Algebra 2: Math Induction

Notes, Examples, and Practice Exercises (with Solutions)

Topics include factoring, sigma notation, exponents, factorials, sequences and series, and more.

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Example: Use induction to prove $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Step 1: Verify it works for $n = 1$ (and, perhaps, a few others)

If $n = 1$ $\quad 1 = \frac{1(1+1)}{2} = 1$

If $n = 2$ $\quad 1 + 2 = \frac{2(2+1)}{2} = 3$

If $n = 3$ $\quad 1 + 2 + 3 = \frac{3(3+1)}{2} = 6$

So, the equation works for these numbers... How do we know if it works for <u>any</u> integer?!?

Step 2: Assume the formula is correct. Then, evaluate for the <u>next</u> term...

If the formula is correct, then $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

Then, if we evaluate the next term, $1 + 2 + 3 + \ldots + n + (n+1) = \frac{(n+1)((n+1)+1)}{2}$ or $\frac{(n+1)(n+2)}{2}$

So, how do we confirm this assumption?!?!

Step 3: Using our confirmed equations, add the next term..

When we tested $n = 1, 2$ and 3, we had the correct solution...

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Let's add the next term to both sides...

$$1 + 2 + 3 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$(n+1) + \sum_{k=1}^{n} k = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

Step 4: Compare and conclusion

Notice, in step 2, when we directly input $(n+1)$ into the formula, we get $\frac{(n+1)(n+2)}{2}$

Then, in step 3, when we added the next term to previous known terms, we get $\frac{(n+1)(n+2)}{2}$

Since both approaches lead to the same answer, the general formula works for every successive integer!

Example: Use induction to prove $\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}$ or, $1+3+6+\ldots+\frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$

Step 1: Verify it works for $n = 1$ (and, perhaps, a few others)

If $n = 1$: $1 = \frac{1(1+1)(1+2)}{6} = 1$

If $n = 2$: $1+3 = \frac{2(2+1)(2+2)}{6} = 4$

If $n = 3$: $1+3+6 = \frac{3(3+1)(3+2)}{6} = 10$

So, the equation works for these numbers... How do we know if it works for <u>any</u> integer?!?

Step 2: Assume the formula is correct. Then, evaluate for the <u>next</u> term...

If the formula is correct, then $1+3+6+\ldots+\frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$

Then, if we evaluate the next term, $1+3+6+\ldots+\frac{k(k+1)}{2}+\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)(k+3)}{6}$

So, how do we confirm this assumption?!?!

Step 3: Using the confirmed equations in step 1, add the next term to the confirmed basis..

When we tested $n = 1$, 2 and 3, we had the correct solution...

$$1+3+6+\ldots+\frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$$

Let's add the next term to both sides...

$$1+3+6+\ldots+\frac{k(k+1)}{2}+\frac{(k+1)(k+2)}{2} = \frac{k(k+1)(k+2)}{6}+\frac{(k+1)(k+2)}{2}$$

common denominator

$$\frac{k(k+1)(k+2)+3(k+1)(k+2)}{6}$$

GCF: $(k+1)(k+2)$

$$\frac{(k+1)(k+2)\cdot(k+3)}{6}$$

Using Summation notation:

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

$$\frac{(n+1)(n+2)}{2}+\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}+\frac{(n+1)(n+2)}{2}$$

$$= \frac{n(n+1)(n+2)}{6}+\frac{3(n+1)(n+2)}{6}$$

$$= \frac{(n+1)(n+2)(n+3)}{6}$$

$$\sum_{k=1}^{n+1} \frac{k(k+1)}{2} = \frac{(n+1)(n+2)(n+3)}{6}$$

Step 4: Compare and conclusion

Notice, in step 2, when we directly input $(k+1)$ into the formula, we get $\frac{(k+1)(k+2)(k+3)}{6}$

Then, in step 3, when we added the next term to previous known terms, we get $\frac{(k+1)(k+2)(k+3)}{6}$

Since both approaches lead to the same answer, the general formula works for every successive integer!

Example: Use induction to prove $n^3 - 4n + 6$ is a *multiple of 3*

Step 1: Verify the base case(s):

if $n = 1$, then $(1)^3 - 4(1) + 6 = 3$

if $n = 2$, then $(2)^3 - 4(2) + 6 = 6$ They are all multiples of 3...

if $n = 3$, then $(3)^3 - 4(3) + 6 = 21$

Step 2: Assume for k:

$k^3 - 4k + 6 = 3Z$ where Z is any integer... so, 3Z is a multiple of 3...

