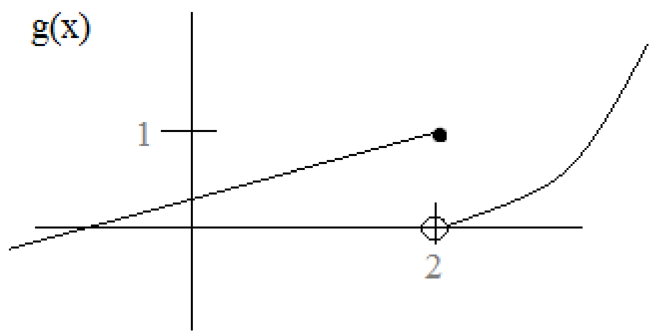
Calculus: Limits and Asymptotes

Notes, examples, & practice quiz (with solutions)



Topics include definitions, greatest integer function, strategies, infinity, slant asymptote, squeeze theorem, and more.

Definition of a Limit

If $f(x)$ gets arbitrarily close to a single number L as x approaches c, then we write

$$\lim_{x \to c} f(x) = L$$

and say that "the limit of $f(x)$, as x approaches c, is L."

> Note -- from the definition:
> 1) The limit is *unique* if it exists.
> (limit from the left = limit from the right)
> 2) The limit does not depend on the actual value of $f(x)$ at c. Instead, it is determined by values of $f(x)$ when x is *near* c

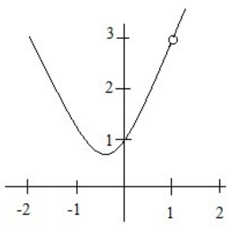
Also, in order for the limit to exist, the values of f must tend to the same number L from the left or the right.

$$\lim_{x \to c^-} f(x) = L \qquad\qquad \lim_{x \to c^+} f(x) = L$$

("left-hand limit of $f(x)$" or "limit from the left")

("right-hand limit of $f(x)$" or "limit from the right")

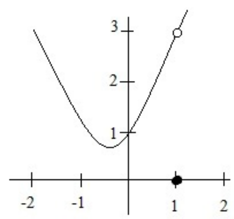
Illustrations and Examples:



$f(x) = \dfrac{x^3 - 1}{x - 1}$

$f(1) = \dfrac{0}{0}$ undefined

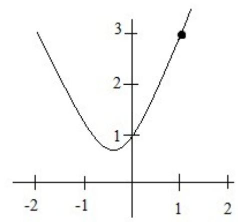
$\lim_{x \to 1} f(x) = 3$



$g(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1} & \text{when } x \neq 1 \\ 0 & \text{when } x = 1 \end{cases}$

$g(1) = 0$

$\lim_{x \to 1} g(x) = 3$



$h(x) = x^2 + x + 1$

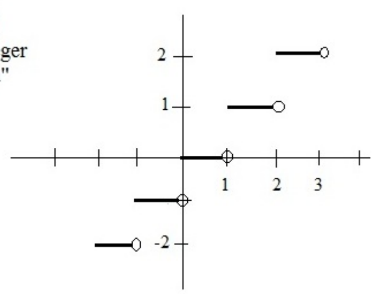
$h(1) = 1 + 1 + 1 = 3$

$\lim_{x \to 1} h(x) = 3$

> Note: The values at 1 are all different, but the limits are all the same, because
> the values of the functions as x gets near 1, approach 3.

$g(x) = [x]$

"greatest integer function"



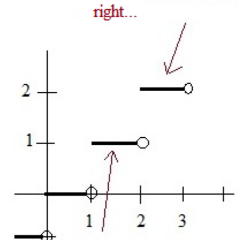
$g(1.5) = 1$

$g(-2.3) = -3$

$g(2) = 2$

$g(1.00034) = 1$

approaching 2 from the right...



approaching 2 from the left...

$\lim_{x \to 1.5} [x] = 1 \qquad \lim_{x \to -2.3} [x] = -3$

> Note: For the 'greatest integer function', the limit exists for all values of x that are NOT integers!

$\lim_{x \to 2} [x] \neq 2$ why is it not equal to 2?

$\lim_{x \to 2^+} [x] = 2$ If you look at the graph, you can see when approaching 2 FROM THE RIGHT, the values are 2...

$\lim_{x \to 2^-} [x] = 1$ But, when approaching 2 FROM THE LEFT, the values are 1..

Since the limits are different, the limit does not exist at 2!!

Example: $\dfrac{(x^2 - 9)}{x^2 + 2x - 3}$

Find the limit as x approaches -3

(algebraically)

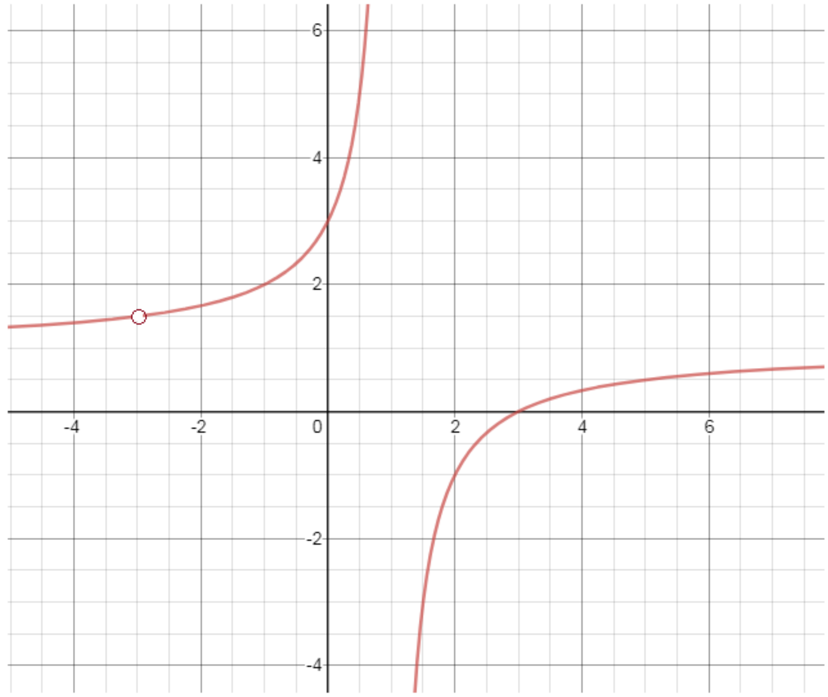
$$\frac{(x + 3)(x - 3)}{(x + 3)(x - 1)}$$

$$\frac{(x - 3)}{(x - 1)}$$

at x = -3

3/2

(graphically)



****The limit of a function is the value that you *approach* from the left and from the right...

Important: you never reach the actual value... the limit is where you're approaching!!

For example, when the limit of a function is 2, you *approach* 2 (or, "get infinitely close to" 2)

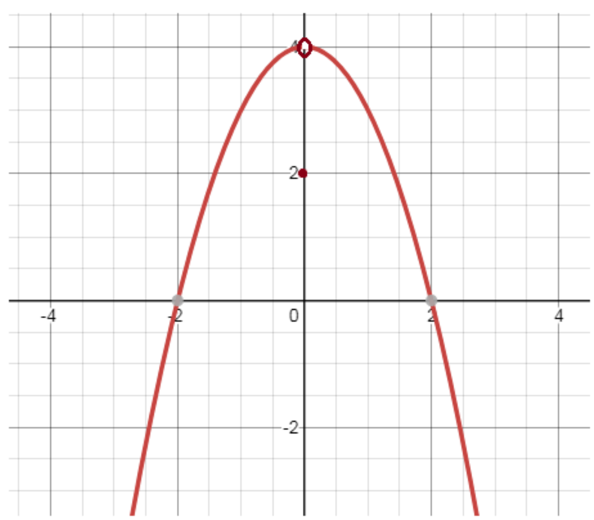
on the number line, you travel -2, -1, 0, 1, 1.5, 1.8, 1.9, 1.99, 1.999, 1.99999..... You never reach 2, because there is always a point in between!

Example:

$$f(x) = \begin{cases} \dfrac{-x^3 + 4x}{x} & \text{if } x = 0 \\ 2 & \text{if } x = 0 \end{cases}$$

$f(0) = 2$

But, the limit as x approaches 0 is 4...



Notice, the limit as you approach x = 0, is 4....
(i.e. as you get infinitely close to x = 0, the value gets closer and closer to 4...)

However, at the exact point x = 0, the output is 2!

Finding Limits: Examples

1) "Plug in the Number" (Direct Substitution)

$$\lim_{x\to 3} x^2+3x-1 = \lim_{x\to 3} x^2 + \lim_{x\to 3} 3x - \lim_{x\to 3} 1$$

$$= 9 + 9 - 1 = 17$$

$$\lim_{x\to 0} 1+x\cos(2x) = \lim_{x\to 0} 1 + \lim_{x\to 0} x \cdot \lim_{x\to 0} \cos(2x)$$

$$= 1 + 0 \cdot 1 = 1$$

$$\lim_{x\to 8} \frac{x}{x-8} = \frac{8}{0}$$ undefined (or, does not exist)

2) "Eliminate the Problem"

$$\lim_{x\to 2} \frac{(x^2-4)}{(x-2)} = \cancel{\frac{0}{0}}$$

However, we can factor the numerator..

