dividers' to separate the books...

Here is one example:

☐☐☐ | ☐☐☐☐☐☐☐ | ☐☐☐☐☐

1st
divider

2nd
divider

3 books on top shelf; 7 books on middle shelf; 5 books on bottom

So, how many ways can be set the dividers?

14 choices for 1st divider... 13 choices for 2nd divider...

Then, divide by 2, because of 'double counts'

☐☐☐ | ☐☐☐☐☐☐☐ | ☐☐☐☐☐

2nd
divider

1st
divider

(this is the same book arrangement as the other)

$$\frac{15! \cdot 14 \cdot 13}{2} = \boxed{118998367488000}$$

Alternate Approach:

First shelf: 15 arrange anywhere from 1 to 13 books
Second shelf: arrange anywhere from 1 to remaining books (minus 1 or more for third shelf)
Third shelf: arrange remaining books

$$\sum_{i=1}^{13}\left(\sum_{j=1}^{14-i}\left(\sum_{k=15-i-j}^{15-i-j}\left({}_{15}P_i \cdot {}_{15-i}P_j \cdot {}_{15-i-j}P_k\right)\right)\right)$$

$$= \boxed{118998367488000}$$

Note: This is an application of 'balls and urns', where
balls are distinguishable, urns are distinguishable,
and, order of the balls in each urn counted.

Example: A company manufactures 4-digit yard address signs.
They include all 10 digits (0 thru 9), but would not set 0 in the first position.
Also, their signs can be rotated 180 degrees, allowing them to produce
fewer signs.

How many different signs do they produce?

First, we'll identify the amount of possible numbers...

1000 - 9999

9	10	10	10

first digit, 9 possible
2nd digit, 10 possible
3rd digit, 10 possible
4th digit, 10 possible

Since 0 cannot be the first digit, there are 9000 possible.

9 x 10 x 10 x 10 = 9000

Then, we'll determine which numbers have rotational symmetry...

0, 1, 6, 8, 9 (2, 3, 4, 5, 7 do not have rotational symmetry...)

And, recognize that 6 and 9 'change' when they are turned
0, 1, and 8 remain as themselves..

How many change when flipped?

Number of ways to change 1st and 4th digits when flipped...

4	5	1	3
1, 6, 8, 9	0, 1, 6, 8, 9	same (or reflection) as 2nd digit	choosing from 1, 6, 8, 9 where doesn't match first digit

4 x 5 x 1 x 3 = 60

Number of ways to change 2nd and 3rd digits when flipped...

4	5	4	1
1, 6, 8, 9	0, 1, 6, 8, 9	digit that doesn't match or reflect 2nd digit	matches (or reflects) the first digit

4 x 5 x 4 x 1 = 80

Number of ways to change all four digits when flipped...

