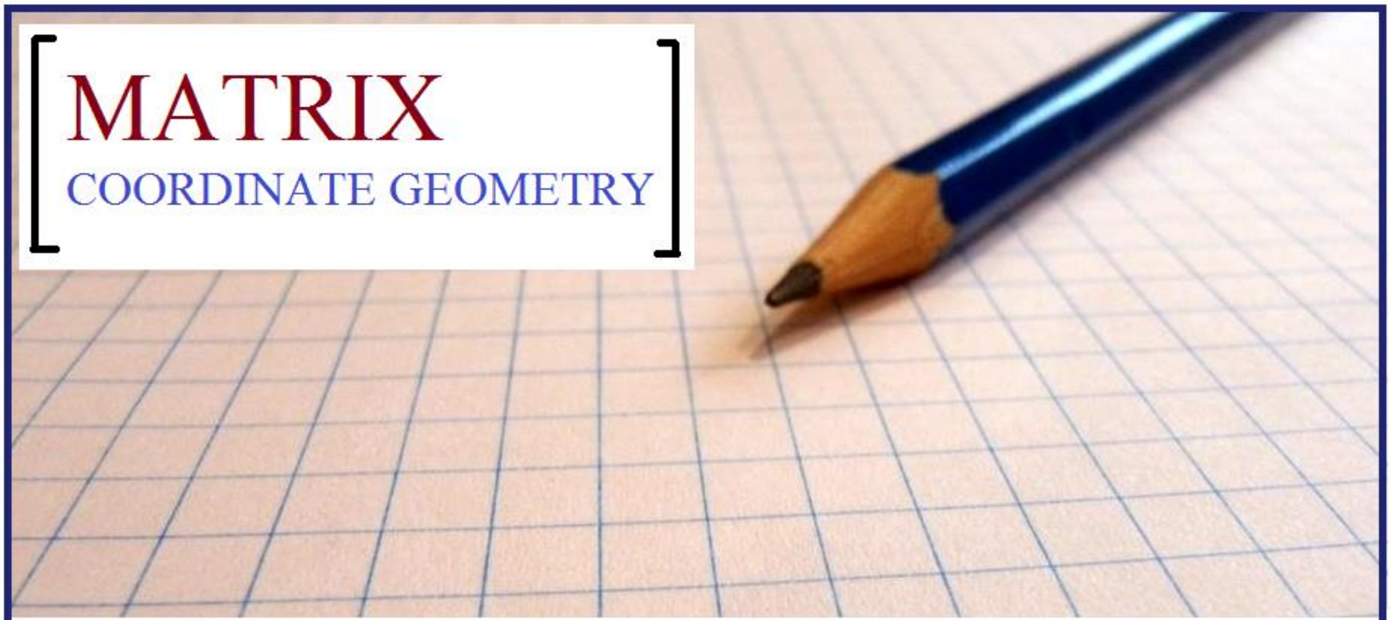
MATRIX

COORDINATE GEOMETRY



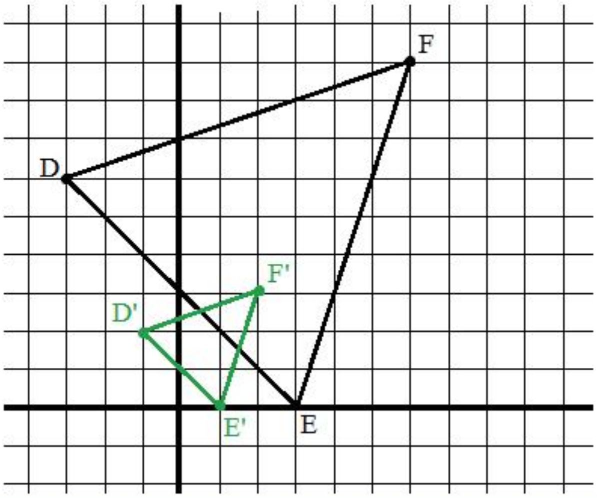
Triangle DEF with vertices

D(-3, 6) E(3, 0) F(6, 9)

$$\begin{bmatrix}-3 & 3 & 6\\ 6 & 0 & 9\end{bmatrix}$$

$$\frac{1}{3}\begin{bmatrix}-3 & 3 & 6\\ 6 & 0 & 9\end{bmatrix}=\begin{bmatrix}-1 & 1 & 2\\ 2 & 0 & 3\end{bmatrix}$$

(scalar multiplication)



Notes, Examples, and Practice Exercises....

$$\begin{bmatrix}\text{Translation} & \text{Rotation}\\ \text{Reflection} & \text{Transformation}\\ \text{Identity} & \text{Inverse}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

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CONTENTS

INTRODUCTION

An important application of matrices is in coordinate geometry. This packet introduces topics such as mapping, translation, and transformation. It is comprised of notes and examples, followed by practice exercises (and solutions).

Some terms may vary – such as ‘enlargement’ instead of ‘dilation’. But, the overall concepts are utilized in math classes. Also, while this packet emphasizes 2x2, 3x2, and 4x2 linear matrices, most of the methods can be applied to matrices of greater dimensions.

Thanks for checking out this packet. (Hope it helps!)

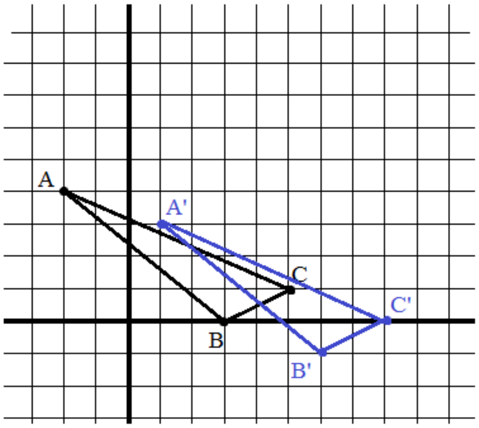
Questions, suggestions, and feedback are appreciated.

Cheers,

Lance

Matrix Coordinate Geometry

Translate triangle ABC with vertices A(-2, 4) B(3, 0) C(5, 1)
where $(x, y) \longrightarrow (x + 3, y - 1)$



This represents a *horizontal* shift 3 units to the right and a *vertical* shift 1 unit down.

