Curve Sketching: Critical Values, Extrema, and Concavity

Notes, Examples, and Exercises (with Solutions)

Topics include max/min, derivatives, points of inflection, charts, graphing, odd/even functions, and more.

Mathplane.com

Example: $f(x) = x - x^{\frac{2}{3}}$

Using first derivative, find the critical values...

$f'(x) = 1 - \frac{2}{3} x^{\frac{-1}{3}}$ Find derivative $f'(x)$

$0 = 1 - \frac{2}{3\sqrt[3]{x}}$ Set $f'(x) = 0$

$\frac{2}{3\sqrt[3]{x}} = 1$

$\sqrt[3]{x} = \frac{2}{3}$

$x = \frac{8}{27} = .296$ (approx) Is $f'(x)$ undefined?

and, equation is undefined at $x = 0$ ("cusp")

Using second derivative, find the critical values ...

$f'(x) = 1 - \frac{2}{3} x^{\frac{-1}{3}}$

$f''(x) = 0 + \frac{2}{9} x^{\frac{-4}{3}}$ Find second derivative $f''(x)$

$0 = \frac{2}{9\sqrt[3]{x^4}}$ Set $f''(x) = 0$

Since the second derivative is never equal to zero, there is no point of inflection...

However, since the second derivative is undefined at x = 0, it verifies the cusp...

Note: $f'(8/27) = 0$ (critical point)

$f''(8/27) > 0$ (concave up)

⇨ $x = \frac{8}{27}$ must be a local minimum!

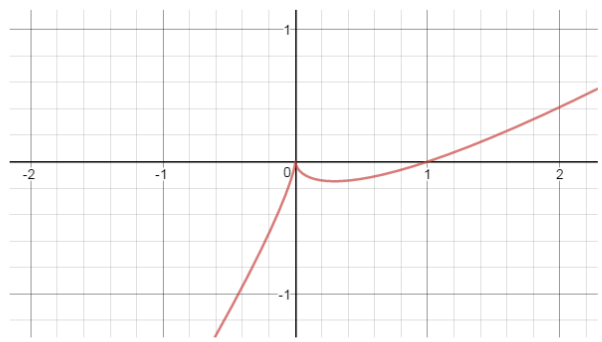
Note: if $x < 0$

$f'(x) > 0$ increasing

if $0 < x < 8/27$

$f'(x) < 0$ decreasing

⇨ x = 0 must be local maximum



Two approaches to finding relative maximum/minimum

Method 1: Find first derivative and second derivative...

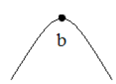
If $f'(a) = 0$

$f''(a) > 0$ local minimum

horizontal tangent line
concave up

If $f'(b) = 0$

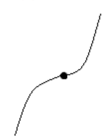
$f''(b) < 0$ local maximum



horizontal tangent line
concave down

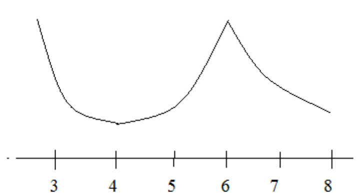
If $f'(c) = 0$

$f''(c) = 0$ 'plateau' or 'pause'



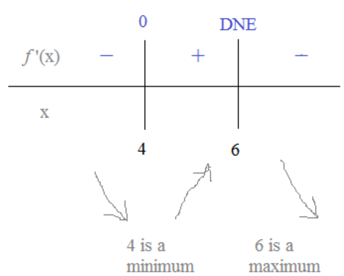
smooth curve that changes
from concave down to concave up

Method 2: Increasing/Decreasing chart (or 'number line')



$f'(4) = 0$

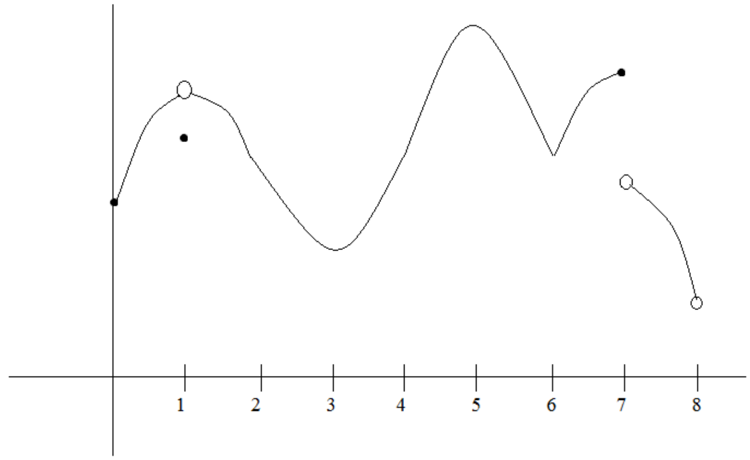
$f'(6)$ is undefined



Critical Values

Example: Where are the absolute (global) maximum and minimum?

Where are the local (relative) maxima and minima?



Local minima occur at x = 1
x = 3
x = 6

Global minimum? None!

