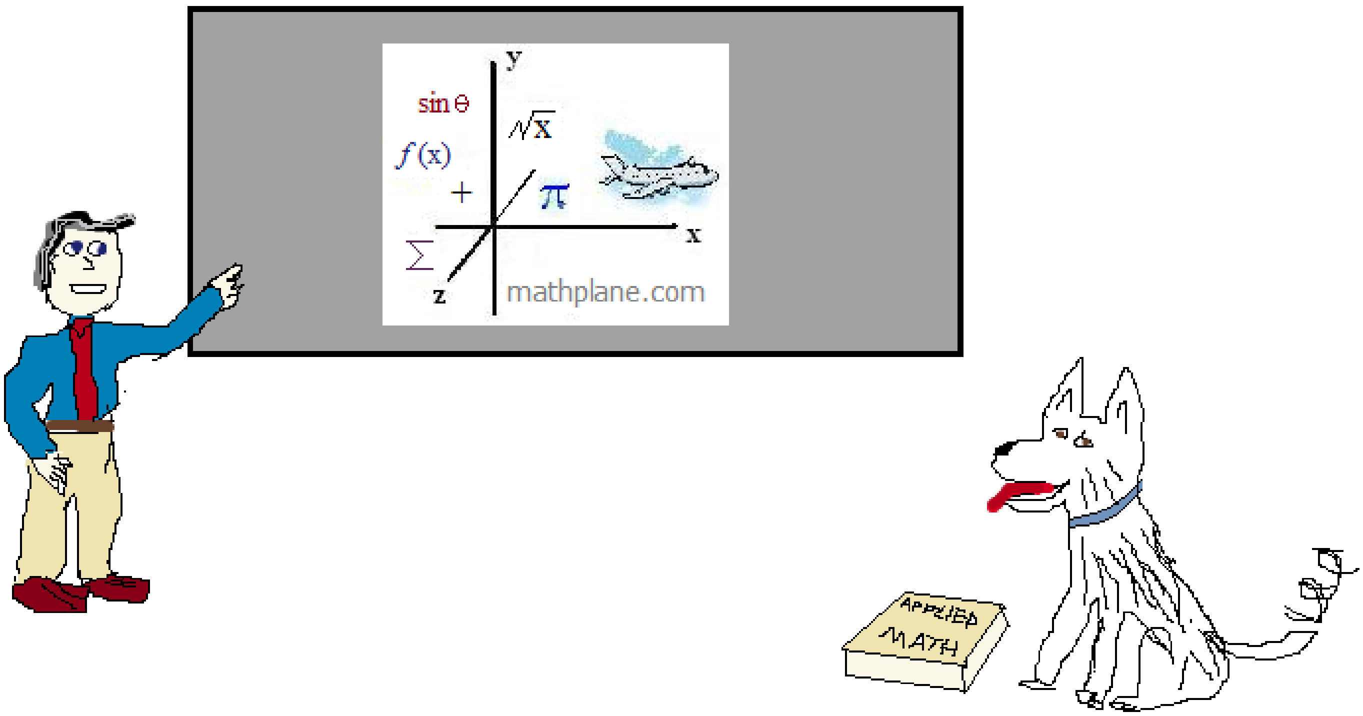
Additional Derivatives: Trigonometry, Logarithms, and Exponents

Formulas, Examples, and Practice Exercises (with Solutions)



Natural Logarithms

Definitions:

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

If u is a differentiable function of x, then

$$\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$$

What does it mean? The derivative of a natural logarithm is $\dfrac{\text{'the derivative of the function'}}{\text{'the function'}}$

Examples: $f(x) = \ln(2x^2 + 4)$

$u = 2x^2 + 4$

(using the above definition) $\dfrac{du}{dx} = 4x$

$$\frac{d}{dx}[\ln(2x^2+4)] = \frac{1}{2x^2+4}(4x)$$

$$= \frac{2x}{x^2+2}$$

Note:

$$f'(x) = \frac{\text{'Derivative of function'}}{\text{'original function'}} = \frac{4x}{(2x^2+4)} = \frac{2x}{x^2+2}$$

$g(x) = \ln\sqrt{x+1}$

"original function" : $\sqrt{x+1}$

"derivative of the function" : $\dfrac{1}{2}(x+1)^{-1/2} = \dfrac{1}{2\sqrt{x+1}}$

$$g'(x) = \frac{\text{'derivative'}}{\text{'original'}} = \frac{\frac{1}{2\sqrt{x+1}}}{\sqrt{x+1}} = \frac{1}{2(x+1)}$$

Trig Functions

Definitions:

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx} \qquad \frac{d}{dx}\cos u = -\sin u\frac{du}{dx} \qquad \frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$$

$$\frac{d}{dx}\csc u = -\csc u(\cot u)\frac{du}{dx} \qquad \frac{d}{dx}\sec u = \sec u(\tan u)\frac{du}{dx} \qquad \frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$$