The output is $A' = (-2 + 3, 4 - 1) = (1, 3)$

$B' = (3 + 3, 0 - 1) = (6, -1)$

$C' = (5 + 3, 1 - 1) = (8, 0)$

This may be expressed as a 1x2 matrix
(1 row/2 columns)

$$\begin{array}{l} \begin{array}{cc} x & y \end{array} \quad [-2\ \ 4] + [3\ \ -1] = [1\ \ 3] \\ \quad [3\ \ 0] + [3\ \ -1] = [6\ \ -1] \\ \quad [5\ \ 1] + [3\ \ -1] = [8\ \ 0] \end{array}$$

1st column is *x values*
2nd column is *y values*

Or, it may be expressed as a 2x1 matrix
(2 rows/1 column)

$$\begin{matrix} x \\ y \end{matrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

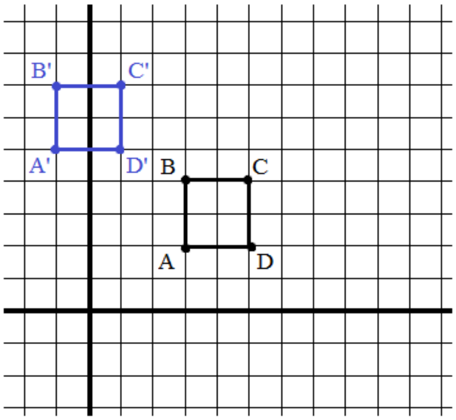
$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

1st row is *x values*
2nd row is *y values*

These individual matrices can be combined and expressed as one matrix.
Then, the translation matrix can be expanded to match the dimensions of the *coordinate matrix*.

Translate the square ABCD by shifting it 4 units to the left and 3 units up.
The vertices are the following: A(3, 2) B(3, 4) C(5, 4) D(5, 2)



$$\begin{array}{cccccc} & \begin{matrix} A & B & C & D \end{matrix} & & & & \begin{matrix} A' & B' & C' & D' \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{bmatrix} 3 & 3 & 5 & 5 \\ 2 & 4 & 4 & 2 \end{bmatrix} & + & \begin{bmatrix} -4 & -4 & -4 & -4 \\ 3 & 3 & 3 & 3 \end{bmatrix} & = & \begin{bmatrix} -1 & -1 & 1 & 1 \\ 5 & 7 & 7 & 5 \end{bmatrix} \\ & \text{2 x 4 matrix} & & \text{2 x 4 matrix} & & \downarrow \end{array}$$

the entire matrix represents the vertices
A'(-1, 5) B'(-1, 7) C'(1, 7) D'(1, 5)

Matrix Coordinate Geometry

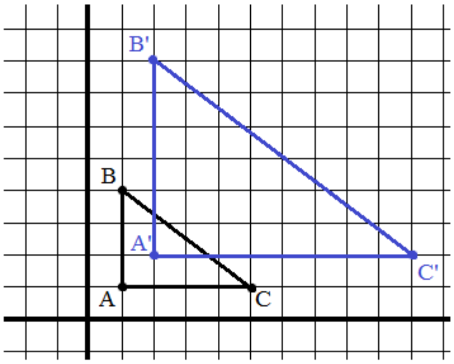
Scalar Multiplication will enlarge (or shrink) the mapped figure by a constant ratio.

Right Triangle ABC A(1, 1) B(1, 4) C(5, 1)

is expressed in the 2x3 matrix $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 4 & 1 \end{bmatrix}$

$$2\begin{bmatrix} 1 & 1 & 5 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 10 \\ 2 & 8 & 2 \end{bmatrix}$$

The triangle dimensions are doubled.
(Perimeter is 2x and area is 2^2x (or 4x))

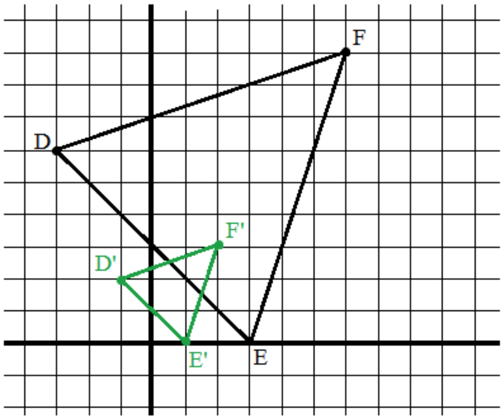


Triangle DEF with vertices

D(-3, 6) E(3, 0) F(6, 9) $\begin{bmatrix} -3 & 3 & 6 \\ 6 & 0 & 9 \end{bmatrix}$

$$\frac{1}{3}\begin{bmatrix} -3 & 3 & 6 \\ 6 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix}$$

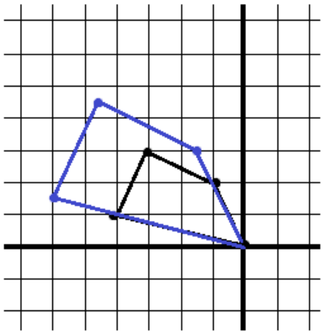
This triangle's dimensions are shrunk to 1/3 the original size.



Observations:

1) If a vertex is on the origin, then the figure will remain on the origin

For any scalar K, $K\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



2) For scalar K:

if K > 1, then the figure grows

if 0 < K < 1, then the figure shrinks

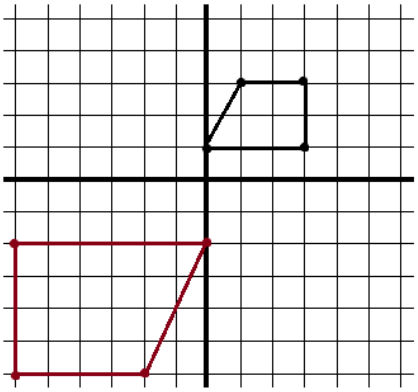
if K = 1, the figure remains the same

if K = 0, the image is transformed into a point on the origin!

$$0\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3) If the scalar is negative, then the image is magnified and reflected over the origin.

$$-2\begin{bmatrix} 0 & 1 & 3 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -6 & -6 \\ -2 & -6 & -6 & -2 \end{bmatrix}$$



Matrix Coordinate Geometry

To discover the reflection matrices, consider the identy matrix I:

$$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

For any matrix M, $I \cdot M = M$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}1a+0c & 1b+0d\\0a+1c & 0b+1d\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & -4 & x\\5 & y & 0 & 10\end{bmatrix} = \begin{bmatrix}1 & 3 & -4 & x\\5 & y & 0 & 10\end{bmatrix}$$

Now, suppose you want to reflect the coordinates (represented by the matrix) over the y-axis...
You need to change all the x terms into -x (without changing the y terms!)