Local maxima occur at x = 5
x = 7

Global maximum? x = 5

Examples: Find critical values (extrema) and points of inflection(s):

$f(x) = 18x^2 - \ln x$

$f'(x) = 36x - \frac{1}{x}$

$f''(x) = 36 + \frac{1}{x^2}$

$0 = 36x - \frac{1}{x}$

$36x^2 = 1$

$x = +/- \ 1/6$ but, it cannot be -1/6

lnx must be positive

XX +
-1/6 0 1/6

XX +
0

$f(x) = \frac{e^x + e^{-x}}{2}$

$f'(x) = \frac{e^x - e^{-x}}{2}$ critical value when x = 0

$f''(x) = \frac{e^x + e^{-x}}{2}$ critical value when x = 0

$g(x) = x + \frac{4}{x}$

$g'(x) = 1 + \frac{-4}{x^2}$

$x = 2, -2$ and $x \neq 0$

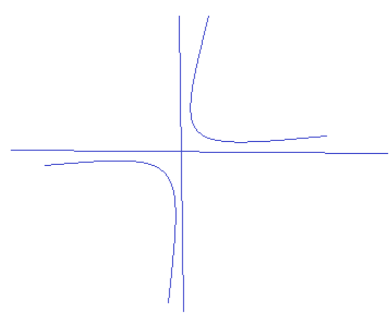
$g''(x) = \frac{8}{x^3}$

$x \neq 0$

(no points of inflection

concave up when > 0

concave down when x < 3



Example:

X	0	1	2	3
f	0	2	0	−2
f'	3	0	DNE	−3
f''	0	−1	DNE	0

X	$0 < X < 1$	$1 < X < 2$	$2 < X < 3$
f	+	+	−
f'	+	−	−
f''	−	−	−

The charts represent the function f (X) on the interval (0, 3)

a) What are the absolute extrema?

b) What are the point(s) of inflection?

c) Sketch the graph of f (X)

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a) The function increases from 0 to 1, then it decreases from 1 to 3. (and, $f' = 0$ at x = 1). Therefore, the absolute maximum in the interval [0, 3] occurs at x = 1 (the coordinate (1, 2))

And, the minimum will occur at either x = 0 or x = 3... Since $f(0) = 0$ and $f(3) = -2$, the absolute minimum occurs at x = 3 (the coordinate (3, -2))

b) A point of inflection occurs when the second derivative equals zero.

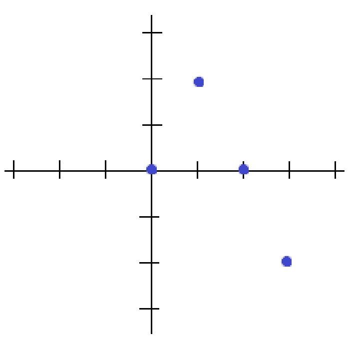
On the interval (0, 3), there are no points of inflection.

If the domain of the function were extended, there would be points of inflection at x = 0 and x = 3

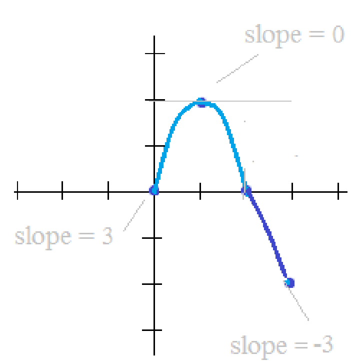
c) to sketch the graph, start with the function:

Coordinates will include

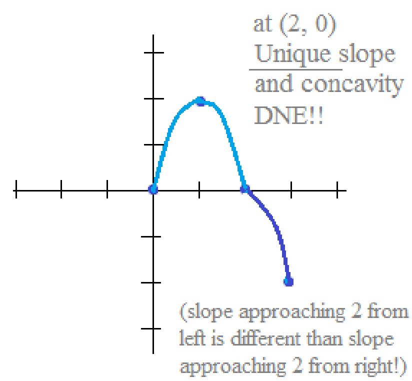
(0, 0) (1, 2) (2, 0) (3, -2)



then, use the first derivative f' to identify the instantaneous slope...



Use the 2 charts and second derivatives to smooth the curves....



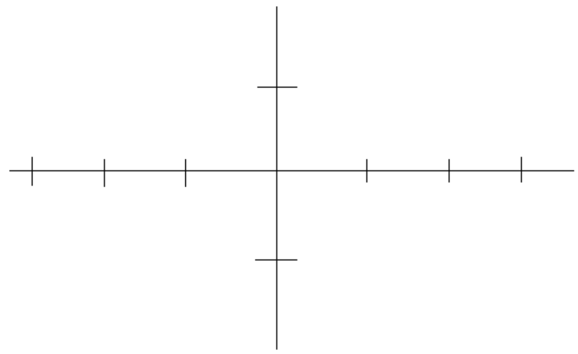
1) The even function $f(x)$ has the following characteristics:

Extrema, Concavity, and other properties...

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	1	positive	0	negative	-1	negative
$f'(x)$	undefined	negative	0	negative	undefined	positive
$f''(x)$	undefined	positive	0	negative	undefined	negative

a) Sketch a possible graph

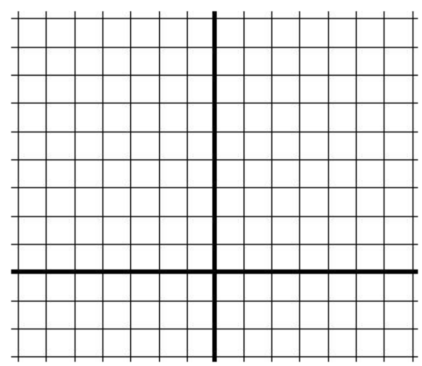
b) Where are the points of inflection ?



c) Where are the local minima? Explain your reasoning.

2) Find the intervals where the function is increasing and decreasing.