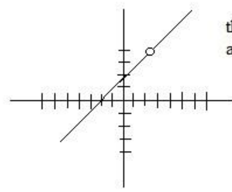
$$\lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)}$$

then, cancel the denominator...

$$\lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

then, solve...

$$\lim_{x\to 2} x+2 = \boxed{4}$$



there is a 'hole' at (2, 4)

$$\lim_{x\to 2} \frac{x^3-8}{x^2-4}$$

Again, at x=2, there is a 0 in the denominator..

So, we factor (with difference of squares and difference of cubes)

$$\lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)}$$

Simplify and plug in 2...

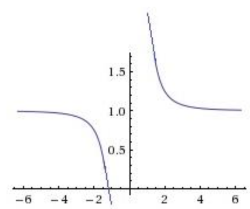
$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{(x+2)\cancel{(x-2)}}$$

$$\frac{(2)^2+2(2)+4}{(2)+2} = \frac{12}{4} = \boxed{3}$$

3) "Extrapolate the Limit"

$$\lim_{x\to\infty} 1+\frac{2}{x^3} = 1$$

x	1/2	1	2	4	10	approaches ∞
f(x)	17	3	1.25	1.031	1.002	approaches 1



$$\lim_{x\to\infty} \frac{2^{-x}}{2^x} = 0$$

As x increases, the numerator is getting smaller and the denominator is getting larger..
Therefore, the function is decreasing toward 0.

"End Behavior of Polynomial"

For polynomials whose (largest) degree ≥ 1

if leading coefficient is positive, then it becomes infinite as x does..
if leading coefficient is negative, then it becomes negatively infinite as x increases to ∞

$$\lim_{x\to\infty} x^2-23x+2 = \infty \qquad \lim_{x\to\infty} 8x-5x^2 = -\infty \qquad \lim_{x\to\infty} .00004x-10^5 = \infty$$

4) "Utilizing the Conjugate"

$$\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}$$ "plug in the number" $$\frac{\sqrt{1+0}-1}{0}=\frac{0}{0}$$ (cannot determine yet)

$$\lim_{h\to 0}\frac{(\sqrt{1+h}-1)}{h}\cdot\frac{(\sqrt{1+h}+1)}{(\sqrt{1+h}+1)}=\frac{\sqrt{1+h}^{\,2}-1^2}{h\,(\sqrt{1+h}+1)}$$

**Important observation: we multiplied the numerator terms, but did not combine the denominator terms

$$=\frac{\cancel{h}}{\cancel{h}\,(\sqrt{1+h}+1)}=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}$$ Try again... $$\boxed{\frac{1}{2}}$$

5) "Expand or Rewrite"

$$\lim_{h\to 0}\frac{(h+2)^3-8}{h}$$ "substitute the 0" $$\frac{(0+2)^3-8}{0}=\frac{0}{0}$$ (cannot determine yet)

Expand the numerator:

$(h+2)(h+2)=h^2+4h+4$

$$\begin{array}{rl} (h^2+4h+4)(h+2)= & h^3+4h^2+4h \\ & \underline{\qquad 2h^2+8h+8} \\ & h^3+6h^2+12h+8 \end{array}$$

$$\lim_{h\to 0}\frac{h^3+6h^2+12h+8-8}{h}=$$

$$\lim_{h\to 0}h^2+6h+12=\boxed{12}$$

$$\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}=\lim_{x\to -4}\frac{\frac{x}{4x}+\frac{4}{4x}}{4+x}=\lim_{x\to -4}\frac{\frac{x+4}{4x}}{x+4}=\lim_{x\to -4}\frac{1}{4x}=\boxed{\frac{-1}{16}}$$

Example: Using 2 methods, find $$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$$

method 1: multiply by the conjugate

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\cdot\frac{(\sqrt{x}+3)}{(\sqrt{x}+3)}=$$

$$\lim_{x\to 9}\frac{x-9}{x-9\,(\sqrt{x}+3)}=$$

$$\lim_{x\to 9}\frac{1}{(\sqrt{x}+3)}=\boxed{\frac{1}{6}}$$

method 2: extrapolate / use a chart

x	8	8.5	8.9	9	9.1	9.5	10
$f(x)$	.171	.169	.167		.166	.164	.162

Example: $\lim_{x \to 0} \frac{1 - \cos x}{\sin x}$

substitute $x = 0$, $\frac{1 - \cos(0)}{\sin(0)} = \frac{0}{0}$ indeterminate...

(Use conjugate) $\frac{1 - \cos x}{\sin x} \cdot \frac{(1 + \cos x)}{(1 + \cos x)} =$

$\frac{1 - \cos^2 x}{\sin x(1 + \cos x)} =$

(Trig Identity)

$\frac{\sin^2 x}{\sin x(1 + \cos x)} = \frac{\sin x}{(1 + \cos x)}$

substitute $x = 0$ $\frac{\sin(0)}{(1 + \cos(0))} = \frac{0}{2} = \boxed{0}$

Example: $\lim_{x \to 0} \frac{\frac{x - 1}{5x - 3} - \frac{1}{3}}{x}$

substitute $x = 0$, and the result is $\frac{0}{0}$ indeterminate...

(Combine numerator terms and simplify)

$$\frac{\frac{3(x - 1)}{3(5x - 3)} - \frac{(5x - 3)(1)}{(5x - 3)(3)}}{x} = \frac{\frac{3x - 3 - (5x - 3)}{(5x - 3)(3)}}{x} = \frac{\frac{-2x}{15x - 9}}{x}$$

$$= \frac{-2}{15x - 9} = \boxed{\frac{2}{9}}$$

Example: $\lim_{x \to 2^-} \frac{x^2(x - 2)(x + 3)}{|x - 2|}$

limit as x approaches 2 from the left side...
at $x = 2$, the equation is 0/0

$\frac{x^2\cancel{(x - 2)}(x + 3)}{\cancel{|x - 2|}}$

put in $x = 2$, and the solution is 20...

$\boxed{\text{-20 from the left!}}$

(20 from the right)..

x	1	1.5	1.9	1.95	2.05	2.1	2.5	3
f(x)	-4	-10.1	-17.7	-18.8	21.2	22.5	34.4	54

Example: $\lim_{x \to 4} \frac{\frac{3}{x + 2} - \frac{1}{x - 2}}{x - 4}$

$$\frac{\frac{3(x - 2) - (x + 2)}{(x - 2)(x + 2)}}{x - 4} = \frac{\frac{3x - 8 - x}{(x - 2)(x + 2)}}{x - 4}$$

$$\frac{2(x - 4)}{(x - 2)(x + 2)} \cdot \frac{1}{x - 4} = \frac{2}{(x - 2)(x + 2)}$$

$$\lim_{x \to 4} \frac{2}{(x - 2)(x + 2)} = \boxed{1/6}$$

Using a graph to determine the limit: The absolute value example

$$h(x) = \frac{x - 3}{|x - 3|}$$

Find $\lim\limits_{x \to 3} h(x)$

Step 1: Use direct substitution

$$\lim_{x \to 3} \frac{x - 3}{|x - 3|} = \frac{(3) + 3}{|(3) + 3|} = \frac{0}{0}$$ Inconclusive!

Step 2: Try to factor or use conjugates

These techniques will not work in this problem...

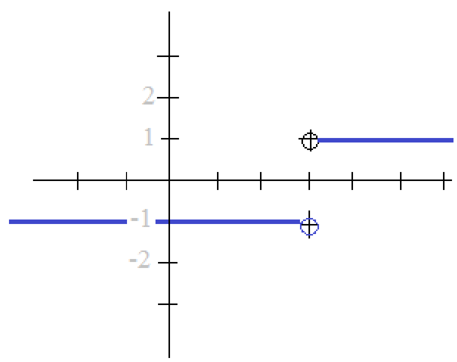
Step 3: Make tables of values and graph

approaching from the left

x	h(x)
2	-1
2.5	-1
2.8	-1
2.9	-1
2.99	-1

approaching from the right

x	h(x)
4	1
3.5	1
3.2	1
3.1	1
3.01	1



Conclusion:

$$\lim_{x \to 3^-} h(x) = -1$$

$$\lim_{x \to 3^+} h(x) = 1$$

$$\lim_{x \to 3} h(x) = \text{Does Not Exist (DNE)}$$

(limit from the left and limit from the right differ)

"Squeeze" Theorem

(Also, called "pinching theorem" or "sandwich theorem")

Brief Summary: If $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

Example: $\lim_{x \to 0} x^2 \cos(\frac{1}{x})$

Try direct substitution: $(0)^2 \cos(\frac{1}{0})$????