Step 3: Prove for next term, (k + 1)

$(k+1)^3 - 4(k+1) + 6 = (k+1)(k+1)(k+1) - 4k - 4 + 6$ expand

$(k^2 + 2k + 1)(k+1) - 4k + 2$ collect like terms

$k^3 + 3k^2 + 3k + 1 - 4k + 2$ rearrange

$k^3 - 4k + 3k^2 + 3k + 1 + 2$ regroup

$k^3 - 4k + 3k^2 + 3k + 3$ ***add multiples of 3...

$k^3 - 4k + 6 + 3k^2 + 3k + 3 + 3k$ 6 and 3k are each divisible by 3...

$3Z + 3(k^2 + 2k + 1)$ Both terms are divisible by 3 (i.e. multiples of 3)...

Note: We know 3k is divisible by 3. So, if x is divisible by 3, then x + 3k must also be divisible by 3!!

Example: $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$

Prove $F_n \leq 2^n$ by induction...

Step 1: Look at the base cases and verify (It's apparent that F is a Fibonacci sequence..)

$F_1 = 1 \leq 2^1$

$F_2 = 1 \leq 2^2$

$F_3 = 2 \leq 2^3$

$F_4 = 3 \leq 2^4$

$F_5 = 5 \leq 2^5$

Step 2: Assume for k

$F_k \leq 2^k$

Step 3: Prove for k + 1

$F_{k+1} \leq 2^{k+1}$

$\leq 2^k \cdot 2^1$

$\leq 2 \cdot 2^k$

$F_{k+1} \leq 2^k + 2^k$

We know (by definition) that $F_{k+1} = F_k + F_{k-1}$

$F_k \leq 2^k$ and, $F_{k-1} \leq 2^k$

Therefore, $F_k + F_{k-1} \leq 2^k + 2^k$

Example: Prove: $8^n - 3^n$ is a multiple of 5

Step 1: Verify a few cases

If $n = 1$, then 5 ✓

If $n = 2$, then $64 - 9 = 55$ ✓

Step 2: Set up Assumption: $8^k - 3^k = 5P$

Since it works for k = 1 and k = 2, prove using induction that it works for the following terms...

$$8^{k+1} - 3^{k+1} =$$

$$8^k \cdot 8^1 - 3^k \cdot 3^1 =$$

$$8^k - 3^k = 5P$$

$$8^k = 5P + 3^k$$

using substitution

$$(5P + 3^k) \cdot 8 - 3 \cdot 3^k =$$

$$40P + 8 \cdot 3^k - 3 \cdot 3^k =$$

$$40P + (8 - 3) \cdot 3^k =$$

$$40P + 5 \cdot 3^k =$$

$$5(8P + 3^k) =$$

must be multiple of 5!!! ✓

Example: Prove $\prod_{i=2}^{n} \frac{i-1}{i} = \frac{1}{n}$

Verify a few cases:

If n = 3,

$$\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$ ✓

If n = 4,

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$$ ✓

We must show that Next term:

$$\prod_{i=2}^{n+1} \frac{i-1}{i} = \frac{1}{n+1}$$

first n terms last term

$$\prod_{i=2}^{n+1} \frac{i-1}{i} = \frac{1}{n} \cdot \frac{(n+1)-1}{n+1}$$

$$= \frac{1}{n} \cdot \frac{n}{n+1}$$

$$= \frac{1}{n+1}$$ ✓

Example: If $a_0 = 1$ $a_1 = 3$ and, $a_n = 2a_{n-1} - a_{n-2}$

show that $a_n = 2n + 1$

So, we want to show $2a_{n-1} - a_{n-2} = 2n + 1$

Basis: $a_0 = 1$ $a_1 = 3$

2n + 1

Test: $a_2 = 2a_1 - a_0 \Rightarrow 2(3) - 1 = 5$ $\quad 2(2) + 1 = 5$ ✓

Test: $a_3 = 2a_2 - a_1 \Rightarrow 2(5) - 3 = 7$ $\quad 2(3) + 1 = 7$ ✓

Assume: $a_k = 2a_{k-1} - a_{k-2}$ for $k > 1$

$a_k = 2k + 1$

for (k + 1): $a_{k+1} = 2a_k - a_{k-1}$

substitution

$= 2(2k + 1) - a_{k-1}$

substitution

$= 2(2k + 1) - [2(k - 1) + 1]$

$= 4k + 2 - [2k - 1]$

$= 2k + 3$ ✓

$a_{k+1} = 2(k + 1) + 1 \Rightarrow 2k + 3$ ✓

Example: Prove by induction that $\sum_{i=1}^{n} \frac{1}{4i^2 - 1} = \frac{n}{2n+1}$ for natural numbers n