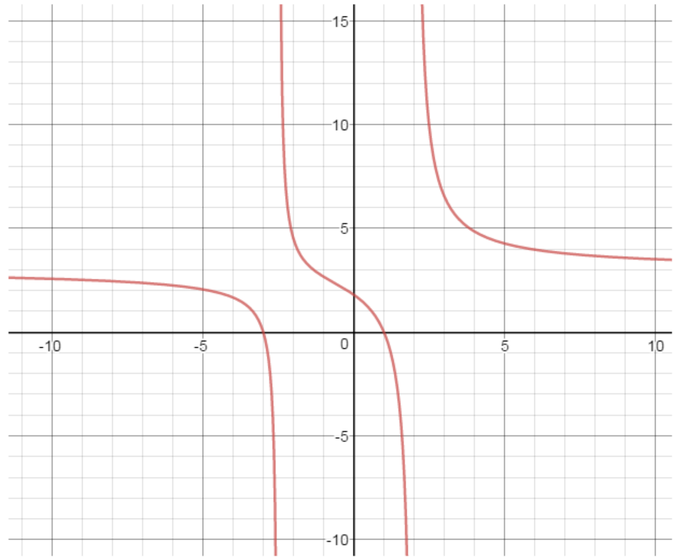
$f(x) = \sqrt{16 - x^2}$



3) Find the absolute ('global') maximum and minimum of $f(x) = 3x^4 - 4x^3$ over the interval [-1, 2]

Interpreting graphs: Critical Values

4) Fill in the charts describing all the critical values and behavior of the graph.



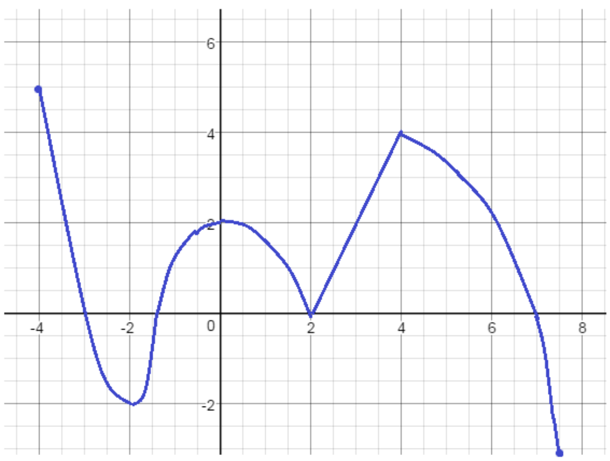
a) Critical Values, Intervals of Increasing and Decreasing

x	
$f'(x)$	

b) Critical Values and Intervals of Concavity

x	
$f''(x)$	

5)



a) Zeros, positive and negative intervals

x	
$f(x)$	

b) Critical Values, Intervals of Increasing and Decreasing

x	
$f'(x)$	

c) Critical Values and Intervals of Concavity

x	
$f''(x)$	

6) Find the critical values for the function $y = \dfrac{x^2}{x - 1}$

Identify increasing and decreasing intervals.
Describe the concavity.
Sketch a graph.

7) $y = x^{1/3}(x^2 - 27)$

Find the critical values, inflection points, and intervals (increasing/decreasing; concave up/down)

8) Find the extrema and increasing/decreasing intervals for the function

$$y = (x^{4/3} - 3x^{1/3})^2$$

9) Find the critical values, extrema, points of inflection, intervals of increasing/decreasing and concave up/concave down.

$f(\theta) = 2 + \cos 2\theta$ on the interval $[0, \pi]$

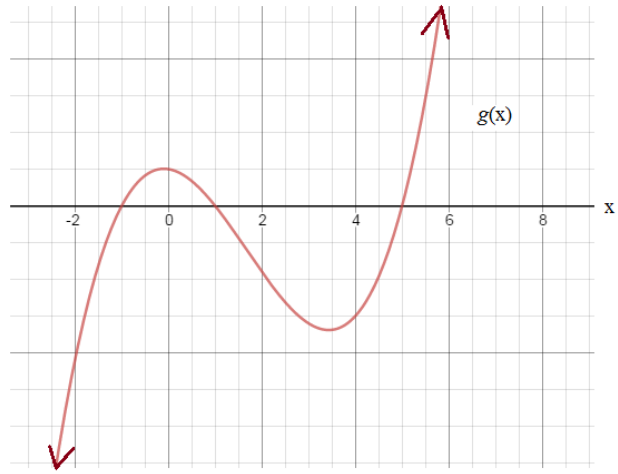
Calculus Derivatives: Critical Values and Interpreting Graphs

Suppose the sketch is the function $g(x)$...

a) What are the zeros?

b) What intervals is $g(x)$ increasing?

c) Identify the point of inflection.

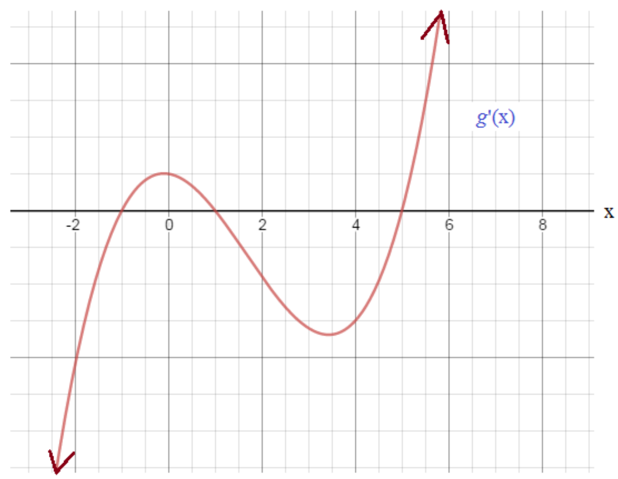


Now, suppose the sketch is $g'(x)$ --- the *derivative* of $g(x)$...

a) What intervals is $g(x)$ increasing?

b) Identify the local extrema of $g(x)$.