But, we know $-1 \le \cos(x) \le 1$ (range of a cosine function)

So, $-1 \le \cos(\frac{1}{x}) \le 1$

multiply each term by x^2

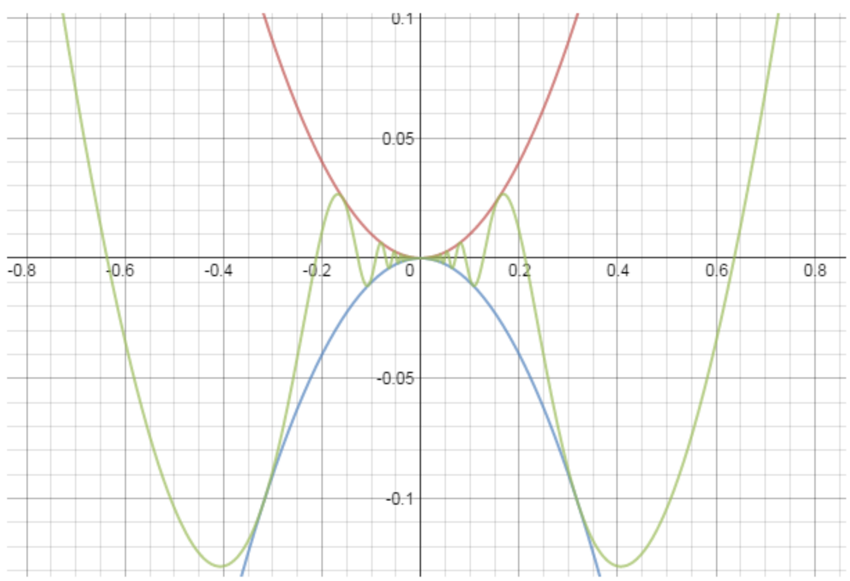
$$-x^2 \le x^2 \cos(\frac{1}{x}) \le x^2$$

then, find the limit of the lower and upper terms.....

$\lim_{x \to 0} -x^2 = 0$

$\lim_{x \to 0} x^2 = 0$

$$\lim_{x \to 0} x^2 \cos(\frac{1}{x})$$



Applying the Squeeze Theorem

Example: If $4x - 11 \le f(x) \le x^2 - 5x + 9$ for all $x \ge 0$

find $\lim\limits_{x \to 5} f(x)$

"limit below" $\lim\limits_{x \to 5} 4x - 11 = 9$

"limit above" $\lim\limits_{x \to 5} x^2 - 5x + 9 = 9$

therefore, using the 'squeeze theorem',

the "limit in the middle" $\lim\limits_{x \to 5} f(x) = 9$

Example:

Verify $\lim\limits_{x \to 0} \sqrt{x^3 + x^2}\left(\sin\frac{\pi}{x}\right) = 0$

Using substitution, we run into a problem: $\sqrt{0^3 + 0^2}\left(\sin\frac{\pi}{0}\right)$

????

Since we can't determine $\sin\frac{\pi}{0}$, we'll utilize the squeeze theorem...

We know $-1 \le \sin \le 1$ ---------> $-1 \le \left(\sin\frac{\pi}{x}\right) \le 1$

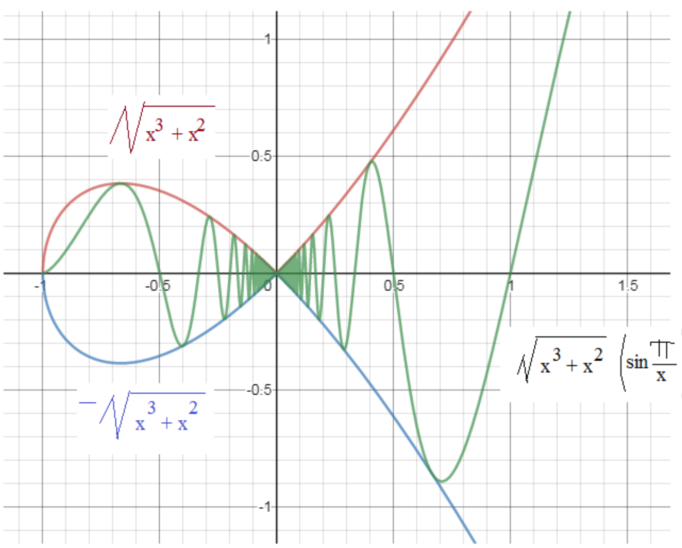
and, we know $\sqrt{x^3 + x^2} \ge 0$ and, -------> $-\sqrt{x^3 + x^2} \le 0$

Therefore, putting it all together: $\sqrt{x^3 + x^2}\left(\sin\frac{\pi}{x}\right) \le \sqrt{x^3 + x^2}$ AND $\sqrt{x^3 + x^2}\left(\sin\frac{\pi}{x}\right) \ge -\sqrt{x^3 + x^2}$

So, $-\sqrt{x^3 + x^2} \le \sqrt{x^3 + x^2}\left(\sin\frac{\pi}{x}\right) \le \sqrt{x^3 + x^2}$

and, the $\lim\limits_{x \to 0} \sqrt{x^3 + x^2} = 0$ and, $\lim\limits_{x \to 0} -\sqrt{x^3 + x^2} = 0$

So, the limit in the middle must be squeezed to zero!!



Find $\lim\limits_{x \to 0} \sqrt{x}\left(1 + \sin^2\left(\frac{2\pi}{x}\right)\right)$

we know $-1 \le \sin \le 1$...

Therefore, $0 \le \sin^2 \le 1$

and, finally, $1 \le 1 + \sin^2 \le 2$

$$\Rightarrow \quad 1\sqrt{x} \le \sqrt{x}\left(1 + \sin^2\left(\frac{2\pi}{x}\right)\right) \le 2\sqrt{x}$$

limit is 0

Definition of Limit $\lim_{x \to c} f(x) = L$

means that for each $\varepsilon > 0$, there must be a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$

Comments: When we choose a ε , there will exist a δ that works.

The δ is not unique, because any positive number less than δ works.

Example: Show that $\lim_{x \to 3} x + 2 = 5$

Let $\varepsilon > 0$. We seek a number $\delta > 0$ such that

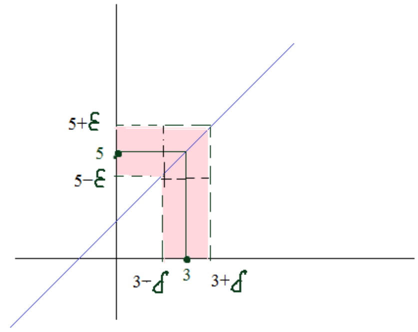
if $0 < |x - 3| < \delta$, then $|(x + 2) - 5| < \varepsilon$

First, let's find the connection...

$$|x - 3| = |(x + 2) - 5|$$

$$|x - 3| = |x - 3|$$

this suggests that we can set $\delta = \varepsilon$



Note: as the δ value shrinks to 0, there are values of ε inside the pink area

eventually, δ gets to 0, then ε goes to 0

this leaves $c = 3$ and $L = 5$

Example: Show that $\lim_{x \to 2} 3x - 1 = 5$

We need to show for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|(3x - 1) - 5| < \varepsilon$ whenever $0 < |x - 2| < \delta$

Since we know that our choice of δ is dependent on ε,

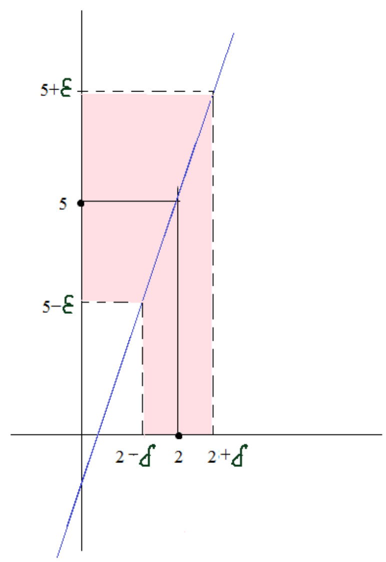
we must establish the connection between them.

$$|(3x - 1) - 5| = |3x - 6| = 3|x - 2| < \varepsilon$$

$$|x - 2| < \frac{\varepsilon}{3}$$

we can choose $\delta < \frac{\varepsilon}{3}$

$$0 < |x - 2| < \delta \leq \frac{\varepsilon}{3}$$



Finding the "limit as X approaches infinity"

General Rule: ("The Rational function Theorem")

Determining the limits at ∞ for functions expressed as a ratio of two polynomials:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0$$

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \dots + b_0$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \begin{cases} \pm\infty & \text{if } n > m & \text{("top heavy")} \\ 0 & \text{if } n < m & \text{("bottom heavy")} \\ \dfrac{a_n}{b_m} & \text{if } n = m & \text{("lead exponents equal"; look at the lead coefficients)} \end{cases}$$

Steps to find a limit (that approaches infinity)

1) (If necessary), expand the equation to reveal the degrees of the polynomials.