Examples: $f(x) = \cos 3x^2$

u "function" $= 3x^2$

$\dfrac{du}{dx}$ "derivative of the function" $= 6x$

$$f'(x) = -\sin(3x^2)(6x) = -6x\sin(3x^2)$$

Derivatives: Absolute Value and Exponents

Absolute Value Rule

Definition:

If $y = |u|$, where u is a differentiable function of x, then

$$\frac{d}{dx}[\,|u|\,] = \frac{u}{|u|}\frac{du}{dx}$$

wherever $u(x) \neq 0$

What does it mean?

If you have an equation inside an absolute value, then the derivative is:

$$\frac{\text{equation without the absolute value}}{\text{equation with the absolute value}}\ (\text{derivative of equation})$$

(and, no zeros in the denominator)

Example: $f(x) = |4 - x^2|$

$$f'(x) = \frac{4 - x^2}{|4 - x^2|}(-2x)$$

$f'(x) = (1)(-2x) = -2x$ when $4 - x^2 > 0$

$f'(x) = (-1)(-2x) = 2x$ when $4 - x^2 < 0$

And, the derivative does not exist at $x = \pm 2$

Exponential Functions

Definitions:

$$\frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}[a^x] = a^x(\ln a)$$

$$\frac{d}{dx}[e^u] = e^u\frac{du}{dx} \qquad \frac{d}{dx}[a^u] = a^u(\ln a)\frac{du}{dx}$$

What does it mean?

The derivative of 'e to an exponent' is "e to the exponent times the derivative of the exponent."

The derivative of 'number to an exponent' is "number to an exponent times ln(number) times the derivative of the exponent."

Examples:

$f(x) = e^{3x - 1}$

$f'(x) = e^{3x - 1}(3) = 3e^{3x - 1}$

$g(x) = 10^{x^2}$

$g'(x) = 10^{x^2}\ln(10)\ (2x)$

Reminder: $e \cong 2.718$

$\ln(e) = \log_e e = 1$

$\ln(4) = \log_e 4 =$ (approx.) 1.386

Derivatives of Trig Functions

Examples:

$g(x) = \sin\sqrt{2x}$

function: $\sqrt{2x}$

derivative of the function: $\frac{1}{2}(2x)^{\frac{-1}{2}}(2) = \frac{1}{\sqrt{2x}}$

$$g'(x) = \cos\sqrt{2x}\left(\frac{1}{\sqrt{2x}}\right) = \frac{\cos\sqrt{2x}}{\sqrt{2x}}$$

$h(x) = \sec^3(2x)$ Let $u = 2x$

$u' = 2$

$h(x) = \sec(u)^3$

$h'(x) = 3\sec(u)^2 \cdot \sec u(\tan u)u'$

$= 3\sec(2x)^2 \cdot \sec(2x)\tan(2x)(2)$

$= 6\tan(2x)\sec^3(2x)$

Note: according to the power rule, derivative of $[\sec(u)]^3$ is $3[\sec(u)]^2 \cdot$ ("derivative of sec(u)")

Find the relative extrema of $f(x) = \sin 2x + 4$ on the interval $[0, 2\pi)$

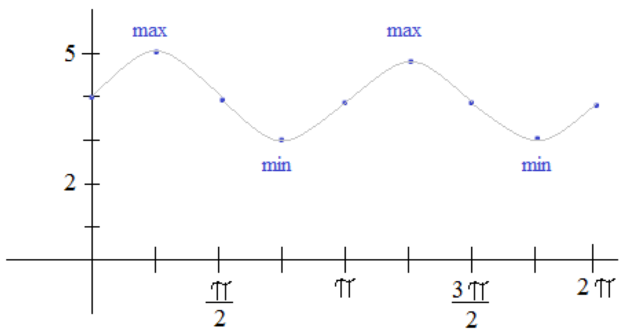
Find $f'(x)$: $\cos(2x)(2) + 0 = 2\cos(2x)$

To find the max/min, set $f'(x) = 0$

$2\cos(2x) = 0$

$\cos(2x) = 0$

$x = \frac{\pi}{4} \quad \frac{3\pi}{4} \quad \frac{5\pi}{4} \quad \frac{7\pi}{4}$



Also, $f''(x) = 0$ will help identify concavity and points of inflection.

Find derivative of $2\cos(2x)$: $2 \cdot [-\sin(2x)(2)] = -4\sin(2x)$

To find points of inflection, set second derivative equal to zero.