If you multipy by a scalar, the x <u>and</u> y terms will change...

Instead, adjust the Identity matrix..

R_y = Reflection matrix over the y–axis: $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

(this changes all the terms in the 1st row; and, the 2nd row remains the same)

Note:
1) the dimensions must be acceptable.
# of columns in R = # of rows in M
2) Reflection matrix is to the *left* of the coordinate matrix

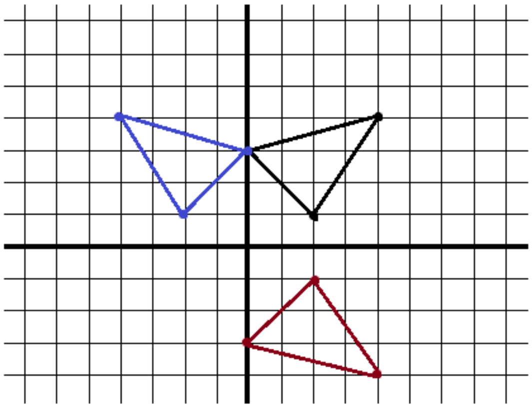
R_x = Reflection matrix over the x-axis: $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

(this changes all the terms in the 2nd row; but, the 1st row remains the same)

Reflect the triangle with vertices ABC over the y-axis:

$$\underset{R_y}{\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}}\underset{M}{\begin{bmatrix}2 & 4 & 0\\1 & 4 & 3\end{bmatrix}} = \underset{M'_y}{\begin{bmatrix}-2 & -4 & 0\\1 & 4 & 3\end{bmatrix}}$$

$$\underset{R_x}{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}\underset{M}{\begin{bmatrix}2 & 4 & 0\\1 & 4 & 3\end{bmatrix}} = \underset{M'_x}{\begin{bmatrix}2 & 4 & 0\\-1 & -4 & -3\end{bmatrix}}$$



R_o = Reflection over the origin: $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

Note: Since all terms are opposites when reflected over the origin, multiplying the coordinate matrix by -1 (scalar) will determine the matrix reflected over the origin.

Matrix Coordinate Geometry

Deriving the reflection matrix (over the y-axis):

Suppose we have a coordinate matrix A, where row 1 is the x-values, row 2 is the y-values $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (represents coordinates (a, c) (b, d))

If we reflect the coordinates over the y-axis, the result would be matrix B $\begin{bmatrix} -a & -b \\ c & d \end{bmatrix}$

What matrix X would reflect $A \rightarrow B$?

We know XA = B where X is the reflection matrix over the y-axis.
Therefore, if we find X, we would discover the reflection matrix!

$$\underset{X}{\begin{bmatrix} X \end{bmatrix}} \underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \underset{B}{\begin{bmatrix} -a & -b \\ c & d \end{bmatrix}}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$XA = B$$
$$XAA^{-1} = BA^{-1}$$
$$XI = BA^{-1}$$
$$X = BA^{-1}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

$$\underset{X}{\begin{bmatrix} X \end{bmatrix}} = \underset{B}{\begin{bmatrix} -a & -b \\ c & d \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}} = \begin{bmatrix} \dfrac{-ad + bc}{ad - bc} & \dfrac{ab - ba}{ad - bc} \\ \dfrac{cd - dc}{ad - bc} & \dfrac{-bc + da}{ad - bc} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

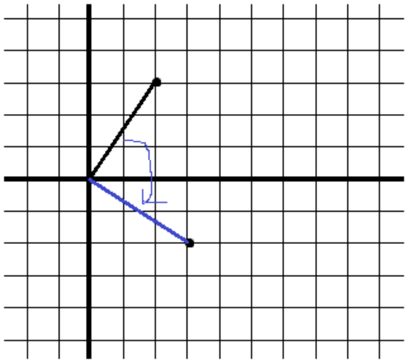
The reflection matrix
(over the y-axis)

Using this method of algebra, matrices, and inverses, we can determine a *rotation matrix.*

Find the rotation matrix -- clock wise 90° about the origin

Notice that (2, 3) translates into (3, -2) when it is rotated 90 degrees clockwise.

In fact, any (x, y) will turn into (y, -x)



Expressed as a coordinate matrix:

$(2, 3) \longrightarrow (3, -2)$
$(a, c) \longrightarrow (c, -a)$
$(b, d) \longrightarrow (d, -b)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} c & d \\ -a & -b \end{bmatrix}$$

Finding a rotation matrix:

Set up the matrix equations,
find the inverse, and solve:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

XA = B

$$\overset{X}{\begin{bmatrix} \;X\; \end{bmatrix}}\overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \overset{B}{\begin{bmatrix} c & d \\ -a & -b \end{bmatrix}}$$

$XAA^{-1} = BA^{-1}$

$$\begin{bmatrix} \;X\; \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix}\begin{bmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

$XI = BA^{-1}$

$$\begin{bmatrix} \;X\; \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{cd - dc}{ad - bc} & \frac{-cb + da}{ad - bc} \\ \frac{-ad + bc}{ad - bc} & \frac{ab - ba}{ad - bc} \end{bmatrix}$$

$X = BA^{-1}$

$$\begin{bmatrix} \;X\; \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Rotation matrix
(90 degrees clockwise
around the origin)

Test the result:

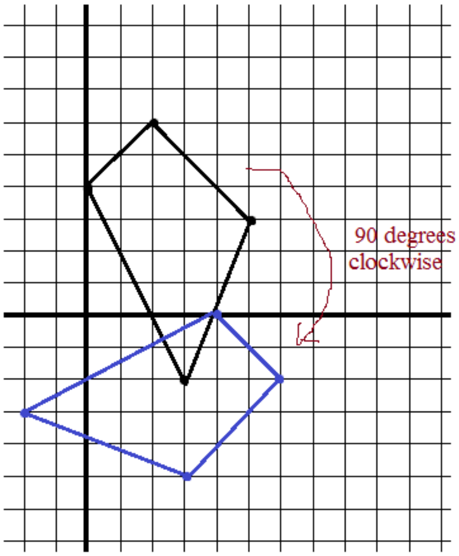
Rotate Quadrilateral ABCD clockwise 90 degrees (around the origin)

