Algebra 2: Parametric Topics

Topics include conics, lines, trig identities, graphing, projectile models, and more.

Brief introduction to Parametric Equations

Equations are often expressed with 2 variables: a dependent variable and an independent variable..

For example, $y = 3x + 5$ For every change in x (the independent variable), y (the dependent variable) increases by 3 units...

The movement of x and y are related directly to each other..

Suppose we introduce a 3rd component, t.. As t moves, x moves in one way and y moves in another way.

Note: x and y are *directly* related to t.. And, they are *indirectly* related to each other..

Application: A plane is flying east at 500 miles per hour. And, the wind is blowing north at 50 miles per hour. Use parametric equations to express the direction of the plane relative to time (t).

$$x = 500t$$
$$y = 50t$$
$$t \geq 0$$

Every 1 hour (t increases by 1), the plane moves 500 miles in (x) direction and 50 miles in (y) direction.

Change parametric equation of line to symmetric equation:

Example:

$$x = 2t + 4 \qquad t = \frac{x - 4}{2}$$

$$y = 3t \qquad \text{solve each equation for t} \qquad t = \frac{y}{3} \qquad \text{Substitution} \qquad \frac{x-4}{2} = \frac{y}{3} = \frac{z+7}{2}$$

$$z = 2t - 7 \qquad t = \frac{z+7}{2}$$

Change $y = f(x)$ into a parametric ('parameterize' the function)

let $x = t$

then, $y = f(t)$

Example: $y = x^2$

$x = t$

$y = t^2$

reminder: in some instances, the domain of t must be defined

Example: Write the equation of a line that passes through (2, 5) and (-3, 4) in parametric form.

$$\text{slope} = \frac{4 - 5}{-3 - 2} = \frac{1}{5}$$

During every increment (t), the y component moves +1, and the x component moves +5 units.

$x = \quad + 5t$

$y = \quad + 1t$

Then, we can choose either point to finish the parametric equations

$x = 2 + 5t$

$y = 5 + t$

Quick check: at $t = 0$, $(x, y) = (2, 5)$ ✓

at $t = -1$, $(x, y) = (-3, 4)$ ✓

Parametric Notes : Shortcuts and formulas

Circle: center: (h, k)

$x = h + r\cos\theta$

$y = k + r\sin\theta$

Example: A circle with center (3, −9) and radius 4

$x = 3 + 4\cos\theta$

$y = -9 + 4\sin\theta$

Line: slope: $\frac{\Delta y}{\Delta x}$

Example: (3, 1) (−2, 11)

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{10}{-5} = \frac{2}{-1}$$

$x = -1t$

$y = 2t$

then, add a point on the line...

$x = -1t + 3$ $\quad$ or $\quad$ $x = -1t - 2$

$y = 2t + 1$ $\quad\quad\quad\;$ $y = 2t + 11$

Parametric Notes: Conics

Example: A circle with center (3, −7) is tangent to the x-axis. Write a parametric equation of the circle.

radius: 7 units
center: (3, -7)

$$(x-3)^2 + (y+7)^2 = 49$$

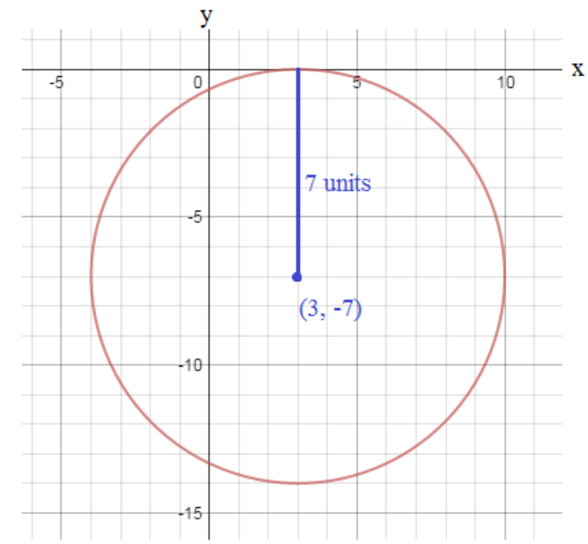
$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{(x-3)^2}{49} + \frac{(y+7)^2}{49} = 1$$

$$\cos^2\theta = \frac{(x-3)^2}{49} \qquad \sin^2\theta = \frac{(y+7)^2}{49}$$

$$\cos\theta = \frac{(x-3)}{7} \qquad \sin\theta = \frac{(y+7)}{7}$$

$$x = 7\cos\theta + 3$$
$$y = 7\sin\theta - 7$$



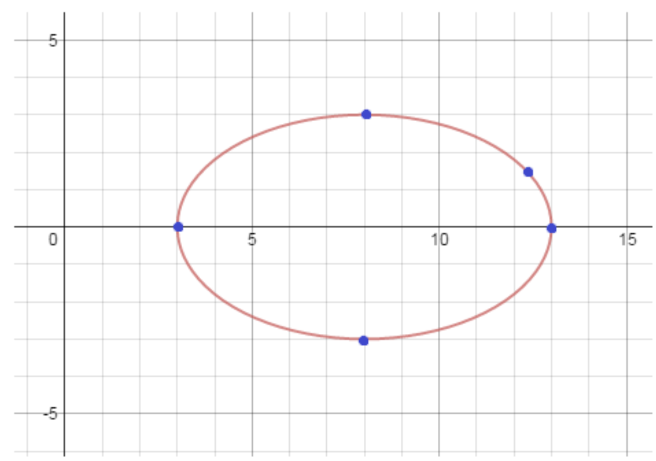
Quick check:

θ	x	y
0	10	-7
90	3	0
180	-4	-7
270	3	-14

Example: $x = 5\cos\theta + 8$
$y = 3\sin\theta$

a) Plot 5 points on an xy-coordinate plane

θ	x	y	
0	13	0	
30	$5\sqrt{3}/2 + 8$	3/2	(12.33, 1.5)
90	8	3	
180	3	0	
270	8	-3	



b) Convert to a rectangular equation

$$\cos\theta = \frac{(x-8)}{5} \qquad \sin\theta = \frac{y}{3}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\left(\frac{(x-8)}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{(x-8)^2}{25} + \frac{y^2}{9} = 1$$

Example: The equations of two intersecting planes are

$$2x + 3y - z = 8$$
$$x - 5y + 2z = -1$$

Use a parametric expression to describe the line of intersection...