$-4\sin(2x) = 0$

$\sin(2x) = 0$ points of inflection at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Derivatives: Natural Logarithms

Example: $y = xe^2 - e^x$ Find y'

(product rule)

$y' = (1)e^2 + (0)e^2(x) - e^x$

derivative of e^u = "derivative of u" $\cdot\ e^u$

$y' = e^2 - e^x$

Example: $y = \ln\frac{1}{x}$ Find dy/dx

method 1:

$$\frac{dy}{dx} = \frac{-1(x)^{-2}}{\frac{1}{x}} = \frac{x}{-x^2} = \frac{-1}{x}$$

derivative of ln: $\dfrac{\text{"derivative of equation"}}{\text{"equation"}}$

method 2: (applying log rules)

$y = \ln(x)^{-1}$

$y = -\ln(x)$ (logarithm power rule)

$$\frac{dy}{dx} = \frac{-1}{x}$$

Example: "The speed of a rumor"

A rumor can be modeled by the function $p(t) = \dfrac{340}{1 + 2^{4-t}}$ where t is time (hours) and p(t) is the number of people who know about the rumor

a) How many students originally heard the rumor?

$$p(0) = \frac{340}{1 + 2^{4-0}} = \frac{340}{17} = 20 \text{ students}$$

t	0	1	2	3	4	5	6
p(t)	20	38	68	113	170	227	272

b) How fast is the rumor spreading after 4 hours?

Find p'(t) to find the rate of change.

Using Quotient rule:

$$p'(t) = \frac{0 - 2^{4-t}(\ln 2)(-1)(340)}{(1 + 2^{4-t})^2}$$

Then, $$p'(4) = \frac{0 - 2^0(\ln 2)(-340)}{(1 + 2^0)^2} = \frac{(\ln 2)(340)}{4} = 58.9$$

Approximately, 59 students/hour

Derivatives: More Examples

Example: $\sin(x) = e^y$ Find $\frac{dy}{dx}$

Method 1: implicit differentiation

$$\cos(x) = e^y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cos(x)}{e^y}$$

$$\sin(x) = e^y$$

$$\frac{dy}{dx} = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

Method 2: logarithm properties

Convert exponential into logartihm form

$$y = \ln(\sin(x))$$

$$\frac{dy}{dx} = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

Example: $y = \log_6(3x)$

Method 1: implicit differentiation

$$6^y = 3x$$

$$\ln(6) \cdot 6^y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\ln(6) \cdot 6^y}$$

$$\frac{dy}{dx} = \frac{3}{\ln(6) \cdot (3x)}$$

Method 2: Apply the formula

$$y = \log_6(3x)$$

$$\frac{dy}{dx} = \frac{\text{"Derivative"}}{\text{"ln of the base"} \quad \text{"itself"}}$$

$$\frac{dy}{dx} = \frac{3}{\ln(6) \cdot (3x)}$$

Example: 6^{5x}

Method 1: Apply formula

"ln of the base" "itself" "derivative"