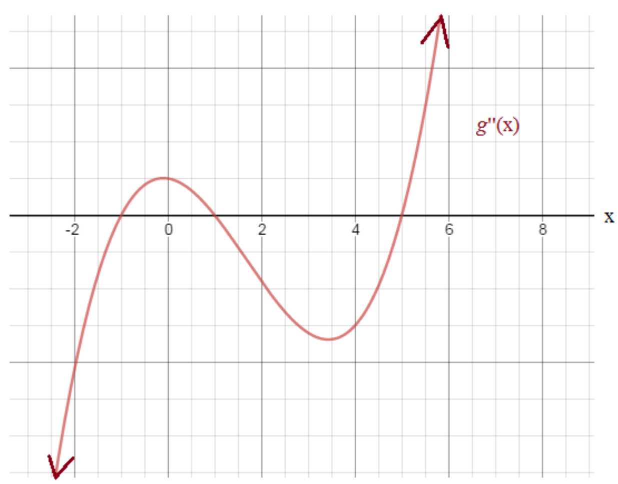
c) Identify the point(s) of inflection of $g(x)$.
When is the function $g(x)$ concave up?



Now, suppose the sketch is $g''(x)$ --- the *2nd derivative of* $g(x)$...

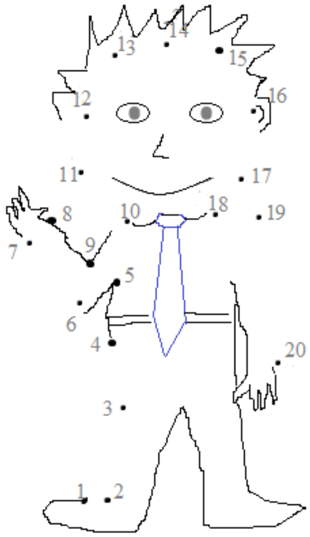
a) When is the function $g(x)$ concave up?

b) Identify the point(s) of inflection of $g(x)$.



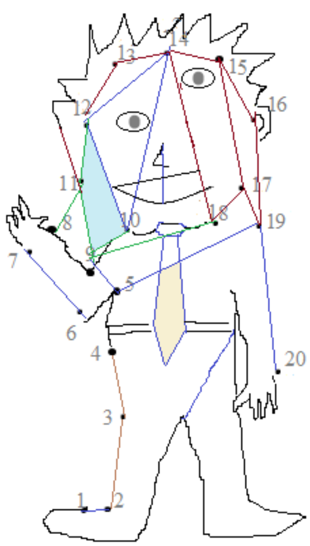
Connect the dots in numerical order
to complete the picture!

Name: ____________



Connect the dots in numerical order
to complete the picture!

Name: *Pablo*



Abstract Art:
Origins of Cubism

Little Picasso fails a math exercise...

(... But, he discovers another interest.)

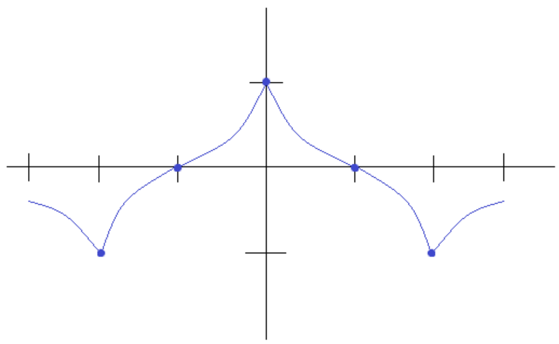
1) The even function $f(x)$ has the following characteristics:

Extrema, Concavity, and other properties...

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$
$f(x)$	1	positive	0	negative	−1	negative
$f'(x)$	undefined	negative	0	negative	undefined	positive
$f''(x)$	undefined	positive	0	negative	undefined	negative

SOLUTIONS

a) Sketch a possible graph



b) Where are the points of inflection ?

at $x = 1$ (because $f''(x) = 0$

and, $x = -1$ (because the function is 'even')

c) Where are the local minima? Explain your reasoning.

The critical values occur at $x = 0, 1,$ and 2
(because $f'(x) = 0$ or is undefined...)

Also, critical values occur at $x = -1$ and -2
(because this is an even function)

Since $f'(1.5) < 0$ (decreasing)
and $f'(2.5) > 0$ (increasing), $x = 2$ is a minimum..

Then, $x = -2$ is also a minimum (even function reflects over y-axis)

Strategy for sketch:

step 1: look at $f(x)$ ---> plot the points...

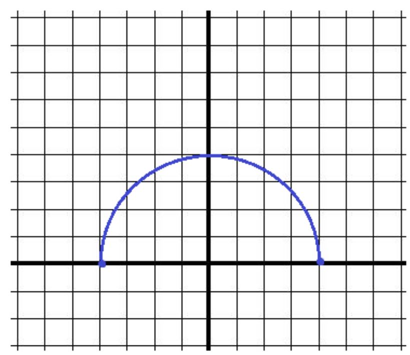
step 2; look at $f'(x)$ ----> note the direction (increasing/decreasing/constant

step 3: look at $f''(x)$ ----> 'bend' the direction to correspond to concavity

step 4; for undefined parts, add kinks, cusps, asymptotes, etc...

2) Find the intervals where the function is increasing and decreasing.

$f(x) = \sqrt{16 - x^2}$ Domain: [-4, 4]

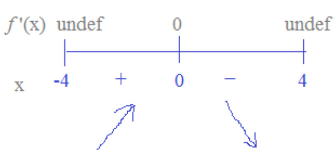


$f(x) = (16 - x^2)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(16 - x^2)^{\frac{-1}{2}}(-2x)$

$= \dfrac{-2x}{2\sqrt{16 - x^2}}$

critical values: -4, 0, 4



critical values occur when $f'(x) = 0$
or $f'(x)$ is undefined...

increasing: (-4, 0)
decreasing: (0, 4)

3) Find the absolute ('global') maximum and minimum of $f(x) = 3x^4 - 4x^3$ over the interval [-1, 2]

Using the first derivative, we can find the critical values....