2) Arrange polynomials with highest degree first.

3a) If the numerator has a higher degree, then the limit is ∞

b) If the denominator has a higher degree, then the limit is 0

c) If the highest degree of the numerator polynomial is the same as the highest degree of the denominator polynomial, then the limit is

$$\frac{\text{lead coefficient of numerator}}{\text{lead coefficient of denominator}}$$

Examples:

$$\lim_{x \to \infty} \frac{x}{x^2 - 1} = \frac{x^1}{x^2 - 1} = 0$$

("bottom heavy")

degree of numerator < degree of denominator
(1) (2)

$$\lim_{x \to \infty} \frac{5x^3 + 2}{(2x+3)^3} = \lim_{x \to \infty} \frac{5x^{\boxed{3}} + 2}{8x^{\boxed{3}} + 36x^2 + 54x + 27} = \frac{5}{8}$$

highest degree of numerator (3) = highest degree of denominator (3)

$$\lim_{x \to \infty} \frac{7 + 3x^2 - 5x^4}{10x^3 - 17} = \lim_{x \to \infty} \frac{-5x^4 + 3x^2 + 7}{10x^3 - 17} = -\infty$$

rewrite polynomial in descending order

degree of numerator > degree of denominator
(4) (3)

("top heavy")

(*Note: a simple graph showing end behavior confirms the limit is negative infinity, rather than positive infinity)

Comparison/Verification:

$$\lim_{x \to \infty} \frac{4 - x^2}{4x^2 - x - 2} = \lim_{x \to \infty} \frac{-x^2 + 4}{4x^2 - x - 2} = \frac{-1}{4}$$

(use "rational function theorem" above)

$$\lim_{x \to \infty} \frac{4 - x^2}{4x^2 - x - 2} = \lim_{x \to \infty} \frac{4/x^2 - 1}{4 - 1/x - 2/x^2} = \frac{0 - 1}{4 - 0 - 0} = \frac{-1}{4}$$

(rewrite and solve) (multiply throughout by $\frac{1}{x^2}$)

Asymptotes: Definitions and Example

The line $y = b$ is a *Horizontal Asymptote* of the graph of $y = f(x)$ if

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b$$

The line $x = a$ is a *Vertical Asymptote* of the graph of $y = f(x)$ if one or more of the following occur:

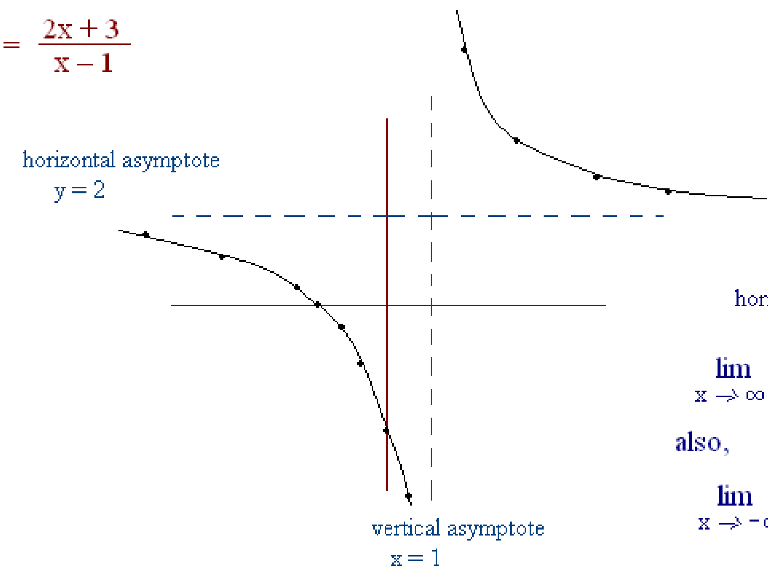
$$\lim_{x \to a^-} f(x) = +\infty \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = -\infty$$

or

$$\lim_{x \to a^-} f(x) = -\infty \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = +\infty$$

$$f(x) = \frac{2x + 3}{x - 1}$$

x	f (x)
-50	1.90
-20	1.76
-5	1.17
-2	0.33
-1	-.5
-1/2	-1.3
0	-3
1/2	-8
1	undefined
2	7
5	3.25
10	2.55
20	2.26
50	2.10



horizontal asymptote:

$$\lim_{x \to \infty} \frac{2x + 3}{x - 1} = 2$$

also,

$$\lim_{x \to -\infty} \frac{2x + 3}{x - 1} = 2$$

vertical asymptote:

$$\lim_{x \to 1^+} \frac{2x + 3}{x - 1} = +\infty$$

$$\lim_{x \to 1^-} \frac{2x + 3}{x - 1} = -\infty$$

Graphs and the *Slant Asymptote*

Asymptotes are very useful when graphing functions.

Examples: $y = \dfrac{x}{x-4}$

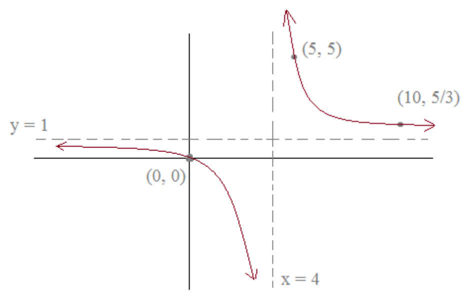
Vertical Asymptote: y is undefined at $x = 4$

Horizontal Asymptote: degree of numerator: 1; degree of denominator: 1 (Since the degrees are equal, look at the coeffients of the lead terms)

therefore $y = 1$

x-intercept: (0, 0)

y-intercept: (0, 0)



Since (0, 0) is below the horizontal asymptote and to the left of the vertical asymptote, sketch the corresponding end behavior.

Then, select a point on the other side of the vertical asymptote. Examples: (5, 5) or (10, 5/3)

Since (5, 5) is above the horizontal asymptote and to the right of the vertical asymptote, sketch the corresponding end behavior.

(note: to check solutions, plug in random values)

$y = \dfrac{x^2+1}{x^2-1}$

Vertical Asymptote: $\dfrac{x^2+1}{x^2-1} = \dfrac{x^2+1}{(x+1)(x-1)}$

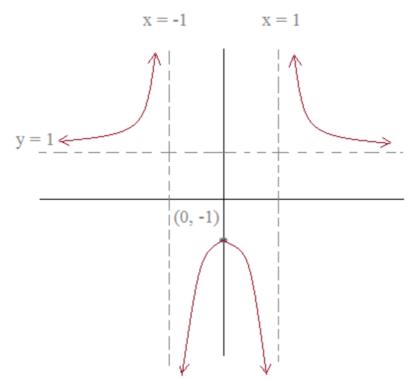
y is undefined at $x = -1$ and $x = 1$

Horizontal Asymptote: $\lim\limits_{x \to \infty} \dfrac{x^2+1}{x^2-1} = 1$

therefore $y = 1$

y-intercept: (0, -1)

x-intercept: none because there is no x value that satisfies $\dfrac{x^2+1}{x^2-1} = 0$



x	−1.1	−1.8	−2	−3	−5	−10
y	$\frac{221}{21}$	$\frac{53}{28}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{13}{12}$	$\frac{101}{99}$

x	2	3	5	10
y	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{13}{12}$	$\frac{101}{99}$

Slant Asymptote: If the highest degree of the numerator is <u>one more</u> than the highest degree of the denominator, then there is a slant asymptote

That asymptote is the quotient <u>without the remainder.</u>

Example: $f(x) = \dfrac{x^2 - 2x - 3}{x + 4}$

$= \dfrac{(x-3)(x+1)}{x+4}$

Vertical Asymptote: $x = -4$

y-intercept: (0, -3/4)

x-intercepts: (3, 0) (-1, 0)

Horizontal Asymptote: None

degree of numerator: 2

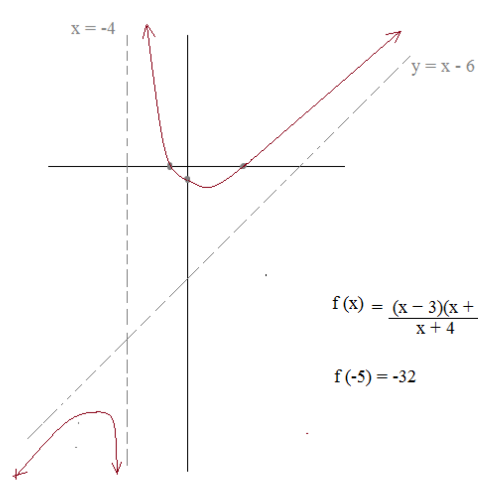
degree of denominator: 1

slant asymptote

$$\begin{array}{r|l} & x - 6 + \frac{21}{x+4} \\ x+4 & x^2 - 2x - 3 \\ & \underline{x^2 + 4x} \\ & -6x - 3 \\ & \underline{-6x - 24} \\ & 21 \end{array}$$

The slant asymptote is $y = x - 6$

As x gets larger and larger, the remainder gets closer and closer to zero. And, the function resembles $x - 6$ more and more.