$$\ln(6) \cdot 6^{5x} \cdot 5$$

Method 2: Use Logrithmic differentiation

$$y = 6^{5x}$$

$$\ln y = \ln 6^{5x}$$

$$\ln y = 5x(\ln 6)$$

derivative of both sides

$$\frac{1}{y}\frac{dy}{dx} = 5 \ln 6$$

$$\frac{dy}{dx} = \ln 6 \cdot 5 \cdot y$$

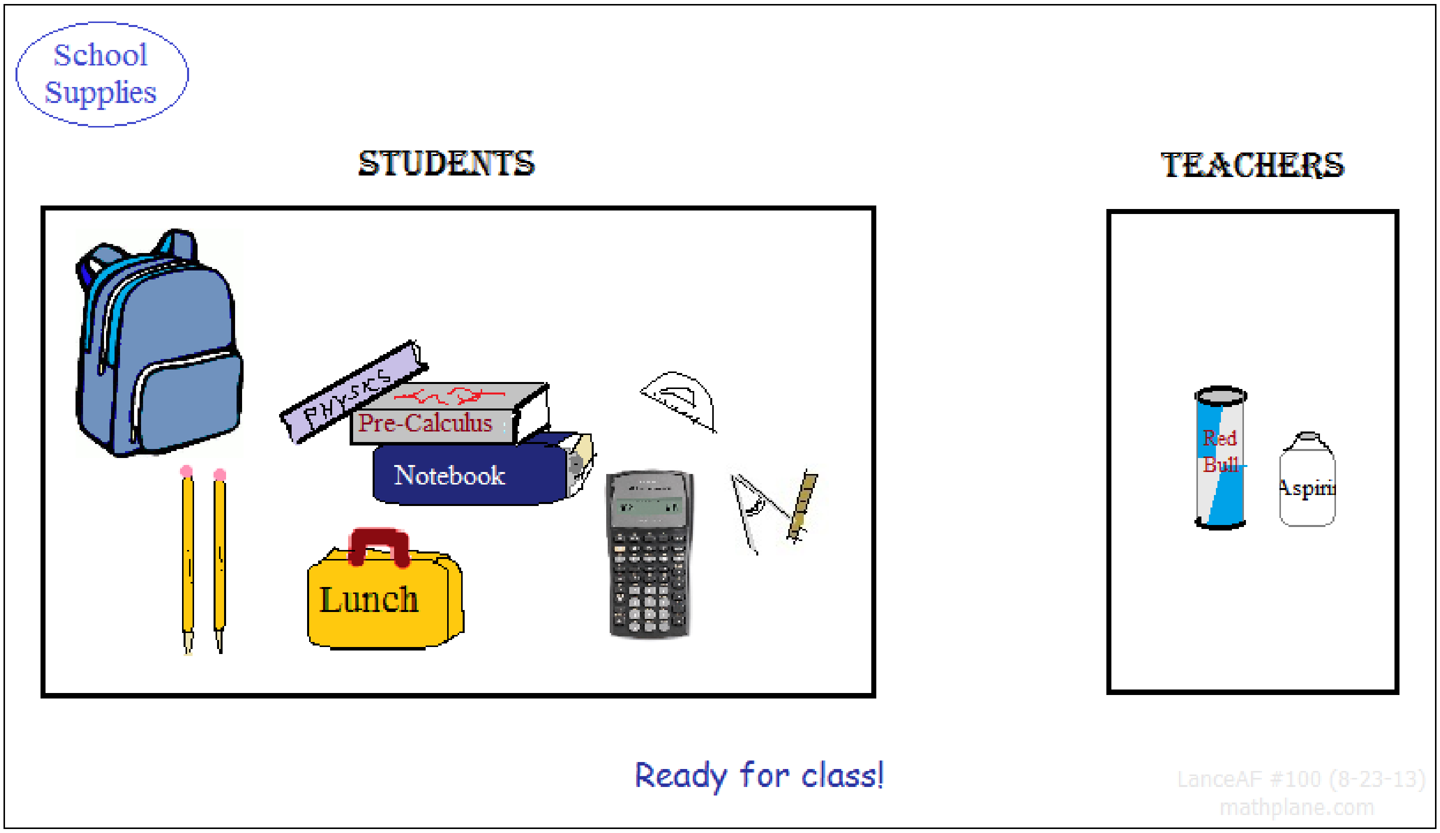
Example: $ln\left(\frac{x^2\sqrt{x^2+9}}{\sin 2x}\right)$

Utilize log properties...

$$lnx^2 + \frac{1}{2} ln(x^2+9) - ln\sin 2x$$

$$\frac{2x}{x^2} + \frac{1}{2}\frac{(2x+0)}{(x^2+9)} - \frac{2\cos 2x}{\sin 2x}$$

$$\frac{2}{x} + \frac{x}{(x^2+9)} - 2\cot(2x)$$



Practice Exercises-→

Differentiation: Trigonometry, logarithm, and exponent functions

Find the 1st derivative of each function.

I. Trigonometry:

a) $f(x) = 2x(\cos x)$

b) $y = \tan^4(3 + 4x)$

c) $g(x) = \dfrac{1 - \sin x}{1 + \sin x}$

II. Logarithms:

a) $y = \ln(x^2)$

b) $h(x) = \dfrac{\ln x}{x^2}$

c) $f(x) = \ln\sqrt{x^2 - 4}$

III. Exponents:

a) $y = x^2 e^x$

b) $y = 3^{x-1}$

c) $f(x) = 4e^{2x}$

IV: More Questions

Differentiation: Trigonometry, logarithm, and exponent functions

a) Find the 1st and 2nd derivatives of the following:

$\sin^2 x$

$\sin(x^2)$

$\sin\sqrt{1 + x^2}$

b) $f(x) = 2\cos^3\left(\frac{3x}{4} + \frac{\pi}{12}\right)$

$f'\left(\frac{11\pi}{9}\right) =$

c) $y = e^x(\sin x + \cos x)$

$\frac{dy}{dx} =$

d) If $y = (\cos x)(\sin x)$, what is the slope of the graph at $\frac{\pi}{3}$

e) $y = x^n + n^x$

find dy/dx

f) Find the equation of the line tangent to $y = \dfrac{e^x}{2x + 1}$ @ $x = 0$

g) Find the equation of the line tangent to $f(x) = \cos^2(2x^3)$ @ $x = \dfrac{\pi}{3}$

h) $y = \sin(x^2)\cos(y)$ dy/dx =

i) $y = \sec^2(5x)$ y' =

j) $y = 5^{(2x - 3)}$ dy/dx =

k) $y = \ln(\cos 2x)$ find the first derivative.