A(0, 4)
B(2, 6)
C(5, 3)
D(3, -2)

$$\begin{array}{ccccc} A & B & C & D & \\ 0 & 2 & 5 & 3 & x \\ 4 & 6 & 3 & -2 & y \end{array}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 5 & 3 \\ 4 & 6 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 3 & -2 \\ 0 & -2 & -5 & -3 \end{bmatrix}$$

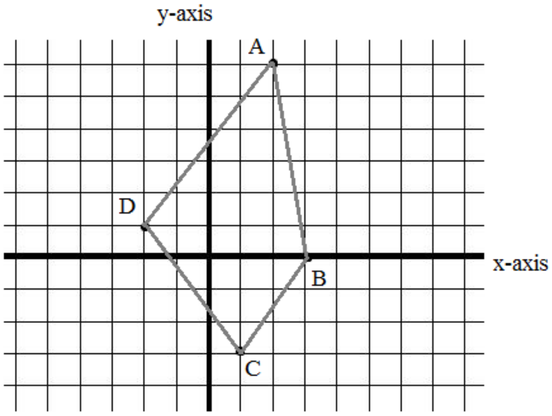
Rotation
Matrix



Matrix Coordinate Geometry Worksheet

I. Mapping

A) Write a 2x4 coordinate matrix representing the polygon ABCD in the xy-plane.



B) In the xy-plane (on the right), graph the triangle whose vertices E, F, and G are expressed by the following (linear) matrix:

$$\begin{bmatrix} 4 & 6 & 5 \\ 1 & 1 & 5 \end{bmatrix}$$

What type of triangle does the matrix describe?

II. Translation

A) What (movement on a Cartesian Plane) does matrix T represent?

AX + T = A'

$$\underset{A}{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} + \underset{T}{\begin{bmatrix} 3 & 3 & 3 \\ -4 & -4 & -4 \end{bmatrix}} = \underset{A'}{\begin{bmatrix} a' & b' & c' \\ d' & e' & f' \end{bmatrix}}$$

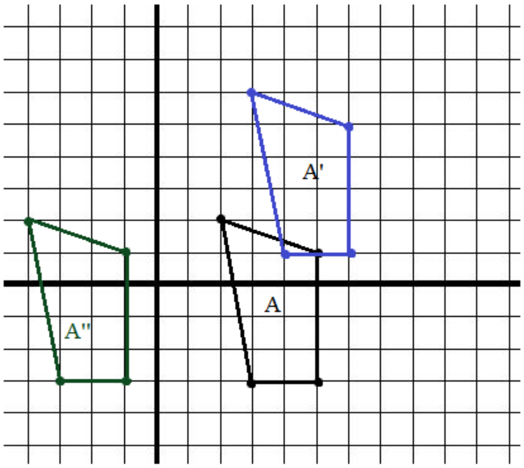
B) Use a 2x4 coordinate matrix to describe each translation in the graph.

A + T = A'

T = $\begin{bmatrix} \quad & \quad \end{bmatrix}$

A + T'' = A''

T'' = $\begin{bmatrix} \quad & \quad \end{bmatrix}$



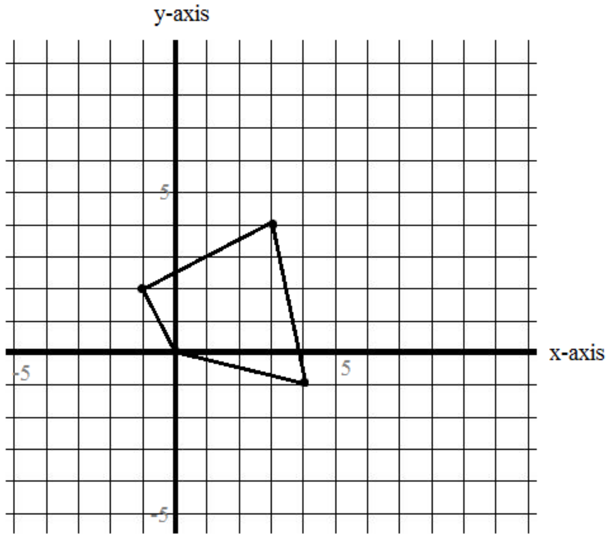
Matrix Coordinate Geometry Worksheet

III. Scalar Transformation

A) Write the 2x4 matrix, listing the four vertices of the shape in the graph.

$$S = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \end{bmatrix}$$

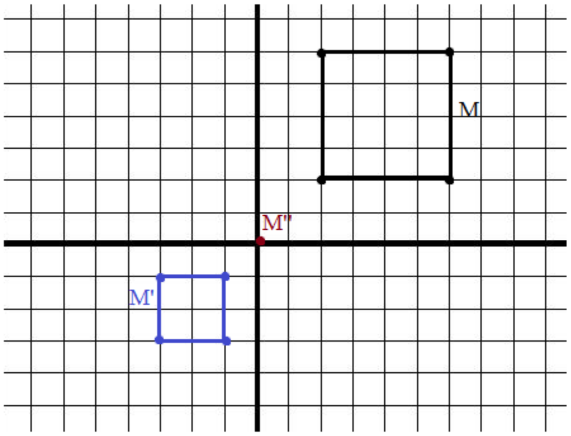
B) What is the matrix 2S?
Map 2S on the graph (on the right).



C) What is the scalar used to transform

1) M $\longrightarrow$ M' ?

2) M $\longrightarrow$ M'' ?



IV. Reflection

Identify (describe) the linear transformation each matrix performs:

1) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

2) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

3) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

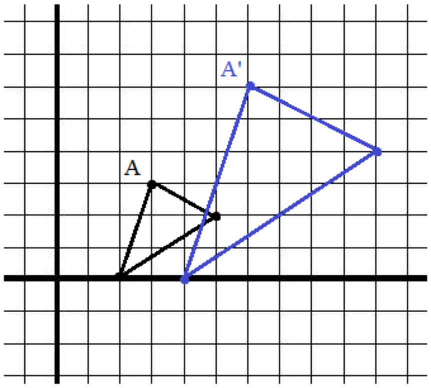
4) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

V. Identifying transformation matrices

For each graph, write the original coordinate matrix M, the transformed matrix M', and the transformed matrix M', and the transformation matrix T.