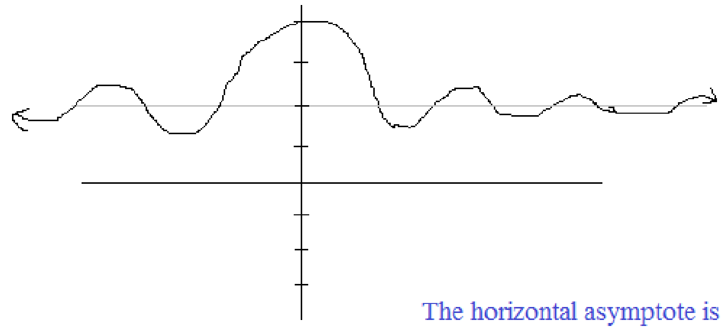


$f(x) = \dfrac{(x-3)(x+1)}{x+4}$

$f(-5) = -32$

Asymptote Note: The limit as x goes to infinity describes *end behavior* of the function.
Although a graph cannot touch the vertical asymptote, it may cross over the horizontal asymptote.

Example:



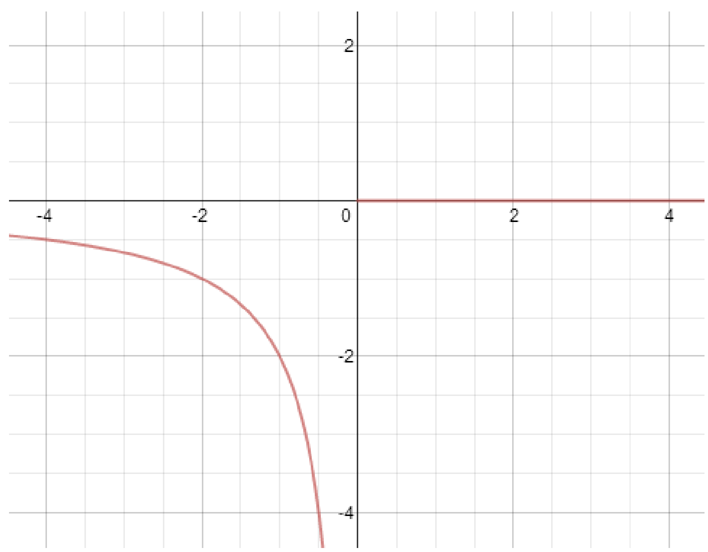
The horizontal asymptote is $y = 2$

There is no vertical asymptote...

Example: $f(x) = \dfrac{1}{x} - \dfrac{1}{|x|}$

Find $\lim\limits_{x \to 0^+} f(x)$ $\qquad \lim\limits_{x \to 0^-} f(x)$ $\qquad \lim\limits_{x \to 0} f(x)$

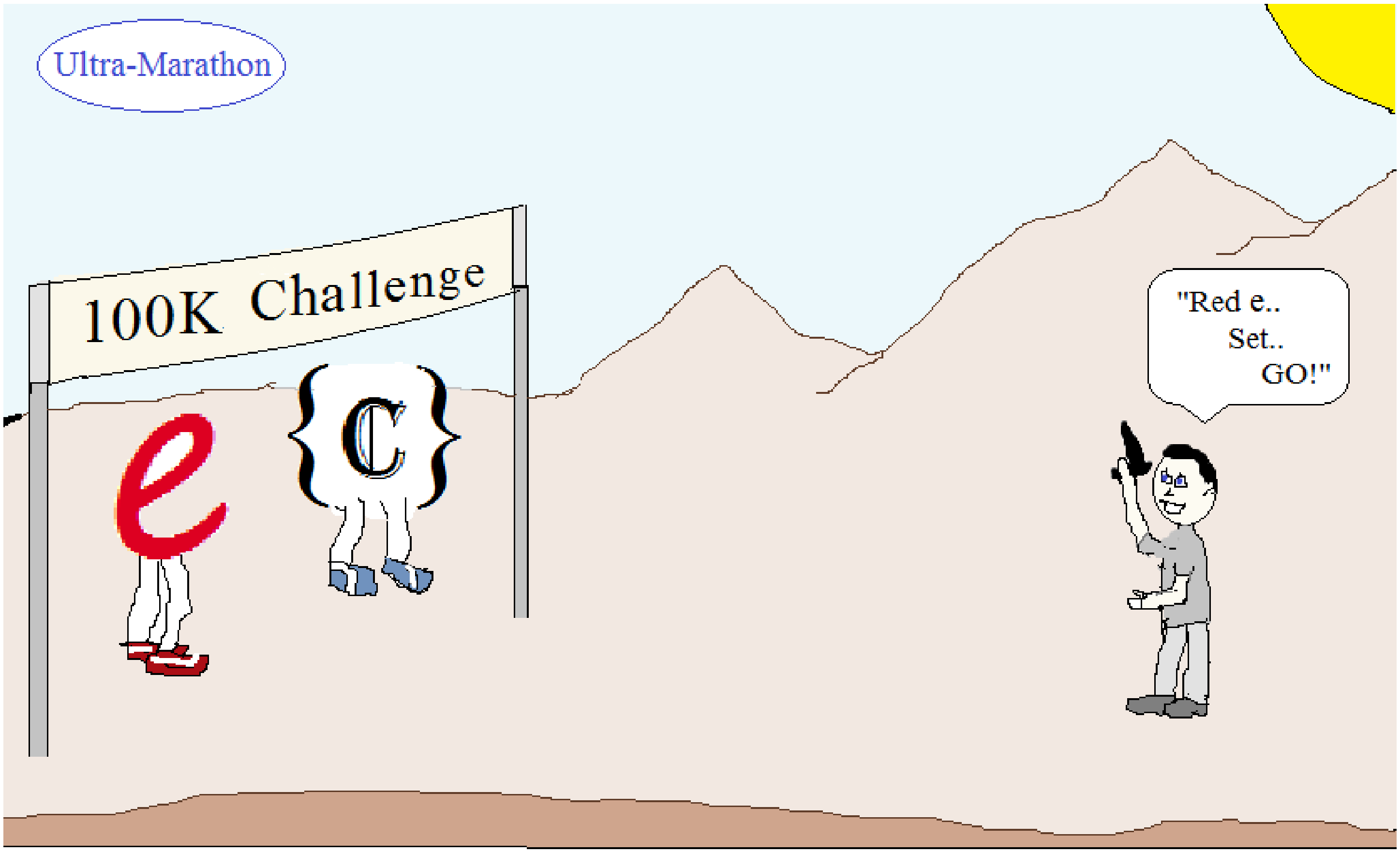
Utilizing the graph of the function at the right, we can easily determine the limits!



$\lim\limits_{x \to 0^+} f(x) = 0$

$\lim\limits_{x \to 0^-} f(x) = -\infty$

$\lim\limits_{x \to 0} f(x)$ DNE
does not exist...



Testing the limits of endurance,
these math figures will run on and on...

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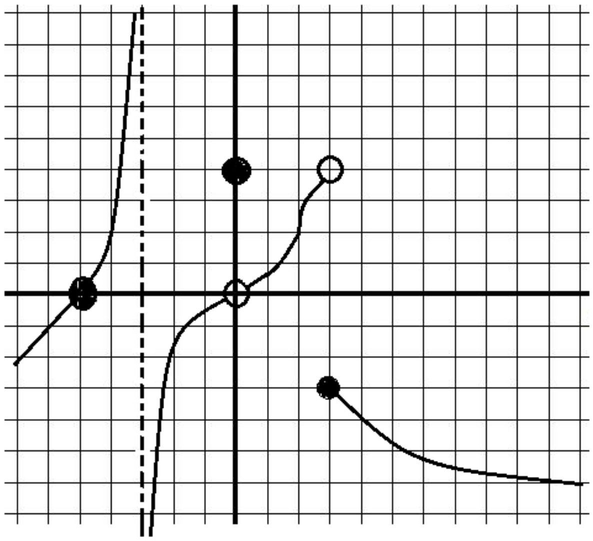
PRACTICE QUIZ