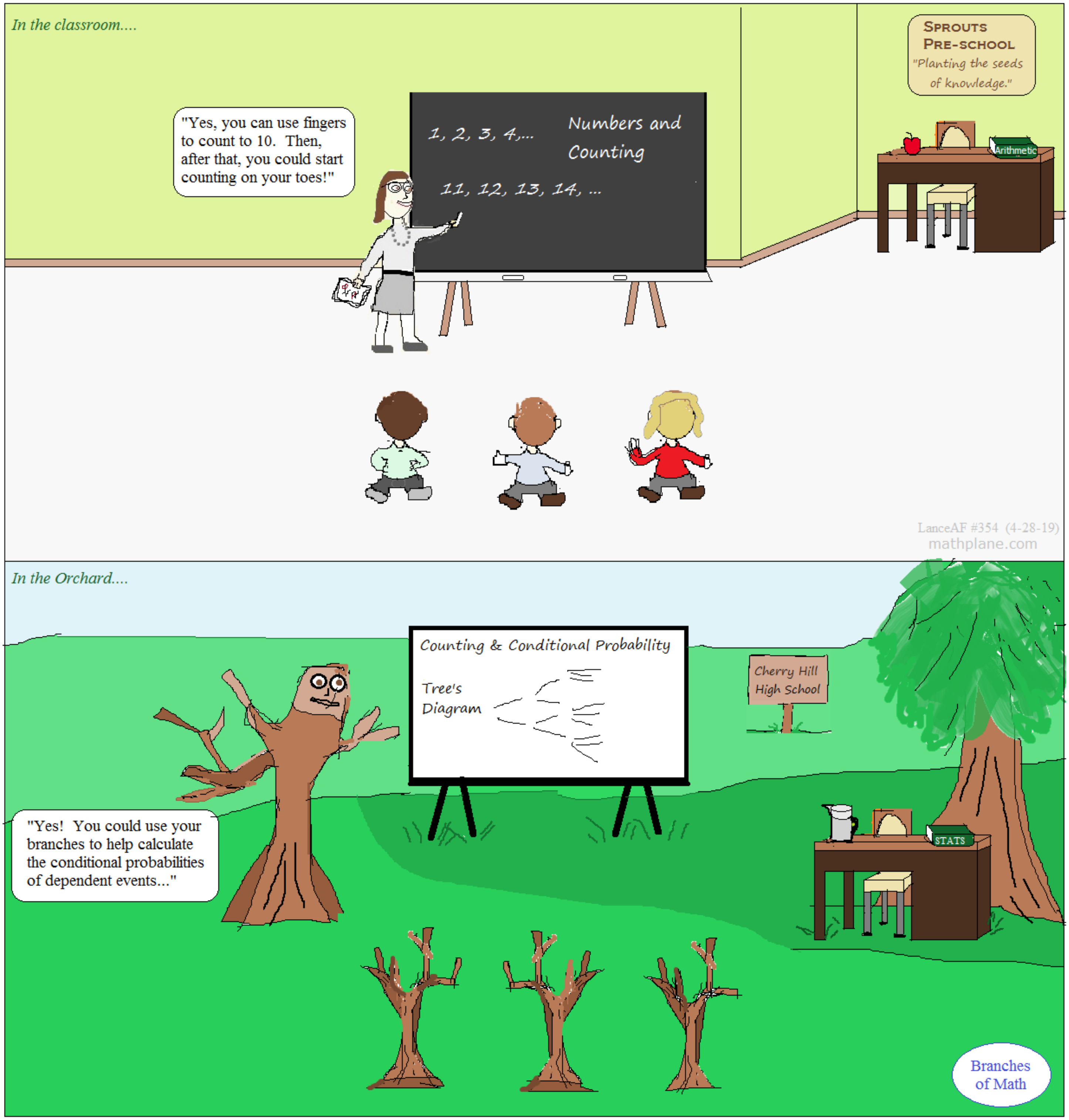
4	5	4	3
1, 6, 8, 9	0, 1, 6, 8, 9	digit that doesn't match or reflect 2nd digit	choosing from 1, 6, 8, 9 where doesn't match first digit

4 x 5 x 4 x 3 = 240

Since there are 380 ways to change a number when flipped...
We can eliminate half of them!

9000 − (380/2) = $\boxed{8810}$

The company only needs to manufacture 8810 numbers...



Practice Questions-→

Counting Questions

1) How many "words" can you make by rearranging the letters in Mississippi?

2) Playing scrabble, you have the following letters: A, A, F, M, R, R, and S

How many different five-letter words could you make?

3) A restaurant's menu offers 5 soups, 6 appetizers, 20 main entrees, and 3 desserts.
Assuming every diner must order at least an entree, how many possible meals could be served?

4) A 6-letter code is made using all 26 letters...

If you alternate vowels (a, e, i, o, u) with consonants,
how many possible codes are there?
(note: letters may be repeated)

5) 12 freshmen and 9 sophomores are picked for positions in a rowing team (1st, 2nd, 3rd and 4th seats)

If you must have 2 freshmen and 2 sophomores rowing, how many possible teams could you make?

6) 7 people arrive at a hotel.
There are 3 available rooms..
1 room has 1 bed, 1 room has 2 beds, and 1 room has 3 beds...

How many possible sleeping arrangements are there (assuming the 7th person will sleep in the car!)

7) "split teams" 10 boys are at a playground, ready to play basketball...

How many ways can they be divided into 5-person teams?

8) For the equation $a + b + c = 14$,

if a, b, and c are separate non-negative numbers, then how many possible values can be assigned?

9) For an octagon, how many triangles can be constructed by connecting any 3 vertices?
(The edges of each triangle will consist of sides or diagonals of the octagon.)

SOLUTIONS

Counting Questions

1) How many "words" can you make by rearranging the letters in Mississippi?

$$\frac{11!}{4!\,4!\,2!} = \frac{11 \times 10 \times \cancel{9} \times \cancel{8} \times 7 \times 6 \times 5}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times 2 \times 1} = \frac{11 \times 10 \times 3 \times 7 \times 6 \times 5}{1 \times 2 \times 1} = \boxed{34650 \text{ words}}$$

2) Playing scrabble, you have the following letters: A, A, F, M, R, R, and S

How many different five-letter words could you make?

If the 7 letters were unique, the answer would be straightforward: $7 \times 6 \times 5 \times 4 \times 3$ (or, ${}_7P_5$)

7 choices for first, 6 choices for second, etc...