V. Derivatives of inverses

a) If $f(x) = x^3 + x^2 + x + 3$ and $g(x) = f^{-1}(x)$

what is the value of $g'(6)$?

b) g is differentiable and $g(x) = f^{-1}(x)$ for all x

$f(-4) = 12$ $f(9) = -4$ $f'(4) = -6$ $f'(9) = 3$

what is $g'(-4)$?

a) 1/3

b) -1/4

c) 1/9

d) -1/6

e) need more information

c) Find the derivative of $3\cos^{-1}\left(\frac{x}{2}\right)$

Differentiation: Trigonometry, logarithm, and exponent functions

Find the 1st derivative of each function.

Answers

I. Trigonometry:

a) $f(x) = 2x(\cos x)$

(product rule) $u = 2x$, $u' = 2$, $v = \cos x$, $v' = -\sin x$

$f'(x) = 2\cdot(\cos x) + (-\sin x)\cdot 2x$

$= 2\cos x - 2x(\sin x)$

$= \boxed{2(\cos x - x\sin x)}$

b) $y = \tan^4(3 + 4x)$

$= [\tan(3 + 4x)]^4$

(power rule) $u = \tan(3 + 4x)$

$u' = \sec^2(3 + 4x)(4)$

$\frac{dy}{dx} = 4\tan^3(3 + 4x)\cdot[\sec^2(3 + 4x)(4)]$

$= \boxed{16\tan^3(3 + 4x)\sec^2(3 + 4x)}$

c) $g(x) = \frac{1 - \sin x}{1 + \sin x}$

(quotient rule) $u = 1 - \sin x$, $u' = -\cos x$, $v = 1 + \sin x$, $v' = \cos x$

$g'(x) = \frac{(-\cos x)(1 + \sin x) - (\cos x)(1 - \sin x)}{(1 + \sin x)^2}$

$= \frac{-\cos x - \cos x\sin x - \cos x + \cos x\sin x}{(1 + \sin x)^2}$

$= \boxed{\frac{-2\cos x}{(1 + \sin x)^2}}$

II. Logarithms:

a) $y = \ln(x^2)$

derivative of natural log: $\frac{\text{"derivative of funtion"}}{\text{"original function"}}$

$u = x^2$, $u' = 2x$

$y' = \frac{2x}{x^2} = \boxed{\frac{2}{x}}$

b) $h(x) = \frac{\ln x}{x^2}$

(quotient rule) $u = \ln x$, $u' = 1/x$, $v = x^2$, $v' = 2x$

$h'(x) = \frac{\frac{1}{x}\cdot x^2 - 2x(\ln x)}{(x^2)^2}$

$= \frac{x - 2x(\ln x)}{x^4}$

$= \boxed{\frac{1 - 2(\ln x)}{x^3}}$

c) $f(x) = \ln\sqrt{x^2 - 4}$

$u = \sqrt{x^2 - 4}$

$\frac{du}{dx} = \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2 - 4}}$

$f'(x) = \frac{\frac{x}{\sqrt{x^2 - 4}}}{\sqrt{x^2 - 4}} = \boxed{\frac{x}{x^2 - 4}}$

III. Exponents:

a) $y = x^2 e^x$

(product rule)

$u = x^2$, $u' = 2x$, $v = e^x$, $v' = e^x$

$\frac{dy}{dx} = 2x(e^x) + e^x(x^2)$

$= \boxed{xe^x(x + 2)}$

b) $y = 3^{x-1}$

$a = 3$, $u = x - 1$, $u' = 1$

$y' = 3^{x-1}(\ln 3)(1)$

$= \boxed{\ln 3(3^{x-1})}$

c) $f(x) = 4e^{2x}$

$f'(x) = 4[e^{2x}](2)$

$= \boxed{8e^{2x}}$

IV: More Questions

SOLUTIONS

Differentiation: Trigonometry, logarithm, and exponent functions

a) Find the 1st and 2nd derivatives of the following:

$\sin^2 x$

$(\sin x)^2$

$y' = 2(\sin x)^1 \cdot \cos x$

$= \sin 2x$

$y'' = 2\cos 2x$

also, $y' = 2\sin x\cos x$

$y'' = 2[\cos x\cos x + (-\sin x)\sin x]$

$2[\cos^2 x - \sin^2 x]$

$2\cos 2x$

$\sin(x^2)$

$y' = \cos(x^2) \cdot 2x$

$= 2x\cos(x^2)$

$y'' = 2 \cdot \cos(x^2) + -\sin(x^2)2x \cdot 2x$

$= 2\cos(x^2) - 4x^2\sin(x^2)$

$\sin\sqrt{1+x^2}$

$y' = \cos(\sqrt{1+x^2}) \cdot \frac{1}{2}(1+x^2)^{\frac{-1}{2}} \cdot 2x$

$= \dfrac{x\cos(\sqrt{1+x^2})}{\sqrt{1+x^2}}$

$y'' = \dfrac{1 \cdot \cos(\sqrt{1+x^2}) - x \cdot \left(-\sin(\sqrt{1+x^2}) \cdot \frac{1}{2}(1+x^2)^{\frac{-1}{2}} \cdot 2x\right)}{(\sqrt{1+x^2})^2}$