Step 1: Combine the equations to simplify

$$2x + 3y - z = 8$$
$$x - 5y + 2z = -1 \quad\rightarrow\quad -2x + 10y - 4z = 2$$

$$13y - 5z = 10 \qquad y = \frac{5z + 10}{13}$$

Step 2: "Parameterize" one of the variables and substitute

$$2x + 3y - z = 8$$

Let $2x = t$

$$\boxed{x = \frac{t}{2}}$$

$$t + 3y - z = 8$$
$$t + 3\left(\frac{5z + 10}{13}\right) - z = 8$$
$$t + \frac{15z}{13} + \frac{30}{13} - \frac{13z}{13} = 8$$
$$\frac{2z}{13} = 8 - \frac{30}{13} - t$$
$$2z = 104 - 30 - 13t$$
$$\boxed{z = 37 - \frac{13}{2}t}$$

$$y = \frac{5z + 10}{13}$$
$$y = \frac{5(37 - \frac{13}{2}t) + 10}{13}$$
$$y = \frac{195 - \frac{65}{2}t}{13}$$
$$\boxed{y = 15 - \frac{5}{2}t}$$

Step 3: Check your solutions

$$2x + 3y - z = 8$$
$$x - 5y + 2z = -1$$

Let $t = 0$

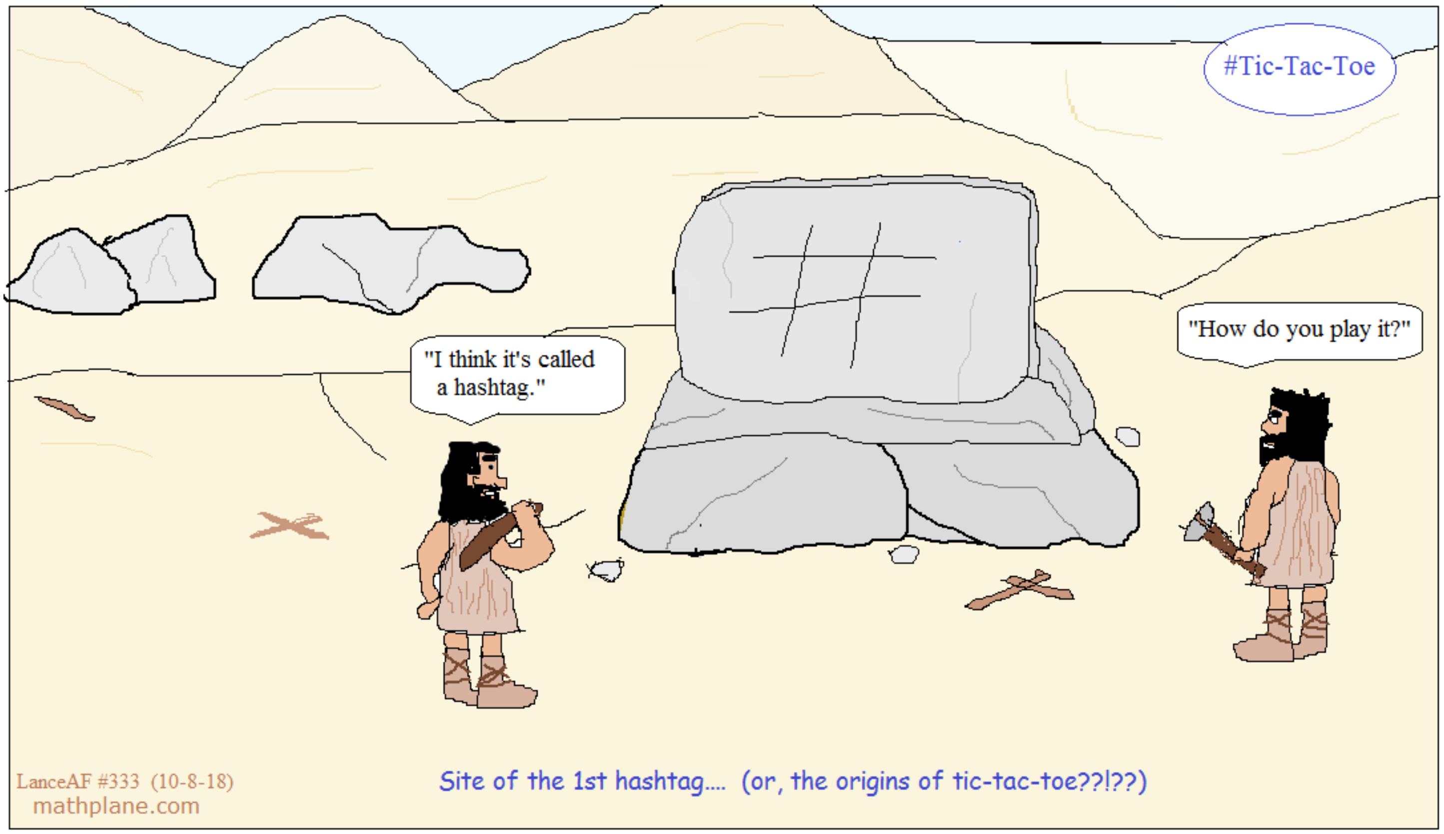
$x = 0$
$y = 15$
$z = 37$

$$2(0) + 3(15) - (37) = 8$$
$$8 = 8 \checkmark$$
$$(0) - 5(15) + 2(37) = -1$$
$$-1 = -1 \checkmark$$

then, let $t = 1$

$x = \frac{1}{2}$
$y = \frac{25}{2}$
$z = \frac{61}{2}$

$$2(\frac{1}{2}) + 3(\frac{25}{2}) - (\frac{61}{2}) = 8$$
$$1 + 37.5 - 30.5 = 8 \checkmark$$
$$(\frac{1}{2}) - 5(\frac{25}{2}) + 2(\frac{61}{2}) = -1$$
$$.5 - 62.5 + 61 = -1 \checkmark$$


#Tic-Tac-Toe
"I think it's called a hashtag."
"How do you play it?"
LanceAF #333 (10-8-18)
mathplane.com
Site of the 1st hashtag.... (or, the origins of tic-tac-toe??!??)

Example: A baseball batter hits a fastball at 100 miles per hour.
The contact occurs 3 feet above home plate at a 15 degree angle.
The fence is 400 feet away and 10 feet high.

Parametric Equations: Projectile Motion

$x = v_0 \cos\theta\, t$

$y = h_0 + v_0 \sin\theta\, t - 16t^2$

where v_0 is initial velocity (feet/second)
θ is angle of contact
t is time (seconds)
h_0 is initial height

a) When is the ball 20 feet above the ground?

This occurs when $y = 20$...

100 miles/1 hour = 528000 feet/3600 seconds
= 146.67 feet/second

$$y = h_0 + v_0 \sin\theta\, t - 16t^2$$

$$20\text{ft} = 3\text{ft} + 146.67 \text{ ft/sec} \cdot \sin(15^\circ)t - 16t^2$$

$$16t^2 - 37.96t + 17 = 0$$

when $t = .60$ and 1.77

b) How high does the ball reach?

We need to find the vertex....
Since the initial height is 3 feet (i.e $y = 3$ when $t = 0$), let's find another point where $y = 3$

$$3 = 3 + 146.67 \cdot \sin(15^\circ)t - 16t^2$$

$$16t^2 - 37.96t = 0$$

$$t = 0 \text{ and } 2.37$$

The midpoint, $t = 1.18$ is when the max height is reached (i.e. vertex)

$$y = h_0 + v_0 \sin\theta\, t - 16t^2$$

$$y = 3 + (146.67)(.2588)(1.18) - 16(1.18)^2$$

$$= 3 + 44.79 - 22.28 \quad = 25.5 \text{ approx.}$$

c) Does the ball clear the fence?