$f'(x) = 12x^3 - 12x^2$

$12x^3 - 12x^2 = 0$

$12x^2(x - 1) = 0$

$x = 0, 1$

$f(0) = 0 - 0 = 0$

$f(1) = 3 - 4 = -1$

Then, we need to identify boundary points....

$f(-1) = 3 + (-4) = 7$

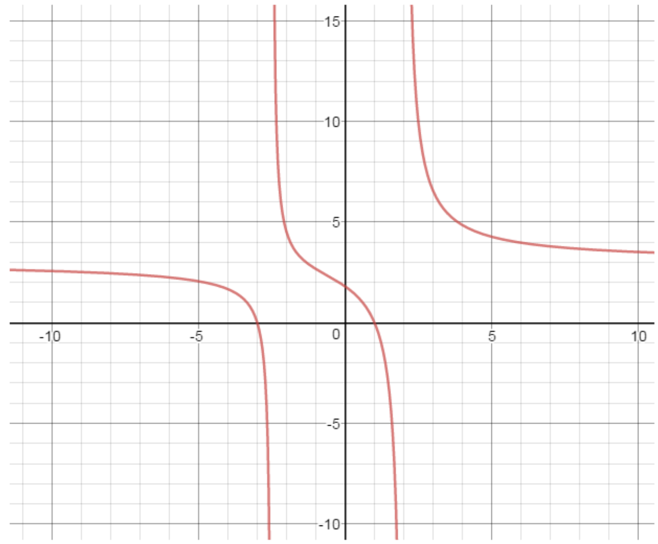
$f(2) = 48 - 32 = 16$

Absolute Minimum: (1, -1)

Absolute Maximum: (2, 16)

4) Fill in the charts describing all the critical values and behavior of the graph.

SOLUTIONS



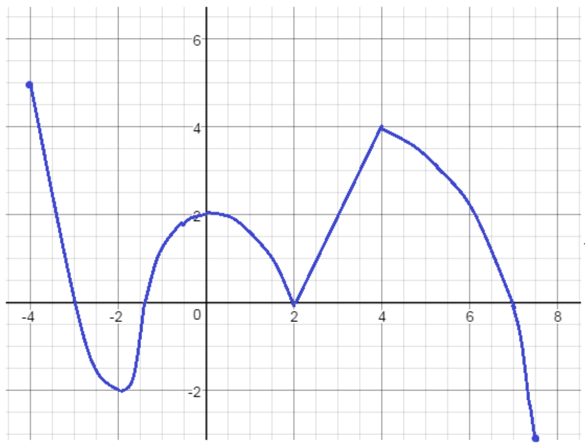
a) Critical Values, Intervals of Increasing and Decreasing

x	x < -2.5	-2.5	-2.5 < x < 2	2	x > 2
$f'(x)$	–	undef.	–	undef.	–

b) Critical Values and Intervals of Concavity

x	x < -2.5	-2.5	-2.5 < x < -.6	-.6	-.6 < x < 2	2	x > 2
$f''(x)$	–	undef.	+	0 Point of Inflection	–	undef.	+

5)



a) Zeros, positive and negative intervals

x	-4 ≤ x < -3	-3	-3 < x < -1.4	-1.4	-1.4 < x < 2	2	2 < x < 7	7	7 < x ≤ 7.5
$f(x)$	+	zero (intercept)	–	zero	+	zero	+	zero	–

b) Critical Values, Intervals of Increasing and Decreasing

x	x = -4	-4 < x < -2	-2	-2 < x < 0	0	0 < x < 2	2	2 < x < 4	4	4 < x < 7.5	7.5
$f'(x)$	endpoint	decreasing —	0	increasing +	0	decreasing –	undefined (cusp)	increasing +	undefined (cusp/kink)	decreasing –	endpoint

c) Critical Values and Intervals of Concavity

x	x = -4	-4 < x < -1.4	-1.4	-1.4 < x < 2	2	2 < x < 4	4	4 < x <7.5	7.5
$f''(x)$	endpoint	concave up +	0 point of inflection	concave down –	undef	constant 0 no concavity	undefined (cusp)	concave down –	endpoint

SOLUTIONS

6) Find the critical values for the function $y = \dfrac{x^2}{x-1}$

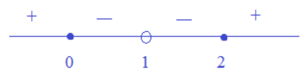
Identify increasing and decreasing intervals.
Describe the concavity.
Sketch a graph.

$$y' = \frac{2x(x-1) - (1)x^2}{(x-1)^2} \Longrightarrow \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

Critical values occur when derivative equals zero or when it's undefined...

Maximum or minimum will occur at x = 0 or 2

Function is undefined and not differentiable at x = 1



first derivative number line displaying slope

$$y'' = \frac{(2x-2)(x-1)^2 - 2(x-1)(x^2-2x)}{(x-1)^4}$$

Since (x − 1) is repeated in each part of the equation, we can simplify that...

$$= \frac{(2x-2)(x-1) - 2(x^2-2x)}{(x-1)^3}$$

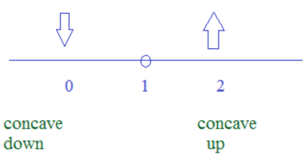
Then, condense the numerator...

$$= \frac{2x^2 - 4x + 2 - 2x^2 + 4x}{(x-1)^3}$$

and, the 2nd derivative simplifies nicely!

$$y'' = \frac{2}{(x-1)^3}$$

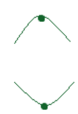
There are no points of inflection..
But, again, it is undefined at x = 1



second derivative number line displaying concavity

Since concave down at x = 0, this must be a maximum..