$= \dfrac{\cos(\sqrt{1+x^2}) + x^2\sin(\sqrt{1+x^2})(1+x^2)^{\frac{-1}{2}}}{1+x^2}$

$\dfrac{\cos(\sqrt{1+x^2}) + \dfrac{x^2\sin(\sqrt{1+x^2})}{\sqrt{1+x^2}}}{1+x^2}$

b) $f(x) = 2\cos^3\left(\frac{3x}{4} + \frac{\pi}{12}\right)$

$f'\left(\frac{11\pi}{9}\right) =$

$f(x) = 2\left[\cos\left(\frac{3x}{4} + \frac{\pi}{12}\right)\right]^3$

$f'(x) = 6\cos\left(\frac{3x}{4} + \frac{\pi}{12}\right)^2 \cdot \left[-\sin\left(\frac{3x}{4} + \frac{\pi}{12}\right) \cdot \frac{3}{4}\right]$

$f'\left(\frac{11\pi}{9}\right) = 6\cos^2\left(\frac{33\pi}{36} + \frac{\pi}{12}\right) \cdot \left[-\sin\left(\frac{33\pi}{36} + \frac{\pi}{12}\right) \cdot \frac{3}{4}\right]$

$6 \cdot (-1)^2 \quad \cdot \quad 0 \quad \cdot \frac{3}{4} = 0$

c) $y = e^x(\sin x + \cos x)$

$\frac{dy}{dx} =$

Use the product rule:

$y' = e^x(\sin x + \cos x) + e^x(\cos x + (-\sin x))$

u' v u v'

$e^x\sin x + e^x\cos x + e^x\cos x - e^x\sin x$

$2e^x\cos x$

d) If $y = (\cos x)(\sin x)$, what is the slope of the graph at $\frac{\pi}{3}$

$y' = (-\sin x)(\sin x) + (\cos x)(\cos x)$

(product rule)

at $\frac{\pi}{3}$ slope is $(-\sin\frac{\pi}{3})(\sin\frac{\pi}{3}) + (\cos\frac{\pi}{3})(\cos\frac{\pi}{3})$

$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2}$ $= 1$

e) $y = x^n + n^x$

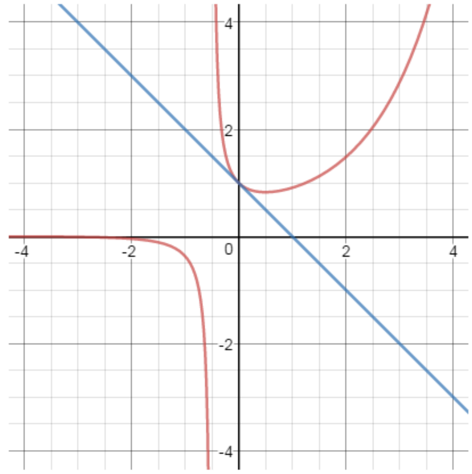
find dy/dx

$dy/dx = n(x)^{n-1} + \ln(n)n^x$

power rule — logarithm/exponent rule

f) Find the equation of the line tangent to $y = \dfrac{e^x}{2x+1}$ @ x = 0

SOLUTIONS



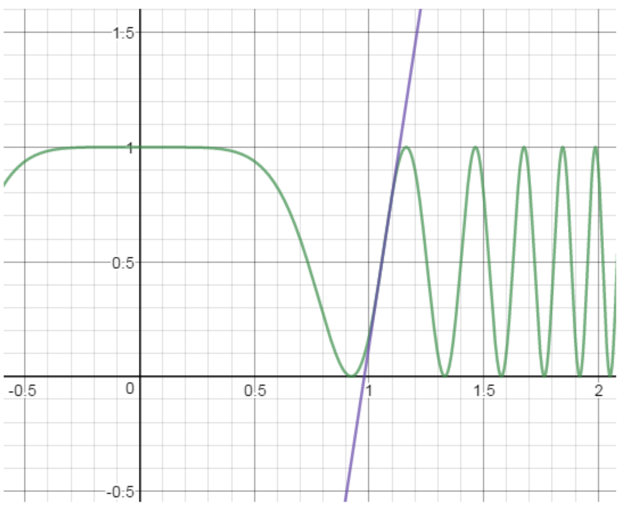
The coordinate of the point is $y = \dfrac{e^0}{2(0)+1} = \dfrac{1}{1} = 1$