But, we have two A's and two R's...

Example: F A R M S could be spelled with either A or either R...
So, we need to eliminate any double counts...

$$\frac{{}_7P_5}{2!\;\;2!} = \boxed{630 \text{ possible words}}$$

3) A restaurant's menu offers 5 soups, 6 appetizers, 20 main entrees, and 3 desserts.
Assuming every diner must order at least an entree, how many possible meals could be served?

$\boxed{20}$ $\boxed{6}$ $\boxed{7}$ $\boxed{4}$ ⇒ $20 \times 6 \times 7 \times 4 = \boxed{3360}$

main entrees

soup choices (5 soups or skip)

appetizer choices (6 apps or skip)

dessert choices (3 desserts or skip)

4) A 6-letter code is made using all 26 letters...

If you alternate vowels (a, e, i, o, u) with consonants,
how many possible codes are there?
(note: letters may be repeated)

5 21 5 21 5 21 — 1157625

AND

21 5 21 5 21 5 — 1157625

$\boxed{2315250}$

5) 12 freshmen and 9 sophomores are picked for positions in a rowing team (1st, 2nd, 3rd and 4th seats)

If you must have 2 freshmen and 2 sophomores rowing, how many possible teams could you make?

$${}_{12}C_2 \times {}_9C_2 \times 4! = 66 \times 36 \times 24 = \boxed{57024 \text{ possible rowing set-ups}}$$

picking 2 from the 12 freshmen

picking 2 from the sophs

arranging the 4 chosen rowers

6) 7 people arrive at a hotel.
There are 3 available rooms..
1 room has 1 bed, 1 room has 2 beds, and 1 room has 3 beds...

How many possible sleeping arrangements are there (assuming the 7th person will sleep in the car!)

$${}_7C_1 \cdot {}_6C_2 \cdot {}_4C_3 \cdot {}_1C_1 = 7 \times 15 \times 4 \times 1 = \boxed{420}$$

7 choices for room 1.... then, 6 choices for room 2/bed 1

5 choices for room 2/bed 2 (divided by 2 because order of beds doesn't matter)

4 choices for room 3/bed 1, 3 choices for room 3/bed 2, and
2 choices for room 3/bed 3 (divided by 3! because order doesn't matter)

remaining choice goes to the car...

7) "split teams" 10 boys are at a playground, ready to play basketball...
How many ways can they be divided into 5-person teams?

Since the 5-person teams are combinations, ${}_{10}C_5$ to pick the first team..

Then, after choosing 5 for one side, the remaining 5 (by default) are the other team...

$\boxed{\text{252 ways to split the players}}$

8) For the equation a + b + c =14,

if a, b, and c are separate non-negative numbers, then how many possible values can be assigned?

In this case, each "urn" a, b, and c, must have at least 1 ball...
since the balls are indistinguishable, we place 1 in each urn...

then, solve for a + b + c = 14

(or, separating identical books onto 3 bookshelves..)

'1st urn picks 0 to 14 balls'

'2nd urn picks from remaining balls'

'the 3rd urn c', takes the remaining balls. It only has 1 choice

$$\sum_{a=0}^{14} \sum_{b=0}^{14-a} \sum_{c=14-a-b}^{14-a-b} (1) = \boxed{120}$$

total possible ways for
(b) indistinguishable balls into (n) distinguishable urns $= \binom{b+n-1}{b}$

or, apply the formula...

$$\binom{b+n-1}{b} = \binom{14+3-1}{14} = \boxed{120}$$

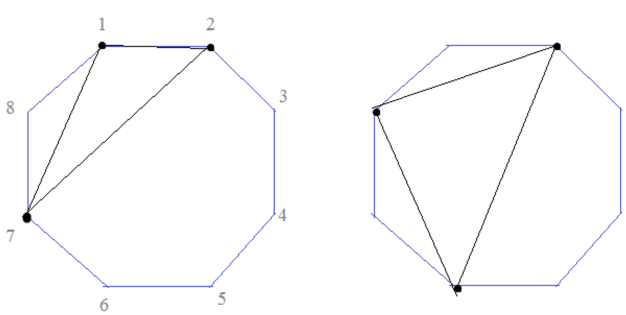
Note: this differs slightly from the bookshelf question...
In the bookshelves, the order of books on each shelf mattered..
In this case, 'the order of the balls in each urn doesn't matter' --
the order of the items in a, b, c are not arranged..

9) For an octagon, how many triangles can be constructed by connecting any 3 vertices?
(The edges of each triangle will consist of sides or diagonals of the octagon.)