Basis: let n = 1 $\quad \sum_{i=1}^{1} \Rightarrow \frac{1}{4(1)^2 - 1} = \frac{1}{3}$ $\qquad \frac{n}{2n+1} = \frac{1}{3}$ ✓

let n = 2 $\quad \sum_{i=1}^{2} \Rightarrow \frac{1}{4(1)^2 - 1} + \frac{1}{4(2)^2 - 1} = \frac{1}{3} + \frac{1}{15} = \frac{2}{5}$ $\qquad \frac{n}{2n+1} = \frac{2}{5}$ ✓

Assume: $\sum_{i=1}^{n} \frac{1}{4i^2 - 1} = \frac{n}{2n+1}$

first n terms $\qquad$ (n + 1) th term

$$\sum_{i=1}^{n+1} \frac{1}{4t^2 - 1} = \sum_{i=1}^{n} \frac{1}{4t^2 - 1} + \frac{1}{4(n+1)^2 - 1}$$

$$= \frac{n}{2n+1} + \frac{1}{4n^2 + 8n + 3} = \frac{n}{2n+1} + \frac{1}{(2n+3)(2n+1)} = \frac{n(2n+3)}{(2n+3)(2n+1)} + \frac{1}{(2n+3)(2n+1)}$$

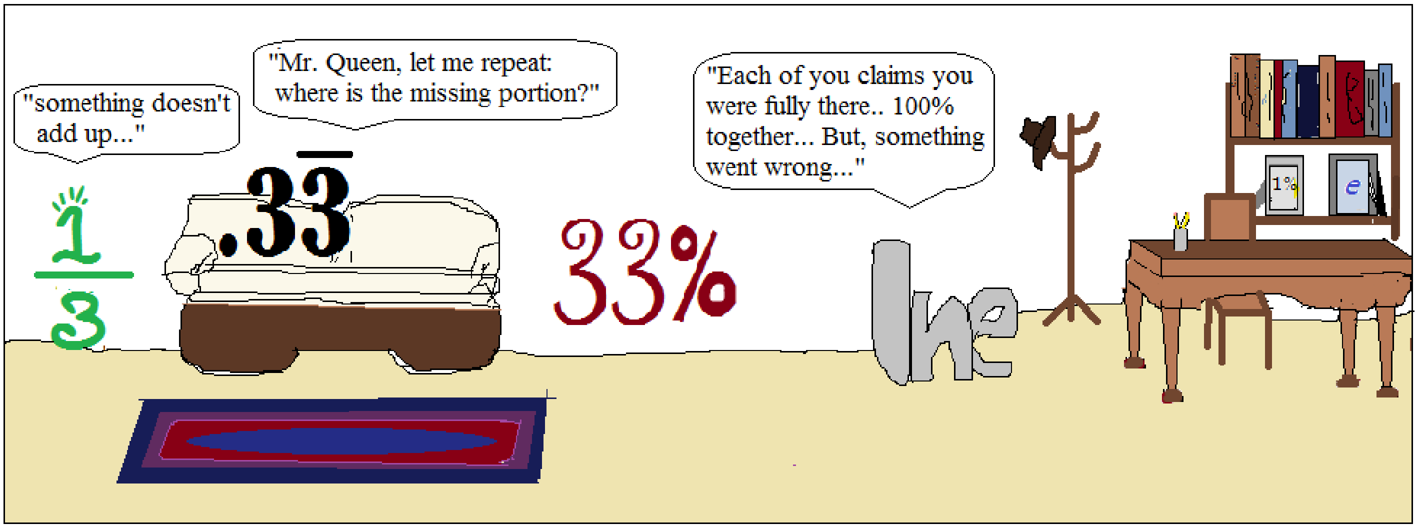
$$= \frac{2n^2 + 3n + 1}{(2n+3)(2n+1)}$$

$$= \frac{(2n+1)(n+1)}{(2n+3)(2n+1)} = \frac{n+1}{2n+3}$$

$$\sum_{i=1}^{n+1} \frac{1}{4t^2 - 1} = \frac{n+1}{2(n+1)+1}$$

(1975) Ine Queen and *The Case of the 3 Portions...*

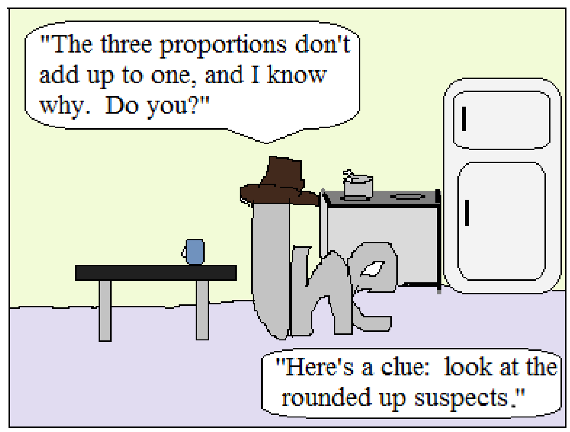
Ellery's Cousin



Ine Queen is the *one* detective for any math mystery (naturally)...

$$.3\overline{3} + 33\% + \frac{1}{3} \neq 1$$

What is the missing piece?