The ball reaches the fence when $x = 400$

$$x = v_0 \cos\theta\, t$$

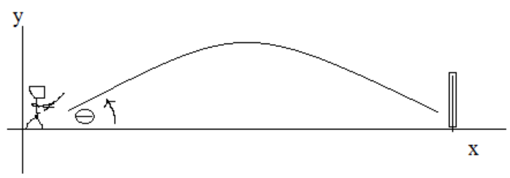
$$400 = 146.67 \cdot \cos(15^\circ)t$$

$$t = \frac{400}{146.67 \cdot .966} = 2.82 \text{ seconds}$$

Then, at $t = 2.82$ seconds,

$$y = 3 + 146.67 \cdot \sin(15^\circ)(2.82) - 16(2.82)^2$$

$$= 3 + 107.0 - 127.2 = -107 < 10 \text{ (foot fence)} \quad \text{NO}$$



d) Suppose the bat's contact occurs at a 25 degree angle.
Would that be a home run (i.e. clear the 10 foot fence) ?

First, find when the ball reaches the fence....

$$x = v_0 \cos\theta\, t$$

$$400 = 146.67 \cdot \cos(25^\circ)t$$

$$t = \frac{400}{146.67(.906)} = 3.01 \text{ seconds}$$

and, the height (y) at $t = 3$ seconds...

$$y = h_0 + v_0 \sin\theta\, t - 16t^2$$

$$y = 3 + 146.67 \cdot \sin(25^\circ)(3) - 16(3)^2$$

$$= 44.96 \text{ feet}$$

YES, it clears the fence!

e) What angle is necessary to clear the fence?

So, 15 degrees is not enough of an angle.. And,
25 degrees easily clears the fence...

To find the height necessary, we must solve for θ

$$x = v_0 \cos\theta\, t \qquad y = h_0 + v_0 \sin\theta\, t - 16t^2$$

$$400 = 146.67 \cdot \cos\theta\, t \qquad 10 = 3 + 146.67 \cdot \sin\theta\, t - 16t^2$$

$$t = \frac{400}{146.67 \cdot \cos\theta}$$

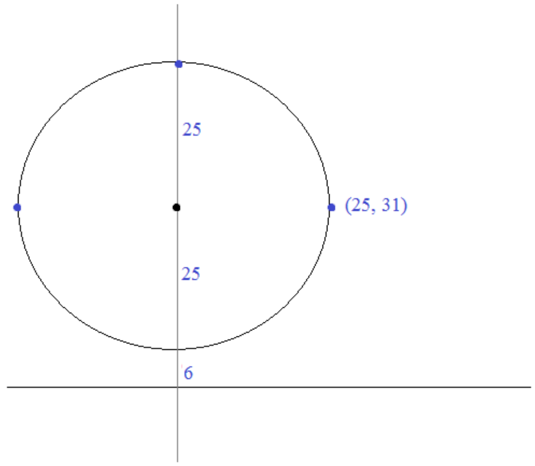
$$7 = 146.67 \cdot \sin\theta \frac{400}{146.67 \cdot \cos\theta} - 16\left(\frac{400}{146.67 \cdot \cos\theta}\right)^2$$

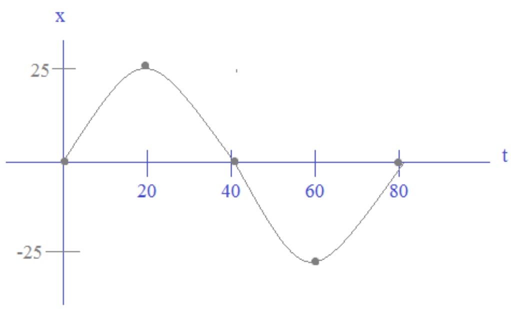
$$7 = 400\tan\theta - \frac{119}{\cos^2\theta} \qquad \theta = 19.38^\circ \text{ (approximately)}$$

Example: Write a parametric model describing a ferris wheel that is 6 feet off the ground, has a span (diameter) of 50 feet, and takes 80 seconds per revolution...

First, let's break up the components:

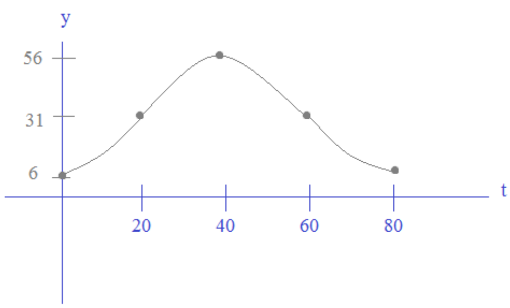
t	(x, y)
t = 0	(0, 6)
t = 20	(25, 31)
t = 40	(0, 56)
t = 60	(-25, 31)
t = 80	(0, 6)





the x-component (oscillates from left to right)

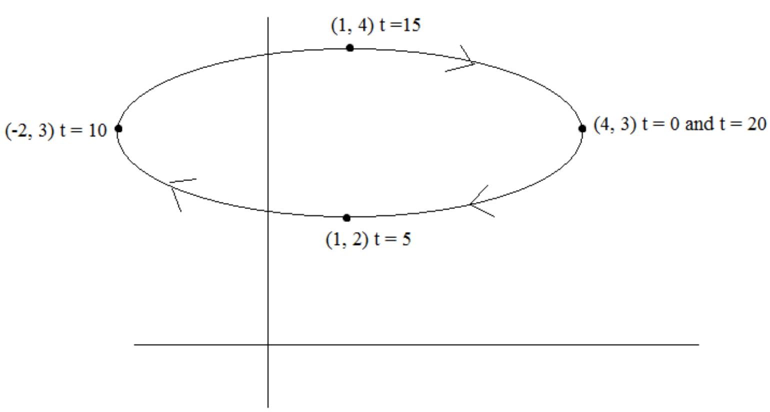
$$x = 25\sin(\frac{\pi}{40})t$$



the y-component (oscillates up and down)

$$y = -25\cos(\frac{\pi}{40})t$$

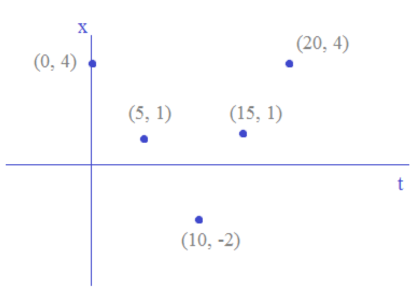
Example: Find the parametric equation:



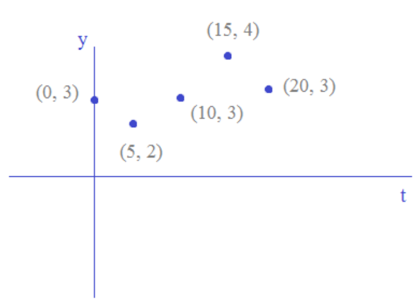
Step 1: Create a table

t	(x, y)
0	(4, 3)
5	(1, 2)
10	(-2, 3)
15	(1, 4)
20	(4. 3)

Step 2: Graph the components



Step 3: Write the equations



$$x = 3\cos(\frac{\pi}{10})t + 1$$

$$y = -\sin(\frac{\pi}{10})t + 3$$

Example: A car's location on a map is modeled by the parametric equation $x = 3 + t$

$y = 4t - 7$

where x is the east/west distance from the town center
y is the north/south distance from the town center

t is time (minutes)

a) After 5 minutes, how far has the car traveled?

Every minute (t), the car travels 1 mile east and 4 miles north...

(using distance formula), the car travels $\sqrt{17}$ miles per minute

after 5 minutes, the car traveled $\boxed{5\sqrt{17}}$

$x = 3 + 1t$
$y = 4t - 7$

rate of change (related to t)
1 east and 4 north

b) When the car is 21 miles north of the town center, how far east is it?