Since any triangle consists of any 3 vertices, it follows that the number of possibilities is ${}_8C_3$ or $\binom{8}{3}$

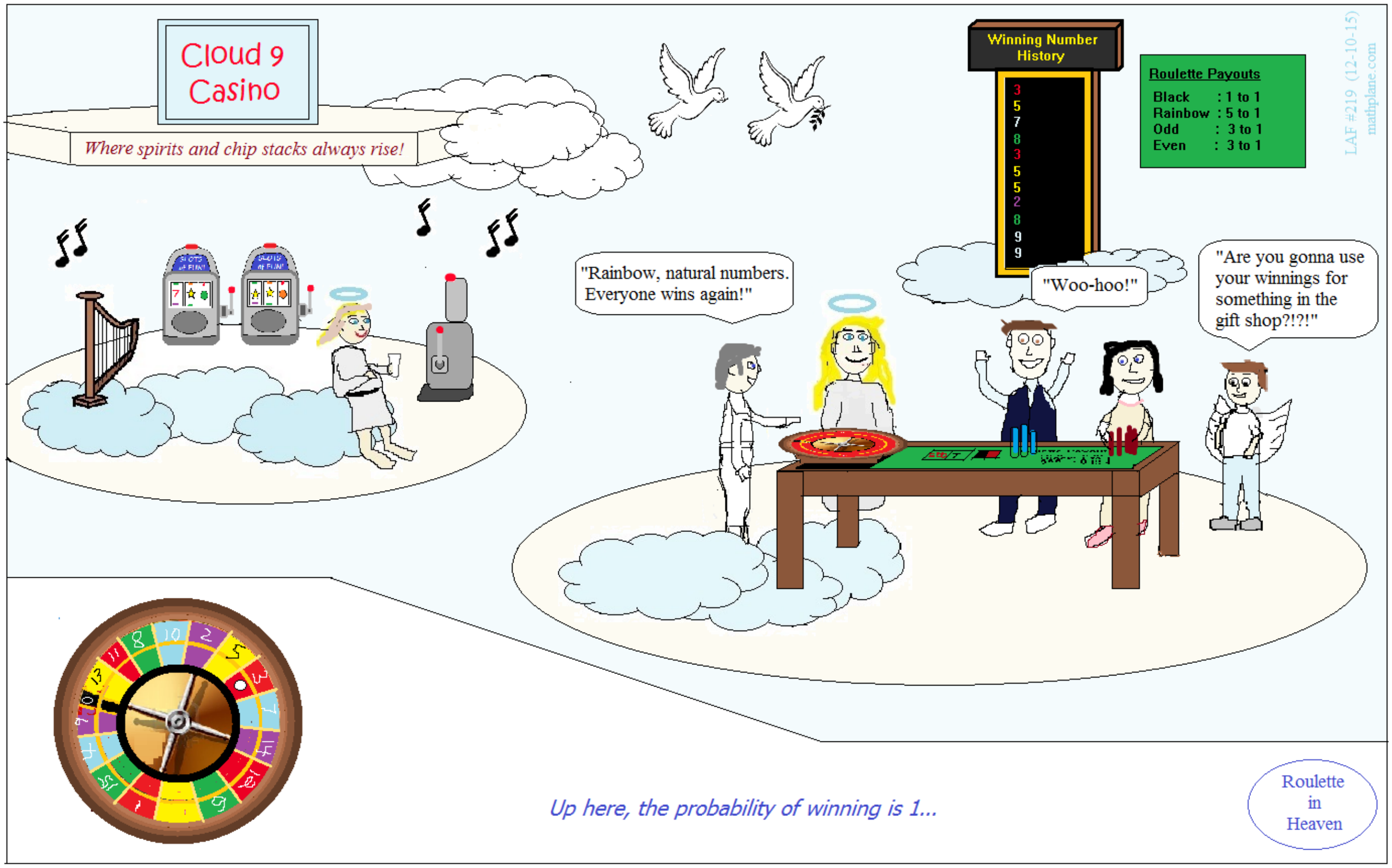
$\boxed{\text{56 possible triangles}}$

Examples:



8 choices for first vertex,
7 choices for second vertex,
6 choices for third vertex...

then, divided by 3! because vertices 1, 2, 7 is same as 7, 1, 2



Balls and Urns, Nested Sums, and more counting approaches…

Distinguishable Balls into Distinguishable Urns

Example: Suppose 10 friends go on vacation, and each makes a reservation at one of 3 hotels. How many ways can the friends stay at the hotels?

Example: I bring home 8 unique toys for my 3 dogs. How many ways can I distribute the toys to my dogs, assuming every dog gets at least one toy?