The slope at that point is

$$y' = \frac{e^x(2x+1) - 2(e^x)}{(2x+1)^2} \quad \overset{@\ x = 0}{\Longrightarrow} \quad \frac{1-2}{1} = -1$$

$$y - 1 = -1(x - 0) \Longrightarrow \boxed{y = -x + 1}$$

g) Find the equation of the line tangent to $f(x) = \cos^2(2x^3)$ @ $x = \dfrac{\pi}{3}$



$[\cos(2x^3)]^2$

$2[\cos(2x^3)]^1 \cdot -\sin(2x^3) \cdot 6x^2$

at $x = \dfrac{\pi}{3}$ $\quad f'(x) = 6.533$

and, at $\dfrac{\pi}{3}$ $\quad f(x) = .441$

slope is 6.533

the point is $\left(\dfrac{\pi}{3}, .441\right)$

$$\boxed{y - .441 = 6.533\left(x - \frac{\pi}{3}\right)}$$

h) $y = \sin(x^2)\cos(y)$ $\quad$ dy/dx =

$$(1)\frac{dy}{dx} = \cos(x^2) \cdot 2x \cdot \cos(y) + \left(-\sin y \frac{dy}{dx}\right) \cdot \sin(x^2)$$

$$\frac{dy}{dx} + \sin(x^2)\sin y \frac{dy}{dx} = \cos(x^2) \cdot 2x \cdot \cos(y)$$

$$\frac{dy}{dx} = \boxed{\frac{\cos(x^2) \cdot 2x \cdot \cos(y)}{1 + \sin(x^2)\sin y}}$$

product rule
chain rule
implicit differentiation
trig derivatives

i) $y = \sec^2(5x)$ $\quad$ y' =

$2\sec(5x) \cdot \sec(5x)\tan(5x) \cdot 5$

$\boxed{10\sec^2(5x)\tan(5x)}$

power rule
chain rule
trig derivatives

j) $y = 5^{(2x-3)}$ $\quad$ dy/dx =

$$\boxed{\frac{dy}{dx} = \ln 5 \cdot 5^{(2x-3)} \cdot 2}$$

"ln of the base" "itself" "derivative of the exponent"

k) $y = \ln(\cos 2x)$ $\quad$ find the first derivative.

"derivative"

$$y' = \frac{-\sin(2x) \cdot 2}{\cos(2x)} = \boxed{-2\tan(2x)}$$

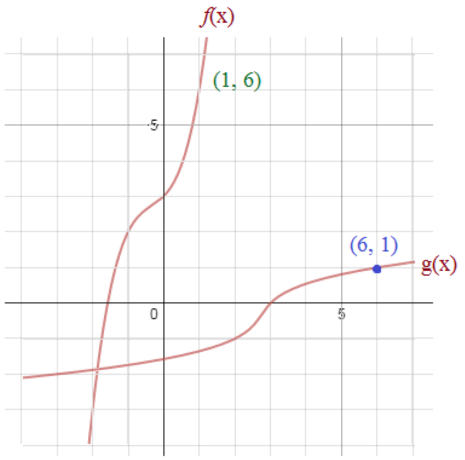
"itself"

V. Derivatives of inverses

SOLUTIONS

a) If $f(x) = x^3 + x^2 + x + 3$ and $g(x) = f^{-1}(x)$

what is the value of $g'(6)$?



Inverses reflect over the line $y = x$, and the coordinates are reversed...

$$g'(x) = \frac{1}{f'(y)}$$

Slope at (6, 1) is the reciprocal of the slope at (1, 6)

For $y = 6$,

$$6 = x^3 + x^2 + x + 3, \qquad x = 1$$

So, the slope at (1, 6) will be the reciprocal of the slope at (6, 1)!

$$f'(x) = 3x^2 + 2x + 1 + 0$$

then, $f'(1) = 3 + 2 + 1 = 6$

therefore, $g'(6) = \dfrac{1}{f'(1)} = \boxed{\dfrac{1}{6}}$

b) g is differentiable and $g(x) = f^{-1}(x)$ for all x

$f(-4) = 12$ $\quad f(9) = -4$ $\quad f'(4) = -6$ $\quad f'(9) = 3$

what is $g'(-4)$?

a) 1/3

b) -1/4

c) 1/9

d) -1/6

e) need more information

g(x) and f(x) are inverses...

So, if $f(9) = -4$, then $g(-4)$ must equal 9.