The car is 21 miles north when $y = 21$..
If $y = 21$, then
$21 = 4t - 7$ and $t = 7$ minutes

When $t = 7$, the car's position east is $x = 3 + (7) = \boxed{10 \text{ miles}}$

NOTE: if we remove the parameter,

$y = 4(x - 3) - 7$

$y = 4x - 19$

(10, 21) is a point on that line

Example: Convert the parametric equations into cartesian coordinate system:

$x = 2 + 3\sec t$

$y = 1 + 4\tan t$

then, graph...

$\sec t = \dfrac{x-2}{3}$ $\tan t = \dfrac{y-1}{4}$

recognize the tangent and secant:

trig identity: $1 + \tan^2 x = \sec^2 x$

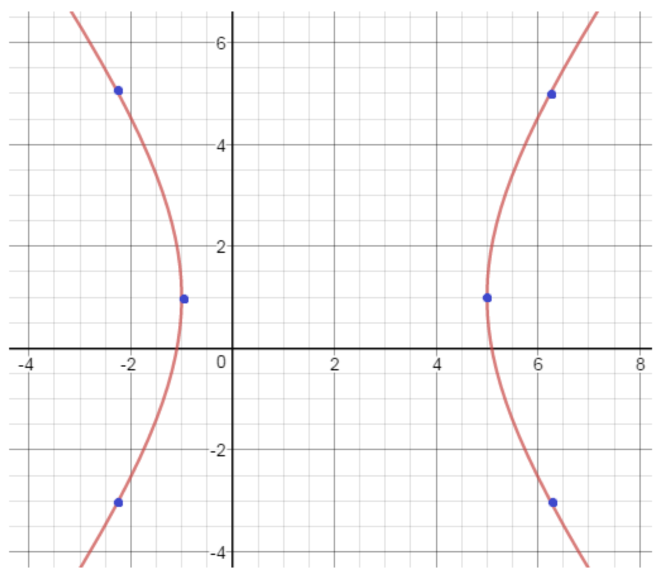
$1 + (\tan t)^2 = (\sec t)^2$

$1 + \left(\dfrac{y-1}{4}\right)^2 = \left(\dfrac{x-2}{3}\right)^2$

$\left(\dfrac{x-2}{3}\right)^2 - \left(\dfrac{y-1}{4}\right)^2 = 1$

Hyperbola!

$\dfrac{(x-2)^2}{9} - \dfrac{(y-1)^2}{16} = 1$

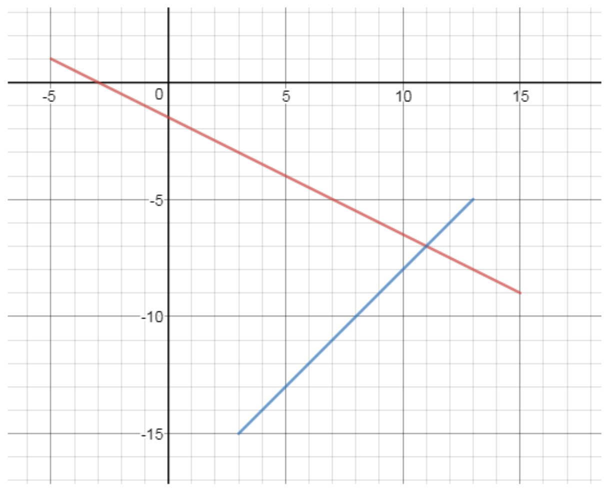


Quick check:	θ	x	y	θ	x	y
	0	5	1	45	6.24	5
	90	undef	undef	135	-2.24	-3
	180	-1	1	225	-2.24	5
	270	undef	undef	315	6.24	-3

Example: The following are 2 parametric equations, describing points on a plane as a function of time (seconds). Where and when do the equations intersect?

$$\begin{cases} x_1 = 2t - 5 \\ y_1 = -t + 1 \end{cases} \qquad \begin{cases} x_2 = t + 3 \\ y_2 = t - 15 \end{cases}$$

parametric graph for $0 < t < 10$



If the curves/lines meet, then $x_1 = x_2$

$$2t - 5 = t + 3$$
$$t = 8$$

So, if they meet, it would occur at $t = 8$

and, at $t = 8$

$x_1 = x_2 = 11$ ✓

$y_1 = y_2 = -7$ ✓

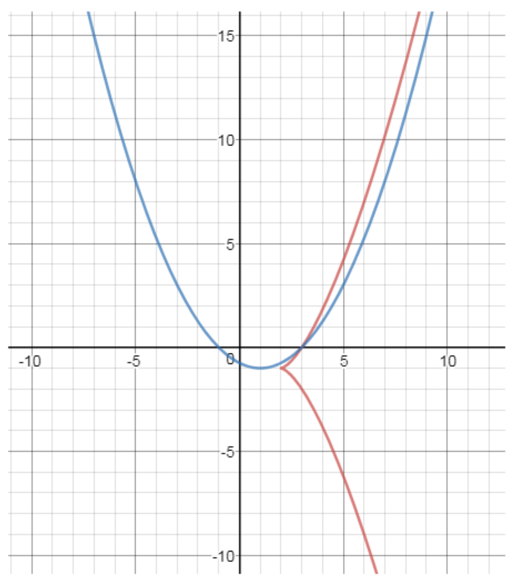
the equations intersect

at $t = 8$ seconds

at (11, -7)

Example: The following are 2 parametric equations, describing points on a plane as a function of time (seconds). Where and when do the equations intersect?

$$\begin{cases} x_1 = t^2 + 2 \\ y_1 = t^3 - 1 \end{cases} \qquad \begin{cases} x_2 = -2t + 1 \\ y_2 = t^2 - 1 \end{cases}$$



If the curves meet, then $x_1 = x_2$ at some time (t)...

$$t^2 + 2 = -2t + 1$$
$$t^2 + 2t + 1 = 0$$
$$(t + 1)(t + 1) = 0$$
$$t = -1$$

If $t = -1$, then $x_1 = x_2 = 3$

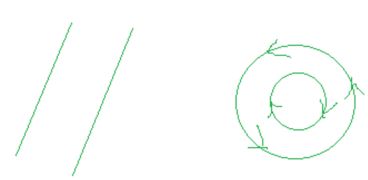
If $t = -1$, then $y_1 = -2$ and $y_2 = 0$

THERE IS NO SOLUTION!!

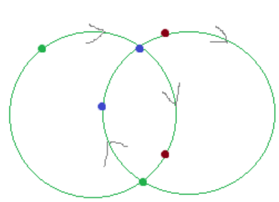
Note: 2 equations may not intersect if they don't overlap....

EX: parallel lines

EX: 2 concentric circles



Also, 2 parametric equations may not intersect, if the points are occupied at different times!



Parametric Conics System

Example: $x_1 = 3\cos t$ $\quad$ $x_2 = -3 + \cos t$

$y_1 = 2\sin t$ $\quad$ $y_2 = 1 + \sin t$

a) Graph the conics, and determine their intersection...

b) If the path of a particle (x_1, y_1) is the first equation, and the path of a particle (x_2, y_2) is the second equation, do the particles collide?