Indistinguishable Balls into Distinguishable Urns

Example: My math class has 30 students. I want to bring in 30 cupcakes for them. The bakery has 5 types of cupcake. If I randomly pick out the cupcakes, how many possible cupcake batches can I purchase for the students?

Example: For the equation $a + b + c = 15$, where a, b, and c are non-negative integers, how many possible values are there for a, b, and c?

Distinguishable Balls into Indistinguishable Urns

Example: Suppose 20 friends go on vacation. How many ways can they be divided among 3 (identical) cabins.

Example: 3 distinguishable gentleman are visiting a town of 3 indistinguishable inns.
How many ways can these 3 visitors stay in the town?

Indistinguishable Balls into Indistinguishable Urns

Example: A bakery packs 12 vanilla cupcakes into 3 identical brown boxes. Assuming each box will have at least 1 cupcake, how many ways can the baker divide the cupcakes?

Counting: Balls and Urns

Nine urns are lined up.
If 3 *distinct* balls are randomly placed in the urns, how many possible arrangements (assuming an urn cannot have more than 1 ball)?

Nine urns are lined up.
If 3 *indistinguishable* balls are randomly placed in the urns, how many possible arrangements (assuming an urn cannot have more than 1 ball)?

Determine and compare the four scenarios of balls and urns:

4 distinguishable balls placed in 3 distinguishable urns

4 distinguishable balls placed in 3 indistinguishable urns

4 indistinguishable balls placed in 3 distinguishable urns

4 indistinguishable balls placed in 3 indistinguishable urns

Distinguishable Balls into Distinguishable Urns

Example: Suppose 10 friends go on vacation, and each makes a reservation at one of 3 hotels. How many ways can the friends stay at the hotels?

Imagine Dave is the first friend..
He has 3 choices of hotels -- Hyatt, Quality Inn, or Candlewood Suites
Then, Tom has 3 choices: H, QI, or C
Then, Jill has 3 choices: H, QI, or C

$$3^{10} = \boxed{59049}$$

$$\sum_{H=0}^{10} \sum_{Q=0}^{10-H} \sum_{C=10-H-Q}^{10-H-Q} \binom{10}{H}\binom{10-H}{Q}\binom{10-H-Q}{C} = \boxed{59049}$$

Using Nested sums: we count all the ways the 3 hotels can pick from the 10 friends

> total possible ways for
> (k) distinguishable balls into (n) distinguishable urns = n^k
>
> or, balls (b) into urns (u) = u^b

Example: I bring home 8 unique toys for my 3 dogs: shepherd, lab, and husky. How many ways can I distribute the toys to my dogs, assuming each dog gets at least one toy?

Each toy has 3 choices: 3^8

However, this includes cases where dogs get no toys.
We have to subtract those...

One dog gets no toys: $3 \cdot (2^8 - 2)$

pick dog that gets left out — 8 toys have 2 choices — scenario where all 8 toys go to only one other dog

And, there are 3 cases where one dog gets all toys. $3 \cdot 1^8$
(husky gets all + shepard gets all + lab gets all)

$$6561 - 762 - 3 = \boxed{5796}$$

Ways to distribute 8 items to 3 dogs — Ways to distribute 8 items to 2 out of 3 dogs — Ways to distribute 8 items to only one of 3 dogs

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Indistinguishable Balls into Distinguishable Urns

Example: My math class has 30 students. I want to bring in 30 cupcakes for them. The bakery has 5 types of cupcake. If I randomly pick out the cupcakes, how many possible cupcake batches can I purchase for the students?

Imagine the 30 students (cupcakes) are the balls.
Then, imagine the 5 types of cupcake are the 5 distinguishable urns.

We can line up the "5 cupcake types" (urns)..
Then, distribute the 30 student/cupcakes among the 5 types

$$\binom{30+5-1}{30} = \boxed{46376 \text{ cupcake order options}}$$

> total possible ways for
> (b) indistinguishable balls into (n) distinguishable urns = $\binom{b+n-1}{b}$

Note: $$\sum_{a=0}^{30} \sum_{b=0}^{30-a} \sum_{c=0}^{30-a-b} \sum_{d=0}^{30-a-b-c} (1) = 46376$$ ✓

1st cupcake chooses up to 30 students — 2nd cupcake type chooses from remaining students — etc... 5th cupcake gets remaining students...

Example: For the equation $a + b + c = 15$, where a, b, and c are non-negative integers, how many possible values are there for a, b, and c?

Method 1: Using nested sums...