Limits Quick Quiz

Part I: Identifying limits and values on a graph



1) $f(0) =$

2) $\lim\limits_{x \to 0} =$

3) $\lim\limits_{x \to -4} =$

4) $\lim\limits_{x \to 3} =$

5) $\lim\limits_{x \to -3^+} =$

6) $f(4) =$

7) $f(-3) =$

Part II: Finding Limits

1) $\lim\limits_{x \to 3} \dfrac{2x - 6}{x^2 - 9}$

2) $\lim\limits_{x \to 4} \dfrac{x^2 + 6x - 40}{3x + 6}$

3) $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

4) $\lim\limits_{x \to \infty} \dfrac{3 - 4x^2}{x^2 + 3x + 2}$

5) $\lim\limits_{x \to 0} \dfrac{|x|}{3x} =$

6) $\lim\limits_{x \to \infty} \dfrac{235}{3x + 2}$

7) $\lim\limits_{x \to 2^+} \dfrac{4}{x - 2}$

8) $\lim\limits_{x \to 2^-} \dfrac{4}{x - 2}$

9) $\lim\limits_{x \to 2} \dfrac{4}{x - 2}$

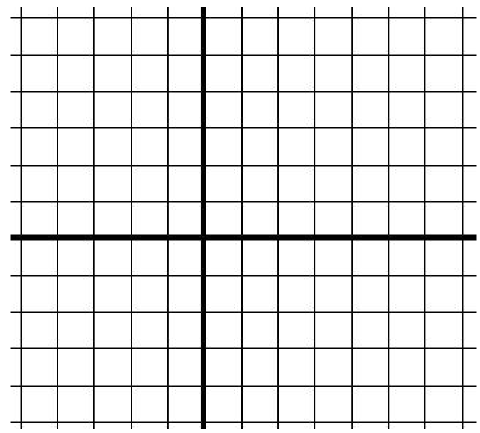
10) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x^2 - 7x + 10}$

11) $\lim\limits_{x \to 0} \dfrac{x^2 + 3x}{x}$

12) $\lim\limits_{x \to \infty} \dfrac{2x^3 + 5x}{-3x^2 + 6}$

Part III: Graphing

1) Graph $f(x) = \begin{cases} 2x + 1 & \text{if} \quad x > 1 \\ 2 - x & \text{if} \quad x \leq 1 \end{cases}$



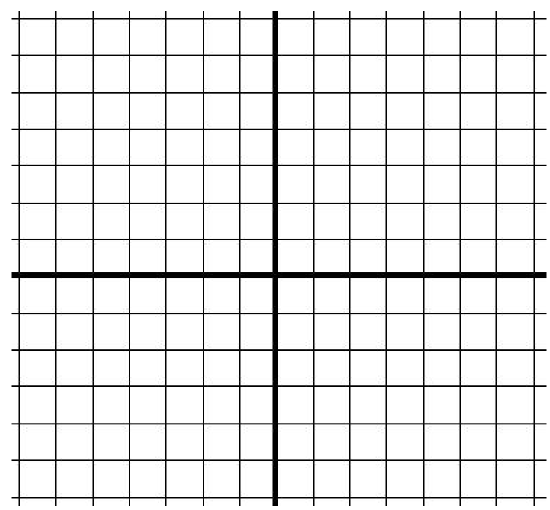
Then, identify: $f(1) =$

$\lim_{x \to 1^+} f(x) =$

$\lim_{x \to 1^-} f(x) =$

$\lim_{x \to 1} f(x) =$

2) Sketch a function with the following properties:



$g(0) = 2$

$\lim_{x \to 2^+} g(x) = \infty$

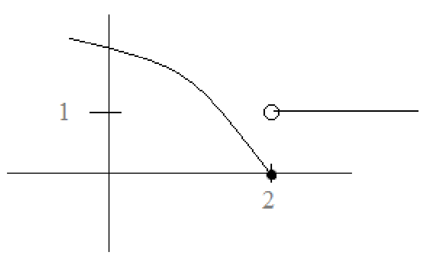
$\lim_{x \to 2^-} g(x) = \infty$

$\lim_{x \to -\infty} g(x) = -1$

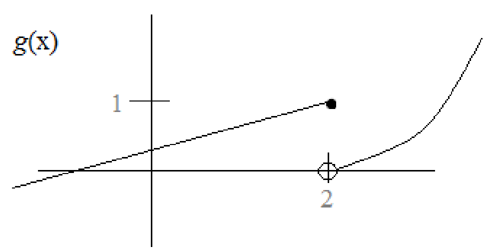
$\lim_{x \to +\infty} g(x) = -1$

3) **Challenge:

$f(x)$



$g(x)$



Answer:

a) $\text{limit}_{x \to 2} f(x) =$

b) $\text{limit}_{x \to 2} g(x) =$

c) $\lim_{x \to 2} (f(x) + g(x)) =$

d) $\lim_{x \to 2} (f(x)g(x)) =$

IV. Miscellaneous Multiple Choice

Limits, Asymptotes, & Continuity

1) As x increases to infinity, the function $f(x) = 2e^{-x}$ gets closer to

a) 0
b) 1/2
c) 2
d) *e*
e) infinity

2) A rational function of the form $y = \dfrac{ax}{x+b}$ has a vertical asymptote at x = 5 and a horizonal asymptote at y = −3

Which is a possible function?

a) $\dfrac{5x}{x-3}$ b) $\dfrac{3x}{x+5}$ c) $\dfrac{-3x}{x-5}$ d) $\dfrac{-5x}{x+3}$ e) $\dfrac{-3x}{x+5}$

3) Let p(x) be a cubic polynomial function, where $p(3) < 0$, $p(7) > 0$, and $p(9) < 0$,
Which statements are true?

statement I: there are 3 zeros
statement II: a zero exists at $x < 3$ OR $x > 9$
statement III: for $p(x) = 0$, there are 2 solutions between 3 and 9

a) I
b) I and II
c) I and III
d) II
e) I, II, and III

4) $\lim\limits_{x \to 3} 9 =$

a) 3
b) 9
c) Does not exist
d) 0
e) 27

5) Find the value of k so g(x) is continuous:

$$g(x) = \begin{cases} k + x & x < 10 \\ xk & x \geq 10 \end{cases}$$

a) 10
b) 0
c) 10/9
d) 1
e) no solution

6) $\lim_{t \to 4} \dfrac{t^2 - 16}{\dfrac{1}{4} - \dfrac{1}{t}}$

a) 4
b) 16
c) 64
d) 128
e) undefined

7) $\lim_{x \to 0} \dfrac{(x+1)^2 - 1}{x}$

a) -1
b) 0
c) 1
d) 2
e) Does not exist

8) $\lim_{x \to 3} \dfrac{x}{x^2 - 9}$

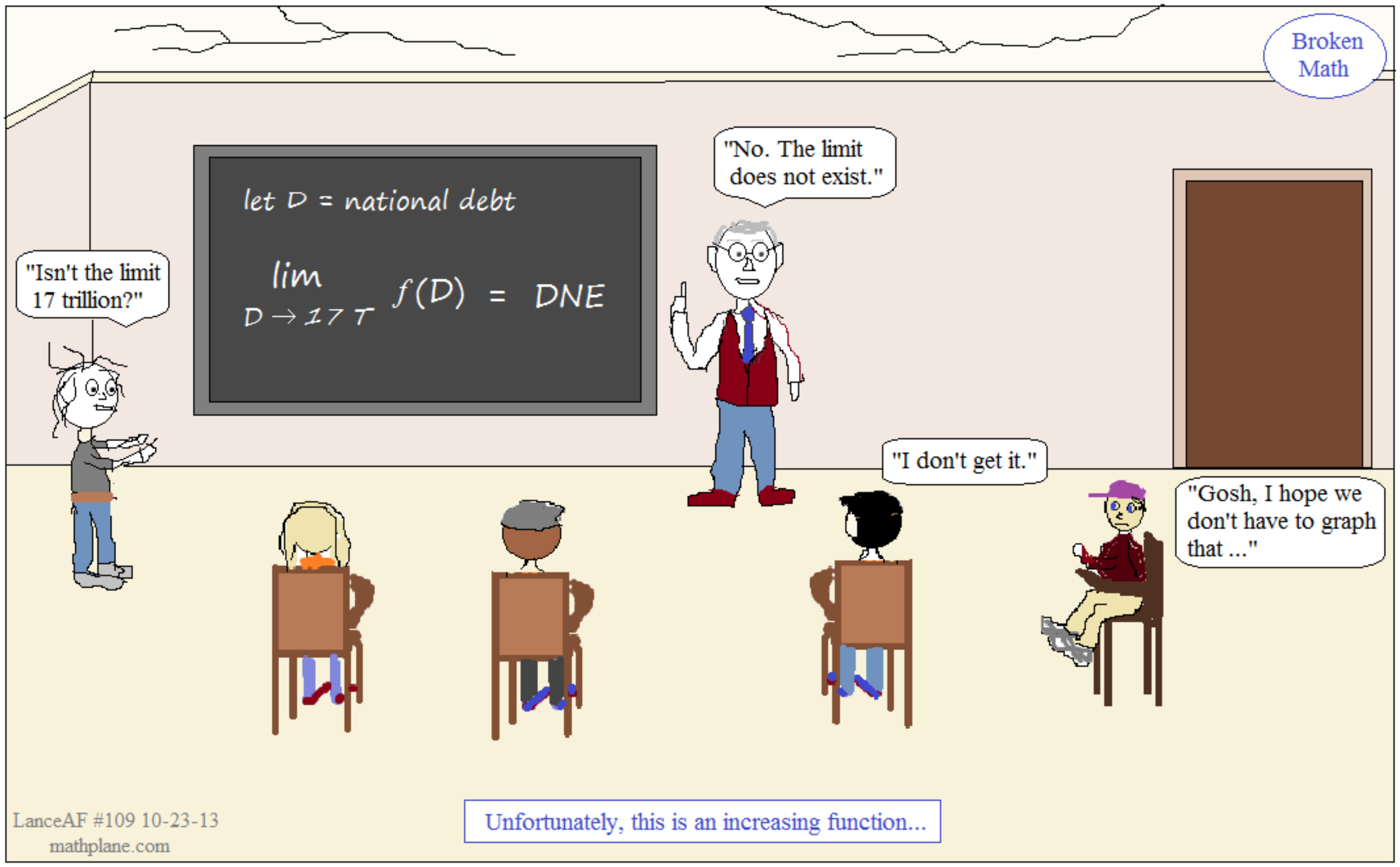
a) 3
b) 9
c) positive infinity
d) negative infinity
e) does not exist