LanceAF #89 6-7-13
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Practice Exercises -→

Use induction to prove the following:

Math Induction Practice

1) $$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

2) $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Use induction to prove the following:

Math Induction Practice

3) $(n - 1)(n)(n + 1)$ is divisible by 6 for all integers n

4) Show that the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges to 2.

5) Prove: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

6) Prove: $3 + 5 + 7 + \ldots + (2n + 1) = n^2 + 2n$

7) Prove $3^{2n} - 1$ is divisible by 8 for $n \geq 0$

8) Prove: $3n^2 + 3n$ is divisible by 6

9) Prove: $1 + 8 + 27 + 64 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

10) Use induction to prove that $n^5 - n$ is a multiple of 5

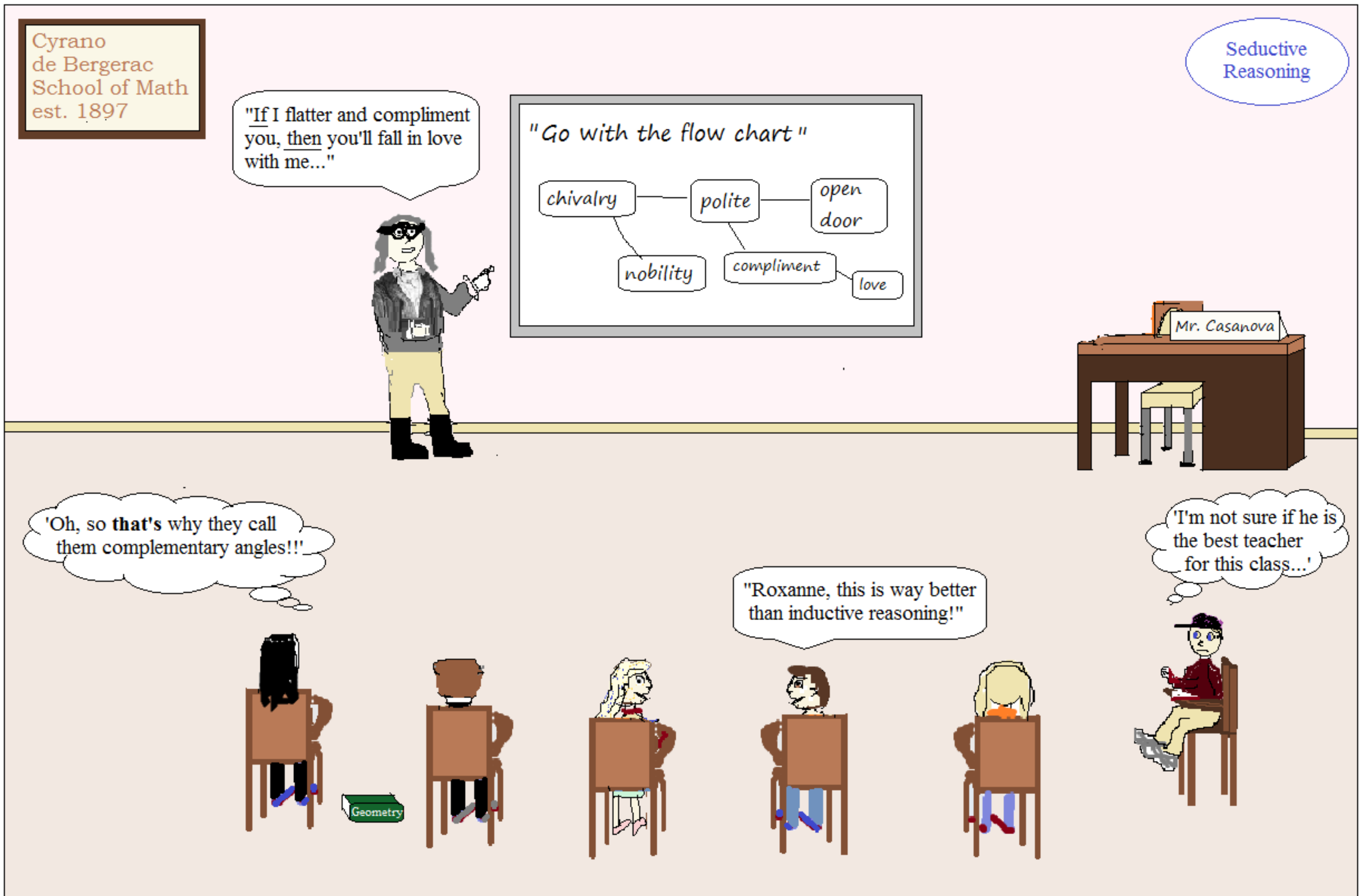
11) Prove $2^{n+2} + 3^{2n+1}$ is a multiple of 7.