'1st urn picks 0 to 15 balls' — '2nd urn picks from remaining balls' — '3rd urn takes remaining balls. note: there is only 1 possiblity'

$$\sum_{a=0}^{15} \sum_{b=0}^{15-a} \sum_{c=15-a-b}^{15-a-b} (1) = \boxed{136}$$

Method 2: Applying combination formula

$$\binom{b+n-1}{b} = \binom{15+3-1}{15} = \boxed{136}$$

> This is like lining up 3 urns (a, b, c)..
> Then, placing 15 balls in the urns...
>
> Each ball has 3 choices, urn a, b, or c...
>
> 3^{15} FLAWED!!!
>
> Why is this approach flawed?
>
> This method double counts...
>
> Example: 1st ball goes into urn a, 2nd ball goes into urn b, and, remaining 13 go into urn c..
> $1 + 1 + 13 = 15$
>
> Then, 1st ball goes into urn b, 2nd ball goes into urn a, and, remaining 13 go into urn c..
> $1 + 1 + 13 = 15$
>
> Same result counted twice!

Example: For the equation $a + b + c = 17$, where a, b, and c are non-negative integers, how many possible values are there for a, b, and c?

Balls and Urns: 3 methods to solve

This is an application of indistinguishable balls into distinguishable urns..

Here we ware placing 17 (identical) balls into 3 (unique) urns a, b, and c...

Method 1: Formula

$$\binom{b+n-1}{b} = \binom{17+3-1}{17} = \boxed{171}$$

Method 2: Nested Sums

$$\sum_{a=0}^{17} \sum_{b=0}^{17-a} \sum_{c=17-a-b}^{17-a-b} (1) = \boxed{171}$$

'a' selects from 17 balls

'b' selects from remaining balls

'c' gets the rest of the balls

Method 3: Arranging balls and dividers

Imagine 17 balls in a line... Then, place 2 dividers to separate them. (Each of the 3 batches represents the 3 integers..

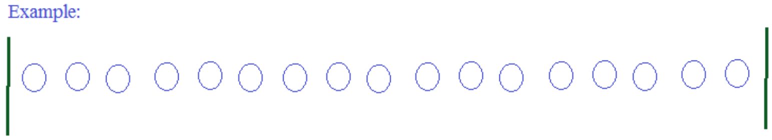
Example:

In this instance, a = 3, b = 9, and c = 5

So, we just have to count the number of ways ot set the 2 dividers...

Since these are *non-negative* numbers, zero is an option...

Example:



In this instance, a = 0, b = 17, and c = 0

First divider can be placed in 18 places... Second divider can be placed in 18 places.

$$\frac{18 \times 18}{2} = \cancel{\boxed{162 \text{ possible ways}}}$$

(It's divided by 2 to get rid of double counts.
EX: 1st divider placed in position 4, 2nd placed in position 8
is same as 1st divider placed in position 8, 2nd placed in position 4)

FLAWED!!!
Since doubles aren't double-counted, they should not be included.
example: dividers in position 6 and 8, then 8 and 6 were counted twice.. But, dividers in position 10 and 10 are not counted twice.
So, it is unnecessary to divide 18 of the divider pairs ..

⇨ place 2 dividers among the 18 slots... $_{18}C_2 = 153$

Then, add 18 for the ways to place dividers next to each other..

$$153 + 18 = \boxed{171}$$

Distinguishable Balls into Indistinguishable Urns

Example: 3 distinguishable gentleman are visiting a town of 3 indistinguishable inns.
How many ways can these 3 visitors stay in the town?

If we apply the formula: $\dfrac{3^3}{3!}$

3^3 — options for each gentleman

$3!$ — since 3 inns are identical

⇒ 27/6 is not possible

$$\cancel{\sum_{a=0}^{3}\ \sum_{b=0}^{3-a}\ \sum_{c=3-a-b}^{3-a-b} {}_{3}C_{a}\cdot {}_{3-a}C_{b}\cdot {}_{3-a-b}C_{c}}$$

Here is the list of scenarios:

Inn	1	2	3
gentleman	A	B	C

6 multiples of same outcome

1	2	3
A	B	C
A	C	B
B	A	C
B	C	A
C	A	B
C	B	A

AB together and C alone (6 multiple counts of same outcome)

1	2	3
AB	C	
AB		C
C	AB	
	AB	C
C		AB
	C	AB

1	2	3
AC	B	
AC		B
B	AC	
	AC	B
B		AC
	B	AC

1	2	3
BC	A	
BC		A
A	BC	
	BC	A
A		BC
	A	BC

All 3 gentlemen stay at same hotel (**Instead of 6, there are only 3 multiple counts.)

1	2	3
ABC		
	ABC	
		ABC

There are only 5 possible outcomes!