Since concave up at x = 2, this must be a minimum



To sketch the rational expression....

$$y = \frac{x^2}{x-1}$$

vertical asymptote at x = 0

slant asymptote at y = x + 1

1	1	0	0
		1	1
	1	1	1 (remainder)

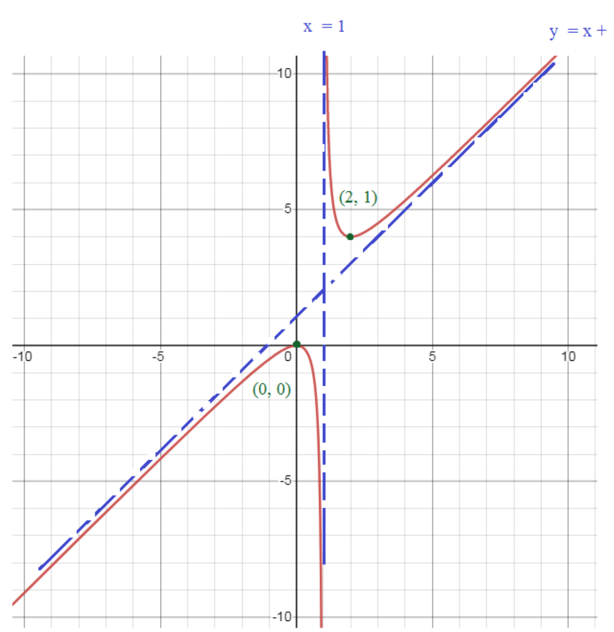
x + 1

local maximum at (0, 0)

local minimum at (2, 1)

concave down for x < 1

concave up for x > 1



7) $y = x^{1/3}(x^2 - 27)$

SOLUTIONS

Find the critical values, inflection points, and intervals (increasing/decreasing; concave up/down)

Instead of using product rule, we'll rewrite first...

$$y = x^{7/3} - 27x^{1/3}$$

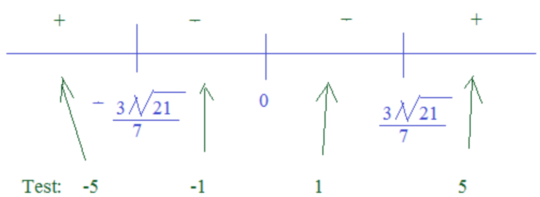
$$y' = \frac{7}{3}x^{4/3} - 9x^{-2/3}$$

To find critical values, we set $y' = 0$

$$\frac{7}{3}x^{4/3} - 9x^{-2/3} = 0$$

$$\frac{7}{3}x^{-2/3}\left(x^2 - \frac{27}{7}\right) = 0$$

$x = \pm \frac{3\sqrt{21}}{7}$ and, at x = 0, it's undefined....



$$\frac{7}{3}x^{-2/3}\left(x^2 - \frac{27}{7}\right) = y'$$

decreasing $\left(-\frac{3\sqrt{21}}{7}, \frac{3\sqrt{21}}{7}\right)$

increasing $\left(-\infty, -\frac{3\sqrt{21}}{7}\right) \cup \left(\frac{3\sqrt{21}}{7}, \infty\right)$

$\frac{3\sqrt{21}}{7} \cong 1.96$

$$y' = \frac{7}{3}x^{4/3} - 9x^{-2/3}$$

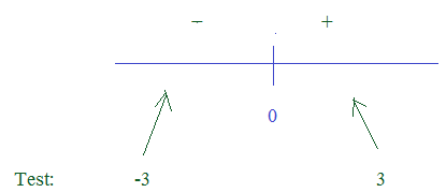
$$y'' = \frac{28}{9}x^{1/3} + 6x^{-5/3}$$

To find inflection points, set $y'' = 0$

$$\frac{28}{9}x^{1/3} + 6x^{-5/3} = 0$$

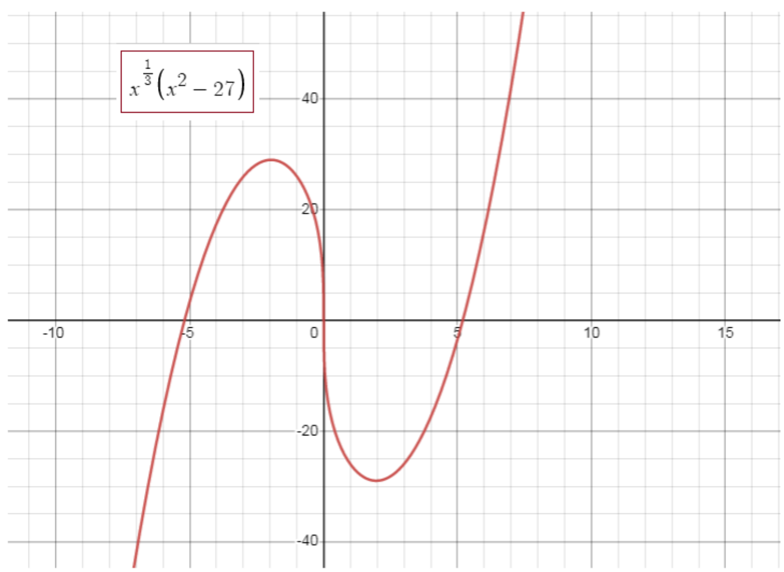
$$\frac{28}{9}x^{-5/3}\left(x^2 + \frac{54}{28}\right) = 0$$

$x = 0$ is critical value --- undefined...



concave down $(-\infty, 0)$

concave up $(0, \infty)$



8) Find the extrema and increasing/decreasing intervals for the function

SOLUTIONS

$$y = (x^{4/3} - 3x^{1/3})^2$$

To find critical values, we take the first derivative (using power rule and chain rule)

$$y' = 2(x^{4/3} - 3x^{1/3})^1 \cdot \left(\frac{4}{3}x^{1/3} - x^{-2/3}\right)$$