12) Prove $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

13) Prove $4^n - 2^n > 3^n$ for $n > 1$

14) Prove $3^n > 2^{n+1}$ for $n > 1$

15) Prove $n^{n/2} \geq 2^n$ for $n > 3$



LanceAF #337 (11-11-18)
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Using (un)conditional (love) statements,
Casanova teaches logic and reasoning to geometry students!

Solutions-→

Use induction to prove the following: | Math Induction Practice

SOLUTIONS

1) $$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

Step 1: Verify

If n = 1, then $1(1+1) = \frac{1(1+1)(1+2)}{3}$

$2 = 2$

If n = 4, then $2 + 6 + 12 + 20 = \frac{4(5)(6)}{3}$

$40 = 40$

So, we'll assume the formula is correct...

Step 2: Find n + 1, using the formula...

$$\sum_{k=1}^{n+1} k(k+1) = \frac{(n+1)(n+1+1)(n+1+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3}$$

Step 3: Use confirmed formula (from step 1) and add the next term..

$$2 + 6 + 12 + 40 \ldots + n(n+1) + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$

$$= \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3}$$ GCF and regroup factors..

$$= \frac{(n+1)(n+2)(n+3)}{3}$$

Using successive term OR using the direct formula get same result.. ✓

2) $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Step 1: Verify for n = 1

$$(1)^3 = \frac{(1)^2(1+1)^2}{4} = 1$$

also, if n = 2 $(1)^3 + (2)^3 = \frac{(2)^2(2+1)^2}{4} = 9$

if n = 3 $(1)^3 + (2)^3 + (3)^3 = \frac{(3)^2(3+1)^2}{4} = 36$

Step 2: Evaluate (using the formula) for n + 1

$$\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2((n+1)+1)^2}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

Step 3: Since we know the formula worked for n = 1 (and n = 2 and 3), we'll assume it's correct. Now, let's add the next term....

$$1 + 9 + 36 + \ldots + n^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + (n+1)^2(n+1)$$ "split the term"

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^2(n+1)}{4}$$ common denominator

$$= \frac{n^2(n+1)^2 + (4n+4)(n+1)^2}{4}$$ condense

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$ regroup and

$$= \frac{(n+1)^2(n+2)^2}{4}$$ factor

The result of step 3 matches the formula result in step 2! ✓

Use induction to prove the following: SOLUTIONS

Math Induction Practice

3) $(n-1)(n)(n+1)$ is divisible by 6 for all integers n

Step 1: Verify it works

if $n = 1$ $(1-1)(1)(1+1) = 0$

if $n = 2$ $(2-1)(2)(2+1) = 6$ each is divisible by 6

if $n = 3$ $(3-1)(3)(3+1) = 24$

Step 2: Assume the statement is true for k

$(k-1)(k)(k+1) = 6Z$ where Z is any integer (6Z must be divisible by 6)

$k^3 - k = 6Z$

Step 3: Prove for the next term $(k+1)$

$((k-1)+1)(k+1)((k+1)+1) =$

$(k)(k+1)(k+2) =$

$(k^2+k)(k+2) =$

$k^3 + 3k^2 + 2k =$

$k^3 + 3k^2 + 3k - k =$

$k^3 - k + 3k^2 + 3k =$

6Z

Then, is $3k^2 + 3k$ a multiple of 6?

if $3k^2 + 3k = 6Z$

then, $k^2 + k = 2Z$

So, is $k^2 + k$ always even?

$k(k+1)$ is always even!!

an even x odd number is even..

4) Show that the geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ converges to 2.