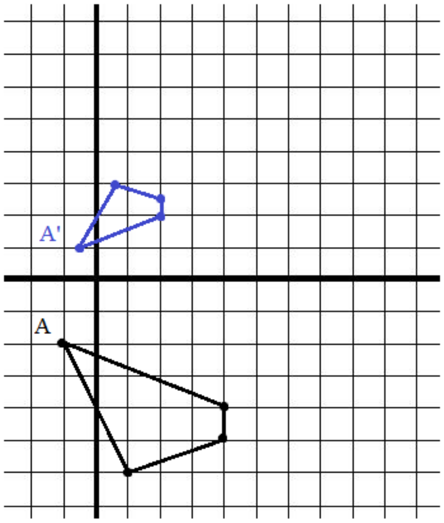
1) M = [ ] M' = [ ]

T = [ ]



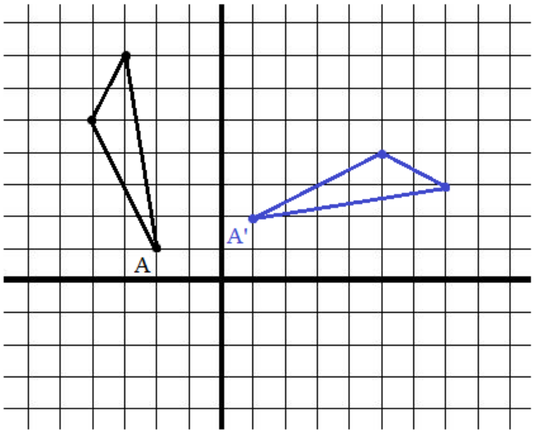
2) M = [ ] M' = [ ]

T = [ ]



3) M = [ ] M' = [ ]

T = [ ]



Matrix Coordinate Geometry Worksheet

SOLUTIONS

I. Mapping

A) Write a 2x4 coordinate matrix representing the polygon ABCD in the xy-plane.

$$\begin{bmatrix} 2 & 3 & 1 & -2 \\ 6 & 0 & -3 & 1 \end{bmatrix}$$

A(2, 6)
B(3, 0)
C(1, -3)
D(-2, 1)

B) In the xy-plane (on the right), graph the triangle whose vertices E, F, and G are expressed by the following (linear) matrix:

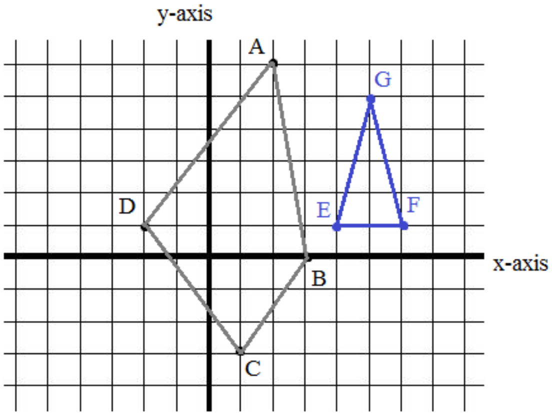
(see graph)

$$\begin{bmatrix} 4 & 6 & 5 \\ 1 & 1 & 5 \end{bmatrix}$$

E(4, 1)
F(6, 1)
G(5, 5)

What type of triangle does the matrix describe?

Isosceles triangle



II. Translation

A) What (movement on a Cartesian Plane) does matrix T represent?

$$AX + T = A'$$

$$\underset{A}{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} + \underset{T}{\begin{bmatrix} 3 & 3 & 3 \\ -4 & -4 & -4 \end{bmatrix}} = \underset{A'}{\begin{bmatrix} a' & b' & c' \\ d' & e' & f' \end{bmatrix}}$$

The figure shifts to the *right 3 units* and *down 4 units*

B) Use a 2x4 coordinate matrix to describe each translation in the graph.

$A + T = A'$

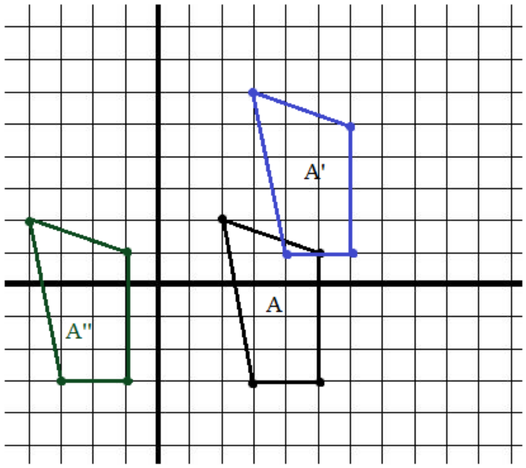
horizontal shift: 1 unit to the right
vertical shift: 4 units up

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

$A + T'' = A''$

$$T'' = \begin{bmatrix} -6 & -6 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

horizontal shift: 6 units to the left
(no vertical shift along the y-axis)



Matrix Coordinate Geometry Worksheet

SOLUTIONS

III. Scalar Transformation

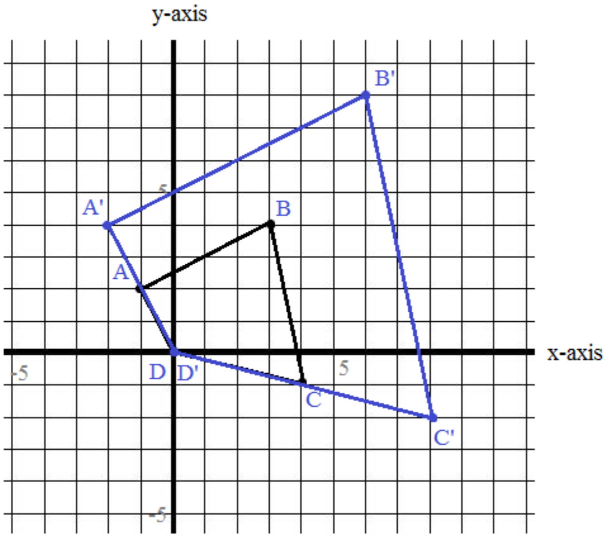
A) Write the 2x4 matrix, listing the four vertices of the shape in the graph.

$$S = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 2 & 4 & -1 & 0 \end{bmatrix}$$

A(-1, 2)
B(3, 4)
C(4, -1)
D(0, 0)

B) What is the matrix 2S?
Map 2S on the graph (on the right).

$$2S = 2\begin{bmatrix} -1 & 3 & 4 & 0 \\ 2 & 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 8 & 0 \\ 4 & 8 & -2 & 0 \end{bmatrix}$$



C) What is the scalar used to transform

1) M $\longrightarrow$ M' ?