Set derivative equal to 0

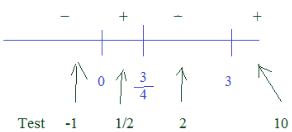
$$0 = (2x^{4/3} - 6x^{1/3})\left(\frac{4}{3}x^{1/3} - x^{-2/3}\right)$$

$$(2x^{4/3} - 6x^{1/3}) = 0 \qquad \left(\frac{4}{3}x^{1/3} - x^{-2/3}\right) = 0$$

$$2x^{1/3}(x - 3) = 0 \qquad x^{-2/3}\left(\frac{4}{3}x - 1\right) = 0$$

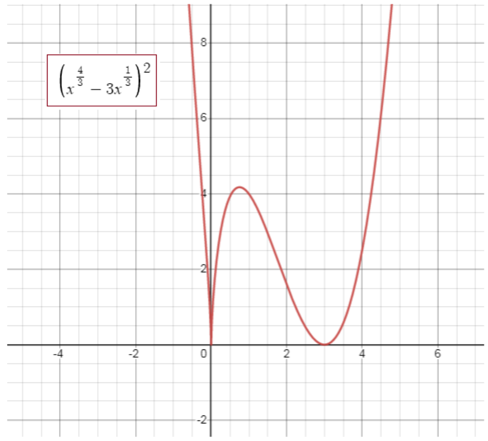
undefined at 0

critical values ⇒ $x = 0,\ 3,\ \frac{3}{4}$



increasing (0, 3/4) U (3, ∞)

decreasing (−∞, 0) U (3/4, 3)



9) Find the critical values, extrema, points of inflection, intervals of increasing/decreasing and concave up/concave down.

$f(\theta) = 2 + \cos 2\theta$ on the interval $[0, \pi]$

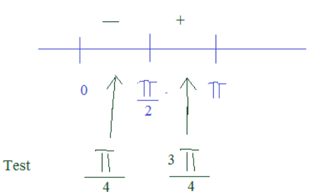
$f'(\theta) = 0 - 2\sin 2\theta$

$-2\sin 2\theta = 0$

$\sin 2\theta = 0$

$2\theta = 0, \pi, 2\pi$, etc...

$\theta = 0, \frac{\pi}{2}, \pi$ in the interval...



$f'(\theta) = -2\sin 2\theta$

decreasing $(0, \frac{\pi}{2})$

increasing $(\frac{\pi}{2}, \pi)$

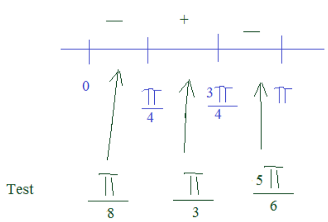
$f''(\theta) = -4\cos 2\theta$

$-4\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc..

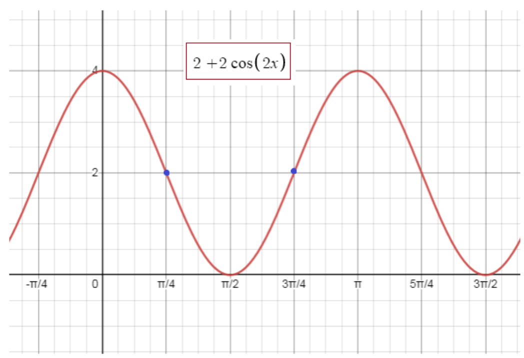
$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ in the interval



$f''(\theta) = -4\cos 2\theta$

concave up $(\frac{\pi}{4}, \frac{3\pi}{4})$

concave down $(0, \frac{\pi}{4})$ U $(\frac{3\pi}{4}, \pi)$

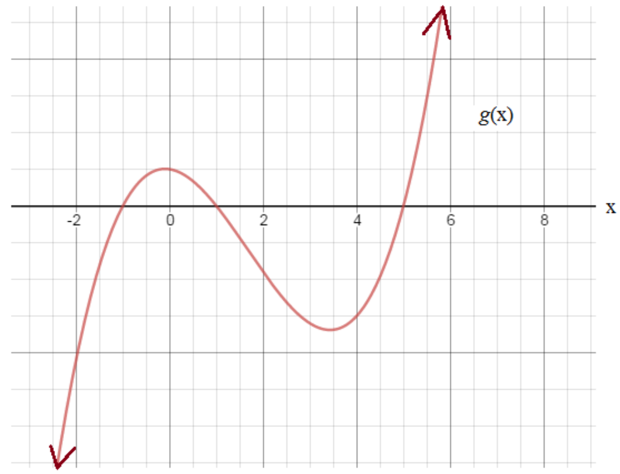


Points of inflection at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$

Calculus Derivatives: Critical Values and Interpreting Graphs

SOLUTIONS

Suppose the sketch is the function g(x)...



a) What are the zeros?

The x-intercepts occur when the curve crosses the x-axis... Zeros are -1, 1, and 5

b) What intervals is g(x) increasing?

g(x) is increasing when the slope is positive..

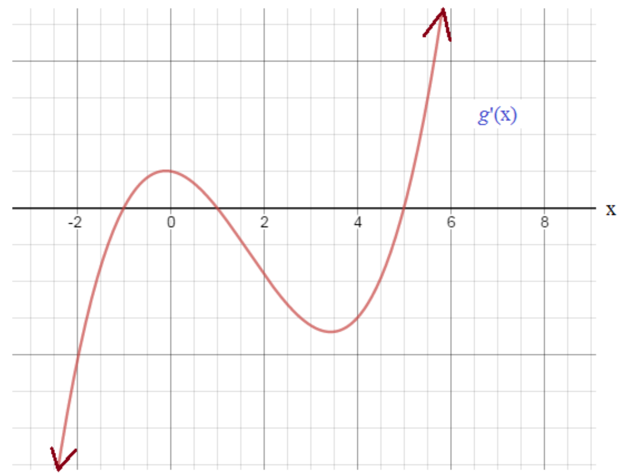
$(-\infty, 0) \cup (3.5, \infty)$

c) Identify the point of inflection.