Let $S = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}\right)$

Then, $\frac{1}{2}S = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n \cdot 2}\right)$

Subtract: $S - \frac{1}{2}S = \lim_{n\to\infty}\left(1 - \frac{1}{2^{n+1}}\right)$

$\frac{1}{2}S = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{2^{n+1}}$

$\frac{1}{2}S = 1 - 0$

$\frac{1}{2}S = 1$

$S = 2$

5) Prove: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

SOLUTIONS

Math Induction Practice

Step 1: Try a few numbers to see if possible...

If $n = 1$, $\frac{1}{2!} = \frac{1}{2}$ $\quad 1 - \frac{1}{(1+1)!} = \frac{1}{2}$ ✓

If $n = 2$, $\frac{1}{2!} + \frac{2}{3!} = \frac{5}{6}$ $\quad 1 - \frac{1}{(2+1)!} = \frac{5}{6}$ ✓

Step 2: Assume for all integers k

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

Step 3: Confirm, if works for next term...

Prove: $$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$

$$1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$$

$$1 - \frac{1}{(k+1)!} \cdot \frac{(k+2)}{(k+2)} + \frac{k+1}{(k+2)!}$$

$$1 - \frac{(k+2)}{(k+2)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$$ ✓

6) Prove: $3 + 5 + 7 + \ldots + (2n+1) = n^2 + 2n$

Step 1: Test assumption

$n = 1 \quad 2(1) + 1 = 3 = (1)^2 + 2(1)$ ✓

$n = 2 \quad 3 + 5 = 8 = (2)^2 + 2(2)$ ✓

$n = 3 \quad 3 + 5 + 7 = 15 = (3)^2 + 2(3)$ ✓

Step 2: Assume for all integers k

$$3 + 5 + 7 + \ldots + (2k+1) = k^2 + 2k$$

Step 3: Prove for the next term....

$$3 + 5 + 7 + \ldots + (2k+1) + (2(k+1)+1) = (k+1)^2 + 2(k+1)$$

$$k^2 + 2k + 2k + 3$$

(split the 3)

$$k^2 + 2k + 1 + 2k + 2$$

$$(k+1)^2 + 2(k+1) = (k+1)^2 + 2(k+1)$$ ✓

7) Prove $3^{2n} - 1$ is divisible by 8 for $n \geq 0$

SOLUTIONS

Mathematical Induction Practice

If $n = 0$, $3^0 - 1 = 0$

If $n = 1$, $3^2 - 1 = 8$

If $n = 2$, $3^4 - 1 = 80$

$3^{2n} - 1 = 8Z \Rightarrow 3^{2n} = 8Z + 1$

$3^{2(n+1)} - 1 = 3^{2n+2} - 1$

$= 3^{2n} \cdot 3^2 - 1$

substitute

$9 \cdot 3^{2n} - 1 \Rightarrow 9 \cdot (8Z + 1) - 1 = 72Z + 9 - 1 = 72Z + 8$ $8(9Z + 1)$ is divisible by 8!!! ✓

8) Prove: $3n^2 + 3n$ is divisible by 6

Let n = 1 ----> 6 ✓

Let n = 2 ----> 18 ✓

Let n = 3 ----> 36 ✓

General assumption: $3k^2 + 3k = 6P$

Prove using induction: next term....

$3(k+1)^2 + 3(k+1) =$

$3k^2 + 6k + 3 + 3k + 3 =$

rearrange

$3k^2 + 3k + 6k + 3 + 3 =$

$6P + 6k + 6$

$6P + 6(k+1)$

6P is divisible by 6 (by assumption)
and
6(k + 1) must be divisible by 6! ✓

9) Prove: $1 + 8 + 27 + 64 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Test: If $n = 2$, $1 + 8 = 9$ $\left(\frac{2(2+1)}{2}\right)^2 = 9$ ✓

If $n = 3$, $1 + 8 + 27 = 36$ $\left(\frac{3(3+1)}{2}\right)^2 = 36$ ✓

General: $1 + 8 + 27 + 64 + ... + k^3 = \left(\frac{k(k+1)}{2}\right)^2$

Induction by adding next term: $1 + 8 + 27 + 64 + ... + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$\frac{k^2(k+1)^2}{2^2} + (k+1)^2(k+1)$$

$$\frac{k^2(k+1)^2}{2^2} + \frac{4(k+1)^2(k+1)}{2^2}$$

$$(k+1)^2 \cdot \left[\frac{k^2 + 4k + 4}{2^2}\right] = (k+1)^2\left[\frac{(k+2)(k+2)}{2^2}\right] = \left(\frac{(k+1)(k+2)}{2}\right)^2$$ ✓