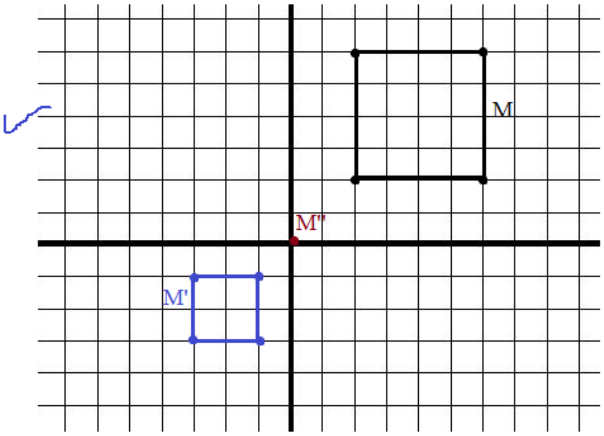
-1/2

$$-1/2\begin{bmatrix} 2 & 2 & 6 & 6 \\ 2 & 6 & 2 & 6 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 & -3 \\ -1 & -3 & -1 & -3 \end{bmatrix}$$ ✓

2) M $\longrightarrow$ M'' ?

0

because $0\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



IV. Reflection

Identify (describe) the linear transformation each matrix performs:

1) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(x, y) $\longrightarrow$ (x, -y)

reflection over the x-axis

2) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(x, y) $\longrightarrow$ (-x, y)

reflection over the y-axis

3) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(x, y) $\longrightarrow$ (-x, -y)

reflection over the origin (or 180° rotation)

4) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

(x, y) $\longrightarrow$ (x, y)

NO transformation (identity matrix)

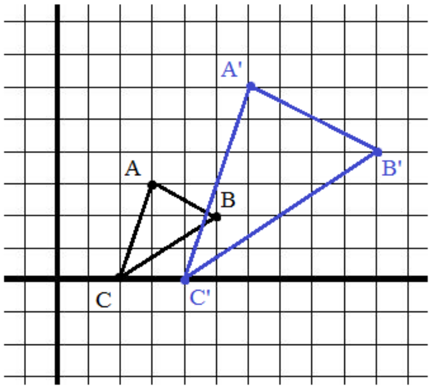
V. Identifying transformation matrices

SOLUTIONS

For each graph, write the original coordinate matrix M, the transformed matrix M', and the transformation matrix T.

1) $M = \begin{bmatrix} 3 & 5 & 2 \\ 3 & 2 & 0 \end{bmatrix}$ $\quad M' = \begin{bmatrix} 6 & 10 & 4 \\ 6 & 4 & 0 \end{bmatrix}$

A(3, 3)
B(5, 2)
C(2, 0)
A'(6, 6)
B'(10, 4)
C'(4, 0)



$$\underset{T}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{M}{\begin{bmatrix} 3 & 5 & 2 \\ 3 & 2 & 0 \end{bmatrix}} = \underset{M'}{\begin{bmatrix} 6 & 10 & 4 \\ 6 & 4 & 0 \end{bmatrix}}$$

$3a + 3b = 6$
$5a + 2b = 10$
$2a + 0b = 4$

$a = 2$
$b = 0$

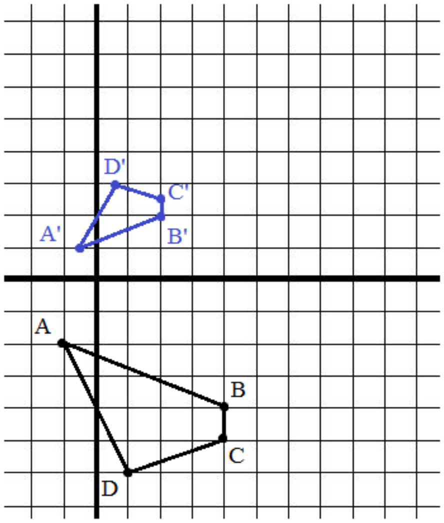
$3c + 3d = 6$
$5c + 2d = 4$
$2c + 0d = 0$

$c = 0$
$d = 2$

note: this is the same as scalar multiplication x 2

$T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

2) $M = \begin{bmatrix} -1 & 4 & 4 & 1 \\ -2 & -4 & -5 & -6 \end{bmatrix}$ $\quad M' = \begin{bmatrix} -1/2 & 2 & 2 & 1/2 \\ 1 & 2 & 5/2 & 3 \end{bmatrix}$



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & 4 & 4 & 1 \\ -2 & -4 & -5 & -6 \end{bmatrix} = \begin{bmatrix} -1/2 & 2 & 2 & 1/2 \\ 1 & 2 & 5/2 & 3 \end{bmatrix}$$

$-a - 2b = -1/2$
$4a - 4b = 2$
$4a - 5b = 2$
$a - 6b = 1/2$

$a = 1/2$
$b = 0$

$-8b = 0 \quad b = 0$

(combination method)

$-c - 2d = 1$
$4c - 4d = 2$
$4c - 5d = 5/2$
$c - 6d = 3$

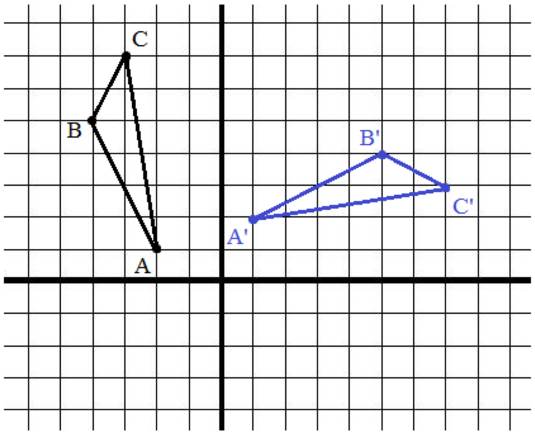
$c = 0$
$d = -1/2$

$-8d = 4 \quad d = -1/2$

$T = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}$

(reduce by 1/2 and reflected over x-axis)