9) $\lim_{x \to 3} \dfrac{1}{(x-3)^2}$

a) 3
b) 9
c) positive infinity
d) negative infinity
e) does not exist

10) $\lim_{x \to 4^+} \dfrac{x}{x^2 - 16}$

a) -16
b) 0
c) 4
d) positive infinity
e) does not exist


Broken Math
let D = national debt
$\lim_{D \to 17T} f(D) = DNE$
"Isn't the limit 17 trillion?"
"No. The limit does not exist."
"I don't get it."
"Gosh, I hope we don't have to graph that ..."
Unfortunately, this is an increasing function...
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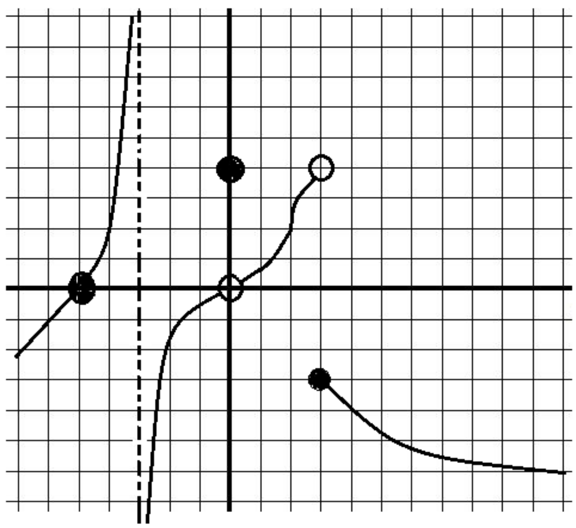
SOLUTIONS-→

Limits Quick Quiz

SOLUTIONS

Part I: Identifying limits and values on a graph



1) $f(0) = 4$

2) $\lim\limits_{x \to 0} = 0$

3) $\lim\limits_{x \to -4} = 2$

4) $\lim\limits_{x \to 3} =$ DNE (Does Not Exist, because limit from the left (4) is not the same as limit from the right (-3))

5) $\lim\limits_{x \to -3^+} = -\infty$

6) $f(4) = -4$

7) $f(-3) =$ Undefined

Part II: Finding Limits

1) $\lim\limits_{x \to 3} \dfrac{2x-6}{x^2-9}$

plug in value: $\dfrac{2(3)-6}{(3)^2-9} = \dfrac{0}{0}$

factor and cancel: $\dfrac{2\cancel{(x-3)}}{(x+3)\cancel{(x-3)}}$

plug in value again: $\dfrac{2}{((3)+3)} = \boxed{\dfrac{1}{3}}$

2) $\lim\limits_{x \to 4} \dfrac{x^2+6x-40}{3x+6}$

plug in value: $\dfrac{(4)^2+6(4)-40}{3(4)+6}$

$\dfrac{0}{18} = \boxed{0}$

3) $\lim\limits_{x \to 9} \dfrac{\sqrt{x}-3}{x-9} \cdot \dfrac{\sqrt{x}+3}{\sqrt{x}+3}$ (multiply by the conjugate)

$\dfrac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)} = \dfrac{1}{(\sqrt{x}+3)}$

limit is $\boxed{\dfrac{1}{6}}$

4) $\lim\limits_{x \to \infty} \dfrac{3-4x^2}{x^2+3x+2}$

lead term of numerator: $-4x^2$ (degree: 2)
lead term of denominator: x^2 (degree: 2)

Since the degrees are the same, use the coefficients: $\dfrac{-4}{1} = \boxed{-4}$

5) $\lim\limits_{x \to 0} \dfrac{|x|}{3x} =$ DNE

$\lim\limits_{x \to 0^-} = \dfrac{-1}{3}$

$\lim\limits_{x \to 0^+} = \dfrac{1}{3}$

Since limits are not equal, then $\boxed{\text{limit does not exist}}$

6) $\lim\limits_{x \to \infty} \dfrac{235}{3x+2}$

The degree of the denominator is greater than the degree of the numerator. Since it is "bottom heavy", $\boxed{\text{the limit is } 0}$

(as x gets larger and larger, the function decreases)

7) $\lim\limits_{x \to 2^+} \dfrac{4}{x-2}$

x	3	2.5	2.1	2.01	2.001
f(x)	4	8	40	400	4000

As x gets closer to 2, f(x) gets larger and larger $\boxed{\infty}$

8) $\lim\limits_{x \to 2^-} \dfrac{4}{x-2}$

x	1	1.5	1.9	1.99	1.999
f(x)	-4	-8	-40	-400	-4000

$\boxed{-\infty}$

9) $\lim\limits_{x \to 2} \dfrac{4}{x-2}$

$\boxed{\text{Does not exist.}}$

limit from the left is negative ∞
limit from the right is positive ∞
Since they are not equal, limit DNE

10) $\lim\limits_{x \to 2} \dfrac{x^2-4}{x^2-7x+10}$

$f(2) = \dfrac{0}{0}$

factor the quadratics:

$\dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x-5)}$ and plug in 2

$\boxed{\dfrac{4}{-3}}$

11) $\lim\limits_{x \to 0} \dfrac{x^2+3x}{x}$

$\lim\limits_{x \to 0} \dfrac{\cancel{x}(x+3)}{\cancel{x}}$

$((0)+3) = \boxed{3}$

12) $\lim\limits_{x \to \infty} \dfrac{2x^3+5x}{-3x^2+6}$

Degree of numerator is greater than degree of denominator, so the equation would go to ∞

("top heavy)... Then, since there is a negative sign, equation goes to $\boxed{-\infty}$

(Note: sometimes, a quick sketch of the function can be helpful; or, it's a nice way to check your answers!)

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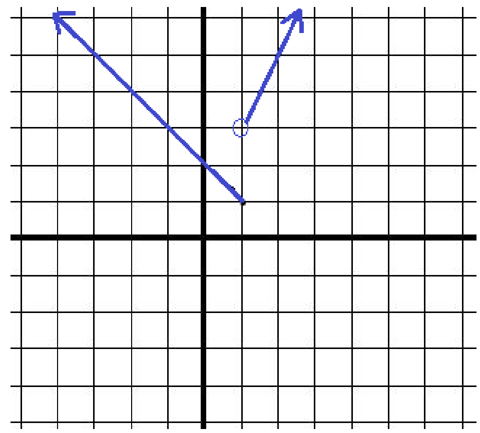
Part III: Graphing SOLUTIONS

Limits Quick Quiz

1) Graph $f(x) = \begin{cases} 2x+1 & \text{if} \quad x > 1 \\ 2-x & \text{if} \quad x \le 1 \end{cases}$



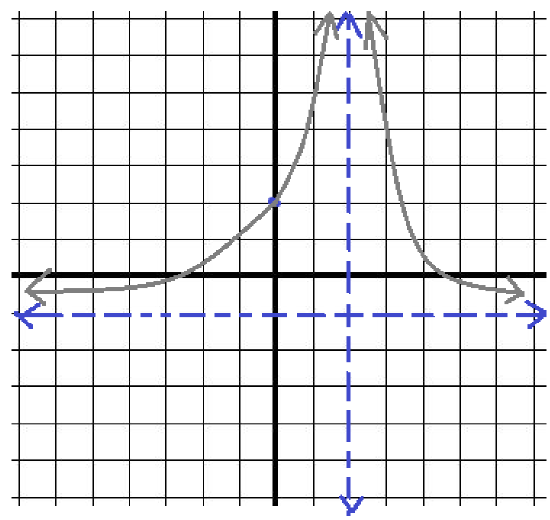
Then, identify: $f(1) = 1$

$\lim_{x \to 1^+} f(x) = 3$

$\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1} f(x) = \text{DNE}$ (Limit from the left and from the right are different, so limit does not exist)

2) Sketch a function with the following properties:



$g(0) = 2$ plot the point (0, 2)

$\lim_{x \to 2^+} g(x) = \infty$

$\lim_{x \to 2^-} g(x) = \infty$

vertical asymptote at 2, and function "goes up" on both sides of the asymptote...