10) Use induction to prove that $n^5 - n$ is a multiple of 5

SOLUTIONS

Mathematical Induction Practice

Basis: $n = 1$ $1^5 - 1 = 0$ ✓

$n = 2$ $2^5 - 2 = 30$ ✓

$n = 3$ $3^5 - 3 = 240$ ✓

hypothesis is $k^5 - k = 5P$

then, $(k+1)^5 - (k+1)$ must be multiple of 5

$k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - (k+1)$

$k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + \not{1} - (k + \not{1})$

rearrange

$k^5 - k$ $\quad + 5k^4 + 10k^3 + 10k^2 + 5k$

$5P$ $\quad 5(k^4 + 2k^3 + 2k^2 + k)$

multiple of 5 multiple of 5 ✓

11) Prove $2^{n+2} + 3^{2n+1}$ is a multiple of 7.

If $n = 0$, then $2^2 + 3^1 = 7$ ✓

If $n = 1$, then $2^3 + 3^3 = 35$ ✓

Assume $2^{k+2} + 3^{2k+1} = 7Z$ where Z is any integer.

Now, show that it is true for the next term k + 1

$2^{(k+1)+2} + 3^{2(k+1)+1}$

$2^{k+3} + 3^{2k+3}$ we're trying to utilize the initial assumption in our proof...

exponent laws

$2 \cdot 2^{k+2} + 3^2 \cdot 3^{2k+1}$

"split" the terms

$2^{k+2} + 2^{k+2} + 8 \cdot 3^{2k+1} + 3^{2k+1}$

7Z

$2^{k+2} + 3^{2k+1} + 2^{k+2} + 8 \cdot 3^{2k+1}$

7Z

"split" terms again..

$2^{k+2} + 3^{2k+1} + 2^{k+2} + 3^{2k+1} + 7 \cdot 3^{2k+1}$ ✓

7Z 7Z

must be a multiple of 7!!

Mathematical Induction Practice

SOLUTIONS

12) Prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Basis:

$n = 1 \quad \sum_{i=1}^{1} i^2 = 1 \qquad \frac{1(1+1)(2(1))+1)}{6} = \frac{6}{6} = 1$ ✓

$n = 2 \quad \sum_{i=1}^{2} i^2 = 1 + 4 = 5 \qquad \frac{2(2+1)(2(2)+1)}{6} = \frac{30}{6} = 5$ ✓

$n = 3 \quad \sum_{i=1}^{3} i^2 = 1 + 4 + 9 = 14 \qquad \frac{3(3+1)(2(3)+1)}{6} = \frac{84}{6} = 14$ ✓

To prove by induction, let's show for all integers k....

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6} \Leftarrow \boxed{\frac{(k+1)(k+2)(2k+3)}{6}}$$

Rewrite by splitting up the left side into basis + (k + 1) term

$$\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\frac{(k+1)}{6} \cdot \frac{k(2k+1)}{1} + \frac{6(k+1)(k+1)}{6}$$

$$\frac{(k+1)}{6}\left(\frac{k(2k+1)}{1} + 6(k+1)\right)$$

$$\frac{(k+1)}{6}\left(2k^2 + k + 6k + 6\right)$$

$$\frac{(k+1)}{6}(2k^2 + 7k + 6) = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\boxed{\frac{(k+1)(2k+3)(k+2)}{6}}$$ ✓

13) Prove $4^n - 2^n > 3^n$ for $n > 1$

Basis: If $n = 2$, $16 - 4 > 9$ ✓

If $n = 3$, $64 - 8 > 27$ ✓

Let's verify that all subsequent integers n will satisfy the inequality of the basis...

$$4^{n+1} - 2^{n+1} > 3^{n+1}$$

$$4 \cdot 4^n - 2 \cdot 2^n > 3^{n+1} \qquad \text{exponent law}$$

$$4\left(4^n - \frac{1}{2} \cdot 2^n\right) > 3^{n+1} \qquad \text{factor out the 4}$$

$$4\left(4^n - 2^{n-1}\right) > 3^{n+1} \qquad \text{apply exponential law}$$

$$\left(4^n - 2^{n-1}\right) > \frac{1}{4} \cdot 3^{n+1} \qquad \text{divide each side by 4}$$

$$\left(4^n - 2^{n-1}\right) > 4^n - 2^n > 3^n > \frac{1}{4} \cdot 3^{n+1}$$ ✓

$$\frac{3}{4} \cdot 3^n$$

Insert basic and compare values

Applying transitive property of inequality

SOLUTIONS

14) Prove $3^n > 2^{n+1}$ for $n > 1$

When $n = 2$, $3^2 > 2^{2+1} \Rightarrow 9 > 8$ ✓

$n = 3$, $3^3 > 2^{3+1} \Rightarrow 27 > 16$ ✓

To prove that all natural numbers that follow....