3) $M = \begin{bmatrix} -2 & -4 & -3 \\ 1 & 5 & 7 \end{bmatrix}$ $\quad M' = \begin{bmatrix} 1 & 5 & 7 \\ 2 & 4 & 3 \end{bmatrix}$



$$\overset{T}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{M}{\begin{bmatrix} -2 & -4 & -3 \\ 1 & 5 & 7 \end{bmatrix}} = \overset{M'}{\begin{bmatrix} 1 & 5 & 7 \\ 2 & 4 & 3 \end{bmatrix}}$$

$-2a + b = 1$
$-4a + 5b = 5$
$-3a + 7b = 7$

substitution method

$b = 2a + 1$

$-3a + 7(2a + 1) = 7$
$-3a + 14a + 7 = 7$
$a = 0$
$b = 1$

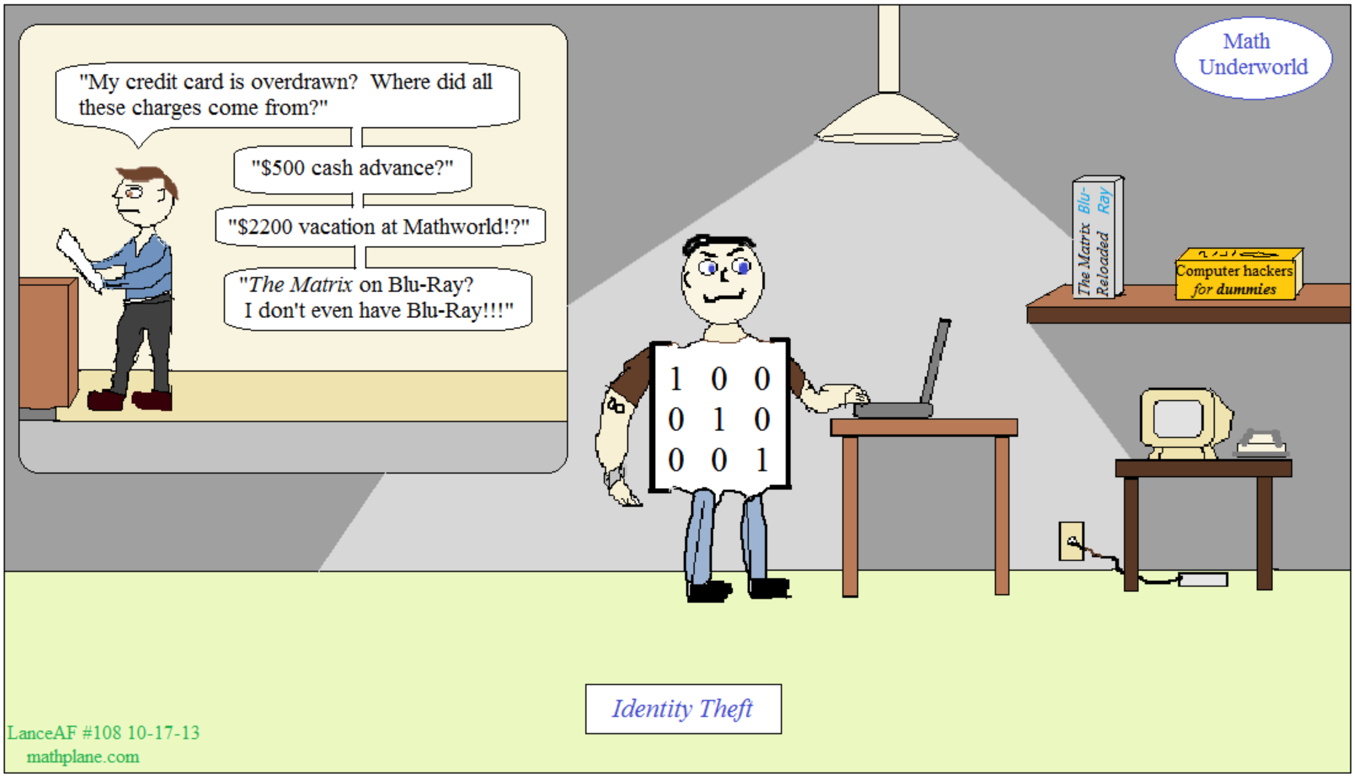
$-2c + d = 2 \longrightarrow d = 2c + 2$
$-4c + 5d = 4$
$-3c + 7d = 3$

$-3c + 7d = 3$
$-3c + 7(2c + 2) = 3$
$11c = -11$
$c = -1$

$d = 0$

$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(90 degree clockwise rotation about the origin)



**Challenge Question:

The *endpoints of a diagonal* of a square drawn in the xy coordinate plane are expressed as

(0, 0) and (-2, 0).

When this square is transformed by a 2x2 matrix, the resulting quadrilateral has coordinates

(0, 0), (5, 1), (6, 4), and (1, 3).

Find the transformation matrix.

Solution on the next page…

Matrix Transformation problem

The endpoints of a diagonal of a square drawn in the x, y coordinate plane are expressed as (-2, 0) and (0, 0). When this square is transformed by a particular 2x2 transformation matrix T, the resulting quadrilateral has coordinates (0, 0) (5, 1) (6, 4) (1, 3). Find the transformation matrix.