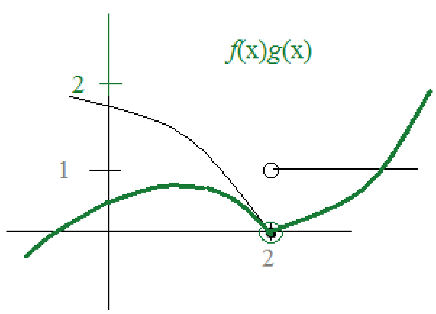
$\lim_{x \to -\infty} g(x) = -1$

$\lim_{x \to +\infty} g(x) = -1$

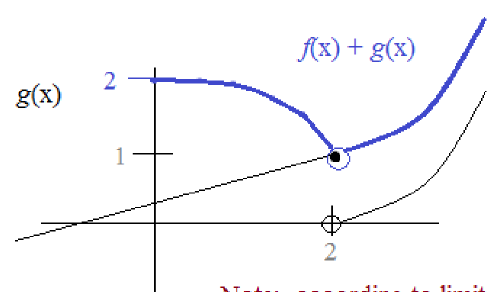
horizontal asymptote at y = -1

3) **Challenge:

$f(x)$



$g(x)$



Answer:

a) $\lim_{x \to 2} f(x) = \text{DNE}$

b) $\lim_{x \to 2} g(x) = \text{DNE}$

c) $\lim_{x \to 2} (f(x) + g(x)) = 1$

d) $\lim_{x \to 2} (f(x)g(x)) = 0$

Note: according to limit theorems:
lim (f + g) = lim(f) + lim(g)
lim(fg) = lim(f)lim(g)
But, that assumes the lim(f) and lim(g) exist.. Since they do not, the theorems cannot be applied!

IV. Miscellaneous Multiple Choice

Limits, Asymptotes, & Continuity

SOLUTIONS

1) As x increases to infinity, the function $f(x) = 2e^{-x}$ gets closer to

a) 0
b) 1/2
c) 2
d) e
e) infinity

rewrite function as $\frac{2}{e^x}$

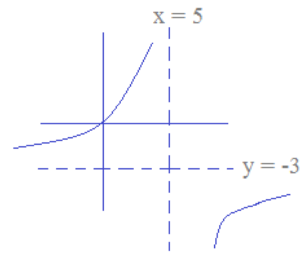
as x gets infinitely larger, e^x goes to infinity...

therefore, $\frac{2}{e^x}$ gets smaller and smaller, approaching 0

2) A rational function of the form $y = \frac{ax}{x + b}$ has a vertical asymptote at x = 5 and a horizontal asymptote at y = −3

Which is a possible function?

a) $\frac{5x}{x-3}$ b) $\frac{3x}{x+5}$ c) $\frac{-3x}{x-5}$ d) $\frac{-5x}{x+3}$ e) $\frac{-3x}{x+5}$



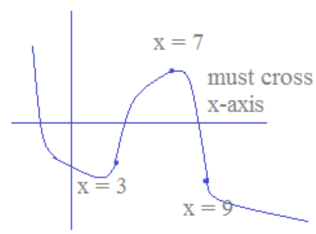
(possible sketch)

3) Let p(x) be a cubic polynomial function, where p(3) < 0, p(7) > 0, and p(9) < 0,
Which statements are true?

statement I: there are 3 zeros
statement II: a zero exists at x < 3 OR x > 9
statement III: for p(x) = 0, there are 2 solutions between 3 and 9

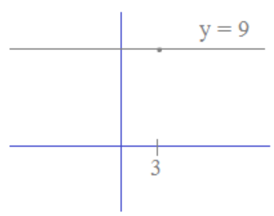
a) I
b) I and II
c) I and III
d) II
e) I, II, and III

polynomial function is continuous...



4) $\lim_{x \to 3} 9 =$ 9

a) 3
b) 9
c) Does not exist
d) 0
e) 27



5) Find the value of k so g(x) is continuous:

$$g(x) = \begin{cases} k + x & x < 10 \\ xk & x \geq 10 \end{cases}$$

a) 10
b) 0
c) 10/9
d) 1
e) no solution

to be continuous, each part of the piecewise function must meet:

k + x = xk

at x = 10:

10 + k = 10k
10 = 9k
$\boxed{k = 10/9}$

SOLUTIONS

6) $\lim\limits_{t \to 4} \dfrac{t^2 - 16}{\dfrac{1}{4} - \dfrac{1}{t}}$

a) 4
b) 16
c) 64
d) $\boxed{128}$
e) undefined

substitute t = 4, and the result is $\frac{0}{0}$ indeterminate

$$\lim_{t \to 4} \frac{(t+4)(t-4)}{\dfrac{t}{4t} - \dfrac{4}{4t}} = \lim_{t \to 4} \frac{(t+4)(t-4)}{\dfrac{(t-4)}{4t}} = \lim_{t \to 4} (t+4)(4t) = \boxed{128}$$

7) $\lim\limits_{x \to 0} \dfrac{(x+1)^2 - 1}{x}$

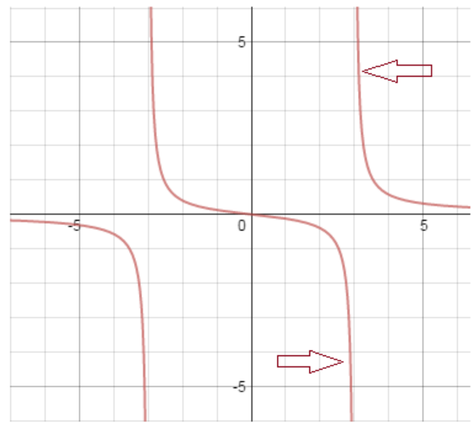
a) -1
b) 0
c) 1
d) $\boxed{2}$
e) Does not exist

Using substitution, we see the result is 0/0 indeterminate

so, we'll try expanding the numerator

$$\lim_{x \to 0} \frac{x^2 + 2x + 1 - 1}{x}$$

$$\lim_{x \to 0} \frac{x^2 + 2x}{x} = \lim_{x \to 0} \frac{\cancel{x}(x+2)}{\cancel{x}} = \boxed{2}$$



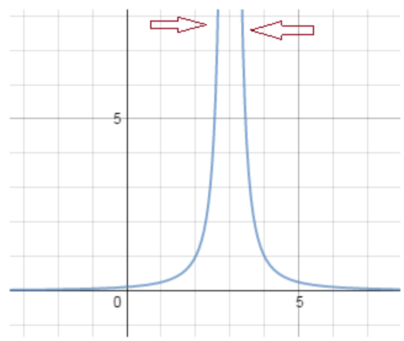
8) $\lim\limits_{x \to 3} \dfrac{x}{x^2 - 9}$

a) 3
b) 9
c) positive infinity
d) negative infinity
e) $\boxed{\text{does not exist}}$

Limit from the left is negative infinity
Limit from the right is positive infinity
Therefore, limit does not exist (DNE)

if x = 3.1, then numerator is positive and denominator is positive..
if x = 2.9, then numerator is positive and denominator is negative...

Note: These are limits; approaching vertical asymptotes



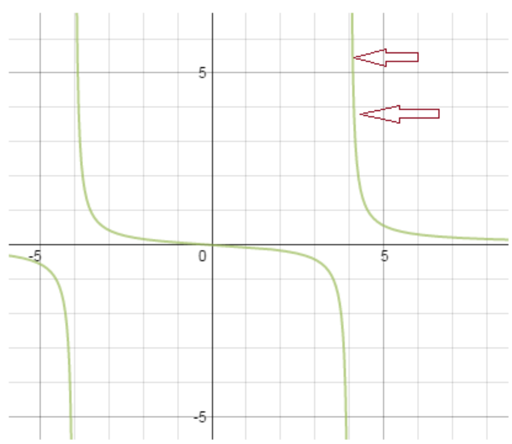
9) $\lim\limits_{x \to 3} \dfrac{1}{(x-3)^2}$

a) 3
b) 9
c) $\boxed{\text{positive infinity}}$
d) negative infinity
e) does not exist

Limit approaching 3 from the right is infinity
Limit approaching 3 from the left is infinity
Therefore, limit is positive infinity

$$\lim_{x \to 3^+} \frac{1}{(x-3)^2} = +\infty$$

$$\lim_{x \to 3^-} \frac{1}{(x-3)^2} = +\infty$$



10) $\lim\limits_{x \to 4^+} \dfrac{x}{x^2 - 16}$

a) -16
b) 0
c) 4
d) $\boxed{\text{positive infinity}}$
e) does not exist

Limit approaching 4 from the right is infinity.
(at 4, the equation is 4/0)

If x = 4.1, then $\dfrac{4.1}{(4.1)^2 - 16} = 5.06$

If x = 4.01, then $\dfrac{4.01}{(4.01)^2 - 16} = 50.06$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers,

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