$$3^{k+1} > 2^{k+1+1}$$

$$3^1 \cdot 3^k > 2^1 \cdot 2^{k+1}$$

$$3^k > \frac{2}{3} \cdot 2^{k+1}$$

$$\underbrace{3^k > 2^{k+1}}_{\text{from basis}} \underbrace{> \frac{2}{3} \cdot 2^{k+1}}_{\text{inequality for any positive numbers}}$$

using the basis inequality, we verify the inequality from induction

$$\underbrace{3^1 \cdot 3^k > 3^k > 2^1 \cdot 2^{k+1}}_{✓} > 2^{k+1}$$

15) Prove $n^{n/2} \geq 2^n$ for $n > 3$

If $n = 4$, $4^{4/2} \geq 2^4 \Rightarrow 16 \geq 16$ ✓

If $n = 5$, $5^{5/2} \geq 2^5 \Rightarrow 56 \geq 32$ ✓

To prove, for subsequent integers,

$$(n+1)^{\frac{n+1}{2}} \geq 2^{n+1}$$

square both sides

$$(n+1)^{n+1} \geq 2^{2(n+1)}$$

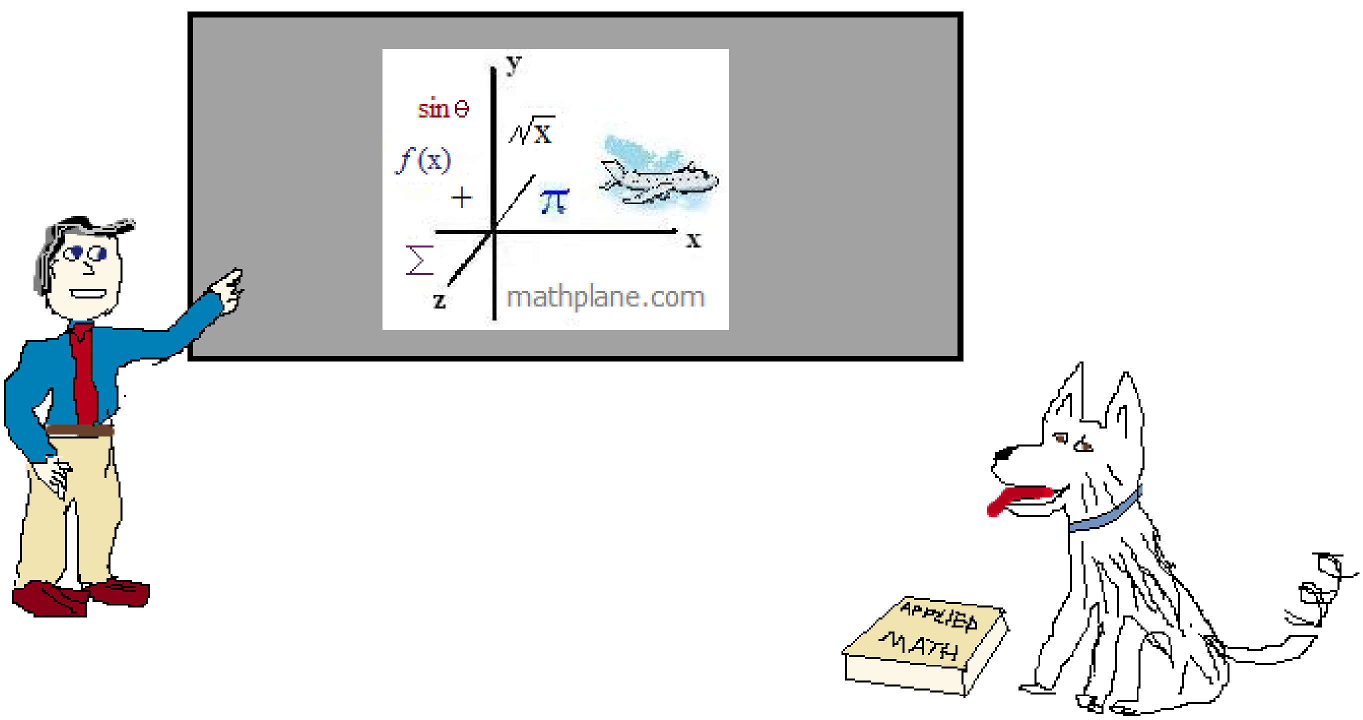
$$(n+1)^{n+1} \geq 4^{(n+1)}$$

since $n \geq 4$, this inequality is always true! ✓

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Thanks.



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