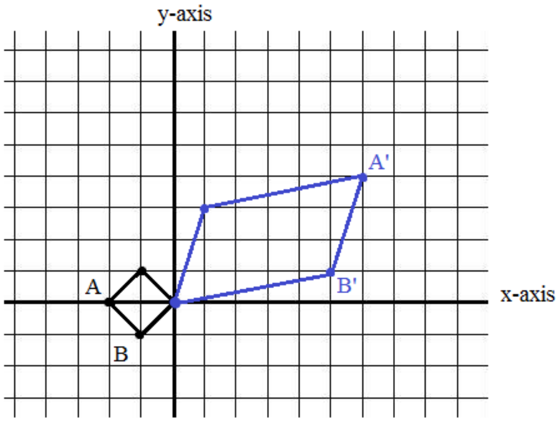
Step 1: Sketch the figures (and, identify coordinates)

1) diagonals of a square are equal and perpendicular; therefore, the other endpoints are (-1, 1) and (-1, -1)

length of each diagonal is 2; they bisect each other at (-1, 0)

2) it appears that the square is "reflected" and "stretched" therefore,

(0, 0) ⟶ (0, 0)
B (-1, -1) ⟶ (5, 1)
A (-2, 0) ⟶ (6, 4)
(-1, 1) ⟶ (1, 3)



Step 2: Express coordinates in matrix form

$$S = \begin{array}{c} \begin{array}{cccc} \quad & B & A & \quad \end{array} \\ \begin{bmatrix} 0 & -1 & -2 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} x \\ y \end{array} \qquad S' = \begin{array}{c} \begin{array}{cccc} \quad & B' & A' & \quad \end{array} \\ \begin{bmatrix} 0 & 5 & 6 & 1 \\ 0 & 1 & 4 & 3 \end{bmatrix} \end{array} \qquad T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

transformation matrix

Step 3: Solve

TS = S'

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 6 & 1 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

T x S (matrix multiplication) S'

row 1/col 1: 0 = 0
row 1/col 2: -a -b = 5
row 1/col 3: -2a + 0 = 6
row 1/col 4: -a + b = 1

row 2/col 1: 0 = 0
row 2/col 2: -c -d = 1
row 2/col 3: -2c + 0 = 4
row 2/col 4: -c + d = 3

(substitution and solve algebraically)

a = -3
b = -2

c = -2
d = 1

Step 4: Check solution

$$T = \begin{bmatrix} -3 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -3 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -2 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\text{square}} = \begin{bmatrix} 0+0 & 3+2 & 6+0 & 3+(-2) \\ 0+0 & 2+(-1) & 4+0 & 2+1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 5 & 6 & 1 \\ 0 & 1 & 4 & 3 \end{bmatrix}}_{\text{transformed quadrilateral}}$$

Finding the area of a polygon:
"Shoelace Method"

Example: The following coordinates are the vertices of a rhombus: (2, 3) (5, 7) (8, 3) (5, -1)...

What is the area?

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 8 & 3 \\ 5 & -1 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 8 & 3 \\ 5 & -1 \\ 2 & 3 \end{bmatrix}$$

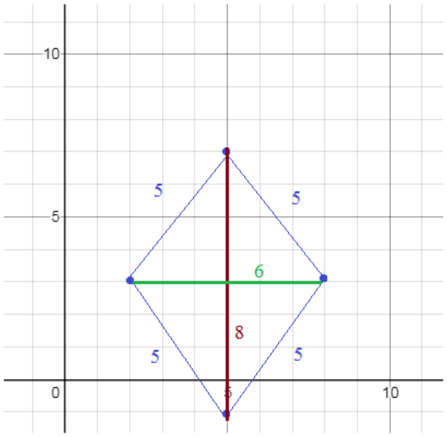
$$2(7) + 5(3) + 8(-1) + 5(3) = 36$$

$$\frac{1}{2}\left| 36 - 84 \right| = 24$$

Using the "shoelace method", we set up a matrix with the vertices listed in consecutive order (and, the first vertex is repeated at the end to close the polygon)

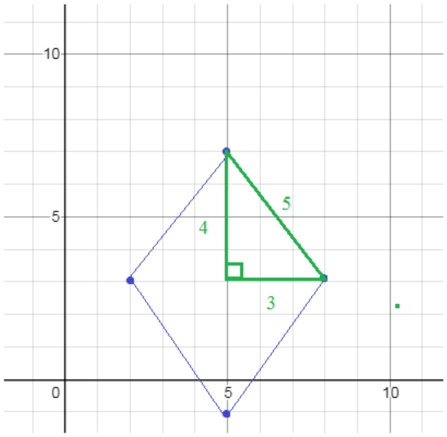
$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 8 & 3 \\ 5 & -1 \\ 2 & 3 \end{bmatrix}$$

$$5(3) + 8(7) + 5(3) + 2(-1) = 84$$



area of rhombus is (1/2)(diagonal 1)(diagonal 2)

(1/2)(6)(8) = 24



The area of 1 triangle = (1/2)(base)(height) = (1/2)(3)(4) = 6

So, area of entire rhombus is 24

Example: Dilate the following equation by a factor of 2

$$y = 2x^2 + 3x + 1$$

Method 1: translate 3 points

Pick 3 points on the curve:		Translate/dilate the 3 points:
(0, 1)	⇨	(0, 2)
(2, 15)	⇨	(4, 30)
(-3, 10)	⇨	(-6, 20)

Determine the equation of the curve going through the 3 points!

$$y = ax^2 + bx + c$$

at (0, 2): $2 = 0a + 0b + c$

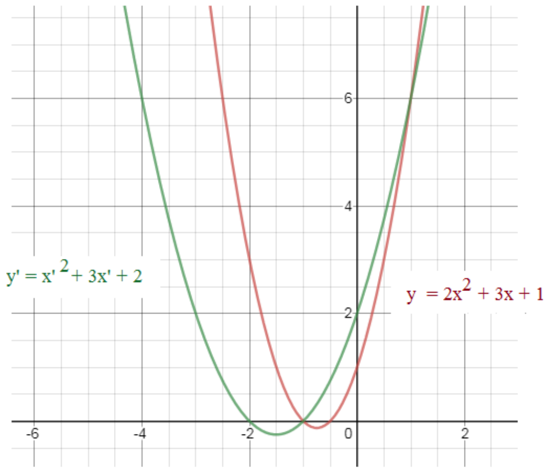
at (4, 30): $30 = 16a + 4b + c$

at (-6, 20): $20 = 36a + -6b + c$

solve the system....

$a = 1 \quad b = 3 \quad c = 2$

$$\Rightarrow \boxed{y = x^2 + 3x + 2}$$



Method 2: Using matrix transformations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and, since $y = 2x^2 + 3x + 1$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ 2x^2 + 3x + 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x \\ 4x^2 + 6x + 2 \end{bmatrix}$$

$y' = 4x^2 + 6x + 2$ and $x' = 2x \Rightarrow$ then, $x = x'/2$

$$\text{so, } y' = 4\left(\frac{x'}{2}\right)^2 + 6\left(\frac{x'}{2}\right) + 2$$

$$\boxed{y' = x'^2 + 3x' + 2}$$