At x = 1.5 (approximately)

when curve changes from concave down to concave up..

Now, suppose the sketch is g'(x) --- the *derivative* of g(x)...



a) What intervals is g(x) increasing?

the sketch represents when the slope (IROC) of the *function.*
So, any place where g'(x) > 0, the function is increasing!

$(-1, 1) \cup (5, \infty)$

b) Identify the local extrema of g(x).

The local extrema occur when g'(x) = 0

x = -1 is local minimum — slope of function goes from negative to positive ----> minimum

x = 1 is local maximum

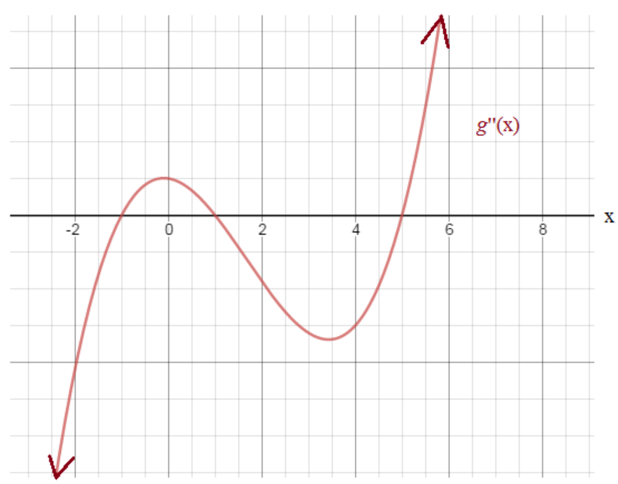
x = 5 is local minimum — slope of function goes from positive to negative ----> maximum

c) Identify the point(s) of inflection of g(x).
When is the function g(x) concave up?

Since this is a sketch of g'(x), the function g(x) is concave up when the curve is increasing...
$(-\infty, 0) \cup (3.5, \infty)$

Points of inflection are at max and min of derivative function!
x = 0 and 3.5

Now, suppose the sketch is g''(x) --- the *2nd derivative of* g(x)...



a) When is the function g(x) concave up?

Since the second derivative graph represents concavity of the original graph,

any place above the x-axis represents a positive value ----> concave up...

$(-1, 1) \cup (5, \infty)$

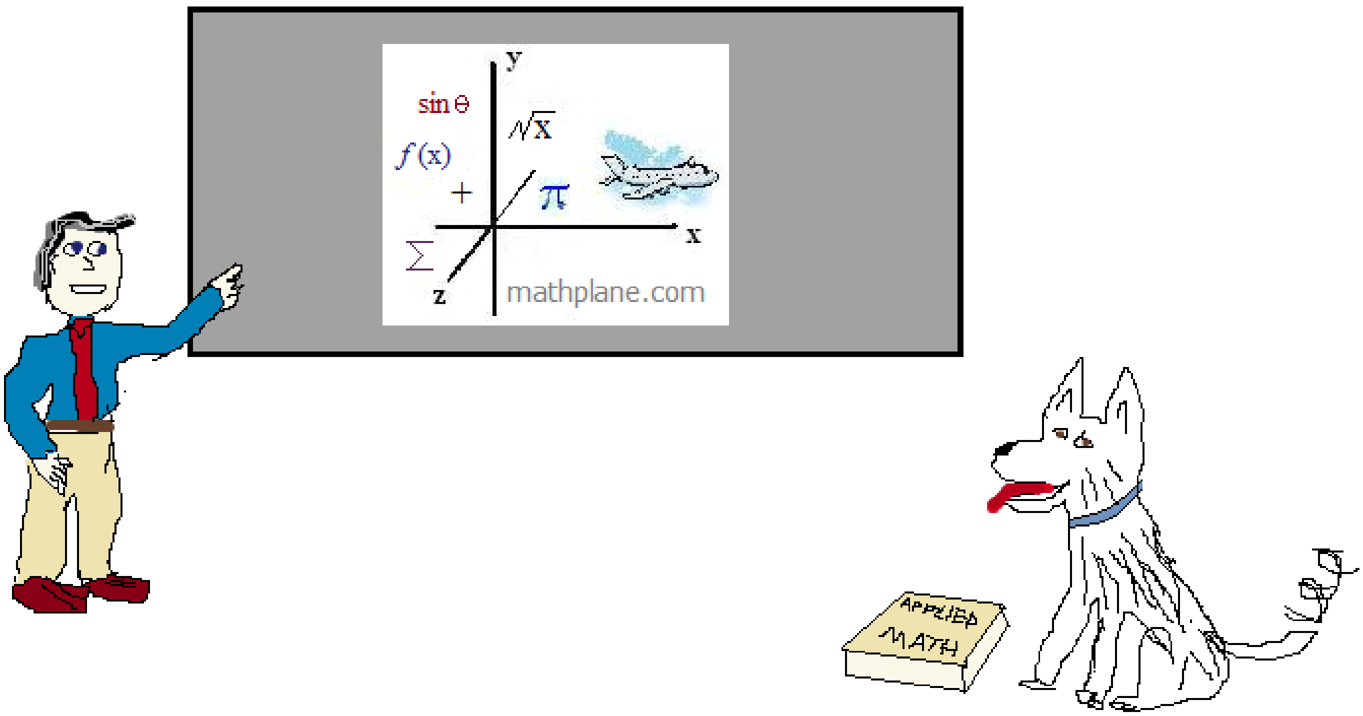
b) Identify the point(s) of inflection of g(x).

Points of inflection occur when concavity changes.
This occurs at x = -1, 1, and 5

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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