A B C alone

AB together and C alone
BC together and A alone
AC together and B alone

ABC all together

> How many ways can the 3 distinguishable gentlemen stay in a town of 6 indistinguishable inns?
>
> The answer is still only 5 outcomes!
>
> all alone, all together, AB together (and C alone)
> AC together (and B alone)
> BC together (and A alone)

Indistinguishable Balls into Indistinguishable Urns

Example: A bakery packs 12 vanilla cupcakes into 3 identical brown boxes. Assuming each box will have at least 1 cupcake, how many ways can the baker divide the cupcakes?

In this case, we can use brute force method and list the possibilities....

10 1 1
9 2 1
8 3 1
8 2 2

7 4 1
7 3 2
6 5 1
6 4 2
6 3 3

5 5 2
5 4 3
4 4 4

12 possible ways

Note: 6, 5, 1 is the same as 1, 5, 6

To help organize, we listed numbers in descending order.

Distinguishable Balls into Indistinguishable Urns : Troubles and Challenges

Example: Suppose 20 friends go on vacation. How many ways can they be divided among 3 (identical) cabins ?

In this case, we just want to know how the 20 friends are divided up. (In other words, who is staying with whom in the cabins.)

First Attempt: total possible ways for (k) distinguishable balls into (n) distinguishable urns $= n^k$

So,

total possible ways for (k) distinguishable balls into (n) indistinguishable urns $= \dfrac{n^k}{n!}$

Each person picks one of 3 cabins... 3^{20}

Then, we eliminate the double counts..

Example: If 2 pick the 1st cabin, 2 pick the 2nd cabin, 16 pick 3rd, that's same as 2 pick 1st, 16 pick 2nd, 2 pick 3rd (because cabins are same...)

$\dfrac{3^{20}}{3!} \Rightarrow$ It's not an integer!?!?!

FLAWED!!

Why is it flawed?

Second Attempt: Arranging with spots and dividers

Arrange the individual 20 friends and 2 dividers to separate them...

$\dfrac{22!}{2!}$ because the dividers are identical

Example:

4 in 1st cabin 9 in 2nd cabin 7 in 3rd cabin

Then, divide by 3! because the cabins are identical..

$\dfrac{22!}{2!\,3!}$

FLAWED!!

Why is it flawed?

In the above method, we're arranging the individuals in the cabin. That is unnecessary..

Example: If cabin 1 has person 2, 5, 7, 9 in one separation and 5, 7, 9, 2 in another arrangement, it's counted twice!

Third attempt:

$\dfrac{3^{20}}{a!\,(20-a)!\,(20-a-b)!}$ each person picks a cabin

where a, b, and c represent the number of people in each cabin...

But, how do you do that for every possible way?

Fourth Approach: Reorder the thinking....

20 distinguishable balls into 3 indistinguishable urns

⇩

3 indistinguishable urns choosing from 20 distinguishable balls

'1st urn picks 0 to 20 balls'

'2nd urn picks from remaining balls'

'3rd urn takes remaining balls. note: there is only 1 possiblity'

$$\sum_{a=0}^{20} \sum_{b=0}^{20-a} \sum_{c=20-a-b}^{20-a-b} {}_{20}C_a \cdot {}_{20-a}C_b \cdot {}_{20-a-b}C_c = 3486784401$$

Then, since the urns (cabins) are indistinguishable, $\dfrac{3486784401}{3!}$

FLAWED

Does this method differentiate the friends?
Example: When 1st urn picks 10 balls, which ones?
Every possible batch of 10 different friends must be counted.

Example: Cabin 1 has friend 2, 6, 7, 10
Cabin 2 has friend 1, 13, 20
Cabin 3 has thirteen others..

Cabin 1 has friend 2, 6, 7, 10
Cabin 2 has thirteen others.
Cabin 3 has friend 1, 13, 20

Cabin 1 has friend 1, 13, 20
Cabin 2 has friend 2, 6, 7, 10
Cabin 3 has thirteen others.

etc...

Counting: Balls and Urns

Nine urns are lined up.
If 3 *distinct* balls are randomly placed in the urns, how many possible arrangements (assuming an urn cannot have more than 1 ball)?

Approach: "view from the balls"

ball 1: 9 choices of urns
ball 2: 8 remaining choices of urns
ball 3: 7 remaining choices of urns

$9 \times 8 \times 7 = \boxed{504}$

Nine urns are lined up.
If 3 *indistinguishable* balls are randomly placed in the urns, how many possible arrangements (assuming an urn cannot have more than 1 ball)?

Approach: "view from the urns"

there are 9 urns... Choose 3 of them...