$x_1 = 3\cos t$ $\quad$ $x_2 = -3 + \cos t$

$y_1 = 2\sin t$ $\quad$ $y_2 = 1 + \sin t$

Set the x's and y's equal to each other

$3\cos t = -3 + \cos t$ $\quad$ $2\sin t = 1 + \sin t$

$2\cos t = -3$ $\quad$ $\sin t = 1$

$\cos t = -3/2$ $\quad$ $t = 90^\circ$

no solution

at $t = 90^0$

$x_1 = 3(0)$ $\quad$ $x_2 = -3 + (0)$

$y_1 = 2(1)$ $\quad$ $y_2 = 1 + (1)$

(0, 2) $\quad$ (-3, 2)

particles will never collide...

$x_1 = 3\cos t$

$y_1 = 2\sin t$

at (-3, 0) ----> $3\cos t = -3$ $\quad$ $t = 180^\circ$ (or π)

$2\sin t = 0$

$x_2 = -3 + \cos t$

$y_2 = 1 + \sin t$

if $t = 180^\circ$ (or π) then $-3 + \cos(180^\circ) = -4$

$1 + \sin(180^\circ) = 1$

(-4, 1)

Removing the Parameter to graph

$x_1 = 3\cos t$ $\quad$ $y_1 = 2\sin t$

$\cos t = \frac{x}{3}$ $\quad$ $\sin t = \frac{y}{2}$

$\sin^2 + \cos^2 = 1$

$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$ $\qquad$ $\frac{x^2}{9} + \frac{y^2}{4} = 1$

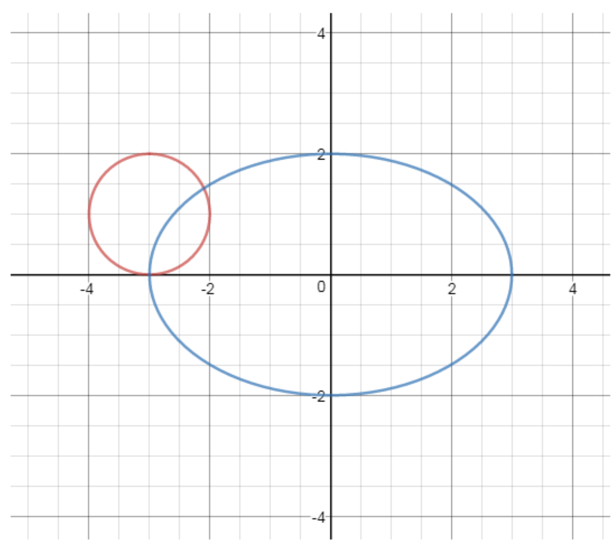
$x_2 = -3 + \cos t$ $\quad$ $y_2 = 1 + \sin t$

$\cos t = x + 3$ $\quad$ $\sin t = y - 1$

$\sin^2 + \cos^2 = 1$ $\qquad$ $(y-1)^2 + (x+3)^2 = 1$

intersection:

(-3, 0) and (-2.1, 1.4)



Parametric Equations Exploration: Domain and Orientation

Compare and contrast the following Parametric Equations:

a) $x = t$, $y = t^2$

b) $x = \sqrt{t}$, $y = t$

c) $x = \sin(t)$, $y = (\sin(t))^2$

d) $x = 3^t$, $y = 3^{2t}$

***NOTE: When you convert the 4 into rectangular coordinate equations, the result is $y = x^2$

But, what about the graphs?!?!?

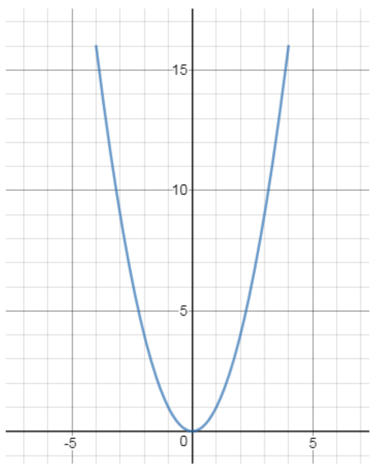
a)

t	x	y
-2	-2	4
-1	-1	1
0	0	0
1	1	1
2	2	4

$x = t$

$y = t^2$

Graph is for
$-4 < t < 4$



Domain: all real numbers

Range: $y \geq 0$

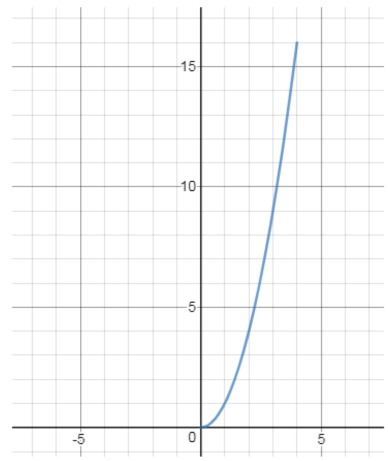
b)

t	x	y
-2	Undefined	-2
-1	Undefined	-1
0	0	0
1	1	1
2	$\sqrt{2}$	2
4	2	4

$x = \sqrt{t}$

$y = t$

Graph is for
$0 < t < 16$



Domain: $x \geq 0$

$y \geq 0$

c)

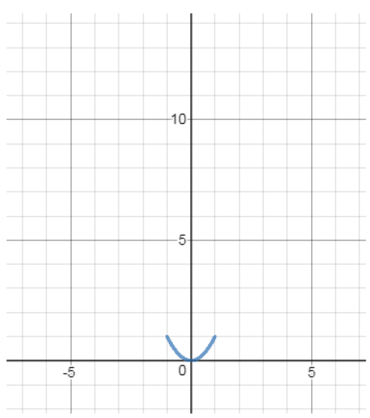
t	x	y
0	0	0
$\frac{\pi}{6}$	1/2	1/4
$\frac{\pi}{2}$	1	1
$\frac{7\pi}{6}$	-1/2	1/4
$\frac{3\pi}{2}$	-1	1

$x = \sin(t)$

$y = (\sin(t))^2$

Graph is for
$-20 < t < 20$

sine must be between
-1 and 1



Domain: $-1 \leq x \leq 1$

Range: $0 \leq x \leq 1$

d)

t	x	y
-3	1/27	1/729
-2	1/9	1/81
-1	1/3	1/9
0	1	1
1	3	9
2	9	81

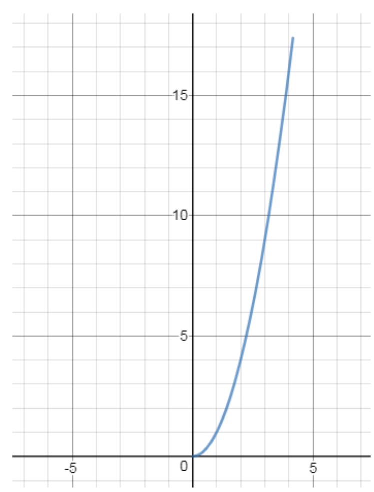
$x = 3^t$

$y = 3^{2t}$

Graph is for
$-20 < t < 1.3$

Note: when $t = 0$,
(x, y) is $(1, 1)$

$(0, 0)$ does not exist



Domain: $x > 0$

Range: $y > 0$

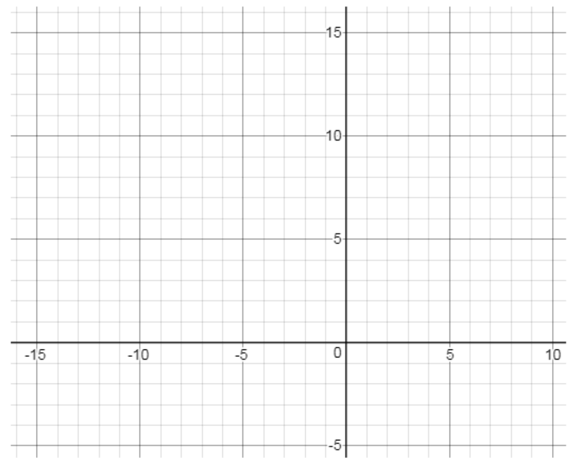
Parametrics Quiz

1) Write 2 different sets of parametric equations for $y = (x-2)^3 + 4$

2) Convert the parametric equations (into rectangular form) and graph:

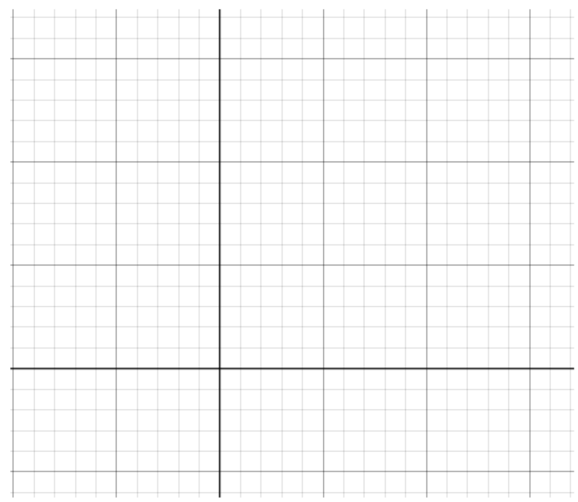
$x = -4 + 3\tan t$

$y = 7 - 2\sec t$



3) Convert and graph the following $x = t^2 + 2$

$y = 4t^2 - 3$



4) An ellipse has vertices at (5, 0) and (-5, 0)
and foci at (3, 0) and (-3, 0)...

What is the equation of the ellipse in parametric form?

Parametrics Quiz

5) Write the symmetric and parametric equations of the line going through (3, -1, 6) and (5, 4, 3)

6) Convert the following parametric equation into rectangular form:

$$\begin{cases} x = 4 + 3\cos(2t) \\ y = 4 + 6\sin(t)\cos(t) \end{cases}$$

7) An enemy cannon mounted 2 meters off the ground, aimed at an angle of elevation of 22 degrees, fires a cannonball at 142 meters/second at a castle 1200 meters away.

If the castle wall is 70 meters high, will the cannonball strike <u>inside</u> the castle?!?!

The parametric model of a projectile is

$$\begin{cases} x = v_0 \cos\theta\, t \\ y = -4.9t^2 + v_0 \sin\theta\, t + h_0 \end{cases}$$

t is time in seconds

h_0 is the height in <u>meters</u> (when launched)

(-16 instead of -4.9, if using feet)

Parametrics Quiz

8) A quarterback throws a pass to a receiver 90 feet away. If he releases the pass 7 feet above the ground, at an angle of 35 degrees,

a) what velocity does he throw the pass?

b) when does the receiver catch the pass? (assume the reception is 4 feet above the ground.)

9) In my backyard, I threw a ball. Its path can be modeled by the parametric equation $x = 2 + s$, $y = 6 + 3s$ for $s \geq 0$

At the same moment, my dog ran in a direction modeled by the parametric equation $x = t - 1$, $y = 2t + 2$ for $t \geq 0$

Sketch the paths of the ball and the dog...

Does that dog catch the ball? Explain...

Parametrics Quiz

10) What is the equation (in rectangular form)?

$x = 4\sec\theta$

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11) Find the intersection of the following:

particle 1: $x = 1 - 3t$
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$y = s$

12) Oscillanting points

a) A point is oscillating between (3, -1) and (5, 11),
starting at the midpoint and traveling to (5, 11) first...
Write a parametric model showing the path....

b) Write a parametric equation of point oscillating (in a straight line) between (-1, 1) and (3, 7)

c) Write a parametric equation of a point oscillating (in a straight line) between (3, 7) and (9, −1)
that starts at (3, 7)

Parametrics Quiz

mathplane.com

SOLUTIONS

1) Write 2 different sets of parametric equations for $y = (x-2)^3 + 4$

1) $y = t$

$t = (x-2)^3 + 4$

$x - 2 = \sqrt[3]{t-4}$

$x = \sqrt[3]{t-4} + 2$

2) $x = t$

$y = (t-2)^3 + 4$

2) Convert the parametric equations (into rectangular form) and graph:

$x = -4 + 3\tan t$

$y = 7 - 2\sec t$

$x + 4 = 3\tan t$ $\qquad$ $y - 7 = -2\sec t$

$\frac{(x+4)}{3} = \tan t$ $\qquad$ $\frac{(y-7)}{-2} = \sec t$

recognize the Pythagorean Trig Identity

$1 + \tan^2 x = \sec^2 x$

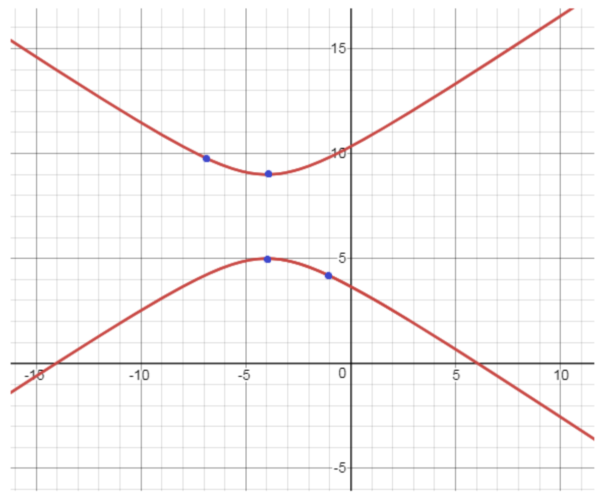
$$1 + \left(\frac{(x+4)}{3}\right)^2 = \left(\frac{(y-7)}{-2}\right)^2$$

$$1 = \left(\frac{(y-7)}{-2}\right)^2 - \left(\frac{(x+4)}{3}\right)^2$$

Hyperbola!!

points on the graph

t	x	y
0	-4	5
$\pi/4$	-1	4.17
$\pi/2$	undef	undef
$3\pi/4$	-7	9.82
π	-4	9



3) Convert and graph the following $x = t^2 + 2$

$y = 4t^2 - 3$

t	-2	-1	0	1	2	3
x	6	3	2	3	6	11
y	13	1	-3	1	13	33

Remove the parameter (t)

$t^2 = x - 2$

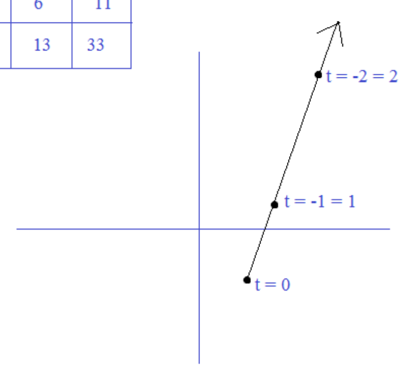
then, substitute into the 2nd equation...