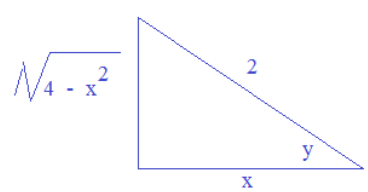
Then, $f'(9) = 3$... therefore, the slope of the inverse $g'(-4) = 1/3$

c) Find the derivative of $3\cos^{-1}\left(\frac{x}{2}\right)$

since 3 is a constant, we'll ignore that for now...

$$\cos^{-1}\left(\frac{x}{2}\right) = y$$

$$\cos y = \frac{x}{2}$$



using implicit diff...

$$-\sin y \frac{dy}{dx} = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}(-\csc y)$$

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{-2}{\sqrt{4 - x^2}}$$

simplify and bring in the constant

$$\boxed{\frac{dy}{dx} = \frac{-3}{\sqrt{4 - x^2}}}$$

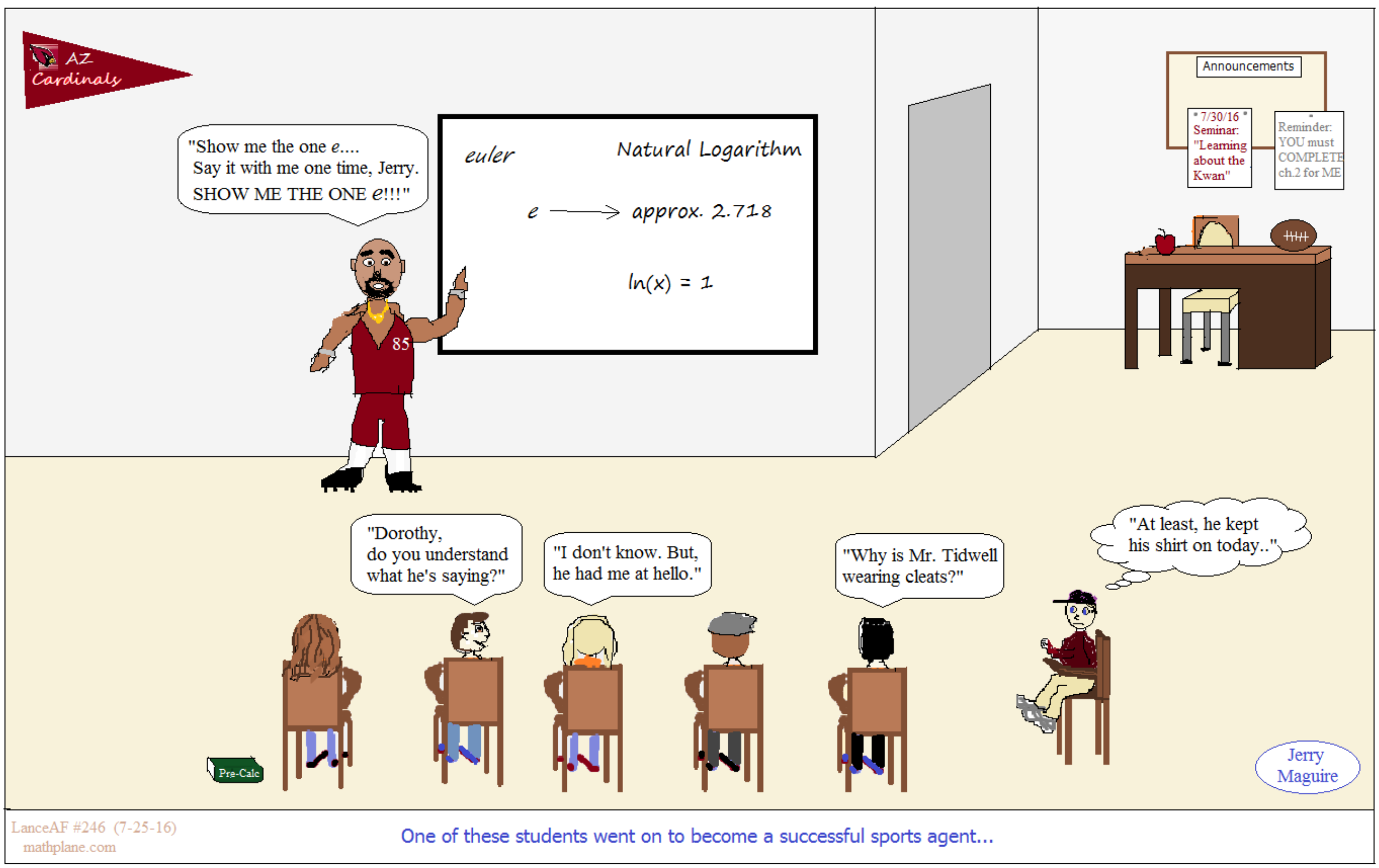
Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers

Also, at Mathplane *Express* for mobile at Mathplane.ORG

And, stores at TeachersPayTeachers and TES...



One of these students went on to become a successful sports agent...