$\frac{9!}{3!\ 6!} = \boxed{\text{84 possible ways}}$

${}_9C_3$

Determine and compare the four scenarios of balls and urns:

4 distinguishable balls placed in 3 distinguishable urns

4 distinguishable balls placed in 3 distinguishable urns

4 different colored balls placed in urn A, urn B, and urn C

red ball has 3 choices; blue ball has 3 choices; yellow ball has 3 choices; green ball has 3 choices...

$3^4 = \boxed{81}$

4 distinguishable balls placed in 3 indistinguishable urns

4 0 0 ⟹ 1 way all are in single urn

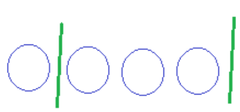
3 1 0 ⟹ 4 choices for separated distinguishable ball chosen

2 2 0 ⟹ AB/CD AC/BD AD/BC three ways to pair them

2 1 1 ⟹ AB AC AD BC BD CD are the pairs that are in one urn

$\boxed{\text{14 ways}}$

4 indistinguishable balls placed in 3 distinguishable urns



5 slots, choose 2 ----> 10
plus, 5 ways the dividers are next to each other

$\boxed{\text{15 total}}$

(same application as non-negative integers $a + b + c = 4$)

$$\binom{b+u-1}{b} = \binom{4+3-1}{4} = \boxed{15}$$

$$\sum_{a=0}^{4}\ \sum_{b=0}^{4-a}\ \sum_{c=4-a-b}^{4-a-b} (1) = \boxed{15}$$

4 indistinguishable balls placed in 3 indistinguishable urns

Since the 4 balls are identical and the 3 urns are identical, there is a lot of overlap (double counts)...
We can just list the possible splits...

4 0 0
3 1 0
2 2 0
2 1 1

$\boxed{\text{4 total}}$

note: all combos are covered...

0 3 1
1 2 1
0 4 0
etc.. can be created by rearranging the identical urns

Nest Sum Formulas

(We will assume each bookshelf can hold anywhere from 0 to all 20 books.)

Example: How many ways can you divide 20 *identical* books onto 3 *distinct* shelves?

$$\sum_{i=0}^{20} \sum_{j=0}^{20-i} \sum_{k=20-i-j}^{20-i-j} (1) = 231$$

How many ways to divide books into 3 shelves

"How many books does each shelf get?"

□ □ □ □ □ | □ □ □ □ □ □ □ | □ □ □ □ □ □ □ □

Number of ways to places 2 dividers: ${}_{21}C_2 + 21 = 231$

ways dividers are next to each other

Example: How many ways can you divide 20 *different* books onto 3 *distinct* shelves?

$$\sum_{i=0}^{20} \sum_{j=0}^{20-i} \sum_{k=20-i-j}^{20-i-j} {}_{20}C_i \cdot {}_{(20-i)}C_j \cdot {}_{(20-i-j)}C_k$$

How many ways to group different books on shelves

"How many, and which books does each shelf get?"

□ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □ □

It is difficult (impossible?) to apply books and dividers because after groups are allocated to shelves,
each group must be divided by n! to get rid of repeats.
(i.e. the allocations vary, so the n! number varies as well...)

Example: How many ways can you *arrange* 20 *different* books onto 3 *distinct* shelves?

$$\sum_{i=0}^{20} \sum_{j=0}^{20-i} \sum_{k=20-i-j}^{20-i-j} {}_{20}P_i \cdot {}_{(20-i)}P_j \cdot {}_{(20-i-j)}P_k$$

$$= 5620003638888803840000$$

How many ways to display different books on shelves

"How many, which books, and how are they displayed?"

□ □ | □ □ □ □ □ | □ □ □ □ □ □ □ □ □ □ □ □ □

Number of ways to arrange books and dividers $\dfrac{(20+2)!}{2}$

because the dividers are not unique

$$= 5620003638888803840000$$

Example: How many ways can you divide 20 *identical* books into 3 *identical* boxes?

List the possiblities:

20 0 0
19 1 0

18 2 0
18 1 1

17 3 0
17 2 1

16 4 0
16 3 1
16 2 2

15 5 0
15 4 1
15 3 2

14 6 0
14 5 1
14 4 2
14 3 3

13 7 0
13 6 1
13 5 2
13 4 3

12 8 0
12 7 1
12 6 2
12 5 3
12 4 4

11 9 0
11 8 1
11 7 2
11 6 3
11 5 4

10 10 0
10 9 1
10 8 2
10 7 3
10 6 4
10 5 5

9 9 2
9 8 3
9 7 4
9 6 5

8 8 4
8 7 5
8 6 6

7 7 6

$\boxed{\text{44 possiblities}}$