$y = 4(x-2) - 3$

$y = 4x - 11$

***Now, we must see if there is a domain restriction!

$x \geq 2$



4) An ellipse has vertices at (5, 0) and (-5, 0)
and foci at (3, 0) and (-3, 0)...

What is the equation of the ellipse in parametric form?

Using properties of ellipses, we can conclude the standard form of equation is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

since $a^2 - b^2 = c^2$

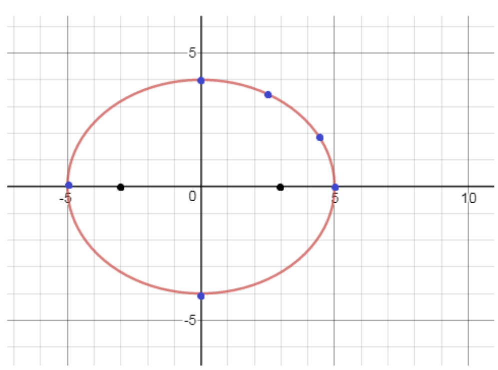
$5^2 - b^2 = 3^2$

$b = 4$

$\cos^2 t + \sin^2 t = 1$

$\cos t = \frac{x}{5}$ $\qquad$ $\sin t = \frac{y}{4}$

$x = 5\cos t$ $\qquad$ $y = 4\sin t$



points on the graph

t	0	$\pi/6$	$\pi/3$	$\pi/2$	π	$3\pi/2$
x	5	4.33	2.5	0	-5	0
y	0	2	3.46	4	0	-4

Parametrics Quiz

SOLUTIONS

5) Write the symmetric and parametric equations of the line going through (3, -1, 6) and (5, 4, 3)

slope of line (i.e. the vector direction)

$\langle 2, 5, -3 \rangle$

parametric equation using (5, 4, 3)

$$\begin{cases} x = 5 + 2t \\ y = 4 + 5t \\ z = 3 - 3t \end{cases}$$

To change to symmetric, set each equation equal to t

$$t = \frac{x-5}{2} \qquad t = \frac{y-4}{5} \qquad t = \frac{z-3}{-3}$$

then, combine

$$\frac{x-5}{2} = \frac{y-4}{5} = \frac{z-3}{-3}$$

Note: to convert from symmetric to parametric, set each part equal to t..
Then, solve for each x, y, z variable...

6) Convert the following parametric equation into rectangular form:

$$\begin{cases} x = 4 + 3\cos(2t) \\ y = 4 + 6\sin(t)\cos(t) \end{cases}$$

$\sin 2x = 2\sin x\cos x$ (trig identity)

$x = 4 + 3\cos(2t)$

$y = 4 + 3\sin(2t)$

$$\cos(2t) = \frac{x-4}{3}$$

$$\sin(2t) = \frac{y-4}{3}$$

$\sin^2 + \cos^2 = 1$ (trig identity)

$$\left(\frac{y-4}{3}\right)^2 + \left(\frac{x-4}{3}\right)^2 = 1$$

$$(x-4)^2 + (y-4)^2 = 9$$ circle

7) An enemy cannon mounted 2 meters off the ground, aimed at an angle of elevation of 22 degrees, fires a cannonball at 142 meters/second at a castle 1200 meters away.

If the castle wall is 70 meters high, will the cannonball strike inside the castle?!?!

The parametric model of a projectile is

$$\begin{cases} x = v_0 \cos\theta\, t \\ y = -4.9t^2 + v_0 \sin\theta\, t + h_0 \end{cases}$$

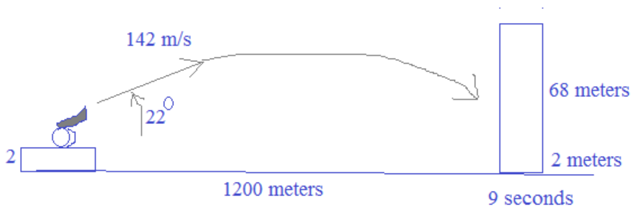
t is time in seconds

h_0 is the height in meters (when launched)

(-16 instead of -4.9, if using feet)

Step 1: Sketch a diagram

Step 2: Find when ball reaches the wall.... i.e. if x = 1200, then what is t?



$x = v_0 \cos t$

$1200 = 142\cos(22)t$

$t = 9.11$ seconds

Step 3: Find the height when cannon ball reaches the wall.... i.e. when t = 9.11, what is height?

$$y = -4.9(9.11)^2 + 142\sin(22)(9.11) + 2 = 79.9 \text{ meters...}$$

so, yes, it will clear the 70 meter wall!!!

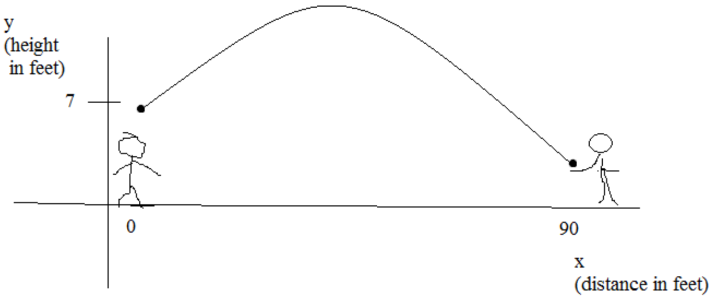
Parametrics Quiz

SOLUTIONS

8) A quarterback throws a pass to a receiver 90 feet away. If he releases the pass 7 feet above the ground, at an angle of 35 degrees,

a) what velocity does he throw the pass?

b) when does the receiver catch the pass? (assume the reception is 4 feet above the ground.)



The ball is released at a height of 7 feet...

at $t = 0$, $y = 7$ and $x = 0$

Since it is released at an angle of 35 degrees,

the y-component is $\sin(35^{\circ}) = .57$

x-component is $\cos(35^{\circ}) = .82$

Therefore, the path of the football is modeled by

$$x = .82V_0t$$

$$y = .57V_0t - 16t^2 + 7$$

t is time

V_0 is the initial velocity

$-16t^2$ is the downward force/acceleration (due to gravity)

Since we know the ball is caught when $x = 90$ and $y = 4$,

$$90 = .82V_0\, t$$

$$4 = .57V_0\, t - 16t^2 + 7$$

after putting this system into a calculator,

(approximately) $V_0 = 54.22$ ft/sec $t = 2.02$ seconds

9) In my backyard, I threw a ball. Its path can be modeled by the parametric equation $x = 2 + s$, $y = 6 + 3s$ for $s \geq 0$

At the same moment, my dog ran in a direction modeled by the parametric equation $x = t - 1$, $y = 2t + 2$ for $t \geq 0$

Sketch the paths of the ball and the dog...

Does that dog catch the ball? Explain...

Although the paths intersect, the dog and ball point of intersection *occur at different times*!

set the x's equal to each other... $t - 1 = 2 + s$ $t - s = 3$

set the y's equal to each other... $2t + 2 = 6 + 3s$ $2t - 3s = 4$

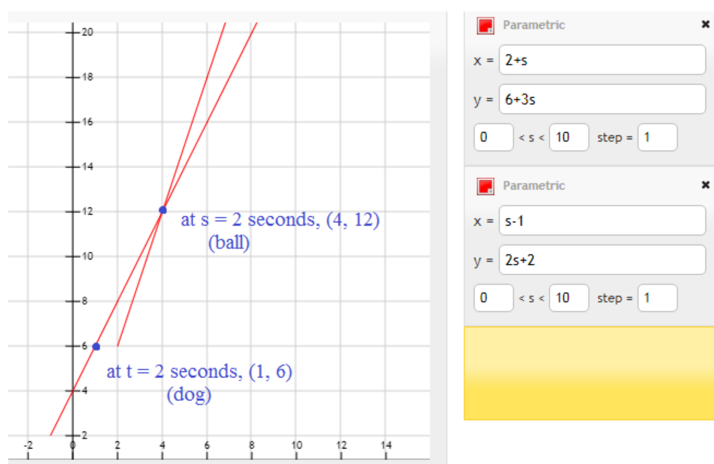
solve the system... $s = 2$ $t = 5$

and, when $s = 2$ and $t = 5$,

$x = 4$ and $y = 12$

So, their PATHS intersect at (4, 12) in the backyard..

HOWEVER, the dog gets there in 5 seconds and the ball gets there in 2 seconds...



Parametrics Quiz

SOLUTIONS

10) What is the equation (in rectangular form)?

$x = 4\sec\theta$

$y = 2\tan\theta$

$\sec\theta = \frac{x}{4}$

$\tan\theta = \frac{y}{2}$

since $1 + \tan^2 = \sec^2$

$1 + \left(\frac{y}{2}\right)^2 = \left(\frac{x}{4}\right)^2$

$$\frac{x^2}{16} - \frac{y^2}{4} = 1$$

Hyperbola!!

11) Find the intersection of the following:

particle 1: $x = 1 - 3t$
$y = -3 + 7t$

particle 2: $x = -1 + 5s$
$y = s$

Set x's equal to each other

$1 - 3t = -1 + 5s$

$5s + 3t = 2$

Set y's equal to each other

$s = -3 + 7t$

$7t - s = 3$

Solve system: $5s + 3t = 2$
$35t - 5s = 15$

$38t = 17$

$t = 17/38$

$s = 5/38$

plug in s and t to get the x and y

(-13/38, 5/38)

12) Oscillanting points

a) A point is oscillating between (3, -1) and (5, 11), starting at the midpoint and traveling to (5, 11) first... Write a parametric model showing the path....

starts at (4, 5) midpoint

slope is 6, where y moves 12 as x moves 2...

$x = 2t + 4 \quad y = 12t + 5$

But, how do we model the oscillation!!?!??

One period is an x movement of 24 (12 one way, 12 back) and, one period is a y movement of 4 (2 one way, 2 back)

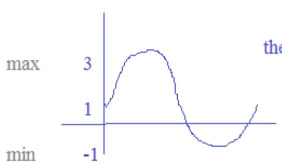
$x = 4 + \sin(\frac{\pi}{2} t)$

$y = 5 + 6\sin(\frac{\pi}{2} t)$

a periodic function...

b) Write a parametric equation of point oscillating (in a straight line) between (-1, 1) and (3, 7)

Since the point is oscillating, we want to use a periodic equation ----> sine or cosine



the x part

the y part..

one possible answer:

$x = 1 + 2\sin(t)$

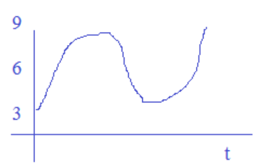
$y = 4 + 3\sin(t)$

c) Write a parametric equation of a point oscillating (in a straight line) between (3, 7) and (9, −1) that starts at (3, 7)

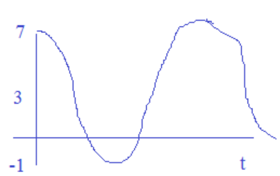
Since we're starting at the endpoint, it is easier to use cosine graph....

$x = 6 - 3\cos t$

$y = 3 + 4\cos t$



the x part

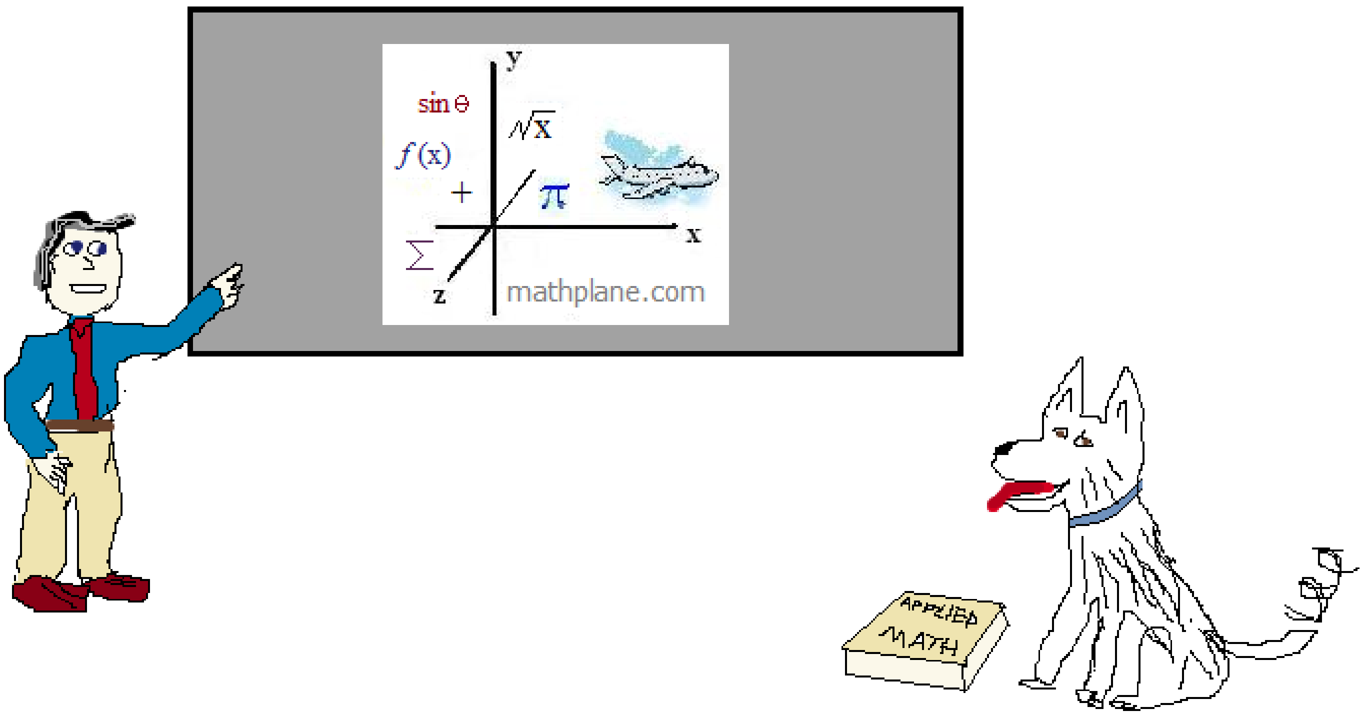


the y part

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or requests, let us know.

Cheers



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